



Computer Tools for the Construction and Analysis of Some Efficient Root-Finding Simultaneous Methods

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Abstract. Using the tools provided by computer algebra system *Mathematica*, we consider two iterative methods of high efficiency for the simultaneous approximation of simple or multiple (real or complex) zeros of algebraic polynomials. The proposed methods are based on the fourth-order Schröder-like methods of the first and second kind. We prove that the order of convergence of both basic total-step simultaneous methods is equal to five. Using corrective approximations produced by methods of order two, three and four for finding a single multiple zero, the convergence order is increased from five to six, seven, and eight, respectively. The increased convergence speed is attained with negligible number of additional arithmetic operations, which significantly increases the computational efficiency of the accelerated methods. Convergence properties of the proposed methods are demonstrated by numerical examples and graphics visualization by plotting trajectories of zero approximations. Flows of iterative processes, presented by these trajectories, point to the stability and robustness of the proposed methods.

1. Introduction

Contemporary powerful computer algebra systems, developed during the last two decades, give a new boost and breakthrough to the development and study of the numerical analysis methods, see, e.g., [1]–[4] and references cited there. In this paper we focus on iterative methods for the simultaneous determination of (real or complex) zeros of algebraic equations, a powerful tool of numerical analysis that is often used in applied mathematics but also for solving various problems of a wide variety of areas of scientific disciplines such as physics, engineering, computer science, economics, biology, astronomy, chemistry, and so on, see, e.g., [5], [6]. Rapid development of digital computers and methods for parallel implementation has led to increased interest for the application of simultaneous zero-finding methods, which is indicated by a huge number of references on this topic; some of them are cited in the books [5], [7], [8], and [9]. Self-validated simultaneous methods, that produce disks containing polynomial zeros and provide the upper error bounds, are of special interest, see, e.g., [7], [10] and [11].

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Recall that the importance of root-finding simultaneous methods arises from their quite convenient features. First, in the course of an iterative process, approximations of zeros affect each other improving them. In this way, even if some approximations move towards undesired directions, the remaining approximations in most cases correct their bad trajectory (so-called *self-correction*). This very convenient property most frequently provides globally convergent behavior of simultaneous methods, important preference over one-point methods that are applied serially in n (*independent*) versions (n is the number of different zeros of a polynomial). As presented in Section 6, the proposed simultaneous methods, especially the Schröder methods of the second kind $(18)_{(r=1)}$, demonstrate almost global convergence. Furthermore, simultaneous methods are executed in n identical versions, which is suitable for parallel implementation in which many calculations are carried out simultaneously. These advantages cannot be utilized by applying n independent sequences of approximations using one-point methods (for example, Newton's or Halley's method) – simply, there is no need for broadcast. Moreover, working serially by applying the successive deflation (removing by linear factor), one-point methods produce inaccurate coefficients of the deflated polynomial and, consequently, further zeros might be wrong. This deflation problem does not occur when simultaneous methods are implemented. In addition, note that the application of simultaneous methods results in approximations of almost the same accuracy compared to the latter once, while deflation produces first approximations of almost the same acceptable accuracy compared to the latter ones, which might even be completely falsified. The described comparison of parallel and serial approaches assumes the same initial conditions concerning localization of zeros, that is, well-separated regions of convergence containing one isolated zero, as done in [12] and [13].

The aim of this paper is to present two new very efficient families of simultaneous methods for finding simple or multiple (real or complex) zeros of an algebraic polynomial. We concentrate on Schröder's methods of the first and second kind since they are the most important general families of root-finding methods. The construction and a deep insight into qualitative analysis of convergence behavior of these methods is provided by computer tools developed during the last two decades. In particular, we have used symbolic computation to perform convergence analysis and multi-precision arithmetic in computer algebra system *Mathematica* to test the performance of the proposed families of methods since they produce approximations of zeros of great accuracy that exceeds the limit of computer arithmetic with IEEE 754-2008 standard. One of the reasons to employ *Mathematica* is its *significance arithmetic* with the property that it keeps track of computed numerical results as well as uses error propagation to control their accuracy (see [14]). Besides, using the advanced computer graphics of *Mathematica*, we have tracked the complete flow of the implemented iterative process, starting from the initial approximations to the zeros that satisfy the given stopping criterion. This type of computer visualization is of particular interest since it gives a novel approach and insight to the global convergence characteristics of simultaneous methods, see the pioneered works [15], [16], [17], the book *Polynomial Root-Finding and Polynomiography* [23] and numerous references cited therein.

Although most simultaneous zero-finding methods, known in the literature, are based on a *zero-relation* of the form

$$\zeta_i = F_i(z_1, \dots, z_n; \zeta_1, \dots, \zeta_n),$$

where z_1, \dots, z_n are approximations to the zeros ζ_1, \dots, ζ_n of a given polynomial P , we will show that fast simultaneous methods can be also constructed starting from some convenient iterative method $z^{(k+1)} = \phi(z^{(k)})$ for finding simple or multiple zero. We show that the iterative methods, known as Schröder's methods of the first and second kind (Section 2), are just appropriate for the construction of simultaneous methods with very high order of convergence, achieved without additional polynomial evaluations.

In Section 3 we show that the order of convergence of two Schröder-like simultaneous methods for finding all (simple or multiple) zeros of a polynomial, constructed by using the fourth-order methods of Schröder's type for a single multiple zero, is five. The use of corrective approximations, produced by one-point methods of order two, three and four, increases the order to six, seven, and eight, respectively, on account of the negligible increase of computational cost. In this way we generate methods of high computational efficiency. Convergence analysis of these methods is presented in Section 4. Numerical results are given in Section 5 to demonstrate convergence rate of the new methods. Finally, in Section 6 we

test convergence properties of the proposed methods from a global point of view by plotting trajectories which start from initial approximations chosen by employing Aberth's selecting procedure.

2. On two families of iterative methods of Schröder's type for multiple roots

The so-called Basic sequence of iterative methods $\{E_r\}$ for finding multiple zeros was extensively studied by Traub [19, Ch 7]. This sequence for simple zeros was studied almost one century ago by E. Schröder [20] and hence, it is usually named in literature as Schröder's method of the *first kind*.

Let f be differentiable sufficiently many times, and let α be its zero of the multiplicity m known in advance. Introduce the abbreviations

$$u(x) = \frac{f(x)}{f'(x)}, \quad A_q(z) = \frac{f^{(q)}(z)}{q!f'(z)} \quad (q = 2, 3, \dots),$$

where $f^{(q)}$ stands for the q -th derivative of f . The members of the Basic sequence $\{E_r\}$ can be generated in a simple manner using Traub's difference-differential relation (see Lemma 7-1 in [19])

$$E_{r+1}(z) = E_r(z) - \frac{m u(z)}{r} E_r'(z), \quad E_2(z) = z - m u(z) \quad (r = 2, 3, \dots). \quad (1)$$

According to (1) we generate the first few E_r :

$$E_2(z) = z - m u(z) \quad (\text{Schröder's method [20], order 2}),$$

$$E_3(z) = z - m u(z) \left[\frac{1}{2}(3 - m) + mA_2(z)u(z) \right],$$

$$E_4(z) = z - m u(z) \left\{ \frac{1}{6}(m^2 - 6m + 11) + m(2 - m)A_2(z)u(z) + m^2[2A_2^2(z) - A_3(z)]u(z)^2 \right\}.$$

We use the notion E_r following Traub's notation from his book [19].

The following assertion was proved in [19, Ch. 7].

Theorem 1. *In sufficiently close vicinity of a simple or multiple root of the equation $f(z) = 0$, the order of convergence of the Basic sequence $\{E_{r+1}\}$ defined by (1) is $r + 1$.*

Schröder's methods of the *second kind* for simple zeros $\{S_r\}$, known also as Schröder-König's methods after the works of Schröder [20] and König [21], can be defined in different way and expressed by various algorithmic schemes. In fact, this class of methods was rediscovered many times, see [22] and the book [23]. In the case of multiple zeros, one of the simplest manners for generating the members of iterative family of Schröder-König's type is based on the following assertion considered in [24] (see Milovanović [25] for more general theorem):

Theorem 2. *Let $\varphi_r(z)$ be an iterative method of order r for finding a simple or multiple zero α of a given function f (sufficiently many times differentiable). Then the iterative method*

$$\varphi_{r+1}(z) := z - \frac{z - \varphi_r(z)}{1 - \frac{1}{r} \varphi_r'(z)} \quad (r \geq 2)$$

has the order of convergence $r + 1$. In particular, if this generating formula starts from Schröder's method of the second order

$$\mathfrak{S}_2(z) := \varphi_2(z) = z - m \frac{f(z)}{f'(z)},$$

then it produces the members of iterative family of Schröder-König's type,

$$\mathfrak{S}_{r+1}(z) = z - \frac{z - \mathfrak{S}_r(z)}{1 - \frac{1}{r} \mathfrak{S}_r'(z)} \quad (r \geq 2) \quad (2)$$

of the order $r + 1$.

In particular, starting from Schröder’s method of the second order $\mathfrak{S}_2(z)$ ($= \varphi_2(z)$), in the first step we obtain Halley-like cubically convergent method for a multiple zero [26] (see, also, [27])

$$\mathfrak{S}_3(z) \equiv \varphi_3(z) = z - \frac{z - \mathfrak{S}_2(z)}{1 - \frac{1}{2} \mathfrak{S}'_2(z)} = z - \frac{mu(z)}{\frac{1}{2}(1 + m) - mA_2(z)u(z)}.$$

Continuing this process, using (2) and the expression of \mathfrak{S}_3 , we obtain the fourth order method

$$\mathfrak{S}_4(z) \equiv \varphi_4(z) = z - \frac{mu(z)[(m + 1)/2 - mA_2(z)u(z)]}{(1 + 3m + 2m^2)/6 - m(1 + m)A_2(z)u(z) + m^2A_3(z)u(z)^2} \tag{3}$$

derived by Farmer and Loizou in [28], etc.

Note that an arbitrary member of the family of Schröder-König’s type \mathfrak{S}_r for multiple zeros can be expressed in the closed form in the following way. Let α be a zero of multiplicity m of an analytic function f . Let us introduce an auxiliary function $F(z) = f(z)^{1/m}$ and define $D_0(z) = 1$, $D_1(z) = F'(z)$, and for each $r \geq 2$

$$D_r(z) = \det \begin{pmatrix} F'(z) & \frac{F''(z)}{2!} & \dots & \frac{F^{(r-1)}(z)}{(r-1)!} & \frac{F^{(r)}(z)}{r!} \\ F(z) & F'(z) & & & \frac{F^{(r-1)}(z)}{(r-1)!} \\ 0 & F(z) & & & \frac{F^{(r-2)}(z)}{(r-2)!} \\ & & & & \vdots \\ 0 & & & & \frac{F''(z)}{2!} \\ 0 & 0 & \dots & F(z) & F'(z) \end{pmatrix}.$$

Then the family of the r -th order of Schröder-König’s type for multiple zeros is given by

$$\mathfrak{S}_r(z) = z - F(z) \frac{D_{r-2}(z)}{D_{r-1}(z)} \quad (r = 2, 3, \dots). \tag{4}$$

For example,

$$D_2(z) = \det \begin{pmatrix} F'(z) & \frac{F''(z)}{2} \\ F(z) & F'(z) \end{pmatrix} = F'(z)^2 - \frac{F(z)F''(z)}{2}$$

and

$$D_3(z) = \det \begin{pmatrix} F'(z) & \frac{F''(z)}{2} & \frac{F'''(z)}{6} \\ F(z) & F'(z) & \frac{F''(z)}{2} \\ 0 & F(z) & F'(z) \end{pmatrix} = F'(z)^3 - F(z)F'(z)F''(z) - \frac{F(z)^2F'''(z)}{6},$$

where

$$F'(z) = \frac{f(z)^{1/m} f'(z)}{mf(z)}, \quad F''(z) = \frac{f(z)^{1/m} (mf(z)f''(z) - (m-1)f'(z)^2)}{m^2 f(z)^2},$$

$$F'''(z) = \frac{f(z)^{1/m} (m^2 f(z)^2 f^{(3)}(z) + (2m^2 - 3m + 1) f'(z)^3 - 3(m-1)mf(z)f'(z)f''(z))}{m^3 f(z)^3}.$$

Substituting these expressions in

$$\mathfrak{S}_4(z) = z - F(z) \cdot \frac{D_2(z)}{D_3(z)},$$

we obtain the fourth-order Farmer-Loizou’s method $\mathfrak{S}_4(z)$ for a multiple zero given by (3).

Remark 1. When we consider methods for multiple zeros, we use the term “methods of Schröder’s type” or “Schröder-like methods” instead of “Schröder’s methods” since Schröder did not consider the case of a multiple zero except for $r = 2$ (see $E_2(z)$ in (1)).

Remark 2. In the recent paper [29], Sugiura and Hasegawa have shown that Schröder’s formula of the second kind of order r converges globally and monotonically to real simple zeros of polynomials on the real line for every odd $r \geq 3$ and they also proved that this formula has the same convergence property for real zeros of entire functions for every odd order $r \geq 5$. This is an important extension of the result related to the global convergence of Halley’s method ($s = 3$) presented in [30].

3. Simultaneous total-step methods based on Schröder-like methods of the first and second kind

In this section we construct two Schröder-like methods for the simultaneous determination of multiple zeros of a complex polynomial

$$P(z) = z^N + a_1z^{N-1} + \dots + a_{N-1}z + a_N$$

of degree N . These methods are based on the fourth-order Schröder-like methods E_4 and Sg_4 given in Section 2. First, we present E_4 and \mathfrak{S}_4 in the following forms for the polynomial P .

Schröder-like method of the first kind E_4 :

$$E_4(z) = z - m \frac{P(z)}{P'(z)} \left\{ \frac{1}{6}(m^2 - 6m + 11) + m(2 - m) \frac{P''(z)}{2P'(z)} \cdot \frac{P(z)}{P'(z)} + m^2 \left[\frac{1}{2} \left(\frac{P''(z)}{P'(z)} \right)^2 - \frac{P'''(z)}{6P'(z)} \right] \left(\frac{P(z)}{P'(z)} \right)^2 \right\}. \tag{5}$$

Schröder-like method of the second kind S_4 :

$$\mathfrak{S}_4(z) = z - m \frac{P(z)}{P'(z)} \cdot \frac{\frac{m+1}{2} - m \frac{P''(z)}{2P'(z)} \cdot \frac{P(z)}{P'(z)}}{\frac{1+3m+2m^2}{6} - m(m+1) \frac{P''(z)}{2P'(z)} \cdot \frac{P(z)}{P'(z)} + m^2 \frac{P'''(z)}{6P'(z)} \left(\frac{P(z)}{P'(z)} \right)^2}. \tag{6}$$

The main idea for transforming the methods (5) and (6) for a single multiple zero of the polynomial P to the corresponding methods for the simultaneous approximation of all multiple zeros of P is based on the use of the rational function $V_i(z)$ defined by

$$V_i(z) = \frac{P(z)}{\prod_{\substack{j=1 \\ j \neq i}}^n (z - v_j)^{m_j}} \quad (i \in I_n := \{1, \dots, n\}), \tag{7}$$

where I_n is the index set and v_1, \dots, v_n are some approximations to the zeros ζ_1, \dots, ζ_n of P of the respective multiplicities m_1, \dots, m_n ($n \leq N$).

Observe that the polynomial P has the same zeros as the function $V_i(z)$. To construct simultaneous methods of Schröder’s type, we replace respectively

$$V_i(z)/V_i'(z), V_i''(z)/V_i'(z), V_i'''(z)/V_i'(z), \text{ instead of } P(z)/P'(z), P''(z)/P'(z), P'''(z)/P'(z)$$

in the above iterative formulas (5) and (6). In this way we obtain

$$E_{4,i}(z) = z - m \frac{V_i(z)}{V'_i(z)} \left\{ \frac{1}{6}(m^2 - 6m + 11) + m(2 - m) \frac{V''_i(z)}{2V'_i(z)} \cdot \frac{V_i(z)}{V'_i(z)} + m^2 \left[\frac{1}{2} \left(\frac{V''_i(z)}{V'_i(z)} \right)^2 - \frac{V'''_i(z)}{6V'_i(z)} \left(\frac{V_i(z)}{V'_i(z)} \right)^2 \right] \right\} \tag{8}$$

and

$$\mathfrak{S}_{4,i}(z) = z - m \frac{V_i(z)}{V'_i(z)} \cdot \frac{\frac{m+1}{2} - m \frac{V''_i(z)}{2V'_i(z)} \cdot \frac{V_i(z)}{V'_i(z)}}{\frac{1+3m+2m^2}{6} - m(m+1) \frac{V''_i(z)}{2V'_i(z)} \cdot \frac{V_i(z)}{V'_i(z)} + m^2 \frac{V'''_i(z)}{6V'_i(z)} \left(\frac{V_i(z)}{V'_i(z)} \right)^2} \tag{9}$$

where $i \in I_n$.

Let us introduce the abbreviations

$$R_{k,i}(z) = \frac{V_i^{(k)}(z)}{V'_i(z)} \quad (k = 0, 1, \dots), \quad \delta_q(z) = \frac{P^{(q)}(z)}{P(z)}, \quad S_{q,i}(z) = \sum_{\substack{j=1 \\ j \neq i}}^n \frac{m_j}{(z - v_j)^q} \quad (q = 1, 2, \dots).$$

Note that $R_{1,i}(z) = 1$. If $z = z_i$ then we will write $\delta_{q,i}$ instead of $\delta_q(z_i)$.

Apart from the set of current approximations $(z_{1,1}, \dots, z_{n,1}) := (z_1, \dots, z_n)$, we also consider the following improved approximations $v_j = z_{j,r}$, where

$$\left\{ \begin{array}{l} z_{j,2} = E_2(z_j) = \mathfrak{S}_2(z_j) = z_j - \frac{m_j}{\delta_{1,j}} \quad (\text{Schröder's method [20], order 2}), \\ z_{j,3} = \mathfrak{S}_3(z_j) = z_j - \frac{2\delta_{1,j}}{\frac{m_j+1}{m_j}\delta_{1,j}^2 - \delta_{2,j}} \quad (\text{Halley-like method [26], order 3}), \\ z_{j,4} = \mathfrak{S}_4(z_j) = z_j - \frac{3m_j(m_j+1)\delta_{1,j}^2 - 3m_j^2\delta_{2,j}}{(1+3m_j+2m_j^2)\delta_{1,j}^3 - 3m_j(m_j+1)\delta_{1,j}\delta_{2,j} + m_j^2\delta_{3,j}} \\ \quad (\text{Farmer-Loizou's method [28], order 4}). \end{array} \right. \tag{10}$$

The second index in $z_{j,r}$ denotes the type of the approximation. Note that any other method for multiple zeros of order three and four can be applied instead of Halley-like method $z_{j,3}$ and Farmer-Loizou's method $z_{j,4}$, but we found that the iterative methods (10) give quite satisfactory results in practice.

Before constructing simultaneous methods based on $E_{4,i}$ and $\mathfrak{S}_{4,i}$, given by (8) and (9), we derive some necessary relations. Using the logarithmic derivative we find from (7)

$$\frac{V'_i(z)}{V_i(z)} = \delta_1(z) - S_{1,i}(z).$$

Hence,

$$V'_i(z) = V_i(z)(\delta_1(z) - S_{1,i}(z)). \tag{11}$$

Applying Leibniz' rule for the derivative of product of two functions $V_i(z)$ and $\delta_1(z) - S_{1,i}(z)$ that appear in (11), we find

$$\begin{aligned} V_i^{(k)}(z) &= \sum_{\lambda=0}^{k-1} \binom{k-1}{\lambda} V_i^{(k-1-\lambda)}(z) (\delta_1(z) - S_{1,i}(z))^{(\lambda)} \\ &= \sum_{\lambda=0}^{k-1} \binom{k-1}{\lambda} V_i^{(k-1-\lambda)}(z) (\delta_1^{(\lambda)}(z) + (-1)^{\lambda+1} \lambda! S_{\lambda+1,i}(z)). \end{aligned} \tag{12}$$

Dividing both sides of the last relation by $V'_i(z)$, from (12) we obtain

$$R_{k,i}(z) = \frac{V_i^{(k)}(z)}{V'_i(z)} = \sum_{\lambda=0}^{k-1} \binom{k-1}{\lambda} R_{k-1-\lambda,i}(z) (\delta_1^{(\lambda)}(z) + (-1)^{\lambda+1} \lambda! S_{\lambda+1,i}(z)),$$

or in the form

$$R_{k,i}(z) = \sum_{\lambda=0}^{k-1} \binom{k-1}{\lambda} R_{k-1-\lambda,i}(z) U_{\lambda,i}(z), \tag{13}$$

where we set

$$U_{\lambda,i}(z) = \delta_1^{(\lambda)}(z) + (-1)^{\lambda+1} \lambda! S_{\lambda+1,i}(z). \tag{14}$$

The first three members of (14) are

$$\begin{aligned} U_{0,i}(z) &= \delta_1(z) - S_{1,i}(z), \\ U_{1,i}(z) &= \delta_2(z) - \delta_1(z)^2 + S_{2,i}(z), \\ U_{2,i}(z) &= \delta_3(z) - 3\delta_1(z)\delta_2(z) + 2\delta_1^3(z) - 2S_{3,i}(z). \end{aligned} \tag{15}$$

Using (11) and (13) we obtain (having in mind that $R_{1,i}(z) = 1$)

$$\begin{aligned} R_{0,i}(z) &= \frac{1}{U_{0,i}(z)} = \frac{1}{\delta_1(z) - S_{1,i}(z)}, \\ R_{1,i} &= 1, \\ R_{2,i}(z) &= U_{0,i}(z) + R_{0,i}(z)U_{1,i}(z), \\ R_{3,i}(z) &= R_{2,i}(z)U_{0,i}(z) + 2U_{1,i}(z) + R_{0,i}(z)U_{2,i}(z), \end{aligned} \tag{16}$$

where $U_{0,i}(z)$, $U_{1,i}(z)$, $U_{2,i}(z)$ are given by (15). Now we are able to construct two simultaneous methods based on Schröder-like methods of the first and second kind (5) and (6).

Simultaneous Schröder-like method of the first kind

Starting from (5) and using (8) and (16) for $z = z_i$ and $m = m_i$, and introducing the iteration index ν , we construct the total-step method for the simultaneous determination of multiple zeros of the polynomial P :

$$\begin{aligned} E_{4,i}(z_i^{(\nu)}) \equiv z_i^{(\nu+1)} &= z_i^{(\nu)} - m_i R_{0,i}^{(\nu)} \left[\frac{1}{6}(m_i^2 - 6m_i + 11) + \frac{1}{2}m_i(2 - m_i)R_{0,i}^{(\nu)}R_{2,i}^{(\nu)} \right] \\ &\quad - m_i^3 (R_{0,i}^{(\nu)})^3 \left[\frac{1}{2}(R_{2,i}^{(\nu)})^2 - \frac{1}{6}R_{3,i}^{(\nu)} \right] \quad (i \in I_n; \nu = 0, 1, \dots). \end{aligned}$$

From computational point of view it is preferable to rewrite the above formula in a more direct form (omitting the argument z_i for simplicity)

$$\begin{aligned} E_{4,i}(z_i^{(\nu)}) \equiv z_i^{(\nu+1)} &= z_i^{(\nu)} - \frac{m_i [11(U_{0,i}^{(\nu)})^4 + 6m_i U_{1,i}^{(\nu)}(U_{0,i}^{(\nu)})^2 + 3m_i^2 (U_{1,i}^{(\nu)})^2 - m_i^2 U_{0,i}^{(\nu)} U_{2,i}^{(\nu)}]}{6(U_{0,i}^{(\nu)})^5} \\ &\quad (i \in I_n; \nu = 0, 1, \dots). \end{aligned} \tag{17}$$

In this way the values $U_{k,i}$ are directly calculated by (14), avoiding (16). Besides, the form is more convenient in convergence analysis. The same is valid for the iterative formula $S_{4,i}$ below.

Simultaneous Schröder-like method of the second kind

In a similar way as in the construction of the simultaneous method (17), we carry out the corresponding substitutions and construct the total-step method for the simultaneous determination of multiple zeros of the polynomial P :

$$\mathfrak{S}_{4,i}(z_i^{(v)}) \equiv z_i^{(v+1)} = z_i^{(v)} - \frac{3m_i R_{0,i}^{(v)}(1 + m_i - m_i R_{0,i}^{(v)} R_{2,i}^{(v)})}{1 + 3m_i + 2m_i^2 - 3m_i(1 + m_i)R_{0,i}^{(v)}R_{2,i}^{(v)} + m_i^2 R_{0,i}^{(v)}R_{3,i}^{(v)}}.$$

As above, we rewrite this iterative form in a convenient form

$$\mathfrak{S}_{4,i}(z_i^{(v)}) \equiv z_i^{(v+1)} = z_i^{(v)} - \frac{3m_i[(U_{0,i}^{(v)})^2 - m_i U_{1,i}^{(v)}]}{(U_{0,i}^{(v)})^3 - 3m_i U_{0,i}^{(v)} U_{1,i}^{(v)} + m_i^2 U_{2,i}^{(v)}} \quad (i \in I_n; v = 0, 1, \dots). \tag{18}$$

To our knowledge, the iterative formulas (17) and (18) are new ones.

Remark 3. Let us emphasize that the relations (12), (13) and (14) are general, which is very useful for the convergence analysis of simultaneous methods obtained from the families of methods (1) and (2) of arbitrary order using the presented approach by the function (7).

Remark 4. Using the $[1/n - 1]$ Padé approximation for the function $V_i(z)$ defined by (7), Sakurai, Torii and Sugiura have derived in [?] the simultaneous family $\mathfrak{S}_{r,i}$ of Schröder-König’s type of arbitrary order for simple zeros.

4. Convergence analysis

Let us introduce the errors $\varepsilon_i^{(v)} = z_i^{(v)} - \zeta_i$, $\varepsilon_{i,r}^{(v)} = z_{i,r}^{(v)} - \zeta_i$ and the null-sequences $\{\varepsilon_i^{(v)}\}$, $\{\varepsilon_{i,r}^{(v)}\}$ ($v = 0, 1, \dots$). For two members of these sequences, say, $\varepsilon_i^{(v)}$, $\varepsilon_j^{(v)}$, for which $|\varepsilon_i^{(v)}| = O(|\varepsilon_j^{(v)}|)$ holds (the same order of moduli), we will write $z = O_M(w)$, where O is the Landau symbol. In what follows we will omit the iteration index v for brevity and write, for example, z_i and \hat{z}_i instead of $z_i^{(v)}$ and $z_i^{(v+1)}$.

Let $|\varepsilon|$ be the absolute value of the error of maximal magnitude, $|\varepsilon| = \max_{1 \leq j \leq n} |\varepsilon_j|$. Assume that magnitudes of all errors $\varepsilon_1, \dots, \varepsilon_n$ are approximately of the same order, then $\varepsilon_i = O_M(\varepsilon_j)$ and $\varepsilon_j = O_M(\varepsilon)$. It is clear that $\varepsilon_{i,1} = \varepsilon_i$ (the case of current approximations, see Section 2). In addition, for convenience, terms and expressions involved in an iterative process will be proclaimed as values of ε -type if they tend to 0 when $v \rightarrow \infty$. According to Theorems 1 and 2 there follows $\varepsilon_{i,2} = O_M(|\varepsilon|^2)$ and $\varepsilon_{i,3} = O_M(|\varepsilon|^3)$. Besides, $\varepsilon_{i,4} = O_M(|\varepsilon|^4)$ for E_4 and \mathfrak{S}_4 . In overall, we have

$$\varepsilon_{i,r} = O_M(|\varepsilon|^r) \quad (r \in \{1, 2, 3, 4\}). \tag{19}$$

Applying the logarithmic derivative to the factorization $P(z) = \prod_{j=1}^n (z - \zeta_j)^{m_j}$, we obtain

$$\delta_1(z) = \frac{P'(z)}{P(z)} = \sum_{j=1}^n \frac{m_j}{z - \zeta_j}. \tag{20}$$

Successive derivation of (20) yields

$$\delta_1^{(\lambda)}(z) = (-1)^\lambda \lambda! \sum_{j=1}^n \frac{m_j}{(z - \zeta_j)^{\lambda+1}} \quad (\lambda = 1, 2, \dots). \tag{21}$$

Using (21) for $z = z_i$ we return to (14) and find

$$\begin{aligned} U_{\lambda,i}(z_i) &= \left((-1)^\lambda \lambda! \sum_{j=1}^n \frac{m_j}{(z_i - \zeta_j)^{\lambda+1}} + (-1)^{\lambda+1} \lambda! \sum_{\substack{j=1 \\ j \neq i}}^n \frac{m_j}{(z_i - z_{j,r})^{\lambda+1}} \right) \\ &= (-1)^\lambda \lambda! \left(\frac{m_i}{\varepsilon_i^{\lambda+1}} + \sum_{\substack{j=1 \\ j \neq i}}^n \frac{m_j}{(z_i - \zeta_j)^{\lambda+1}} - \sum_{\substack{j=1 \\ j \neq i}}^n \frac{m_j}{(z_i - z_{j,r})^{\lambda+1}} \right). \end{aligned}$$

Hence

$$U_{\lambda,i}(z_i) = \frac{(-1)^\lambda \lambda!}{\varepsilon_i^{\lambda+1}} \left[m_i - \varepsilon_i^{\lambda+1} \sum_{\substack{j=1 \\ j \neq i}}^n \varepsilon_{j,r} B_{ij}^{(\lambda)} \right], \tag{22}$$

where

$$B_{ij}^{(\lambda)} = m_j \cdot \frac{\sum_{\eta=0}^{\lambda} (z_i - z_{j,r})^{\lambda-\eta} (z_i - \zeta_j)^\eta}{\left[(z_i - \zeta_j)(z_i - z_{j,r}) \right]^{\lambda+1}}.$$

There are no terms (in the form of differences) involved in the expression of $B_{ij}^{(\lambda)}$ of ε -type; all of them tend to some constants as $\nu \rightarrow \infty$. Therefore, we conclude that $B_{ij}^{(\lambda)} = O_M(1)$ for $|\varepsilon|$ small enough (as ν approaches ∞), which means that these terms do not influence the order of convergence if they appear as coefficients (multipliers) in the ε -expressions such as, for example, (23).

Introduce the abbreviation

$$\mathcal{B}_{i,r}^{(\lambda)} = \sum_{\substack{j=1 \\ j \neq i}}^n \varepsilon_{j,r} B_{ij}^{(\lambda)}.$$

It is obvious that

$$\mathcal{B}_{i,r}^{(\lambda)} = \sum_{\substack{j=1 \\ j \neq i}}^n \varepsilon_{j,r} B_{ij}^{(\lambda)} = O_M(|\varepsilon|^r) \quad (\lambda = 0, 1, 2, \dots). \tag{23}$$

From (22) we have

$$U_{0,i}(z_i) = \frac{m_i - \varepsilon_i \mathcal{B}_{i,r}^{(0)}}{\varepsilon_i}, \quad U_{1,i}(z_i) = -\frac{m_i - \varepsilon_i^2 \mathcal{B}_{i,r}^{(1)}}{\varepsilon_i^2}, \quad U_{2,i}(z_i) = \frac{2(m_i - \varepsilon_i^3 \mathcal{B}_{i,r}^{(2)})}{\varepsilon_i^3}. \tag{24}$$

Now we are able to state the convergence theorems for the simultaneous methods (17) and (18). The subscript r in notations $(17)_r$ and $(18)_r$ means that the iterative formulas (17) and (18) deal with approximations $z_{1,r}, \dots, z_{n,r}$.

Theorem 3. *If the initial approximations $z_1^{(0)}, \dots, z_n^{(0)}$ are sufficiently close to the respective zeros ζ_1, \dots, ζ_n of the polynomial P , then the order of convergence of the iterative methods $(17)_r$ is $r + 4$, $r \in \{1, 2, 3, 4\}$.*

Proof. Substituting the expressions (24) in (17), we obtain the error-relation of the method (17)

$$\begin{aligned} \varepsilon_i = \hat{z}_i - \zeta_i &= \frac{\varepsilon_i^4}{6(m_i - \varepsilon_i \mathcal{B}_{i,r}^{(0)})^5} \left\{ -6\varepsilon_i^2 (\mathcal{B}_{i,r}^{(0)})^5 + 19m_i \varepsilon_i (\mathcal{B}_{i,r}^{(0)})^4 \right. \\ &\quad - 2m_i^2 (\mathcal{B}_{i,r}^{(0)})^2 (8\mathcal{B}_{i,r}^{(0)} + 3\varepsilon_i \mathcal{B}_{i,r}^{(1)}) \\ &\quad \left. + m_i^3 (12\mathcal{B}_{i,r}^{(1)} \mathcal{B}_{i,r}^{(0)} - 3\varepsilon_i (\mathcal{B}_{i,r}^{(1)})^2 + 2\varepsilon_i \mathcal{B}_{i,r}^{(0)} \mathcal{B}_{i,r}^{(2)} - 2m_i^4 \mathcal{B}_{i,r}^{(2)}) \right\}. \end{aligned} \tag{25}$$

Having in mind (23), we obtain from (25)

$$\hat{\varepsilon}_i = \frac{\varepsilon_i^4}{6\left(m_i - \varepsilon_i \sum_{\substack{j=1 \\ j \neq i}}^n \varepsilon_{j,r} \mathcal{B}_{ij}^{(0)}\right)^5} \left\{ -2m_i^4 \sum_{\substack{j=1 \\ j \neq i}}^n \varepsilon_{j,r} \mathcal{B}_{ij}^{(2)} + O_M(|\varepsilon|^{2r}) \right\}. \quad (26)$$

Since the denominator of (26) is bounded and tends to $6m_i^5$ when $|\varepsilon| \rightarrow 0$, it is sufficient to consider only the nominator of (26). In view of (23), from the error-relation (26) we conclude that

$$|\hat{\varepsilon}_i| = O_M(|\varepsilon|^{4+r}) \quad (r \in \{1, 2, 3, 4\}).$$

This completes the proof of the theorem. \square

Theorem 4. *If the initial approximations $z_1^{(0)}, \dots, z_n^{(0)}$ are sufficiently close to the respective zeros ζ_1, \dots, ζ_n of the polynomial P , then the order of convergence of the iterative methods $(18)_r$ is $r + 4$, $r \in \{1, 2, 3, 4\}$.*

Proof. Substituting the expressions (24) in (18), we get the error-relation of the method (18)

$$\hat{\varepsilon}_i = \hat{z}_i - \zeta_i = \frac{\varepsilon_i^4 \left(-(\mathcal{B}_{i,r}^{(0)})^3 + 3m_i \mathcal{B}_{i,r}^{(0)} \mathcal{B}_{i,r}^{(1)} - 2m_i \mathcal{B}_{i,r}^{(2)} \right)}{Q_i}, \quad (27)$$

where

$$Q_i = 6m_i^3 - m_i^2 \varepsilon_i \left(6\mathcal{B}_{i,r}^{(0)} + \varepsilon_i \left(3\mathcal{B}_{i,r}^{(1)} + 2\varepsilon_i \mathcal{B}_{i,r}^{(2)} \right) \right) + 3m_i \varepsilon_i^2 \mathcal{B}_{i,r}^{(0)} \left(\mathcal{B}_{i,r}^{(0)} + \varepsilon_i \mathcal{B}_{i,r}^{(1)} \right) - \varepsilon_i^3 \left(\mathcal{B}_{i,r}^{(0)} \right)^3.$$

According to (23), we get from (27)

$$\hat{\varepsilon}_i = \frac{\varepsilon_i^4 \left(-2m_i^4 \sum_{\substack{j=1, j \neq i}}^n \varepsilon_{j,r} \mathcal{B}_{ij}^{(2)} + O_M(|\varepsilon|^{2r}) \right)}{6m_i^3 + O_M(|\varepsilon|^{r+1})}, \quad (28)$$

and hence, using again (23),

$$|\hat{\varepsilon}_i| = O_M(|\varepsilon|^{4+r}) \quad (r \in \{1, 2, 3, 4\}),$$

which proves Theorem 4. \square

5. Numerical examples

In most published papers the estimate of the quality of simultaneous methods for approximating polynomial zeros has been carried out using two methodologies: (i) numerical examples which deal with sufficiently good initial approximations to the zeros or (ii) theoretical models based on the R -order of convergence of tested methods and their computational costs. In both cases there is a need for very good initial approximations, which is a difficult task. The accuracy of obtained approximations of zeros and convergence speed strictly depend on the choice of initial approximations and can vary to a great extent (model (i)). In the case of the model (ii) the computational cost is calculated using CPU execution times of arithmetic operations of the employed computer, which means that the cost can vary significantly using different computers (see, e.g. [7, Ch. 6]). Certain improvements can be attained by the normalization/scaling of CPU times and then comparing rating lists created for all employed computers. Hence, estimating a proper convergence performances of simultaneous methods, it turns out that the models (i) and (ii) are not of great importance in practice. On the other hand, numerical examples can be beneficial if the user wants to estimate the convergence speed of tested method. In this section we present results performed by

numerical examples. A model based on an empirical methodology, which gives more reliable estimate of the quality of simultaneous methods under real conditions, is considered in Section 6.

In practice, the simultaneous determination of all polynomial zeros to the desired accuracy requires the application of a complete procedure consisting of a three-stage *globally convergent composite algorithm* (see, e.g., [32]):

- (a) find an inclusion region (most frequently disk or rectangle with sides parallel to coordinate axes) in the complex plane, that includes all zeros of a given polynomial;
- (b) apply a *slowly convergent* search subdividing algorithm to get separated initial intervals (disks or rectangles) of reasonably small size so that each of them contains only one zero, and compute the order of multiplicity of that zero;
- (c) take the centers of these intervals as initial zero approximations and improve them by applying a *rapidly convergent* iterative method (such as $E_{4,i}$ or $S_{4,i}$) to the required accuracy defined by the stopping criterion.

The steps (a) and (b), described in detail in the literature (cf. [9, &1.2], [32], [33], [35]), give sufficiently small rectangles or disks; their centers, taken as initial approximations, provide the convergence of the locally convergent iterative methods. To realize the step (a) it is preferable to use some of the many formulas for finding the radius R of inclusion disk of relatively small size; 47 formulas for R are given in the book [5, pp. 28–31]; see, also, the book [23, pp. 345–359]. In this paper we have applied Henrici's inclusion disk, see Remark 5. Note that a related problem of stating initial conditions (depending on initial approximations and polynomial coefficients) that guarantee the convergence of the considered simultaneous method has been studied in [9], [31], [36]–[40], and references cited there.

The convergence speed of the proposed methods (17) and (18) has been tested on a large number of polynomial equations. Corrective approximations have been calculated by Schröder's method \mathfrak{S}_2 ($= E_2$), Halley's method \mathfrak{S}_3 and Farmer-Loizou's method \mathfrak{S}_4 , given by (10). We stress that the calculation of these corrections is carried out using the already calculated values of P , P' , P'' , P''' at the points z_1, \dots, z_n . This means that the convergence rate of these iterative methods is accelerated with negligible additional computational cost. Actually, the methods with corrections require only few additional arithmetic operations per iteration so that the CPU time of each accelerated method with corrections is slightly greater compared to the CPU time necessary for the execution of the basic method (without corrections). Therefore, the applied approach with corrections provides a *high computational efficiency* of the proposed methods with corrections.

Numerical experiments have demonstrated very fast convergence of the proposed methods. To illustrate the effectiveness of the proposed methods, among many tested algebraic polynomials we have selected two examples. We have calculated Euclid's norm

$$e^{(v)} := \|z^{(v)} - \zeta\|_2 = \left(\sum_{i=1}^n |z_i^{(v)} - \zeta_i|^2 \right)^{1/2} \quad (v = 0, 1, \dots)$$

as a measure of accuracy of approximations obtained in the v -th iteration.

When testing any root-finding method it is always useful to examine its convergence behavior in practical implementation and compare the obtained data with theoretical results. For this reason we have calculated the so-called *computational order of convergence* (COC, for brevity) ρ_c using the formula

$$\rho_c = \frac{\log(e^{(v)}/e^{(v-1)})}{\log(e^{(v-1)}/e^{(v-2)})} \quad (29)$$

(see Tables 2, 4, 6 and 8). The last formula was given in [41] in metric space so it can be used for simultaneous methods.

Results of the third iteration in Tables 2–9 are included to demonstrate very fast convergence of the new families of zero-finding methods and good matching of their computational orders of convergence with the theoretical ones given in Theorems 3 and 4 (see the last row in Tables 2, 4, 6 and 8). In practice, two iterations are usually sufficient.

To prevent the loss of significant digits of the produced approximations, we have implemented the considered methods using computer algebra system *Mathematica* with multi-precision arithmetic.

Example 1. The total step methods $(17)_{|r=1,2,3,4}$ and $(18)_{|r=1,2,3,4}$ have been applied for approximating all zeros of the polynomial of degree $N = 31$

$$P_1(z) = (z - 4)^2(z + 1)^4(z^4 - 16)^3(z^2 + 9)^3(z^2 - 2z + 5)(z + 3)^2(z + 4)^3.$$

The values of the zeros of this polynomial and respective multiplicities can be easily recognized from the factorization of $P_1(z)$. The following initial approximations have been chosen:

$$\begin{aligned} z_1^{(0)} &= 4.2 + 0.1i, & z_2^{(0)} &= -1.1 + 0.2i, & z_3^{(0)} &= 2.2 + 0.1i, & z_4^{(0)} &= -2.2 - 0.1i, \\ z_5^{(0)} &= 0.1 + 2.2i, & z_6^{(0)} &= 0.1 - 2.2i, & z_7^{(0)} &= 0.2 + 3.2i, & z_8^{(0)} &= 0.2 - 3.2i, \\ z_9^{(0)} &= 1.1 + 2.2i, & z_{10}^{(0)} &= 1.1 - 2.2i, & z_{11}^{(0)} &= -3.1 - 0.3i, & z_{12}^{(0)} &= -3.9 - 0.2i. \end{aligned}$$

Methods	$(17)_{r=1}$	$(17)_{r=2}$	$(17)_{r=3}$	$(17)_{r=4}$
$e^{(1)}$	7.60(-3)	1.06(-3)	5.91(-4)	1.65(-4)
$e^{(2)}$	4.12(-12)	3.01(-20)	4.96(-25)	3.88(-33)
$e^{(3)}$	4.43(-60)	3.87(-119)	7.39(-171)	1.44(-260)
COC ρ_c (29)	5.177	5.979	6.967	7.944

Table 1: Example 1: Euclid’s error norms – total-step methods (17)

Methods	$(18)_{r=1}$	$(18)_{r=2}$	$(18)_{r=3}$	$(18)_{r=4}$
$e^{(1)}$	3.18(-3)	1.24(-3)	5.03(-4)	1.65(-4)
$e^{(2)}$	2.55(-14)	5.51(-20)	2.00(-25)	4.27(-33)
$e^{(3)}$	1.32(-69)	1.91(-117)	3.25(-174)	2.50(-260)
COC ρ_c (29)	4.983	5.960	6.953	7.949

Table 2: Example 1: Euclid’s error norms – total-step methods (18)

All tested methods have started with the same initial approximations with $e^{(0)} \approx 0.812$. The error norms $e^{(v)}$ ($v = 1, 2, 3$) are displayed in Tables 1 and 2, where $A(-q)$ means $A \times 10^{-q}$. The same notation is used in Tables 3 and 4.

Example 2. Applying the same methods as in Example 1, we have calculated the approximations to the zeros of the polynomial of degree $N = 43$

$$P_2(z) = (z^2 - 1)^3(z^2 + 1)^4(z^2 + 2z + 5)^2(z^4 - 81)^3(z^2 - 4z + 13)^3(z - 5)^7.$$

The values of the zeros of this polynomial and respective multiplicities can be detected from the factorization form of $P_2(z)$. The following initial approximations have been chosen:

$$\begin{aligned} z_1^{(0)} &= 1.2 + 0.3i, & z_2^{(0)} &= -1.2 + 0.3i, & z_3^{(0)} &= 0.3 + 1.2i, & z_4^{(0)} &= 0.3 - 1.2i, \\ z_5^{(0)} &= -1.3 + 2.2i, & z_6^{(0)} &= -1.3 - 2.2i, & z_7^{(0)} &= 3.2 - 0.3i, & z_8^{(0)} &= -3.2 - 0.3i, \\ z_9^{(0)} &= 0.3 + 3.2i, & z_{10}^{(0)} &= 0.3 - 3.2i, & z_{11}^{(0)} &= 2.3 + 2.8i, & z_{12}^{(0)} &= 2.3 - 2.8i, \\ z_{13}^{(0)} &= 5.2 + 0.3i. \end{aligned}$$

Methods	(17) _{r=1}	(17) _{r=2}	(17) _{r=3}	(17) _{r=4}
$e^{(1)}$	1.01(−2)	3.15(−3)	1.19(−3)	3.53(−4)
$e^{(2)}$	1.34(−12)	7.04(−18)	2.71(−23)	1.33(−30)
$e^{(3)}$	3.63(−62)	2.41(−105)	5.41(−160)	1.91(−242)
COC ρ_c (29)	5.020	5.970	6.959	8.017

Table 3: Example 2: Euclid’s error norms – total-step methods (17)

Methods	(18) _{r=1}	(18) _{r=2}	(18) _{r=3}	(18) _{r=4}
$e^{(1)}$	6.06(−3)	2.80(−3)	1.08(−3)	3.61(−4)
$e^{(2)}$	1.70(−13)	4.29(−18)	1.15(−23)	1.39(−30)
$e^{(3)}$	6.70(−66)	1.88(−106)	5.77(−163)	2.59(−242)
COC ρ_c (29)	4.964	5.964	6.974	8.016

Table 4: Example 2: Euclid’s error norms – total-step methods (18)

The error norms $e^{(v)}$ ($v = 1, 2, 3$) are given in Tables 1–4. All tested methods have started with the same initial approximations with $e^{(0)} = 1.3$.

From Tables 1–4 and a number of tested polynomial equations we have concluded that the proposed methods $E_{4,i}$ and $\mathfrak{S}_{4,i}$ converge very fast, especially those with corrections. Two iterative steps are most frequently sufficient in solving most practical problems when initial approximations are reasonably good and polynomials are well-conditioned. From our experiments we have observed that the simultaneous methods $\mathfrak{S}_{4,i}$ of Schröder-König’s type have produced more accurate approximations to the zeros in most examples compared to $E_{4,i}$. This advantage can be explained by widely adopted conjecture (not proved yet in general) that iterative root-finding formulas in the form of a rational approximation give better results than iterative formula of polynomial type. Observe that Schröder’s methods (4) of the second kind \mathfrak{S}_r have the form of a rational function while Schröder’s methods (1) of the first kind E_r have polynomial form. A detailed comparison study of Schröder’s methods of the first and second kind, based on computer visualization by basins of attraction, has been presented in [42].

6. Analysis of global convergence characteristics by Aberth-like trajectories

As discussed in Section 5, applying any simultaneous method for finding polynomial zeros, its convergence performance decisively depends on the choice of initial approximations. This difficult task was considered only in few references, see, e.g. [32], [35], [43], [44]. To provide a better insight into global convergence characteristics of the proposed methods, we have employed an empirical methodology based on computer visualization of the flow of iterative process. For simplicity, we have considered only polynomials with simple zeros and the basic methods (17)_(r=1) and (18)_(r=1) of order five. In our experiments we have used the *Aberth-like distribution* of initial approximations [45], presented below:

(i) Find an *inclusion disk* C in the complex plane that encloses all zeros of the polynomial $P(z) = z^n + a_1z^{n-1} + \dots + a_{n-1}z + a_n$ expressed by its coefficients and the degree n of P .

(ii) Initial approximations are determined using Aberth’s formula [45]

$$z_i^{(0)} = -\frac{a_1}{n} + R \cdot \exp\left(i\left(\frac{\pi}{n}(2k - 3/2)\right)\right) \quad (k = 1, \dots, n),$$

where R is the *radius* of the *inclusion disk* C .

(iii) Iterate the basic methods (17)_(r=1) and (18)_(r=1) until the fulfilment of the termination criterion

$$\max_{1 \leq i \leq n} |P(z_i^{(v)})| < 10^{-5} \text{ for some } v \leq 100.$$

If ν exceeds 100, stop further iterating and proclaim the iterative process unsuccessful.

Remark 5. Note that Aberth did not request the enclosure of zeros in his original paper [45] so that we use the term “Aberth-like distribution.” It is evident from (ii) that the initial approximations are equidistantly spaced along the circle

$$C = \{z : |z + a_1/n| = R\}.$$

The circle C is centered at the point $-a_1/n$, the barycenter of the zeros of ζ_1, \dots, ζ_n of the polynomial P (because of $\zeta_1 + \dots + \zeta_n = -a_1$). The inclusion radius R of relatively small size can be determined using some of numerous formulas given in the books [5, pp. 28–31] and [23, pp. 345–359]. One of the simplest is Henrici’s formula for the inclusion radius [33, p. 457]

$$R = 2 \max_{1 \leq \lambda \leq n} |a_\lambda|^{1/\lambda},$$

which gives satisfactory results in practice. Note that the last formula was known to Fujiwara [34].

Performing the visualization of the flow of iterative processes defined by (17)_($r=1$) and (18)_($r=1$), we have marked positions of approximations in the course of iterative procedure by the points in the complex plane creating trajectories. For demonstration, we have displayed in Figures 1–16 the trajectories for the polynomials $P_3 - P_9$, given below. To get as far more convincing results, we paid attention to select polynomials of various types, see Remark 6.

Remark 6. The polynomial P_3 is Wilkinson’s ill-conditioned polynomial of degree $n = 18$, often a hard nut to crack for most methods. P_4 has two rings of zeros, P_5 has a cluster of zeros close to 0, P_6 has three rings of zeros, P_7 has, among others, zeros lying on two rings centered at the origin, P_8 is a polynomial of Mignotte’s type

$$P(z) = z^n - (az - 1)^2, \quad (n = 25, a = 9).$$

P_9 is the random polynomial with coefficients that randomly take the values -1 or $+1$. Since the zeros of P_9 are clustered and distributed on a circle centered at 0, the final part of trajectories are zoomed in Figures 8 (for $E_{4,i}$) and 16 (for $\mathfrak{S}_{4,i}$) for better insight.

$$P_3(z) = \prod_{m=1}^{18} (z - m)$$

$$P_4(z) = z^{21} + 7z^{20} - 9765626z^{11} - 68359382z^{10} + 9765625z + 68359375$$

$$P_5(z) = z^{25} + (1 + 12i)z^{20} + (1 - 12i)z^{15} + (2 + 5i)z^{10} + (2 - 5i)z^5 + 10$$

$$P_6(z) = z^{18} - 666z^{12} - 45991z^6 + 46656$$

$$\begin{aligned} P_7(z) = & z^{23} + z^{22} - 8z^{21} - 335z^{20} - 480z^{19} + 2496z^{18} - 2599z^{17} + 44823z^{16} \\ & + 84816z^{15} - 50128z^{14} + 745423z^{13} + 83160z^{12} + 167711z^{11} - 553569z^{10} \\ & + 615952z^9 - 10733808z^8 - 21778432z^7 + 12767232z^6 - 190304000z^5 + 665600z^4 \\ & - 11476736z^3 - 21864192z^2 + 12644352z - 189665280 \end{aligned}$$

$$P_8(z) = z^{25} - (9z - 1)^2$$

$$\begin{aligned} P_9(z) = & z^{30} + z^{29} - z^{28} - z^{27} + z^{26} - z^{25} - z^{24} + z^{23} - z^{22} + z^{21} + z^{20} - z^{19} + z^{18} - z^{16} \\ & + z^{15} - z^{14} + z^{13} + z^{12} - z^{11} + z^{10} + z^9 - z^8 - z^7 - z^6 + z^5 + z^4 - z^3 + z^2 + z - 1 \end{aligned}$$

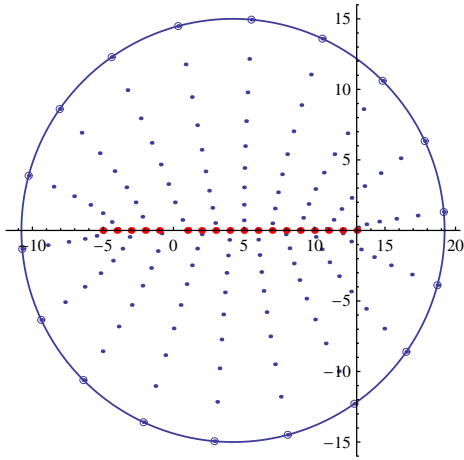


Fig. 1 $E_{4,i} - P_3(z)$, 10 iterations

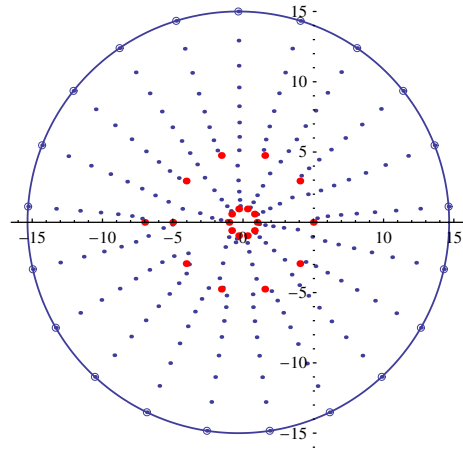


Fig. 2 $E_{4,i} - P_4(z)$, 13 iterations

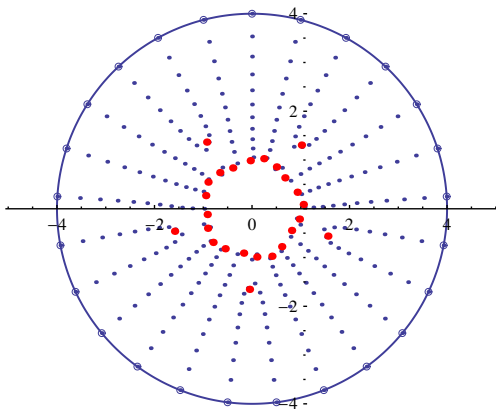


Fig. 3 $E_{4,i} - P_5(z)$, 13 iterations

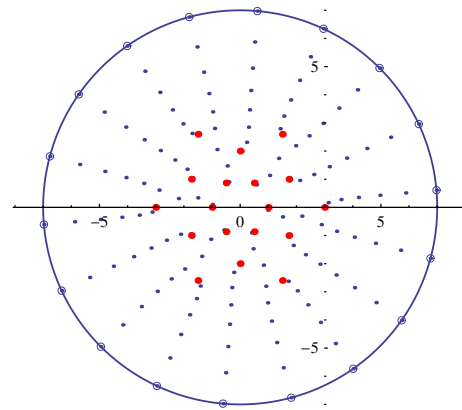


Fig. 4 $E_{4,i} - P_6(z)$, 10 iterations

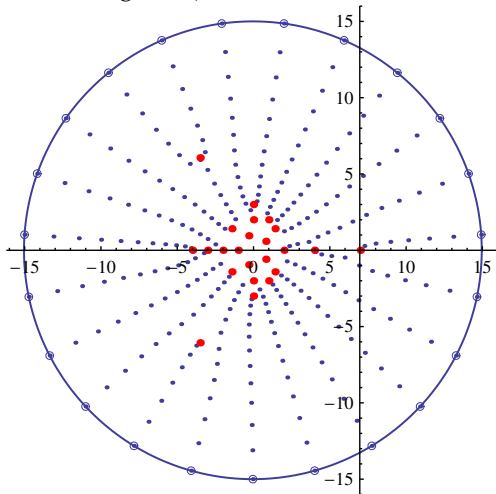


Fig. 5 $E_{4,i} - P_7(z)$, 13 iterations

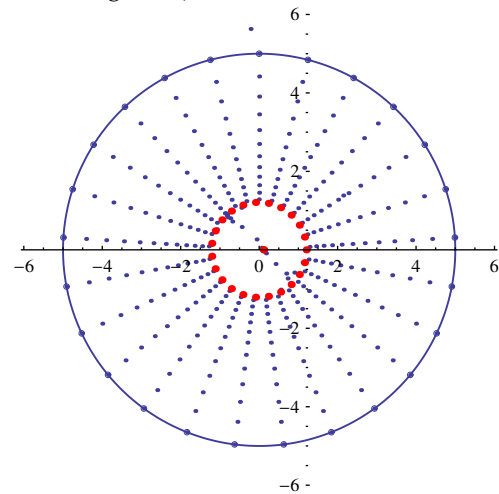


Fig. 6 $E_{4,i} - P_8(z)$, 17 iterations

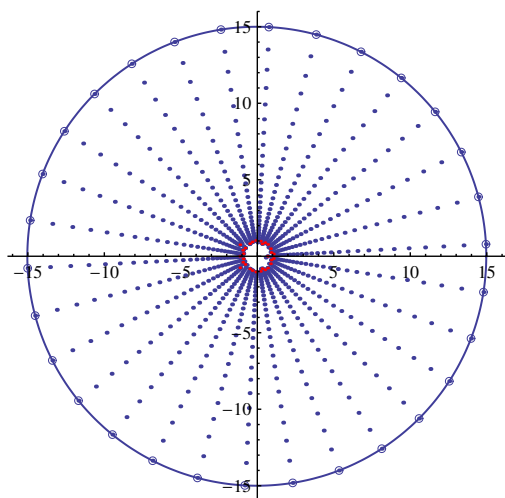


Fig. 7 $E_{4,i} - P_9(z)$, 30 iterations

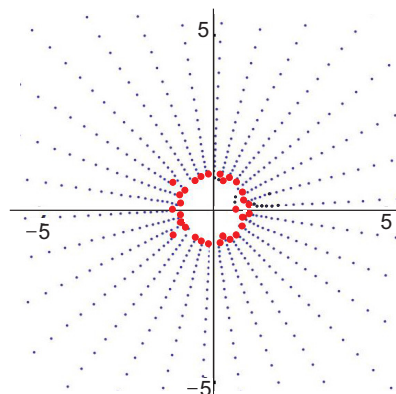


Fig. 8 $E_{4,i} - P_9(z)$ – zoomed

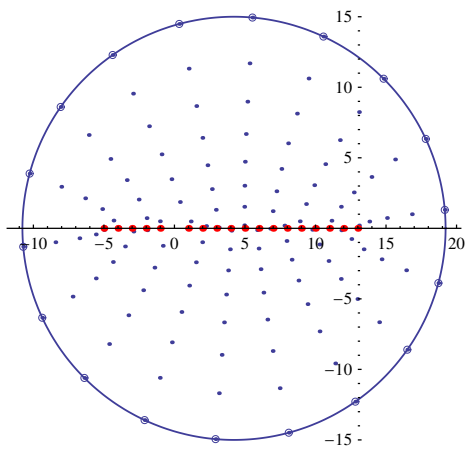


Fig. 9 $S_{4,i} - P_3(z)$, 9 iterations

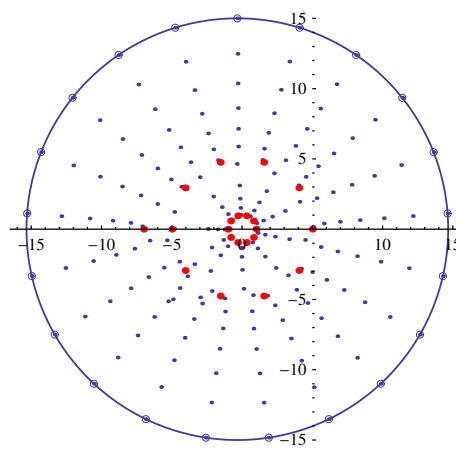


Fig. 10 $S_{4,i} - P_4(z)$, 11 iterations

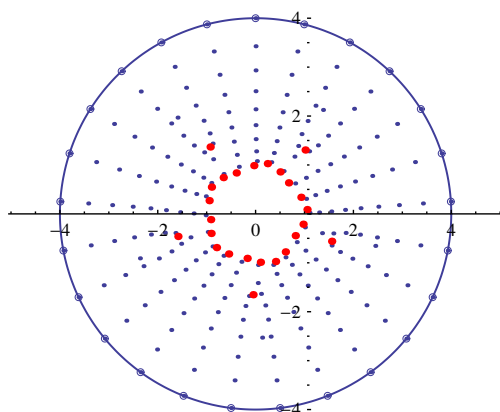


Fig. 11 $S_{4,i} - P_5(z)$, 11 iterations

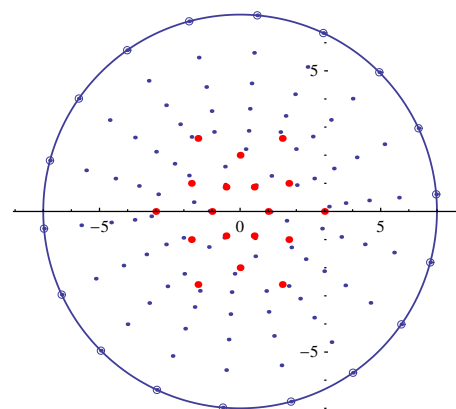


Fig. 12 $S_{4,i} - P_6(z)$, 8 iterations

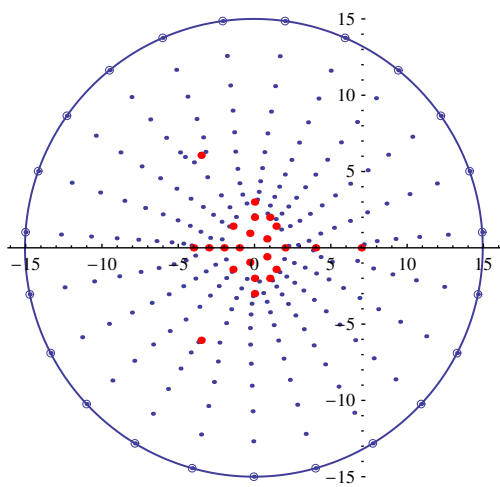


Fig. 13 $\mathfrak{S}_{4,i} - P_7(z)$, 11 iterations

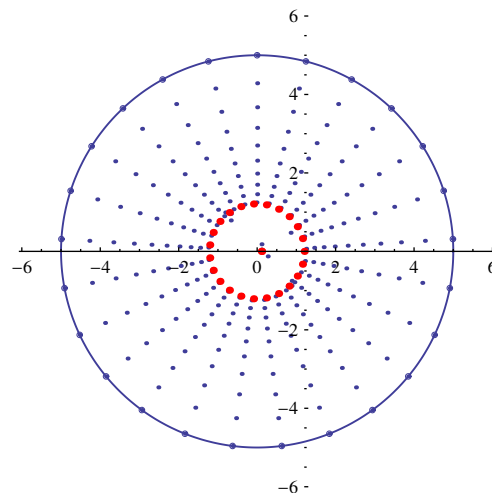


Fig. 14 $\mathfrak{S}_{4,i} - P_8(z)$, 15 iterations

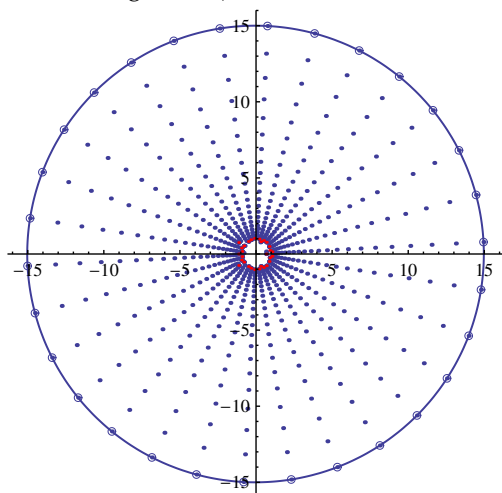


Fig. 15 $\mathfrak{S}_{4,i} - P_9(z)$, 25 iterations

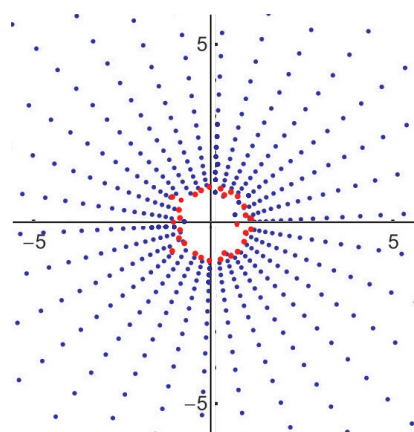


Fig. 16 $\mathfrak{S}_{4,i} - P_9(z)$ – zoomed

From Figures 1–7 for the method $E_{4,i}$ and Figure 9–15 for the method $\mathfrak{S}_{4,i}$, we observe that the trajectories for all seven polynomials $P_3 - P_9$ are almost radially distributed and have rather regular paths. To present a better insight into final iterations in the case of the polynomial P_9 , we have zoomed them in Figure 8 (for $E_{4,i}$) and Figure 16 (for $\mathfrak{S}_{4,i}$). The paths are directed straightforwardly towards the zeros during the iteration process. At the beginning (starting from the initial points equidistantly spaced on the circle), the convergence of the tested methods $(17)_{(r=1)}$ and $(18)_{(r=1)}$ is (super)linear and becomes very fast when approximations reach the neighborhoods of the target – zeros; in fact, both methods attain the proper order (equal to the theoretical value) only in the last two or three iterations. The current approximations of zeros, marked by the points, are almost directly striving to the zeros with very small variations, demonstrating excellent convergence behavior. It is important to say that the method $(18)_{(r=1)}$ reaches the stopping criterion in less iterations than the method $(17)_{(r=1)}$. The number of iterations is displayed in the presented figures. This fact confirms the advantage of the method $(18)_{(r=1)}$ related to $(17)_{(r=1)}$, which was also demonstrated by numerical examples in Section 5.

In our experiment we have tested about 20 polynomial equations and always faced very good convergence performances. According to these results, one could say that the methods $(17)_{(r=1)}$ and $(18)_{(r=1)}$ show globally convergent behavior for the set of chosen polynomials. However, a theoretical proof of global convergence is very difficult task and it remains as one of the most challenging open problems in the theory

of root-finding algorithms. Recall that, at present, this very important problem has been proved only for the Weierstrass-Dochev method [46]

$$z_i^{(v+1)} = z_i^{(v)} - \frac{P(z_i^{(v)})}{\prod_{\substack{j=1 \\ j \neq i}}^n (z_i^{(v)} - z_j^{(v)})} \quad (v = 0, 1, 2, \dots)$$

for the polynomial equations $z^2 + a = 0$ and $z^3 = 0$.

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References

- [1] I. Petković, B. Neta, On an application of symbolic computation and computer graphics to root-finders: The case of multiple roots of unknown multiplicity, *J. Comp. Appl. Math.* 308 (2016) 215–230.
- [2] I. Petković, Đ. Herceg, Symbolic computation and computer graphics as tools for developing and studying new root-finding methods, *Appl. Math. Comput.* 295 (2017) 95–113.
- [3] I. Petković, Đ. Herceg, Computer methodologies for comparison of computational efficiency of simultaneous methods for finding polynomial zeros, *J. Comp. Appl. Math.* 368 (2020) 112513.
- [4] I. Petković, Đ. Herceg, Computers in mathematical research: the study of three-point root-finding methods, *Numer. Algor.* online doi.org/10.1007/s11075-019-00796-6 (2019).
- [5] J. M. McNamee, *Numerical Methods for Roots of Polynomials, Part I*, Elsevier, Amsterdam, 2007.
- [6] J. M. McNamee, V. Pan, *Numerical Methods for Roots of Polynomials, Part II*, Elsevier, Amsterdam, 2013.
- [7] M. S. Petković, *Iterative Methods for Simultaneous Inclusion of Polynomial Zeros*, Springer-Verlag, Berlin-Heidelberg-New York, 1989.
- [8] Bl. Sendov, A. Andreev, N. Kjurkchiev, *Numerical Solution of Polynomial Equations*, Handbook of Numerical Analysis (eds P. Ciarlet, J. Lions), Vol.III, Amsterdam, Elsevier, 1994, pp. 625–778.
- [9] M. S. Petković, *Point Estimation of Root Finding Methods*, Springer, Berlin-Heidelberg, 2008.
- [10] G. Alefeld, J. Herzberger, *Introduction to Interval Computation*, Academic Press, New York, 1983.
- [11] C. Carstensen, M. S. Petković, On iteration methods without derivatives for the simultaneous determination of polynomial zeros, *J. Comput. Appl. Math.* 45 (1993) 251–266.
- [12] D. A. Bini, Numerical computation of polynomial zeros by means of Aberth's method, *Numer. Algor.* 13 (1996) 179–200.
- [13] M. Farmer, *Computing the Zeros of Polynomials using the Divide and Conquer Approach* (doctoral dissertation), Department of Computer Science and Information Systems, Birkbeck, University of London, 2013.
- [14] <http://mathworld.wolfram.com/SignificanceArithmetic.html>
- [15] E. R. Vrscaj, W.J. Gilbert, Extraneous fixed points, basin boundaries and chaotic dynamics for Schröder and König rational functions, *Numer. Math.* 52 (1988) 1–16.
- [16] B. D. Stewart, *Attractor Basins of Various Root-finding Methods*, M.S. thesis, Naval Postgraduate School, Department of Applied Mathematics, Monterey, CA, June 2001.
- [17] J. L. Varona, Graphic and numerical comparison between iterative methods, *Math. Intelligencer* 24 (2002) 37–46.
- [18] B. Kalantari, *Polynomial Root-Finding and Polynomiography*, World Scientific, New Jersey, 2008.
- [19] J. F. Traub, *Iterative Methods for the Solution of Equations*, Prentice Hall, New York, 1964.
- [20] E. Schröder, Über unendlich viele Algorithmen zur Auflösung der Gleichungen, *Math. Ann.* 2 (1870) 317–365.
- [21] J. König, Über eine Eigenschaft der Potenzreihen, *Math. Anal.* 23 (1884) 447–449.
- [22] M. S. Petković, Đ. Herceg, On rediscovered iteration methods for solving equations, *J. Comput. Appl. Math.* 107 (1999) 275–284.
- [23] B. Kalantari, *Polynomial Root-Finding and Polynomiography*, World Scientific, New Jersey, 2009.
- [24] B. Jovanović, A method for obtaining iterative formulas of higher order, *Mat. Vesnik* 9 (1972) 365–369.
- [25] G. V. Milovanović, A method to accelerated iterative processes in Banach space, *Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz.*, No 461–No 479 (1974) 67–71.
- [26] E. Bodewig, Sur la méthode Laguerre pour l'approximation des racines de certaines équations algébriques et sur la critique d'Hermite, *Indag. Math.* 8 (1946) 570–580.
- [27] M. S. Petković, L. D. Petković, Đ. Herceg, On Schröder's families of root-finding methods, *J. Comp. Appl. Math.* 233 (2010) 1755–1762.
- [28] M. R. Farmer, G. Loizou, An algorithm for the total, or partial, factorization of a polynomial, *Math. Proc. Camb. Phil. Soc.* 82 (1977) 427–437.
- [29] H. Sugiura, T. Hasegawa On the global convergence of Schröder's iteration formula for real zeros of entire functions, *J. Comp. Appl. Math.* 358 (2019) 136–145.
- [30] M. Davies, B. Dawson, On the global convergence of Halley's iteration formula, *Numer. Math.* 24 (1975) 133–135.

- [31] M. S. Petković, G. V. Milovanović, A note on some improvements of the simultaneous methods for determination of polynomial zeros, *J. Comput. Math. Appl.* 9 (1983) 65–69.
- [32] M. R. Farmer, G. Loizou, Locating multiple zeros interactively, *Comput. Math. Appl.* 11 (1985) 595–603.
- [33] P. Henrici, *Applied and Computational Complex Analysis*, Vol. I. John Wiley and Sons, New York, 1974.
- [34] M.F. Fujiwara, Über die obere Schranke des absoluten Betrages der Wurzeln einer algebraischen Gleichung, *Tohoku Math. J.* 10 (1916) 167–171.
- [35] M. Farmer, *Computing the Zeros of Polynomials using the Divide and Conquer Approach* (Ph.D. thesis), Department of Computer Science and Information Systems, Birkbeck, University of London, 2013.
- [36] P. D. Proinov, General local convergence theory for a class of iterative processes and its applications to Newton's method, *J. Complexity*, 25 (2009) 38–62.
- [37] P. D. Proinov, New general convergence theory for iterative processes and its applications to Newton-Kantorovich type theorem, *J. Complexity* 26 (2010) 3–42.
- [38] P. D. Proinov, Relationships between different types of initial conditions for simultaneous root finding methods, *Appl. Math. Lett.* 52 (2016) 102–111.
- [39] P. D. Proinov, General convergence theorem for iterative processes and applications to the Weierstrass root-finding methods, *J. Complexity* 33 (2016) 118–144.
- [40] P. D. Proinov, Unified convergence analysis for Picard iteration in n -dimensional vector spaces, *Calcolo* 55 (2018), no. 1, Art 6, 21 pages.
- [41] L. A. Jay, A note on q -order of convergence, *BIT* 41 (2001) 422–429.
- [42] M. S. Petković, L. D. Petković, Dynamic study of Schröder's families of first and second kind, *Numer. Algor.* 78 (2018) 847–865.
- [43] M. Farmer, G. Loizou, S. Maybank, Iteration functions re-visited, *J. Comput. Appl. Math.* 311 (2017) 484–496.
- [44] N. V. Kyurkchiev, *Initial Approximations and Root Finding Methods*, Wiley-VCH, Berlin, 1998.
- [45] O. Aberth, Iteration methods for finding all zeros of a polynomial simultaneously, *Math. Comp.* 27 (1973) 339–344.
- [46] K. Dochev, A modified Newton method for simultaneous approximate solution of algebraic equations, *Phys. Math. J. Bulg. Acad. Sci* 5 (1962) 136–139.