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Two-Step Modulus-Based Matrix Splitting Iteration Method for Horizontal Linear Complementarity Problems

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Abstract. The modulus-based matrix splitting iteration has received substantial attention as a momentous tool for complementarity problems. For the purpose of solving the horizontal linear complementarity problem, we introduce the two-step modulus-based matrix splitting iteration method. We also show the theoretical analysis of the convergence. Numerical experiments illustrate the effectiveness of the proposed approach.

1. Introduction

The horizontal linear complementarity problem, which is a generalization of linear complementarity problem [1], is to find a pair of vectors (z, w) such that

$$Az - Bw = q,\ z \geq 0,\ w \geq 0,\ z^T w = 0,$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times n}$ are known matrices and $q \in \mathbb{R}^n$ is the known vector. This problem is abbreviated as HLCP(A, B, q), which has many applications in noncooperative games, optimization problems, economic equilibrium problems, and traffic assignment problems [2].

The HLCP is one type of significant complementarity problems. Many scholars have studied this problem. There are several strategies to solve it such as interior point approaches [3–5], neural networks [6], homotopy [7] and verification [8] methods. In [9], Mezzadri and Galligani introduced a class of projected splitting methods. Recently, modulus-based matrix splitting is a popular method, which is applied on many kinds of complementarity problems, especially the linear complementarity problem and nonlinear complementarity problem. In 2019, Mezzadri and Galligani presented the modulus-based matrix splitting method to solve the HLCP [10]. Furthermore, the convergence of the methods proposed in [10] has been enlarged by Zheng and Vong[11].

When the matrix B is a unit matrix, the horizontal linear complementarity problem reduces to the standard linear complementarity problem, for which numerous strategies exist. When B is nonsingular, the

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HLCP(A, B, q) is equivalent to the LCP ($B^{-1}A$, $B^{-1}q$) [12]. In 2010, Bai first proposed the modulus-based matrix splitting iteration method by reformulating the LCP into the corresponding modulus equation [13]. Since then, many scholars devised other different modified modulus-based matrix splitting iteration methods, such as general methods [14–17], accelerated methods [18], preconditioned [19, 20] and multisplitting methods [21–25]. Zheng et al. [26] introduced its relaxation variant which generalized the modulus-based matrix splitting iteration method. Based on two matrix splittings of the system matrix, Zhang [27] presented the two-step modulus-based matrix splitting iteration method to improve the rate of convergence. Bai and Zhang [28] proposed modulus-based multigrid methods to solve the LCP via employing modulus-based matrix splitting iteration methods as smoothers.

In this paper, for the sake of improving the iteration rate of the modulus-based matrix splitting, we propose the two-step modulus-based matrix splitting to solve the HLCP(A, B, q). We can acquire a series of its relaxation methods via choosing different matrix splittings. The convergence analysis is given when system matrices are either positive definite matrices or H_+-matrices, respectively. Numerical experiments depict that our proposed method outperforms some existing methods.

The remanent sections are organized as follows. We show several essential lemmas and symbolic representations in Section 2. We give our proposed method, which is the two-step modulus-based matrix splitting, in Section 3. We demonstrate the convergent theorems when the matrices A and B are positive definite matrices and H_+-matrices in Section 4. Numerical examples are given to illustrate the efficiency of our method in section 5. The last section gives a short conclusion.

2. Preliminary results

In the subsequent section, we review several essential notations and indispensable lemmas.

Denote $M = (m_{ij})$, then we show several special matrices as follows:

• The matrix M is called a Z-matrix iff $m_{ij} \le 0$ for any $i \ne j$.
• The Z-matrix M is called an M-matrix iff $M^{-1} \ge 0$.
• The matrix M is called an H-matrix iff its comparison matrix $\langle M\rangle = \langle m_{ij}\rangle$ is an M-matrix, where

$$\langle m_{ij}\rangle = \begin{cases} |m_{ij}|, \; for\; i = j, \\ -|m_{ij}|, \; for\; i \ne j, \end{cases} \quad i, j = 1, 2, \cdots, n.$$

• The matrix M is called an H_+-matrix iff M is an H-matrix with the diagonal elements being positive; see [29].

D_M and $-B_M$ means the diagonal and off-diagonal matrices of the matrix M. Denote the spectral radius of the matrix M by $\rho(M)$. $M = E - F$ is an H-compatible splitting iff it holds that $\langle A\rangle = \langle E\rangle - |F|$. For any two vectors $\mathbf{x} = (x_1, x_2, \cdots, x_n)^T$ and $\mathbf{y} = (y_1, y_2, \cdots, y_n)^T$, we denote $\mathbf{x} \ge \mathbf{y}$ ($\mathbf{x} > \mathbf{y}$) provided that the corresponding elements satisfy $x_i \ge y_i$ ($x_i > y_i$). $|\mathbf{x}|$ means the absolute value of the vector $\mathbf{x}$, and $\mathbf{x}^T$ is its transpose. The notations of matrix is the similar to the aforementioned.

Lemma 2.1. *[10] Let A, $B \in \mathbb{R}^{n\times n}$, let $q \in \mathbb{R}^n$ and let $A = M_A - N_A$ and $B = M_B - N_B$ be splittings of A and of B, respectively. Furthermore, let Γ, Ω be two positive diagonal matrices of order n.*
(i) If (z, w) is a solution of the HLCP(A, B, q) in (1), then $x = \frac{1}{2}(\Gamma^{-1}z - \Omega^{-1}w)$ satisfies

$$(M_A\Gamma + M_B\Omega)x = (N_A\Gamma + N_B\Omega)x + (B\Omega - A\Gamma)|x| - q. \tag{1}$$

(ii) If x satisfies (1), then

$$z = \Gamma(|x| + x)\, and\, w = \Omega(|x| - x) \tag{2}$$

is a solution of the HLCP(A, B, q).

Lemma 2.2. *[30] Let $A, B \in \mathbb{R}^{n\times n}$ be Hermitian and let the eigenvalues $\lambda_l(A)$, $\lambda_l(B)$, and $\lambda_l(A + B)$ be arranged in increasing order, i.e., $\lambda_{\min} \le \lambda_1 \le \lambda_2 \le \cdots \le \lambda_{n-1} \le \lambda_n = \lambda_{\max}$. For each $k = 1, 2, \cdots, n$, we have*

$$\lambda_k(A) + \lambda_1(B) \le \lambda_k(A + B) \le \lambda_k(A) + \lambda_n(B).$$

Lemma 2.3. *[31] Let $B \in \mathbb{R}^{n\times n}$ be a strictly diagonal dominant matrix. Then*

$$\|B^{-1}C\| \leq \max_{1\leq i\leq n} \frac{(|C|e)_i}{(\langle B\rangle e)_i}$$

holds for arbitrary matrix $C \in \mathbb{R}^{n\times n}$, where $e = (1, 1, \cdots, 1)^T$.

Lemma 2.4. *[32] Let A be an H-matrix, then $|A^{-1}| \leq \langle A\rangle^{-1}$.*

Lemma 2.5. *[33] For a nonnegative matrix $A \in \mathbb{R}^{n\times n}$, if there exists a positive vector $y \in \mathbb{R}^n$ such that $Ay < y$, then $\rho(A) < 1$.*

3. The two-step modulus-based matrix splitting

In the subsequent section, we will consider the scheme of the two-step modulus-based matrix splitting iteration method. First, we review the MMS iteration method, see [10] for details.

Method 3.1. [10] (The modulus-based matrix splitting iteration method for the HLCP(A, B, q))(MMS)
Let A, $B \in \mathbb{R}^{n\times n}$ and x, $q \in \mathbb{R}^n$. Starting from an initial guess $x^{(0)} \in \mathbb{R}^n$, let the $(k+1)$-th iterate $x^{(k+1)}$ be the solution of the linear system

$$(M_A + M_B\Omega)x^{(k+1)} = (N_A + N_B\Omega)x^{(k)} + (B\Omega - A)|x^{(k)}| + \gamma q, \tag{3}$$

with γ positive constant, Ω positive diagonal matrix of order n, $A = M_A - N_A$ and $B = M_B - N_B$. The complementarity vectors at each iteration of methods have, then, the form

$$z^{(k+1)} = \frac{1}{\gamma}(|x^{(k+1)}| + x^{(k+1)}), \;\; w^{(k+1)} = \frac{1}{\gamma}\Omega(|x^{(k+1)}| - x^{(k+1)}).$$

For the sake of achieving higher computation efficiency via making full use of the information contained in the matrices A and B, we present the two-step modulus-based matrix splitting iteration method to solve the HLCP.

Method 3.2. *(The two-step modulus-based matrix splitting iteration method for the HLCP(A, B, q))(TMMS)*
Let $A = M_A^{'} - N_A^{'} = M_A^{''} - N_A^{''}$ and $B = M_B^{'} - N_B^{'} = M_B^{''} - N_B^{''}$.

Step 1. *Given $\varepsilon > 0$. Choose an initial vector $x^{(0)} \in \mathbb{R}^n$ and set $k := 0$;*

Step 2. *For $k = 0, 1, 2, \cdots$, calculate $x^{(k+1)}$ by solving the linear systems:*

$$\begin{cases} (M_A^{'} + M_B^{'}\Omega)x^{(k+\frac{1}{2})} = (N_A^{'} + N_B^{'}\Omega)x^{(k)} + (B\Omega - A)|x^{(k)}| + \gamma q, \\ (M_A^{''} + M_B^{''}\Omega)x^{(k+1)} = (N_A^{''} + N_B^{''}\Omega)x^{(k+\frac{1}{2})} + (B\Omega - A)|x^{(k+\frac{1}{2})}| + \gamma q. \end{cases} \tag{4}$$

Step 3. *Set $z^{(k+1)} = \frac{1}{\gamma}(|x^{(k+1)}| + x^{(k+1)})$ and $w^{(k+1)} = \frac{1}{\gamma}\Omega(|x^{(k+1)}| - x^{(k+1)})$.*

Step 4. *If $RES = \min\left(z^{(k+1)}, w^{(k+1)}\right) < \varepsilon$, then terminate. Otherwise, set $k := k + 1$ and return to Step 2.*

Remark 3.3. *With the different matrix splittings of the matrices A and B, Method 3.2 yields a series of two-step modulus-based relaxation methods. Consider the following splittings.*

$$M_A^{'} = \frac{1}{\alpha}(D_A - \beta L_A), \; N_A^{'} = \frac{1}{\alpha}((1-\alpha)D_A + (\alpha-\beta)L_A + \alpha U_A),$$

$$M_A^{''} = \frac{1}{\alpha}(D_A - \beta U_A), \; N_A^{''} = \frac{1}{\alpha}((1-\alpha)D_A + (\alpha-\beta)U_A + \alpha L_A),$$

$$M_B^{'} = \frac{1}{\alpha}(D_B - \beta L_B),\ N_B^{'} = \frac{1}{\alpha}((1-\alpha)D_B + (\alpha-\beta)L_B + \alpha U_B),$$
$$M_B^{''} = \frac{1}{\alpha}(D_B - \beta U_B),\ N_B^{''} = \frac{1}{\alpha}((1-\alpha)D_B + (\alpha-\beta)U_B + \alpha L_B),$$

Method 3.2 reduces to the two-step modulus-based AOR (TMAOR) iteration method.

$$\begin{cases} x^{(k+\frac{1}{2})} =(D_B\Omega + D_A - \beta(L_B\Omega + L_A))^{-1}[((1-\alpha)(D_A + D_B\Omega) \\ \qquad + (\alpha-\beta)(L_A + L_B\Omega) + \alpha(U_A + U_B\Omega))x^{(k)} + \alpha(B\Omega - A)|x^{(k)}| + \gamma\alpha q], \\ x^{(k+1)} =(D_B\Omega + D_A - \beta(U_B\Omega + U_A))^{-1}[((1-\alpha)(D_A + D_B\Omega) \\ \qquad + (\alpha-\beta)(U_A + U_B\Omega) + \alpha(L_A + L_B\Omega))x^{(k+\frac{1}{2})} + \alpha(B\Omega - A)|x^{(k+\frac{1}{2})}| + \gamma\alpha q]. \end{cases} \tag{5}$$

Then, we have the two-step modulus-based SOR (TMSOR) iteration method ($\alpha = \beta$), the two-step modulus-based Gauss-Seidel (TMGS) iteration method ($\alpha = \beta = 1$) and the two-step modulus-based Jacobi (TMJ) iteration method ($\alpha = 1, \beta = 0$).

4. Convergence theorems

In this section, we will give the convergence analysis when the matrices A and B are positive definite matrices or H_+-matrices.

4.1. A and B are positive definite matrices

Theorem 4.1. *Let A and B be positive definite matrices. Suppose $A = M_A^{'} - N_A^{'} = M_A^{''} - N_A^{''}$ and $B = M_B^{'} - N_B^{'} = M_B^{''} - N_B^{''}$ are two splittings of the matrices A, B, respectively. Suppose γ is a positive parameter and Ω is a diagonal matrix with positive elements. Suppose the matrices $M_A^{'}$, $M_A^{''}$, $M_B^{'}\Omega$ and $M_B^{'}\Omega$ are positive definite. Set*

$$\zeta_1^{'}(\Omega) = \|(M_A^{'} + M_B^{'}\Omega)^{-1}N_A^{'}\|,\ \zeta_1^{''}(\Omega) = \|(M_A^{''} + M_B^{''}\Omega)^{-1}N_A^{''}\|,$$
$$\zeta_2^{'}(\Omega) = \|(M_A^{'} + M_B^{'}\Omega)^{-1}N_B^{'}\Omega\|,\ \zeta_2^{''}(\Omega) = \|(M_A^{''} + M_B^{''}\Omega)^{-1}N_B^{''}\Omega\|,$$
$$\zeta_3^{'}(\Omega) = \|(M_A^{'} + M_B^{'}\Omega)^{-1}(M_B^{'}\Omega - M_A^{'})\|,\ \zeta_3^{''}(\Omega) = \|(M_A^{''} + M_B^{''}\Omega)^{-1}(M_B^{''}\Omega - M_A^{''})\|,$$

then the iteration sequence $\{z^{(k)}\}_{k=0}^{+\infty} \subset \mathbb{R}^n$ generated by Method 3.2 converges to the unique solution $z^* \in \mathbb{R}^n$ of the HLCP(A, B, q) for any initial vector $x^{(0)} \in \mathbb{R}^n$ provided that Ω satisfies

$$(2\zeta_1^{'}(\Omega) + 2\zeta_2^{'}(\Omega) + \zeta_3^{'}(\Omega))(2\zeta_1^{''}(\Omega) + 2\zeta_2^{''}(\Omega) + \zeta_3^{''}(\Omega)) < 1. \tag{6}$$

Proof. Assume that x^* is the exact solution, which satisfies the following systems:

$$\begin{cases} (M_A^{'} + M_B^{'}\Omega)x^* = (N_A^{'} + N_B^{'}\Omega)x^* + (B\Omega - A)|x^*| + \gamma q, \\ (M_A^{''} + M_B^{''}\Omega)x^* = (N_A^{''} + N_B^{''}\Omega)x^* + (B\Omega - A)|x^*| + \gamma q, \end{cases} \tag{7}$$

Subtracting (7) from (4), we can get

$$\begin{cases} (M_A^{'} + M_B^{'}\Omega)(x^{(k+\frac{1}{2})} - x^*) = (N_A^{'} + N_B^{'}\Omega)(x^{(k)} - x^*) \\ \qquad\qquad + (B\Omega - A)(|x^{(k)}| - |x^*|), \\ (M_A^{''} + M_B^{''}\Omega)(x^{(k+1)} - x^*) = (N_A^{''} + N_B^{''}\Omega)(x^{(k+\frac{1}{2})} - x^*) \\ \qquad\qquad + (B\Omega - A)(|x^{(k+\frac{1}{2})}| - |x^*|). \end{cases} \tag{8}$$

It is obvious that $M_A^{'} + M_B^{'}\Omega$ and $M_A^{''} + M_B^{''}\Omega$ are nonsingular. Write the norm of the error (8) as

$$\begin{cases} \|x^{(k+\frac{1}{2})} - x^*\| = \|(M_A^{'} + M_B^{'}\Omega)^{-1}[(N_A^{'} + N_B^{'}\Omega)(x^{(k)} - x^*) \\ \qquad\qquad + (B\Omega - A)|x^{(k)}| - |x^*|]\|, \\ \|x^{(k+1)} - x^*\| = \|(M_A^{''} + M_B^{''}\Omega)^{-1}[(N_A^{''} + N_B^{''}\Omega)(x^{(k+\frac{1}{2})} - x^*) \\ \qquad\qquad + (B\Omega - A)|x^{(k+\frac{1}{2})}| - |x^*|]\|. \end{cases}$$

According to the properties of norm and $\||x^{(k)}| - |x^*|\| \le \|x^{(k)} - x^*\|$, the error can be estimated as

$$\begin{cases} \|x^{(k+\frac{1}{2})} - x^*\| \le (2\|(M_A^{'} + M_B^{'}\Omega)^{-1}N_A^{'}\| + 2\|(M_A^{'} + M_B^{'}\Omega)^{-1}N_B^{'}\Omega\| \\ \qquad\qquad + \|(M_A^{'} + M_B^{'}\Omega)^{-1}(M_B^{'}\Omega - M_A^{'})\|)\|x^{(k)} - x^*\|, \\ \|x^{(k+1)} - x^*\| \le (2\|(M_A^{''} + M_B^{''}\Omega)^{-1}N_A^{''}\| + 2\|(M_A^{''} + M_B^{''}\Omega)^{-1}N_B^{''}\Omega\| \\ \qquad\qquad + \|(M_A^{''} + M_B^{''}\Omega)^{-1}(M_B^{''}\Omega - M_A^{''})\|)\|x^{(k+\frac{1}{2})} - x^*\|, \end{cases}$$

which means

$$\|x^{(k+1)} - x^*\| \le (2\zeta_1^{'}(\Omega) + 2\zeta_2^{'}(\Omega) + \zeta_3^{'}(\Omega))(2\zeta_1^{''}(\Omega) + 2\zeta_2^{''}(\Omega) + \zeta_3^{''}(\Omega))\|x^{(k)} - x^*\|.$$

Hence, $(2\zeta_1^{'}(\Omega)+2\zeta_2^{'}(\Omega)+\zeta_3^{'}(\Omega))(2\zeta_1^{''}(\Omega)+2\zeta_2^{''}(\Omega)+\zeta_3^{''}(\Omega)) < 1$ implies $\lim_{k\to\infty} x^{(k)} = x^*$, completing the proof. □

In particular, when $\Omega = \omega I \in \mathbb{R}^{n\times n}$ is a positive scalar matrix and $M_A^{'}$, $M_A^{''}$, $M_B^{'}$ and $M_B^{''}$ are symmetric positive definite matrices, Theorem 4.1 results in the subsequent convergent conditions.

Corollary 4.2. *Let A and B be positive definite matrices. Suppose γ is a positive parameter and $\Omega = \omega I$ with $\omega > 0$. Assume $A = M_A^{'} - N_A^{'} = M_A^{''} - N_A^{''}$ and $B = M_B^{'} - N_B^{'} = M_B^{''} - N_B^{''}$ are two splittings of the matrices A, B, respectively. $M_A^{'}$, $M_A^{''}$, $M_B^{'}$and $M_B^{''}$ are symmetric. Suppose the matrices $M_A^{'}$, $M_A^{''}$, $M_B^{'}\Omega$ and $M_B^{''}\Omega$ are positive definite. Denote the maximum eigenvalues of the matrices $M_A^{'}(M_A^{''})$ and $M_B^{'}(M_B^{''})$ as $\lambda_{\max}^{'}(\lambda_{\max}^{''})$ and $\mu_{\max}^{'}(\mu_{\max}^{''})$. Denote the minimum eigenvalues of the matrices $M_A^{'}$ $(M_A^{''})$ and $M_B^{'}$ $(M_B^{''})$ as $\lambda_{\min}^{'}(\lambda_{\min}^{''})$ and $\mu_{\min}^{'}(\mu_{\min}^{''})$. Define $\tau_A^{'} = \|M_A^{'-1}N_A^{'}\|$, $\tau_A^{''} = \|M_A^{''-1}N_A^{''}\|$, $\tau_B^{'} = \|M_B^{'-1}N_B^{'}\|$ and $\tau_B^{''} = \|M_B^{''-1}N_B^{''}\|$. Then the iteration sequence $\{z^{(k)}\}_{k=0}^{+\infty} \subset \mathbb{R}^n$ generated by Method 3.2 converges to the unique solution $z^* \in \mathbb{R}$ of the HLCP(A, B, q) for any initial vector $x^{(0)} \in \mathbb{R}^n$ provided that the parameter ω satisfies one of the subsequent cases:*

(1) When $\omega \ge \max\{\frac{\lambda_{\min}^{'}+\lambda_{\max}^{'}}{\mu_{\min}^{'}+\mu_{\max}^{'}}, \frac{\lambda_{\min}^{''}+\lambda_{\max}^{''}}{\mu_{\min}^{''}+\mu_{\max}^{''}}\}$ and ω satisfies

$$a_2\omega^2 + a_1\omega + a_0 > 0, \tag{9}$$

where $a_2 = \mu_{\min}^{'}\mu_{\min}^{''} - (2\tau_B^{'} + 1)(2\tau_B^{''} + 1)\mu_{\max}^{'}\mu_{\max}^{''}$, $a_1 = \lambda_{\min}^{'}\mu_{\min}^{''} + \mu_{\min}^{''}\lambda_{\min}^{''} - (2\tau_B^{'} + 1)\mu_{\max}^{'}(2\lambda_{\max}^{''}\tau_A^{''} - \lambda_{\min}^{''}) - (2\tau_B^{''} + 1)\mu_{\max}^{''}(2\lambda_{\max}^{'}\tau_A^{'} - \lambda_{\min}^{'})$ and $a_0 = \lambda_{\min}^{'}\lambda_{\min}^{''} - (2\lambda_{\max}^{'}\tau_A^{'} - \lambda_{\min}^{'})(2\lambda_{\max}^{''}\tau_A^{''} - \lambda_{\min}^{''})$.

(2) When $\frac{\lambda_{\min}^{'}+\lambda_{\max}^{'}}{\mu_{\min}^{'}+\mu_{\max}^{'}} < \omega < \frac{\lambda_{\min}^{''}+\lambda_{\max}^{''}}{\mu_{\min}^{''}+\mu_{\max}^{''}}$ and ω satisfies

$$b_2\omega^2 + b_1\omega + b_0 > 0, \tag{10}$$

where $b_2 = \mu_{\min}^{''}\mu_{\min}^{'} - (2\mu_{\max}^{''}\tau_B^{''} - \mu_{\min}^{''})(2\tau_B^{'} + 1)\mu_{\max}^{'}$, $b_1 = \lambda_{\min}^{''}\mu_{\min}^{'} + \mu_{\min}^{''}\lambda_{\min}^{'} - \lambda_{\max}^{''}\mu_{\max}^{'}(2\tau_A^{''} + 1)(2\tau_B^{'} + 1) - (2\mu_{\max}^{''}\tau_B^{''} - \mu_{\min}^{''})(2\lambda_{\max}^{'}\tau_A^{'} - \lambda_{\min}^{'})$ and $b_0 = \lambda_{\min}^{''}\lambda_{\min}^{'} - \lambda_{\max}^{''}(2\tau_A^{''} + 1)(2\lambda_{\max}^{'}\tau_A^{'} - \lambda_{\min}^{'})$.

(3) When $\frac{\lambda_{\min}^{''}+\lambda_{\max}^{''}}{\mu_{\min}^{''}+\mu_{\max}^{''}} < \omega < \frac{\lambda_{\min}^{'}+\lambda_{\max}^{'}}{\mu_{\min}^{'}+\mu_{\max}^{'}}$, and ω satisfies

$$c_2\omega^2 + c_1\omega + c_0 > 0, \tag{11}$$

where $c_2 = \mu_{\min}^{'}\mu_{\min}^{''} - (2\mu_{\max}^{'}\tau_B^{'} - \mu_{\min}^{'})(2\tau_B^{''} + 1)\mu_{\max}^{''}$, $c_1 = \lambda_{\min}^{'}\mu_{\min}^{''} + \mu_{\min}^{'}\lambda_{\min}^{''} - \lambda_{\max}^{'}\mu_{\max}^{''}(2\tau_A^{'} + 1)(2\tau_B^{''} + 1) - (2\mu_{\max}^{'}\tau_B^{'} - \mu_{\min}^{'})(2\lambda_{\max}^{''}\tau_A^{''} - \lambda_{\min}^{''})$ and $c_0 = \lambda_{\min}^{'}\lambda_{\min}^{''} - \lambda_{\max}^{'}(2\tau_A^{'} + 1)(2\lambda_{\max}^{''}\tau_A^{''} - \lambda_{\min}^{''})$.

(4) When $\omega \le \min\{\frac{\lambda_{\min}^{'}+\lambda_{\max}^{'}}{\mu_{\min}^{'}+\mu_{\max}^{'}}, \frac{\lambda_{\min}^{''}+\lambda_{\max}^{''}}{\mu_{\min}^{''}+\mu_{\max}^{''}}\}$ and ω satisfies

$$d_2\omega^2 + d_1\omega + d_0 > 0, \tag{12}$$

where $d_2 = \mu_{\min}^{'}\mu_{\min}^{''} - (2\mu_{\max}^{'}\tau_B^{'} - \mu_{\min}^{'})(2\mu_{\max}^{''}\tau_B^{''} - \mu_{\min}^{''})$, $d_1 = \lambda_{\min}^{'}\mu_{\min}^{''} + \mu_{\min}^{'}\lambda_{\min}^{''} - \lambda_{\max}^{'}(2\tau_A^{'} + 1)(2\mu_{\max}^{''}\tau_B^{''} - \mu_{\min}^{''}) - \lambda_{\max}^{''}(2\tau_A^{''} + 1)(2\mu_{\max}^{'}\tau_B^{'} - \mu_{\min}^{'})$ and $d_0 = \lambda_{\min}^{'}\lambda_{\min}^{''} - \lambda_{\max}^{'}\lambda_{\max}^{''}(2\tau_A^{'} + 1)(2\tau_A^{''} + 1)$.

Proof. On the basis of Theorem 4.1, we merely need to certify the sufficient condition (6). Owing to the assumptions that $M_A^{'}$ and $M_B^{'}$ are symmetric positive definite matrices and $\Omega = \omega I$ is a positive scalar matrix, we have

$$\begin{aligned}\zeta_1^{'}(\Omega) &= \|(M_A^{'} + M_B^{'}\Omega)^{-1}N_A^{'}\| = \|(M_A^{'} + M_B^{'}\omega)^{-1}M_A^{'}M_A^{'-1}N_A^{'}\| \\ &\leq \|(M_A^{'} + M_B^{'}\omega)^{-1}\|\|M_A^{'}\|\|M_A^{'-1}N_A^{'}\|.\end{aligned}$$

Based on Lemma 2.2, we can obtain

$$\zeta_1^{'}(\Omega) \leq \frac{\lambda_{\max}^{'}\tau_A^{'}}{\lambda_{\min}^{'} + \omega\mu_{\min}^{'}}.$$

Similarly, we have

$$\zeta_2^{'}(\Omega) = \|(M_A^{'} + M_B^{'}\Omega)^{-1}N_B^{'}\Omega\| \leq \frac{\omega\mu_{\max}^{'}\tau_B^{'}}{\lambda_{\min}^{'} + \omega\mu_{\min}^{'}}.$$

As for $\zeta_3^{'}(\Omega)$, we can obtain

$$\begin{aligned}\zeta_3^{'}(\Omega) &= \|(M_A^{'} + M_B^{'}\Omega)^{-1}(M_B^{'}\Omega - M_A^{'})\| \\ &\leq \|(M_A^{'} + M_B^{'}\Omega)^{-1}\|\|M_B^{'}\Omega - M_A^{'}\| \\ &\leq \begin{cases} \frac{\omega\mu_{\max}^{'} - \lambda_{\min}^{'}}{\lambda_{\min}^{'} + \omega\mu_{\min}^{'}} & \text{if } \omega \geq \frac{\lambda_{\min}^{'} + \lambda_{\max}^{'}}{\mu_{\min}^{'} + \mu_{\max}^{'}}, \\ \frac{\lambda_{\max}^{'} - \omega\mu_{\min}^{'}}{\lambda_{\min}^{'} + \omega\mu_{\min}^{'}}, & \text{if } \omega < \frac{\lambda_{\min}^{'} + \lambda_{\max}^{'}}{\mu_{\min}^{'} + \mu_{\max}^{'}}. \end{cases}\end{aligned}$$

Hence, we can obtain the inequalities of the following two cases

(a_1) When $\omega \geq \frac{\lambda_{\min}^{'} + \lambda_{\max}^{'}}{\mu_{\min}^{'} + \mu_{\max}^{'}}$,

$$2\zeta_1^{'}(\Omega) + 2\zeta_2^{'}(\Omega) + \zeta_3^{'}(\Omega) \leq \frac{2\lambda_{\max}^{'}\tau_A^{'} + 2\omega\mu_{\max}^{'}\tau_B^{'} + \omega\mu_{\max}^{'} - \lambda_{\min}^{'}}{\lambda_{\min}^{'} + \omega\mu_{\min}^{'}};$$

(a_2) When $\omega < \frac{\lambda_{\min}^{'} + \lambda_{\max}^{'}}{\mu_{\min}^{'} + \mu_{\max}^{'}}$,

$$2\zeta_1^{'}(\Omega) + 2\zeta_2^{'}(\Omega) + \zeta_3^{'}(\Omega) \leq \frac{2\lambda_{\max}^{'}\tau_A^{'} + 2\omega\mu_{\max}^{'}\tau_B^{'} + \lambda_{\max}^{'} - \omega\mu_{\min}^{'}}{\lambda_{\min}^{'} + \omega\mu_{\min}^{'}};$$

Analogously, we have

$$\zeta_1^{''}(\Omega) \leq \frac{\lambda_{\max}^{''}\tau_A^{''}}{\lambda_{\min}^{''} + \omega\mu_{\min}^{''}}, \ \zeta_2^{''}(\Omega) \leq \frac{\omega\mu_{\max}^{''}\tau_B^{''}}{\lambda_{\min}^{''} + \omega\mu_{\min}^{''}},$$

$$\zeta_3^{''}(\Omega) \leq \begin{cases} \frac{\omega\mu_{\max}^{''} - \lambda_{\min}^{''}}{\lambda_{\min}^{''} + \omega\mu_{\min}^{''}}, & \text{if } \omega \geq \frac{\lambda_{\min}^{''} + \lambda_{\max}^{''}}{\mu_{\min}^{''} + \mu_{\max}^{''}}, \\ \frac{\lambda_{\max}^{''} - \omega\mu_{\min}^{''}}{\lambda_{\min}^{''} + \omega\mu_{\min}^{''}}, & \text{if } \omega < \frac{\lambda_{\min}^{''} + \lambda_{\max}^{''}}{\mu_{\min}^{''} + \mu_{\max}^{''}}. \end{cases}$$

Therefore, we have the inequalities of the following two cases

(b_1) When $\omega \geq \frac{\lambda_{\min}^{''} + \lambda_{\max}^{''}}{\mu_{\min}^{''} + \mu_{\max}^{''}}$,

$$2\zeta_1^{''}(\Omega) + 2\zeta_2^{''}(\Omega) + \zeta_3^{''}(\Omega) \leq \frac{2\lambda_{\max}^{''}\tau_A^{''} + 2\omega\mu_{\max}^{''}\tau_B^{''} + \omega\mu_{\max}^{''} - \lambda_{\min}^{''}}{\lambda_{\min}^{''} + \omega\mu_{\min}^{''}};$$

(b_2) When $\omega < \frac{\lambda''_{\min}+\lambda''_{\max}}{\mu''_{\min}+\mu''_{\max}}$,

$$2\zeta''_1(\Omega) + 2\zeta''_2(\Omega) + \zeta''_3(\Omega) \le \frac{2\lambda''_{\max}\tau''_A + 2\omega\mu''_{\max}\tau''_B + \lambda''_{\max} - \omega\mu''_{\min}}{\lambda''_{\min} + \omega\mu''_{\min}}.$$

Finally, it holds that

(1) When $\omega \ge \max\{\frac{\lambda'_{\min}+\lambda'_{\max}}{\mu'_{\min}+\mu'_{\max}}, \frac{\lambda''_{\min}+\lambda''_{\max}}{\mu''_{\min}+\mu''_{\max}}\}$,

$$\frac{2\lambda'_{\max}\tau'_A + 2\omega\mu'_{\max}\tau'_B + \omega\mu'_{\max} - \lambda'_{\min}}{\lambda'_{\min} + \omega\mu'_{\min}} \cdot \frac{2\lambda''_{\max}\tau''_A + 2\omega\mu''_{\max}\tau''_B + \omega\mu''_{\max} - \lambda''_{\min}}{\lambda''_{\min} + \omega\mu''_{\min}} < 1;$$

(2) When $\frac{\lambda'_{\min}+\lambda'_{\max}}{\mu'_{\min}+\mu'_{\max}} < \omega < \frac{\lambda''_{\min}+\lambda''_{\max}}{\mu''_{\min}+\mu''_{\max}}$,

$$\frac{2\lambda'_{\max}\tau'_A + 2\omega\mu'_{\max}\tau'_B + \omega\mu'_{\max} - \lambda'_{\min}}{\lambda'_{\min} + \omega\mu'_{\min}} \cdot \frac{2\lambda''_{\max}\tau''_A + 2\omega\mu''_{\max}\tau''_B + \lambda''_{\max} - \omega\mu''_{\min}}{\lambda''_{\min} + \omega\mu''_{\min}} < 1;$$

(3) When $\frac{\lambda''_{\min}+\lambda''_{\max}}{\mu''_{\min}+\mu''_{\max}} < \omega < \frac{\lambda'_{\min}+\lambda'_{\max}}{\mu'_{\min}+\mu'_{\max}}$,

$$\frac{2\lambda'_{\max}\tau'_A + 2\omega\mu'_{\max}\tau'_B + \lambda'_{\max} - \omega\mu'_{\min}}{\lambda'_{\min} + \omega\mu'_{\min}} \cdot \frac{2\lambda''_{\max}\tau''_A + 2\omega\mu''_{\max}\tau''_B + \omega\mu''_{\max} - \lambda''_{\min}}{\lambda''_{\min} + \omega\mu''_{\min}} < 1;$$

(4) When $\omega \le \min\{\frac{\lambda'_{\min}+\lambda'_{\max}}{\mu'_{\min}+\mu'_{\max}}, \frac{\lambda''_{\min}+\lambda''_{\max}}{\mu''_{\min}+\mu''_{\max}}\}$,

$$\frac{2\lambda'_{\max}\tau'_A + 2\omega\mu'_{\max}\tau'_B + \lambda'_{\max} - \omega\mu'_{\min}}{\lambda'_{\min} + \omega\mu'_{\min}} \cdot \frac{2\lambda''_{\max}\tau''_A + 2\omega\mu''_{\max}\tau''_B + \lambda''_{\max} - \omega\mu''_{\min}}{\lambda''_{\min} + \omega\mu''_{\min}} < 1.$$

Obviously, when the parameter ω satisfies one of the conditions (9), (10), (11) and (12), we can obtain the condition (6) holds, which implies the Method 3.2 converges to the unique solution. This completes the proof. □

4.2. A and B are H_+-matrices

Theorem 4.3. *Let A and B be H_+-matrices. Suppose $A = M'_A - N'_A = M''_A - N''_A$ and $B = M'_B - N'_B = M''_B - N''_B$ are two H-compatible splittings of the matrices A, B, respectively. Let $M'_A + M'_B\Omega$ and $M''_A + M''_B\Omega$ are H_+-matrices. Suppose Ω is a diagonal matrix with positive elements satisfying $\Omega \ge D_A D_B^{-1}$. If there exists an arbitrary small number ε s.t. $\rho(T_\varepsilon) < 1$ with $T_\varepsilon = D_A^{-1}|B_{B\Omega}| + D_A^{-1}|B_A| + \varepsilon ee^T$ and $e = (1, 1, \cdots, 1)^T \in \mathbb{R}^n$. Then the iteration sequence $\{z^{(k)}\}_{k=0}^{+\infty} \subset \mathbb{R}^n$ generated by Method 3.2 converges to the unique solution $z^* \in \mathbb{R}^n$ of the HLCP(A, B, q) for any initial vector $x^{(0)} \in \mathbb{R}^n$.*

Proof. Assume that x^* is the exact solution, which satisfies the following systems:

$$\begin{cases} (M'_A + M'_B\Omega)x^* = (N'_A + N'_B\Omega)x^* + (B\Omega - A)|x^*| + \gamma q, \\ (M''_A + M''_B\Omega)x^* = (N''_A + N''_B\Omega)x^* + (B\Omega - A)|x^*| + \gamma q, \end{cases} \tag{13}$$

Subtracting (13) from (4), we can get

$$\begin{cases} (M'_A + M'_B\Omega)(x^{(k+\frac{1}{2})} - x^*) = (N'_A + N'_B\Omega)(x^{(k)} - x^*) \\ \qquad\qquad + (B\Omega - A)(|x^{(k)}| - |x^*|), \\ (M''_A + M''_B\Omega)(x^{(k+1)} - x^*) = (N''_A + N''_B\Omega)(x^{(k+\frac{1}{2})} - x^*) \\ \qquad\qquad + (B\Omega - A)(|x^{(k+\frac{1}{2})}| - |x^*|). \end{cases} \tag{14}$$

Since both $M_A^{'} + M_B^{'}\Omega$ and $M_A^{''} + M_B^{''}\Omega$ are H_+-matrices, based on Lemma 2.4, we can obtatin

$$0 \le |(M_A^{'} + M_B^{'}\Omega)^{-1}| \le \langle M_A^{'} + M_B^{'}\Omega\rangle^{-1},$$

$$0 \le |(M_A^{''} + M_B^{''}\Omega)^{-1}| \le \langle M_A^{''} + M_B^{''}\Omega\rangle^{-1}.$$

Then, we take the absolute value of the first equation of (14) as

$$\begin{aligned} &|x^{(k+\frac{1}{2})} - x^*| \\ =&\left|(M_A^{'} + M_B^{'}\Omega)^{-1}\left[(N_A^{'} + N_B^{'}\Omega)(x^{(k)} - x^*) + (B\Omega - A)(|x^{(k)}| - |x^*|)\right]\right| \\ \le&|(M_A^{'} + M_B^{'}\Omega)^{-1}|(|N_A^{'} + N_B^{'}\Omega| + |B\Omega - A|)|x^{(k)} - x^*| \\ \le&\langle M_A^{'} + M_B^{'}\Omega\rangle^{-1}(|N_A^{'} + N_B^{'}\Omega| + |B\Omega - A|)|x^{(k)} - x^*|. \end{aligned} \tag{15}$$

Similar to (15), we obtain

$$\begin{aligned} &|x^{(k+1)} - x^*| \\ =&\left|(M_A^{''} + M_B^{''}\Omega)^{-1}\left[(N_A^{''} + N_B^{''}\Omega)(x^{(k+\frac{1}{2})} - x^*) + (B\Omega - A)(|x^{(k+\frac{1}{2})}| - |x^*|)\right]\right| \\ \le&|(M_A^{''} + M_B^{''}\Omega)^{-1}|(|N_A^{''} + N_B^{''}\Omega| + |B\Omega - A|)|x^{(k+\frac{1}{2})} - x^*| \\ \le&\langle M_A^{''} + M_B^{''}\Omega\rangle^{-1}(|N_A^{''} + N_B^{''}\Omega| + |B\Omega - A|)|x^{(k+\frac{1}{2})} - x^*|. \end{aligned} \tag{16}$$

Combining (15) and (16), we have

$$|x^{(k+1)} - x^*| \le \Re(\Omega)\Re_g(\Omega)|x^{(k)} - x^*|,$$

where $\Re_g(\Omega) = \langle M_A^{'} + M_B^{'}\Omega\rangle^{-1}(|N_A^{'} + N_B^{'}\Omega| + |B\Omega - A|)$ and $\Re(\Omega) = \langle M_A^{''} + M_B^{''}\Omega\rangle^{-1}(|N_A^{''} + N_B^{''}\Omega| + |B\Omega - A|)$.

Obiviously, the iteration sequence generated by Method 3.2 is convergent if $\rho(\Re(\Omega)\Re_g(\Omega))$ is less than one. Then, in the following dissusion, we only need to find the conditions of $\rho(\Re(\Omega)\Re_g(\Omega)) < 1$.

Regarding the iteration matrix $\Re(\Omega)\Re_g(\Omega)$, we have

$$\begin{aligned} \Re(\Omega) =&\langle M_A^{''} + M_B^{''}\Omega\rangle^{-1}(|N_A^{''} + N_B^{''}\Omega| + |B\Omega - A|) \\ =&I - \langle M_A^{''} + M_B^{''}\Omega\rangle^{-1}(\langle M_A^{''} + M_B^{''}\Omega\rangle - |N_A^{''} + N_B^{''}\Omega| - |B\Omega - A|) \\ \le&I - \langle M_A^{''} + M_B^{''}\Omega\rangle^{-1}(\langle M_A^{''}\rangle + \langle M_B^{''}\Omega\rangle - |N_A^{''}| - |N_B^{''}\Omega| - |B\Omega - A|) \\ =&I - \langle M_A^{''} + M_B^{''}\Omega\rangle^{-1}(\langle A\rangle + \langle B\Omega\rangle - |B\Omega - A|) \\ \le&I - 2\langle M_A^{''} + M_B^{''}\Omega\rangle^{-1}(D_A - |B_{B\Omega}| - |B_A|) + 2\langle M_A^{''} + M_B^{''}\Omega\rangle^{-1}D_A\varepsilon ee^T \\ =&I - 2\langle M_A^{''} + M_B^{''}\Omega\rangle^{-1}D_A(I - D_A^{-1}|B_{B\Omega}| - D_A^{-1}|B_A| - \varepsilon ee^T) \\ :=&I - 2\langle M_A^{''} + M_B^{''}\Omega\rangle^{-1}D_A(I - T_\varepsilon) \end{aligned} \tag{17}$$

and

$$\begin{aligned} \Re_g(\Omega) =&\langle M_A^{'} + M_B^{'}\Omega\rangle^{-1}(|N_A^{'} + N_B^{'}\Omega| + |B\Omega - A|) \\ \le&I - 2\langle M_A^{'} + M_B^{'}\Omega\rangle^{-1}D_A(I - T_\varepsilon), \end{aligned} \tag{18}$$

where $e = (1, 1, \cdots, 1)^T \in \mathbb{R}^n$ and $T_\varepsilon = D_A^{-1}|B_{B\Omega}| + D_A^{-1}|B_A| + \varepsilon ee^T$. Obviously, T_ε is a positive matrix and by the Perron-Frobenius theorem[34], there exists a positive vector $y \in \mathbb{R}^n$ such that $T_\varepsilon y = \rho(T_\varepsilon)y$. Based on (17) and (18) and $\rho(T_\varepsilon) < 1$, we can obtain the inequality as follow

$$\begin{aligned} \Re(\Omega)\Re_g(\Omega)y &\le \Re(\Omega)\left[I - 2\langle M_A^{'} + M_B^{'}\Omega\rangle^{-1}D_A(I - T_\varepsilon)\right]y < \Re(\Omega)y \\ &\le \left[I - 2\langle M_A^{''} + M_B^{''}\Omega\rangle^{-1}D_A(I - T_\varepsilon)\right]y < y, \end{aligned} \tag{19}$$

Since $M'_A + M'_B\Omega$ is an H_+-matrix, $\langle M'_A + M'_B\Omega\rangle$ is an M-matrix, i.e., $\langle M'_A + \Omega\rangle^{-1}$ is a positive matrix. Analogously, $\langle M''_A + M''_B\Omega\rangle$ is also a positive matrix. Hence, $\Re(\Omega)\Re_g(\Omega)$ is a nonnegative matrix. According to Lemma 2.5, we directly have $\rho(\Re(\Omega)\Re_g(\Omega)) < 1$, which means the iteration is convergent. □

Corollary 4.4. *Let A and B be H_+-matrices in $\mathbb{R}^{n\times n}$, and Ω be known positive diagonal matrices. Let $A = D_A - L_A - U_A := D_A - B_A$ and $B\Omega = D_{B\Omega} - L_{B\Omega} - U_{B\Omega} := D_{B\Omega} - B_{B\Omega}$. When the parameters α and Ω satisfy one of the subsequent cases:*
(1) $D_{B\Omega} > D_A$ and $2\alpha D_A e > \alpha|B_A + B_{B\Omega}|e + \alpha|B_{B\Omega} - B_A|e$;
(2) $D_{B\Omega} < D_A$ and $2\alpha D_{B\Omega} e > \alpha|B_A + B_{B\Omega}|e + \alpha|B_{B\Omega} - B_A|e$.
Then, for arbitrary initial vector, the TMSOR iteration method is convergent for $0 < \alpha < 1$.

Proof. Let

$$M'_A = \frac{1}{\alpha}(D_A - \alpha L_A),\ N'_A = \frac{1}{\alpha}((1-\alpha)D_A + \alpha U_A),$$

$$M''_A = \frac{1}{\alpha}(D_A - \alpha U_A),\ N''_A = \frac{1}{\alpha}((1-\alpha)D_A + \alpha L_A),$$

$$M'_B = \frac{1}{\alpha}(D_B - \alpha L_B),\ N'_B = \frac{1}{\alpha}((1-\alpha)D_B + \alpha U_B),$$

$$M''_B = \frac{1}{\alpha}(D_B - \alpha U_B),\ N''_B = \frac{1}{\alpha}((1-\alpha)D_B + \alpha L_B),$$

According to 4.3, we need to certify $\rho(\Re(\Omega)\Re_g(\Omega)) < 1$.

$$\begin{aligned}\|\Re_g(\Omega)\| &= \|\langle M''_A + M''_B\Omega\rangle^{-1}(|N''_A + N''_B\Omega| + |B\Omega - A|)\| \\ &= \frac{(|(1-\alpha)(D_A + D_{B\Omega}) + \alpha(L_A + L_{B\Omega})| + \alpha|B\Omega - A|)e}{(D_A + D_{B\Omega} - \alpha|U_A + U_{B\Omega}|)e}.\end{aligned}$$

Hence, $\|\Re_g(\Omega)\| < 1$ holds when the parameters $\alpha < 1$ and Ω satisfy one of the subsequent cases:
(1) $D_{B\Omega} > D_A$ *and* $2\alpha D_A e > \alpha|B_A + B_{B\Omega}|e + \alpha|B_{B\Omega} - B_A|e$;
(2) $D_{B\Omega} < D_A$ *and* $2\alpha D_{B\Omega} e > \alpha|B_A + B_{B\Omega}|e + \alpha|B_{B\Omega} - B_A|e$.
Analogously, we can get $\|\Re(\Omega)\| < 1$. Therefore, $\rho(\Re(\Omega)\Re_g(\Omega)) < \|\Re(\Omega)\| \cdot \|\Re_g(\Omega)\| < 1$, which prove the TMSOR iteration method is convergent. □

5. Numerical results

For the sake of demonstrating the effectiveness of the suggested approaches. We are going to do several experiments, which were performed in Matlab (R2017a) on an Intel(R) Core(TM) i5-4210U, where the memory is 4.00GB and RAM the CPU is 1.70GHz.

Set the initial vector $x^{(0)}$ as $(2, 2, \cdots, 2)^T$. All experiment results include three aspects: the elapsed CPU time in seconds (CPU), the norm of absolute residual vectors (RES), and the number of iteration steps (IT), respectively. 'RES' is defined as

$$RES := \min\left(z^{(k)}, w^{(k)}\right).$$

In the following experiments, when the prescribed iteration number $k_{\max} = 2000$ is exceeded or the residual vector satisfies RES $\leq 10^{-6}$, all runs are terminated. We show five methods to compare and give their abbreviations in Table 1. Set $\Omega = D_A D_B^{-1}$ and $\gamma = 2$. We will consider the problems with four dimensions, i.e., $n = 100,\ 400,\ 900,\ 1600$. The parameters α and β in the TMSOR, TMAOR, MSOR and MAOR methods are tentative optimal parameters.

Example 5.1. *[10] Let m be a prescribed positive integer and $n = m^2$. Consider the HLCP (A, B, q), in which $A = \hat{A} + \mu I$ with μ real parameter is block tridiagonal matrix and $B = \hat{B} + \nu I$ with ν real parameter is block diagonal matrix*

Table 1: The method abbreviations

Method	Description
MJ	the modulus-based Jacobi method
MSOR	the modulus-based successive over-relaxation method
MAOR	the modulus-based accelerated over-relaxation method
TMSOR	the two-step modulus-based successive over-relaxation method
TMAOR	the two-step modulus-based accelerated over-relaxation method

Table 2: Numerical results with $\mu = 0$ and $\nu = 4$ for Example 5.1

Algorithm		m=10	m=20	m=30	m=40
MJ	CPU	2.4043	3.9201	134.7908	1.2804e+03
	RES	7.7037e-07	9.5067e-07	8.8820e-07	8.2581e-07
	IT	42	48	51	53
MSOR	α	1.1	1.2	1.2	1.2
	CPU	0.0603	2.2441	86.9993	804.5363
	RES	6.1557e-09	6.8200e-09	6.9415e-09	5.8503e-09
	IT	28	31	32	33
MAOR	(α,β)	(1.1,1.1)	(1.1,1.1)	(1.1,1.1)	(1.1,1.1)
	CPU	0.0569	2.4601	91.8171	857.4322
	RES	6.1557e-09	5.3277e-09	9.3713e-09	9.7681e-09
	IT	28	33	34	35
TMSOR	α	1.2	1.2	1.1	1.1
	CPU	0.0507	1.8282	51.2272	461.1408
	RES	3.9544e-09	5.5995e-09	3.4988e-09	6.5509e-09
	IT	17	18	18	18
TMAOR	(α,β)	(1.1,1.3)	(1.0,1.3)	(1.1,1.3)	(1.1,1.2)
	CPU	0.0503	1.8676	53.4272	459.9878
	RES	7.8232e-09	4.6771e-09	7.5234e-09	5.1579e-09
	IT	16	18	18	18

$$\hat{A} = \begin{pmatrix} S & -I & 0 & \cdots & 0 & 0 \\ -I & S & -I & \cdots & 0 & 0 \\ 0 & -I & S & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & S & -I \\ 0 & 0 & 0 & \cdots & -I & S \end{pmatrix}, \ \hat{B} = \begin{pmatrix} S & 0 & \cdots & 0 & 0 \\ 0 & S & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & S & 0 \\ 0 & 0 & \cdots & 0 & S \end{pmatrix},$$

and the block diagonal matrix S is defined as a tridiagonal matrix

$$S = tridiag(-1, 4, -1) = \begin{pmatrix} 4 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 4 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 4 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 4 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 4 \end{pmatrix} \in \mathbb{R}^{m\times m}.$$

The vector $q = Az^* - Bw^*$, where z^* and w^* are defined as

$$z^* = (0, 1, 0, \cdots, 1)^T \text{ and } w^* = (1, 0, 1, \cdots, 0)^T,$$

respectively.

From Table 2, we can observe that two-step modulus-based matrix splitting method is sensitive to solve the HLCP. The numeric results which contain the MJ, MSOR, MAOR, TMSOR, TMAOR methods illustrate TSOR and TAOR methods are more efficient than MJ, MSOR and MAOR methods concerning CPU and IT. In particular, the larger the matrix dimension, the more obvious the convergence rate is faster.

Table 3: Numerical results with $\mu = 0$ and $\nu = 4$ for Example 5.2

Algorithm		m=10	m=20	m=30	m=40
MJ	CPU	0.1426	3.0295	129.5049	1.2349e+03
	RES	8.7690e-07	6.6058e-07	9.1828e-07	9.6072e-07
	IT	37	47	50	52
MSOR	α	1.1	1.1	1.1	1.1
	CPU	0.0488	1.5605	61.9086	603.6123
	RES	4.1740e-09	4.5977e-09	6.6385e-09	5.2225e-09
	IT	20	23	24	25
MAOR	(α,β)	(1.1,1.2)	(1.1,1.2)	(1.1,1.2)	(1.1,1.2)
	CPU	0.0466	1.5078	53.5993	539.3606
	RES	5.2093e-09	3.1855e-09	6.3434e-09	2.6760e-09
	IT	18	21	22	23
TMSOR	α	1.1	1.1	1.1	1.1
	CPU	0.0401	1.4519	45.7516	433.8235
	RES	3.7805e-09	2.6089e-09	6.6863e-09	3.0073e-09
	IT	14	16	16	17
TMAOR	(α,β)	(1.1,1.0)	(1.1,1.0)	(1.1,1.1)	(1.1,1.0)
	CPU	0.0409	1.4100	45.6885	405.9570
	RES	8.4003e-09	4.7463e-09	6.6863e-09	5.8979e-09
	IT	13	15	16	16

Example 5.2. *[10] Let m be a prescribed positive integer and $n = m^2$. Consider the HLCP (A, B, q), in which $A = \hat{A} + \mu I$ with μ real parameter is block tridiagonal matrix and $B = \hat{B} + \nu I$ with ν real parameter is block diagonal matrix*

$$\hat{A} = \begin{pmatrix} S & -0.5I & 0 & \cdots & 0 & 0 \\ -1.5I & S & -0.5I & \cdots & 0 & 0 \\ 0 & -1.5I & S & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & S & -0.5I \\ 0 & 0 & 0 & \cdots & -1.5I & S \end{pmatrix}, \quad \hat{B} = \begin{pmatrix} S & 0 & \cdots & 0 & 0 \\ 0 & S & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & S & 0 \\ 0 & 0 & \cdots & 0 & S \end{pmatrix},$$

and the block diagonal matrix S is defined as a tridiagonal matrix

$$S = \begin{pmatrix} 4 & -0.5 & 0 & \cdots & 0 & 0 \\ -1.5 & 4 & -0.5 & \cdots & 0 & 0 \\ 0 & -1.5 & 4 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 4 & -0.5 \\ 0 & 0 & 0 & \cdots & -1.5 & 4 \end{pmatrix} \in \mathbb{R}^{m\times m}.$$

The vector $q = Az^* - Bw^*$, where z^* and w^* are defined as

$$z^* = (0, 1, 0, \cdots, 1)^T \text{ and } w^* = (1, 0, 1, \cdots, 0)^T,$$

respectively.

In Table 3, it contains the results of the MJ, MSOR, MAOR, TMSOR, TMAOR methods, which show that two-step modulus-based matrix splitting method is sensitive to solve the HLCP. The numeric results illustrate TSOR and TAOR methods converge faster than MJ, MSOR and MAOR methods concerning CPU and IT to obtain the same residuals for distinct dimension.

Table 4: Numerical results with $\mu = 0$ and $\nu = 4$ for Example 5.3

Algorithm		m=10	m=20	m=30	m=40
MJ	CPU	0.0455	1.7594	73.1118	857.2154
	RES	7.4681e-07	1.9743e-07	3.4746e-07	5.2083e-07
	IT	17	31	43	54
MSOR	α	1.0	1.0	1.0	1.0
	CPU	0.0336	1.4213	53.6197	618.6461
	RES	2.6600e-09	2.3520e-09	9.3923e-09	8.4486e-09
	IT	15	23	30	38
MAOR	(α,β)	(1.0,1.0)	(1.0,1.0)	(1.0,1.1)	(1.0,1.1)
	CPU	0.0278	1.3892	53.2101	588.8837
	RES	2.6600e-09	2.3520e-09	2.9635e-09	7.5920e-09
	IT	15	23	29	35
TMSOR	α	1.0	1.0	1.0	1.0
	CPU	0.0186	1.1394	33.7205	362.1086
	RES	3.5576e-10	7.5188e-11	2.4680e-09	5.0083e-09
	IT	8	13	17	21
TMAOR	(α,β)	(1.0,1.0)	(1.0,1.1)	(1.0,1.0)	(1.0,1.1)
	CPU	0.0167	1.0623	34.0975	361.1106
	RES	1.3766e-10	8.1566e-09	2.4680e-09	1.7272e-09
	IT	8	12	17	20

Example 5.3. *Let m be a prescribed positive integer and $n = m^2$. Consider the HLCP (A, B, q), in which $A = \hat{A} + \mu I$ with μ real parameter is block tridiagonal matrix and $B = \hat{B} + \nu I$ with ν real parameter is block diagonal matrix*

$$\hat{A} = \begin{pmatrix} S & -I & -I & \cdots & 0 & 0 \\ 0 & S & -I & \cdots & 0 & 0 \\ 0 & 0 & S & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & S & -I \\ 0 & 0 & 0 & \cdots & 0 & S \end{pmatrix}, \quad \hat{B} = \begin{pmatrix} S & 0 & \cdots & 0 & 0 \\ 0 & S & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & S & 0 \\ 0 & 0 & \cdots & 0 & S \end{pmatrix},$$

and the block diagonal matrix S is defined as a tridiagonal matrix

$$S = \begin{pmatrix} 4 & -1 & -1 & \cdots & 0 & 0 \\ 0 & 4 & -1 & \cdots & 0 & 0 \\ 0 & 0 & 4 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 4 & -1 \\ 0 & 0 & 0 & \cdots & 0 & 4 \end{pmatrix} \in \mathbb{R}^{m \times m}.$$

The vector $q = Az^* - Bw^*$, where z^* and w^* are defined as

$$z^* = (0, 1, 0, \cdots, 1)^T \text{ and } w^* = (1, 0, 1, \cdots, 0)^T,$$

respectively.

In Table 4, it contains the results of the MJ, MSOR, MAOR, TMSOR, TMAOR methods, which show that two-step modulus-based matrix splitting method is sensitive to solve the HLCP. The numeric results illustrate the convergence rate is faster obviously as the matrix dimension is increasing. It implies that the TSOR and TAOR methods have an advantage over the MJ, MSOR and MAOR methods concerning CPU and IT.

In short, these above-mentioned numerical results illustrate that the two-step modulus-based method is more efficient than some of existing methods concerning the CPU time and the IT steps under certain conditions. Hence, our proposed method might be more appropriate to solve the HLCP(A, B, q).

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