Filomat 34:7 (2020), 2141–2150
https://doi.org/10.2298/FIL2007141G

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Inclusion and Convolution Features of Univalent Meromorphic Functions Correlating with Mittag-Leffler Function

F. Ghanim[a], Hiba F. Al-Janaby[b]

[a] *Department of Mathematics, College of Science, University of Sharjah, Sharjah, United Arab Emirates*
[b] *Department of Mathematics, College of Science, University of Baghdad, Baghdad-Iraq*

Abstract. The so-called Mittag-Leffler function (M-LF) provides solutions to the fractional differential or integral equations with numerous implementations in applied sciences and other allied disciplines. During the previous century, the interest in M-LF has significantly developed and a variety of extensions and generalizations forms of the M-LF have been posed. Moreover, M-LF played a distinguished and important role in Geometric Function Theory (GFT). The intent of the current study is to reveal various inclusion and convolution features for a specific subclass of univalent meromorphic functions correlating with the integrodifferential operator containing an extended generalized M-LF. Some consequences of the major geometric outcomes are also presented.

1. Introduction

The Mittag-Leffler Function (M-LF) has consistently been a theme that inspired numerous investigators due to its significant role in diverse problems in fractional calculus, operator theory, mathematical analysis and other fields allied with sciences and engineering. Therefore, researchers were more curious of M-LF conduct and consequently extended their outcomes to the complex domain. In Geometric Function Theory (GFT), M-LF is a fabulous gadget that has been hired to propose a lot of new operators and discuss several geometric features of regulatory (analytical) functions. This function is considered as a natural popularization of the exponential function. It is formulated of 1-parameter in 1903, initiated by Mittag-Leffler who was one of the front rank mathematicians, see [19], as:

$$\mathfrak{M}_{\varrho}(z) = \sum_{\kappa=0}^{\infty} \frac{z^{\kappa}}{\Gamma(\varrho\kappa + 1)} \quad (\Re\{\varrho\} > 0,\ z \in \mathbb{C}).$$

Since then, the study of M-LF has become one of the pivotal sorts in Special Function Theory (SFT), as lots of intensive studies have been conducted in this area. In 1905, the first generalized M-LF of 2-parameters was proposed by Wiman [[42] and [43]] as follow:

$$\mathfrak{M}_{\varrho,\tau}(z) = \sum_{\kappa=0}^{\infty} \frac{z^{\kappa}}{\Gamma(\varrho\kappa + \tau)} \quad (\tau \in \mathbb{C},\ \Re(\varrho) > 0,\ z \in \mathbb{C}).$$

2010 *Mathematics Subject Classification*. Primary 30C45; Secondary 30C50, 30C10
Keywords. ittag-Leffler function, univalent functions, meromorphic function, differential subordination, starlike function, convex function
Received: 27 May 2019; Revised: 31 December 2019; Accepted: 14 January 2020
Communicated by Hari M. Srivastava
Email addresses: fgahmed@sharjah.ac.ae (F. Ghanim), fawzihiba@yahoo.com (Hiba F. Al-Janaby)

Apart from the extensions provided by Srivastava [28] in 1968, Prabhakar [22] in 1971 given a formula of generalized M-LF of 3-parameters, namely Prabhakar function, as:

$$\mathfrak{M}^{a}_{\varrho,\tau}(z) = \sum_{\kappa=0}^{\infty} \frac{(a)_\kappa}{\Gamma(\varrho\kappa+\tau)} \frac{z^\kappa}{\kappa!} \quad (\tau,\ a \in \mathbb{C},\ \Re(\varrho) > 0,\ z \in \mathbb{C}).$$

In this context, Srivastava contributed significantly to the study of various features, extensions, generalizations and implementations of M-LF. For their main contributions, in 2009, Srivastava and Tomovski [29] studied a further generalization of M-LF as:

$$\mathfrak{M}^{a}_{\varrho,\tau}(z) = \sum_{\kappa=0}^{\infty} \frac{(a)_{\mathsf{n}\kappa}}{\Gamma(\varrho\kappa+\tau)} \frac{z^\kappa}{\kappa!} \quad (\tau,\ a \in \mathbb{C},\ \Re(\varrho) > \max\{0, \Re(\mathsf{n})-1\},\ \Re(\mathsf{n}) > 0\ z \in \mathbb{C}).$$

The authors also investigated a generalized fractional integral operator involving this more general type of M-LF. Later, in 2010, Tomovskia, Hilferb and Srivastava [39] derived several compositional features, which correlate with specializing the corresponding M–LF considered in [29] and Hardy-type inequalities for the generalized fractional derivative operator. In 2016, Srivastava [32] presented a review of some recent evolutions including different classes of the M-LF type which are connected with several families of generalized Riemann-Liouville and other relevant fractional derivative operators. The following year, Srivastava and Bansal [33] studied a certain class of q-M-LF and investigated sufficient conditions under which it is close-to-convex. At the same time, Srivastava *et al.* [34] imposed a new convolution operator in the form of the generalized M-LF based on the extensively-studied Fox-Wright function in the right-half of the open unit disk where $\Re(z) > 0$. In 2018, Kumar, Choi and Srivastava [17] investigated the solution of the generalized fractional kinetic equation involving another generalized M-LF by applying the Sumudu transform and the Laplace transform techniques. That same year, Srivastava *et al.* [36] introduced a general fractional integral operator that contains the generalized multi-index M-LF type defined by Saxena and Nishimoto [26]. The authors additionally obtained several results which generalize the corresponding results acquired earlier by Kilbas *et al.* [16] and Srivastava and Tomovski [29] . Recently, in 2019, Srivastava *et al.* [38] provided new relations between the M–LF of one, two and three parameters by using Riemann–-Liouville fractional calculus. Ur Rehman, Ahmad, Srivastava and Khan [40] determined lower bounds for the ratio of several normalized q-M-LF and their sequences of partial sums. Actually, the analysis of M-LF and its various formulas have attracted considerable interest, and many researchs has thereafter emerged on this topic. For instance, Srivastava *et al.* ([31], [35]), Rahman *et al.* [24], Agarwal *et al.* [1], Al-Janaby [3], Al-Janaby and Ahmad [4], Al-Janaby and Darus [5] and others.

Related to this line of study, in 2015, Parmar [23] presented a gorgeously new global family of the extended generalized M-LF as:

$$\mathfrak{M}^{(\{a_\iota\}_{\iota\in\mathbb{N}_0};\sigma)}_{\varrho,\tau}(z;s) = \sum_{\kappa=0}^{\infty} \frac{\mathcal{B}^{(\{a_\iota\}_{\iota\in\mathbb{N}_0})}(\sigma+\kappa, 1-\sigma; s)}{B(\sigma, 1-\sigma)} \frac{z^\kappa}{\Gamma(\varrho\kappa+\tau)} \tag{1}$$

$$(\sigma,\ \varrho,\ \tau, \in \mathbb{C};\ \Re(\sigma) > 0,\ \Re(\varrho) > 0,\ \Re(\tau) > 0;\ s \geq 0;\ z \in \mathbb{C}).$$

Here, $\mathbb{C}$ is the complex plane, $B(\sigma, 1-\sigma)$ is the classical Beta function, and $\mathcal{B}^{(\{a_\iota\}_{\iota\in\mathbb{N}_0})}(\sigma+\kappa, 1-\sigma; s)$ is the extended classic Beta function given by [23] as:

$$\mathcal{B}^{(\{a_\iota\}_{\iota\in\mathbb{N}_0})}(\varrho, \tau; s) = \int_0^1 \psi^{\varrho-1}(1-\psi)^{\tau-1}\Theta\left(\{a_\iota\}_{\iota\in\mathbb{N}_0}; -\frac{s}{\psi(1-\psi)}\right)d\psi$$

$$(\min\{\Re(\varrho), \Re(\tau)\} > 0;\ \Re(s) \geq 0),$$

where $\Theta(\{a_\iota\}_{\iota\in\mathbb{N}_0}; z)$ refers to the function of an appropriately bounded sequence $\{a_\iota\}_{\iota\in\mathbb{N}_0}$ of arbitrary complex or real numbers, which is considered by Srivastava *et al.* [30] as:

$$\Theta(\{a_\iota\}_{\iota\in\mathbb{N}_0}; z) = \begin{cases} \sum_{j=0}^{\infty} a_\iota \frac{z^j}{j!} & (|z| < R;\ 0 < R < \infty; a_\iota = 1), \\ \mathrm{M}_0\, z^{\upsilon}\, exp(z)\left[1 + O\left(\frac{1}{z}\right)\right] & (\Re(z) \to \infty;\ \mathrm{M}_0 > 0;\ \upsilon \in \mathbb{C}). \end{cases}$$

On the other hand, several general classes of various types of special functions were discussed and presented recently by Srivastava *et al.* [37]. A class of the relatively more general incomplete *H*-functions as well as their special cases as the incomplete Fox-Wright generalized hypergeometric functions are introduced. Moreover, various features of incomplete *H*-functions, including decomposition and reduction formulas, derivative formulas, some integral transforms, computational representations, and so on are studied.

In the treatment of univalent regular functions in open unit disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$, it is sufficient to consider class $\mathcal{A}$ which involves regular functions defined on Δ, normalized under the stipulations $\mathfrak{L}(0) = \mathfrak{L}'(0) - 1 = 0$. A function $\mathfrak{L}$ has a (Taylor) series of model (see, [7] and [14])

$$\mathfrak{L}(z) = z + \sum_{\kappa=2}^{\infty} c_\kappa z^\kappa \qquad (z \in \Delta). \tag{2}$$

The pivotal property of functions in class $\mathcal{A}$ is the image domain $\mathfrak{L}(\Delta)$ which describes various outstanding geometric properties, such as convex and starlike, [14]. In 1913, Study introduced a convexity property of $\mathfrak{L} \in \mathcal{A}$, see [14]. It states that, $\xi_1, \xi_2 \in \mathfrak{L}(\Delta)$ and $0 \le \delta \le 1$ imply $\delta\xi_1 + (1-\delta)\xi_2 \in \mathfrak{L}(\Delta)$; that is, the regular function $\mathfrak{L} \in \mathcal{A}$ is called convex if $\mathfrak{L}(\Delta)$ is convex. The regulatory stipulation equivalent to convexity property is:

$$\Re\left(\frac{z\mathfrak{L}''(z)}{\mathfrak{L}'(z)} + 1\right) > 0 \quad (z \in \Delta). \tag{3}$$

Let $\mathcal{S}$ refer to a subclass of $\mathcal{A}$, which includes univalent functions.

Recall that any two regular functions ϑ, φ in Δ. The regular function ϑ is called subordinate to regular function φ, symbolized by $\vartheta \prec \varphi$, if there is a regular function θ in Δ with $\theta(0) = 0$ and $|\theta(z)| < 1$, $z \in \Delta$ provided that $\vartheta(z) = \varphi(\theta(z))$. Especially, if regular function φ is univalent in Δ, thus $\vartheta \prec \varphi$ if and only if $\vartheta(0) = \varphi(0)$ and $\vartheta(\Delta) \subseteq \varphi(\Delta)$, [14].

The notion of the differential subordination was imposed by Miller and Mocanu [18] as:
Let $\Gamma : \mathbb{C}^2 \times \Delta \to \mathbb{C}$ and ν be univalent in Δ. If $\mathfrak{L}$ is regular in Δ and achieves the following differential subordination

$$\Gamma(\mathfrak{L}(z), z\mathfrak{L}'(z), z) \prec \mathfrak{J}(z), \tag{4}$$

then $\mathfrak{L}$ is called the solution of (4). The univalent function $\mathfrak{F}$ is so-called a dominant of the solutions of (4), if $\mathfrak{L} \prec \mathfrak{F}$ for all $\mathfrak{L}$ achieves (4).

Closely related to $\mathcal{S}$ is the class Ξ of meromorphic functions defined as:

$$\mathfrak{G}(z) = z + \upsilon_0 + \sum_{\kappa=1}^{\infty} \upsilon_\kappa z^{-\kappa}, \tag{5}$$

that are regular and univalent in the exterior of disk $\nabla = \{z \in \mathbb{C} : 1 < |z| < \infty\}$. The subclass Ξ_0 including all functions $\mathfrak{G} \in \Xi$ with $\upsilon_0 \neq 0$ in ∇, i.e.

$$\Xi_0 = \{\mathfrak{G} \in \Xi : \upsilon_0 \neq 0,\ z \in \nabla\}. \tag{6}$$

The transformation

$$\mathfrak{G}(z) = \frac{1}{\mathfrak{L}(1/z)} \tag{7}$$

yields $\mathfrak{G}$ in Ξ_0 if $\mathfrak{L} \in \mathcal{S}$. Conversely, it shows $\mathfrak{L}$ is in $\mathcal{S}$ if $\mathfrak{G} \in \Xi_0$, ([7] and [14]). In 1914, Gronwall [15] who first introduced the outcome with respect to the coefficient problem. the study state that, if $\mathfrak{G} \in \Xi$ then $\sum_{\kappa=1}^{\infty} \kappa |\upsilon_\kappa|^2 \leq 1$. It is popularly known as the Area Theorem. The interest of the class Ξ arose from an application of the Area Theorem in the proof of the well-known conjecture so-called the Bieberbach's conjecture for the second coefficient.

The transformation

$$\mathfrak{A}(z) = \mathfrak{G}(1/z) \quad (|1/z| > 1)\,, \tag{8}$$

takes each $\mathfrak{G}$ in Ξ into a function $\mathfrak{A}$ of the model:

$$\mathfrak{A}(z) = \frac{1}{z} + \sum_{\kappa=1}^{\infty} \upsilon_\kappa z^\kappa \quad (0 < |z| < 1)\,. \tag{9}$$

Therefore, Σ is referred to the class of univalent meromorphic functions $\mathfrak{A}$ given by (9) which are defined on the punctured disk $\Delta^* = \{z \in \mathbb{C} : 0 < |z| < 1\}$. A function $\mathfrak{A} \in \Sigma$ is convex if it is univalent and the complement of $\mathfrak{A}(\Delta^*)$ is convex. It is equivalent to the regulatory stipulation

$$\Re\left(\frac{(z\mathfrak{A}'(z))'}{\mathfrak{A}'(z)}\right) < 0 \quad (z \in \Delta^*). \tag{10}$$

Various studies dealing with meromorphic functions were made in [[6],[9],[10],[11],[13] and [31]].

By using the extended generalized M-LF given by (1), we consider a new meromorphic function $\mathcal{Q}_{\varrho,\tau}^{(\{a_\iota\}_{\iota\in\mathbb{N}_0};\sigma)}(z;s)$ as:

$$\begin{aligned}\mathcal{Q}_{\varrho,\tau}^{(\{a_\iota\}_{\iota\in\mathbb{N}_0};\sigma)}(z;s) &= \frac{B(\sigma,1-\sigma)\,\Gamma(\varrho+\tau)}{\mathcal{B}^{(\{a_\iota\}_{\iota\in\mathbb{N}_0})}(\sigma+1,1-\sigma;s)}\left[\mathfrak{M}_{\varrho,\tau}^{(\{a_\iota\}_{\iota\in\mathbb{N}_0};\sigma)}(z;s) - \frac{\mathcal{B}^{(\{a_\iota\}_{\iota\in\mathbb{N}_0})}(\sigma,1-\sigma;s)}{B(\sigma,1-\sigma)\,\Gamma(\tau)} + \frac{\mathcal{B}^{(\{a_\iota\}_{\iota\in\mathbb{N}_0})}(\sigma+1,1-\sigma;s)}{B(\sigma,1-\sigma)\,\Gamma(\varrho+\tau)}\,\frac{1}{z}\right]\\ &= \frac{1}{z} + \sum_{\kappa=1}^{\infty} \frac{\Gamma(\varrho+\tau)\,\mathcal{B}^{(\{a_\iota\}_{\iota\in\mathbb{N}_0})}(\sigma+\kappa,1-\sigma;s)}{\mathcal{B}^{(\{a_\iota\}_{\iota\in\mathbb{N}_0})}(\sigma+1,1-\sigma;s)\,\Gamma(\varrho\kappa+\tau)}\,z^\kappa \qquad (11)\\ &= \frac{1}{z} + \sum_{\kappa=1}^{\infty} \Lambda_\kappa\, z^\kappa\end{aligned}$$

where

$$\Lambda_\kappa = \frac{\Gamma(\varrho+\tau)\,\mathcal{B}^{(\{a_\iota\}_{\iota\in\mathbb{N}_0})}(\sigma+\kappa,1-\sigma;s)}{\mathcal{B}^{(\{a_\iota\}_{\iota\in\mathbb{N}_0})}(\sigma+1,1-\sigma;s)\,\Gamma(\varrho\kappa+\tau)}.$$

Since then, by employing newly meromorphic function (11), we introduce a new differential operator by the formula: for $\ell \geq 0$ and $1 \geq \mu \geq 0$

$$\begin{aligned} \mathcal{D}_{\mu}^{0}\,\mathfrak{A}(z) &= \mathfrak{A}(z),\\ \mathcal{D}_{\mu}^{1}\,\mathfrak{A}(z) &= \mathcal{D}_{\mu}\,\mathfrak{A}(z) = \mathcal{Q}_{\varrho,\tau}^{(\{a_\iota\}_{\iota\in\mathbb{N}_0};\sigma)}(z;s)\,\mathfrak{A}(z) + \mu\left[z\left(\mathcal{Q}_{\varrho,\tau}^{(\{a_\iota\}_{\iota\in\mathbb{N}_0};\sigma)}(z;s)\right)'\mathfrak{A}(z) + \frac{1}{z}\right]\\ &= \frac{1}{z} + \sum_{\kappa=1}^{\infty}(1+\mu\,\kappa)\,\Lambda_\kappa\,\upsilon_\kappa\,z^\kappa\\ &\;\vdots\\ \mathcal{D}_{\mu}^{\ell}\,\mathfrak{A}(z) &= \mathcal{D}_{\mu}(\mathcal{D}_{\mu}^{\ell-1}\mathfrak{A}(z)) = \frac{1}{z} + \sum_{\kappa=1}^{\infty}\left[(1+\mu\,\kappa)\,\Lambda_\kappa\right]^{\ell}\upsilon_\kappa\,z^\kappa. \end{aligned} \tag{12}$$

Definition 1.1. *If the function $\mathfrak{A} \in \Sigma$ achieves the following subordination stipulation*

$$(1+\zeta)\,z\left(\mathcal{D}_{\mu}^{\ell}\,\mathfrak{A}(z)\right) + \zeta z^2\left(\mathcal{D}_{\mu}^{\ell}\,\mathfrak{A}(z)\right)' \prec \wp(z)\,, \tag{13}$$

then, $\mathfrak{A}$ is in the class $\Sigma_{\mu}^{\ell}(\zeta,\wp)$, where $\zeta \in \mathbb{C}$ and $\wp(z) \in \Omega$. Where Ω is the class of regular functions $\wp(z)$ with $\wp(0) = 1$, which are convex and univalent in $\Delta = \Delta^* \cup \{0\}$.

Based on posed operator (12), for $\mathfrak{A}_\jmath$, $\mathfrak{X}_\jmath \in \Sigma$, $\jmath = 1,2,..,m$, $c > 0$, and $\eta_\jmath$, $\gamma_\jmath \geq 0$, $\jmath = 1,2,..,m$, we provide the following integrodifferential operator $\Psi_m : \Sigma^m \to \Sigma$ by

$$\Psi_m(z) = \frac{c}{z^c}\int_0^z \omega^{c-1}\prod_{\jmath=1}^{m}\left(\omega\,\mathcal{D}_{\mu}^{\ell}\,\mathfrak{A}_\iota(\omega)\right)^{\eta_\jmath}\left(-\omega^2\,\mathcal{D}_{\mu}^{\ell+1}\,\mathfrak{X}_\jmath(\omega)\right)^{\gamma_\jmath}d\omega. \tag{14}$$

The operator Ψ_m is a generalization of several integral operators which were investigated by some authors (cf., e.g., [12], [20], and [21]). Recall that, for some $0 < \beta < 1$, $\mathcal{S}^*(\beta)$ denotes the class of all starlike functions of order β defined on Δ^*. A regular function $\mathfrak{A}(z) \in \mathcal{A}$ is called a prestarlike of order β in Δ^*, if

$$\frac{z}{(1-z)^{2(1-\beta)}} * \mathfrak{A}(z) \in \mathcal{S}^*(\beta)\,,$$

where $*$ is the convolution product of two regular functions defined on Δ^*, denoted by $\mathfrak{C}(\beta)$ (see [25] and [41]). Besides, a regular function $\mathfrak{A}(z) \in \mathcal{A}$ is in the class $\mathfrak{C}(0)$ if and only if $\mathfrak{A}(z)$ is convex univalent defined on Δ^* and

$$\mathcal{R}\left(\frac{1}{2}\right) = \mathcal{S}^*\left(\frac{1}{2}\right).$$

2. Preliminary Outcomes

Lemma 2.1. [18] *Let $\mathfrak{J}(z)$ and $\wp(z)$ be two regular functions in Δ, $\wp(z)$ convex univalent with $\wp(0) = \mathfrak{J}(0)$. If,*

$$\mathfrak{J}(z) + \frac{1}{c}z\,\mathfrak{J}'(z) \prec \wp(z), \tag{15}$$

where $\Re c \geq 0$ and $c \neq 0$, then

$$\mathfrak{J}(z) \prec \widetilde{\wp}(z) = cz^{-c}\int_0^z \omega^{c-1}\wp(\omega)\,d\omega \prec \wp(z)$$

and $\widetilde{\wp}(z)$ is the best dominant of (15).

Lemma 2.2. [25] *Let* $\lambda < 1$, $\mathfrak{A}(z) \in \mathcal{S}^*(\lambda)$ *and* $\mathfrak{J}(z) \in \mathfrak{C}(\beta)$. *For any regular function* $F(z)$ *in* Δ^*, *then*

$$\frac{\mathfrak{J} * (\mathfrak{A}F)}{\mathfrak{J} * \mathfrak{A}} (\Delta^*) \subset \overline{co}\,(F(\Delta^*))\,,$$

where $\overline{co}\,(F(\Delta^*))$ *represents the convex hull of* $F(\Delta^*)$.

3. Inclusion Features

Theorem 3.1. *Let* $0 \leq \zeta_1 < \zeta_2$. *Then*

$$\Sigma_\mu^\ell\,(\zeta_2;\ \wp) \subset \Sigma_\mu^\ell\,(\zeta_1;\ \wp)$$

Proof. Let $0 \leq \zeta_1 < \zeta_2$ and consider that:

$$\mathfrak{J}\,(z) = z\left(\mathcal{D}_\mu^\ell\ \mathfrak{A}(z)\right) \tag{16}$$

for $\mathfrak{A}(z) \in \Sigma_\mu^\ell\,(\zeta_2;\ \wp)$. Thus the function $\mathfrak{J}(z)$ is regular in Δ^* with $\mathfrak{J}(0) = 1$. By differentiating both sides of (16) with respect to z and utilizing (13), we achive

$$(1+\zeta_2)\, z\left(\mathcal{D}_\mu^\ell\ \mathfrak{A}(z)\right) + \zeta_2 z^2 \left(\mathcal{D}_\mu^\ell\ \mathfrak{A}(z)\right)' = \mathfrak{T}(z) + \zeta_2 z \mathfrak{T}'\,(z) \prec \wp\,(z)\,. \tag{17}$$

Thus an implementation of Lemma 2.1 with $c = \frac{1}{\zeta_2} > 0$ gives:

$$\mathfrak{J}\,(z) \prec \wp(z). \tag{18}$$

Notice that $0 \leq \frac{\zeta_1}{\zeta_2} < 1$ and that $\wp(z)$ is convex univalent in Δ^*, thus (16), (17) and (18), yield

$$(1+\zeta_1)\, z\left(\mathcal{D}_\mu^\ell\ \mathfrak{A}(z)\right) + \zeta_1 z^2 \left(\mathcal{D}_\mu^\ell\ \mathfrak{A}(z)\right)'$$

$$= \frac{\zeta_1}{\zeta_2}\left[(1+\zeta_2)\, z\left(\mathcal{D}_\mu^\ell\ \mathfrak{A}(z)\right) + \zeta_2 z^2 \left(\mathcal{D}_\mu^\ell\ \mathfrak{A}(z)\right)'\right] + \left(1 - \frac{\zeta_1}{\zeta_2}\right)\mathfrak{T}(z) \prec \wp(z).$$

Hence, $\mathfrak{A}(z) \in \Sigma_\mu^\ell\,(\zeta_1;\ \wp)$ and the proof of Theorem 3.1 is complete.

Theorem 3.2. *Let* $\zeta > 0$, $\alpha > 0$ *and* $\mathfrak{A}(z) \in \Sigma_\mu^\ell\,(\zeta;\ \alpha\,\wp + 1 - \alpha)$. *If* $\alpha \leq \alpha_0$, *where*

$$\alpha_0 = \frac{1}{2}\left(1 - \frac{1}{\zeta}\int_0^1 \frac{u^{\frac{1}{\zeta}-1}}{1+u}du\right)^{-1} \tag{19}$$

then $\mathfrak{A}(z) \in \Sigma_\mu^\ell\,(\zeta;\ h)$.

Proof. Let us define,

$$\mathfrak{J}\,(z) = z\left(\mathcal{D}_\mu^\ell\ \mathfrak{A}(z)\right) \tag{20}$$

for $\mathfrak{A}(z) \in \Sigma_\mu^\ell\,(\zeta;\ \alpha\, h + 1 - \alpha)$ with $\zeta > 0$, and $\alpha > 0$. Hence we gain:

$$\mathfrak{J}\,(z) + \zeta z \mathfrak{J}'\,(z) = (1+\zeta)\, z\left(\mathcal{D}_\mu^\ell\ \mathfrak{A}(z)\right) + \zeta z^2 \left(\mathcal{D}_\mu^\ell\ \mathfrak{A}(z)\right)' \prec \alpha\,(\wp(z) - 1) + 1.$$

Thus an implementation of Lemma 2.1 yields:

$$\mathfrak{J}\,(z) \prec \frac{\alpha}{\zeta} z^{-\frac{1}{\zeta}} \int_0^z t^{\frac{1}{\zeta}-1}\wp(t)dt + 1 - \alpha = (\wp * \Psi)\,(z)\,, \tag{21}$$

where

$$\Psi(z) = \frac{\alpha}{\zeta} z^{-\frac{1}{\zeta}} \int_0^z \frac{t^{\frac{1}{\zeta}-1}}{1-t} dt + 1 - \alpha. \tag{22}$$

If $0 < \alpha \leq \alpha_0$, where $\alpha_0 > 1$ is defined by (19), hence from (22) we obtain:

$$\Re\Psi(z) = \frac{\alpha}{\zeta} \int_0^1 u^{\frac{1}{\zeta}-1} \Re\left(\frac{1}{1-uz}\right) du + 1 - \alpha > \frac{\alpha}{\zeta} \int_0^1 \frac{u^{\frac{1}{\zeta}-1}}{1+u} du + 1 - \alpha \geq \frac{1}{2} \qquad (z \in \Delta^*).$$

Now, by utilizing the Herglotz representation for $\Psi(z)$, (20) and (21) yield

$$z\left(\mathcal{D}_\mu^\ell \mathfrak{A}(z)\right) \prec (\wp * \Psi)(z) \prec \wp(z),$$

and because $\wp(z)$ is convex univalent in Δ^*, it follows that $\mathfrak{A}(z) \in \Sigma_\mu^\ell(\zeta; \wp)$. For $\wp(z) = \frac{1}{1-z}$ and $\mathfrak{A}(z) \in \Sigma$, we have

$$z\left(\mathcal{D}_\mu^\ell \mathfrak{A}(z)\right) = \frac{\alpha}{\zeta} z^{-\frac{1}{\zeta}} \int_0^z \frac{t^{\frac{1}{\zeta}-1}}{1-t} dt + 1 - \alpha,$$

and consequently,

$$(1+\zeta)\, z\left(\mathcal{D}_\mu^\ell \mathfrak{A}(z)\right) + \zeta z^2 \left(\mathcal{D}_\mu^\ell \mathfrak{A}(z)\right)' = \alpha\, (\wp(z) - 1) + 1.$$

Thus, $\mathfrak{A}(z) \in \Sigma_\mu^\ell(\zeta; \alpha\, \wp + 1 - \alpha)$. Also, for $\alpha > \alpha_0$, it yields:

$$\Re\, z\left(\mathcal{D}_\mu^\ell \mathfrak{A}(z)\right) \to \frac{\alpha}{\zeta} \int_0^1 \frac{u^{\frac{1}{\zeta}-1}}{1+u} du + 1 - \alpha < \frac{1}{2} \qquad (z \to -1),$$

and therefore $\mathfrak{A}(z) \notin \Sigma_\mu^\ell(\zeta; \wp)$.

4. Convolution Features

Theorem 4.1. *Let* $\mathfrak{A}(z) \in \Sigma_\mu^\ell(\zeta; \wp)$, $\mathfrak{J}(z) \in \Sigma$ *and*

$$\Re\,(z\mathfrak{J}(z)) > \frac{1}{2} \qquad (z \in \Delta^*). \tag{23}$$

Then,

$$(\mathfrak{A} * \mathfrak{J})(z) \in \Sigma_\mu^\ell(\zeta; \wp).$$

Proof. For $\mathfrak{A}(z) \in \Sigma_\mu^\ell(\zeta; \wp)$ and $\mathfrak{J} \in \Sigma$, we acquire:

$$(1+\zeta)\, z\left(\mathcal{D}_\mu^\ell (\mathfrak{A} * \mathfrak{J})(z)\right) + \zeta z^2 \left(\mathcal{D}_\mu^\ell (\mathfrak{A} * \mathfrak{J})(z)\right)'$$

$$= (1+\zeta)\, z\mathfrak{J}(z) * z\left(\mathcal{D}_\mu^\ell \mathfrak{A}(z)\right) + \zeta z\mathfrak{J}(z) * z^2 \left(\mathcal{D}_\mu^\ell \mathfrak{A}(z)\right)' = z\mathfrak{J}(z) * \Psi(z) \tag{24}$$

where

$$\Psi(z) = (1+\zeta)\, z\left(\mathcal{D}_\mu^\ell \mathfrak{A}(z)\right) + \zeta z^2 \left(\mathcal{D}_\mu^\ell \mathfrak{A}(z)\right)' \prec \wp(z). \tag{25}$$

In view of (23), the function $z\mathfrak{J}(z)$ has the Herglotz representation:

$$z\mathfrak{J}(z) = \int_{|x|=1} \frac{dc(x)}{1-xz} \qquad (z \in \Delta^*), \tag{26}$$

where $c(x)$ is a probability measure defined on the unit circle $|x| = 1$ and

$$\int_{|x|=1} dc(x) = 1.$$

Since $\wp(z)$ is convex univalent defined on Δ^*, from (24), (25) and (26), we get

$$(1+\zeta)\, z\left(\mathcal{D}_{\mu}^{\ell}\mathfrak{A}(z)\right) + \zeta z^2\left(\mathcal{D}_{\mu}^{\ell}\mathfrak{A}(z)\right)' = \int_{|x|=1} \Psi(xz)\, dc(x) \prec \wp(z).$$

Therefore $\mathfrak{A}(z) \in \Sigma_{\mu}^{\ell}(\zeta;\wp)$ and the theorem is proved.

Corollary 4.2. *Let $\mathfrak{A}(z) \in \Sigma_{\mu}^{\ell}(\zeta;\wp)$ be defined as (9) and suppose that*

$$\omega_m(z) = \frac{1}{z} + \sum_{n=1}^{m-1} a_n z^{n-1} \qquad (m \in \mathbb{N}\setminus\{1\}).$$

Then the function:

$$\sigma_m(z) = \int_0^1 t\omega_m(tz)\, dt$$

is also in $\Sigma_{\mu}^{\ell}(\zeta;\wp)$.

Proof. We have:

$$\sigma_m(z) = \frac{1}{z} + \sum_{n=1}^{m-1} \frac{a_n}{n+1} z^{n-1} = (\mathfrak{A} * \mathfrak{I}_m)(z) \qquad (m \in \mathbb{N}\setminus\{1\}), \tag{27}$$

where

$$\mathfrak{A}(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^{n-1} \in \Sigma_{\mu}^{\ell}(\zeta;\wp)$$

and

$$\mathfrak{I}_m(z) = \frac{1}{z} + \sum_{n=1}^{m-1} \frac{z^{n-1}}{n+1} \in \Sigma.$$

Furthermore, for $m \in \mathbb{N}\setminus\{1\}$, it is gained from [27] that:

$$\Re\{z\mathfrak{I}_m(z)\} = \Re\left\{1 + \sum_{n=1}^{m-1} \frac{z^n}{n+1}\right\} > \frac{1}{2} \qquad (z \in \Delta^*). \tag{28}$$

From (27) and (28), an implementation of Theorem 4.1 leads to $\sigma_m(z) \in \Sigma_{\mu}^{\ell}(\zeta;\wp)$.

Theorem 4.3. *Let $\mathfrak{A}(z) \in \Sigma_{\mu}^{\ell}(\zeta;\wp)$, $\mathfrak{I}(z) \in \Sigma$ and*

$$z^2\,\mathfrak{I}(z) \in \mathcal{R}(\beta) \qquad (\beta < 1).$$

Then,

$$(\mathfrak{A} * \mathfrak{I})(z) \in \Sigma_{\mu}^{\ell}(\zeta;\wp).$$

Proof. For $\mathfrak{A}(z) \in \Sigma_{\mu}^{\ell}(\zeta; \wp)$ and $\mathfrak{J}(z) \in \Sigma$, from (24) (utilized in the proof of Theorem 4.1), we can write:

$$(1+\zeta)\, z\left(\mathcal{D}_{\mu}^{\ell}(\mathfrak{A} * \mathfrak{J})(z)\right) + \zeta z^{2}\left(\mathcal{D}_{\mu}^{\ell}(\mathfrak{A} * \mathfrak{J})(z)\right)'$$

$$= \frac{z^{2}\mathfrak{J}(z) * z\Psi(z)}{z^{2}\mathfrak{J}(z) * z} \qquad (z \in \Delta^{*}), \tag{29}$$

where $\Psi(z)$ is given by (25).
Since $\wp(z)$ is convex univalent defined on Δ^{*}, $\Psi(z) \prec \wp(z)$, $z^{2}\mathfrak{J}(z) \in \mathcal{R}(\mu)$ and

$$z \in \mathcal{S}^{*}(\beta) \qquad (\beta < 1),$$

From (29) and Lemma 2.2 , the outcome is obtained.

The following result is gained by setting $\beta = 0$ and $\beta = \frac{1}{2}$ in Theorem 4.3

Corollary 4.4. *Let $\mathfrak{A}(z) \in \Sigma_{\mu}^{\ell}(\zeta; \wp)$ and let $\mathfrak{J}(z) \in \Sigma$ achieve either of the following stipulations:*
(i) $z^{2}\mathfrak{J}(z)$ is convex univalent in Δ^{*} or
(ii) $z^{2}\mathfrak{J}(z) \in \mathcal{S}^{*}\left(\frac{1}{2}\right)$. Thus,

$$(\mathfrak{A} * \mathfrak{J})(z) \in \Sigma_{\mu}^{\ell}(\zeta; \wp).$$

References

[1] P. Agarwal, Q. Al-Mdallal, Y. J. Cho and S. Jain, Fractional differential equations for the generalized Mittag-Leffler function, *Adv. Difference Equ.* 58 (2018) 1–8.
[2] J. W. Alexander, Functions which map the interior of the unit circle upon simple regions, *Ann. Math.* 17 (1) (1915) 12–22.
[3] H. F. Al-Janaby, On certain subclass of complex harmonic functions involving a differential operator, *J. Adv. Res. Dyn. Control. Syst.* 10 (2) (2018) 27–36.
[4] H. F. Al-Janaby and M. Z. Ahmad, Differential inequalities related to Sălăgean type integral operator involving extended generalized Mittag-Leffler function, *J. Phys.: Conf. Ser.* 1132 (012061) (2019) 63–82.
[5] H. F. Al-Janaby and M. Darus, Differential Subordination results for Mittag-Leffler type functions with bounded turning property, *Math. Slovaca* 69 (3) (2019) 1–10.
[6] K. A. Challab, M. Darus and F. Ghanim, On subclass of meromorphically univalent functions defined by a linear operator associated with lambda-generalized Hurwitz-Lerch zeta function and q-hypergeometric function, *Italian Journal of Pure and Applied Mathematics* 39 (2018) 410–423.
[7] P. L. Duren, Univalent functions, New York: Springer, 1983.
[8] F. Ghanim, Certain Properties of Classes of Meromorphic Functions Defined by a Linear Operator and Associated with the Hurwitz-Lerch Zeta Function, *Advanced Studies in Contemporary Mathematics (ASCM)* 27 (2) (2017), 175–180.
[9] F. Ghanim and M. Darus, Some properties on a certain class of meromorphic functions related to cho-kwon-srivastava operator, *Asian-European Journal of Mathematics* 5 (04) (2012) 1–9.
[10] F. Ghanim and M. Darus, New subclass of multivalent hypergeometric meromorphic functions, *International Journal of Pure and Applied Mathematics* 61 (3) (2010) 269–280.
[11] F. Ghanim and R. M. El-Ashwah, Inclusion Properties of Certain Subclass of Univalent Meromorphic Functions Defined by a Linear Operator Associated with Hurwitz-Lerch Zeta Function, *Asian-European Journal of Mathematics* 10(3) (2017) 1–10.
[12] F. Ghanim and Hiba F. Al-Janaby, A Certain Subclass of Univalent Meromorphic Functions Defined by a Linear Operator Associated with the Hurwitz-Lerch Zeta Function, *Rad HAZU, Matematicke znanosti (Rad Hrvat. Akad. Znan. Umjet. Mat. Znan.)* 23 (2019) 71–83.
[13] Hiba F. Al-Janay, F. Ghanim and M. Darus, Some Geometric Properties of Integral Operators Proposed by Hurwitz-Lerch Zeta Function, *IOP Conf. Series: Journal of Physics: Conf. Series* 1212 (2019) 1–6.
[14] A. W. Goodman, Univalent functions, I. Tampa, Florida, USA: Mariner, 1983.
[15] T. H. Gronwall, Some remarks on conformal representation, *Ann. of Math.* 12 (1-4) (1914) 72–76.
[16] A. A. Kilbas, M. Saigo and R. K. Saxena, Generalized Mittag-Leffler function and generalized fractional calculus operators, *Integral Transforms Spec. Funct.* 15 (2004) 31–49.
[17] D. Kumar, J. Choi and H. M. Srivastava, Solution of a general family of fractional kinetic equations associated with the generalized Mittag-Leffler function, *Nonlinear Funct. Anal. Appl.* 23 (2018) 455–471.
[18] S.S. Miller. P.T. Mocanu, Differential subordinations and univalent functions, *Michigan Math.J.* 28 (1981) 157–171.
[19] G. M. Mittag-Leffler, Sur la nouvelle fonction $E_{\alpha}(x)$, *C.R. Acad. Sci. Paris.* 137 (1903) 554–558.
[20] A. Mohammed and M. Darus, Integral operators on new families of meromorphic functions of complex order, *J. Inequal. App.* 2011 (2011) 1–12.

[21] A. Mohammed and M. Darus, Starlikeness properties of a new integral operator for meromorphic functions, *J. App. Math.* 2011 (2011) 1–8.

[22] T. R. Prabhakar, A singular integral equation with a generalized Mittag-Leffler function in the kernel. *Yokohama Mah. J.* 19 (1971) 7–15.

[23] R. K. Parmar, A class of extended Mittag-Leffler functions and their properties related to integral transforms and fractional calculus, *Mathematics* 3 (2015) 1069–1082.

[24] G. Rahman, D. Baleanub, M. Al-Qurashid, S. D. Purohite, S. Mubeen and M. Arshada, The extended Mittag-Leffler function via fractional calculus, *J. Nonlinear Sci. Appl.* 10 (2017) 4244–4253.

[25] St. Ruscheweyh, Convolutions in Geometric Function Theory, Sem. Math. Sup. 83, Presses Univ. Montreal 1982.

[26] R. K. Saxena and K. Nishimoto, N-fractional calculus of generalized Mittag-Leffler functions, *J. Fract. Calc.* 37 (2010) 43–52.

[27] R. Singh, S. Singh, Convolution properties of a class of starlike functions, *Proc. Am. Math. Soc.* 106 (1989) 145-152.

[28] H. M. Srivastava, On an extension of the Mittag-Leffler function, *Yokohama Math. J.* 16 (1968) 77–88.

[29] H. M. Srivastava and Ž. Tomovski, Fractional calculus with an integral operator containing a generalized Mittag-Leffler function in the kernel, *J.Appl. Math. Comput.* 211 (2009) 198–210.

[30] H. M. Srivastava, R. K. Parmar and P. Chopra, A class of extended fractional derivative operators and associated generating relations involving hypergeometric functions, *Axioms* 1 (2012) 238–258.

[31] H. M. Srivastava, S. Gaboury and F. Ghanim, Partial sums of certain classes of meromorphic functions related to the Hurwitz-Lerch zeta function, *Moroccan J. Pure Appl. Anal.* 1 (2015) 1-13.

[32] H. M. Srivastava, Some families of Mittag-Leffler type functions and associated operators of fractional calculus (survey), *TWMS J. Pure Appl. Math.* 7 (2016) 123–145.

[33] H. M. Srivastava and D. Bansal, Close-to-convexity of a certain family of q-Mittag-Leffler type functions, *J. Nonlinear Var. Anal.* 1 (2017) 61–69.

[34] H. M. Srivastava, A. Kılıçman, Z. E. Abdulnaby and R. W. Ibrahime, Generalized convolution properties based on the modified Mittag-Leffler function, *J. Nonlinear Sci. Appl* 10 (2017) 4284–4294.

[35] H. M. Srivastava, B. A. Frasin and V. Pescar, Univalence of integral operators involving Mittag-Leffler functions, *Appl. Math. Inf. Sci.* 3 (2017) 635–641.

[36] H. M. Srivastava, M. Bansal and P. Harjule, A study of fractional integral operators involving a certain generalized multi-index Mittag-Leffler function, *Math Meth Appl Sci.* 41 (2018) 6108–6121.

[37] H. M. Srivastava, R. K. Saxen and R. K. Parmar, Some families of the incomplete H-Functions and the incomplete $\bar{H}$-functions and associated integral transforms and operators of fractional calculus with applications, *Russian J. Math. Phys.* 25 (2018) 116–138.

[38] H. M. Srivastava, A. Fernandez and D. Baleanu, Some new fractional-calculus connections between Mittag-Leffler functions, *Mathematics* 7 (2019) 1–10.

[39] Ž Tomovski, R. Hilfer and H. M. Srivastava, Fractional and operational calculus with generalized fractional derivative operators and Mittag-Leffler type functions, *Integral Transforms Spec. Funct.* 21 (2010) 797–814.

[40] M. S. Ur Rehman, Q. Z. Ahmad, H. M. Srivastava, B. Khan and N. Khan, Partial sums of generalized q-Leffler functions functions, *AIMS Mathematics* 5 (2019) 408–420.

[41] D. G. Yang, J. L. Liu, multivalent functions associated with a linear operator, *Appl. Math. Comput.* 204 (2008) 862-871.

[42] A. Wima, Über den Fundamental satz in der Theorie der Funcktionen $E_\alpha(x)$, *Acta Math.* 29 (1905) 191-201.

[43] A. Wima, Über die Nullstellun der Funktionen $E_\alpha(x)$, *Acta Math.* 29 (1905) 217-234.