



Uniform Boundedness of Szász–Mirakjan–Kantorovich Operators in Morrey Spaces with Variable Exponents

Yoshihiro Sawano^a, Xinxin Tian^b, Jingshi Xu^b

^aDepartment of Mathematics, Chuo University, 1-13-27, Kasuga, Bunkyo-ku, 112-8551, Tokyo, Japan

^bSchool of Mathematics and Computing Science, Guilin University of Electronic Technology, Guilin 541004, China

Abstract. The Szász–Mirakjan–Kantorovich operators and the Baskakov–Kantorovich operators are shown to be controlled by the Hardy–Littlewood maximal operator. The Szász–Mirakjan–Kantorovich operators and the Baskakov–Kantorovich operators turn out to be uniformly bounded in Lebesgue spaces and Morrey spaces with variable exponents when the integral exponent is global log–Hölder continuous.

1. Introduction

The Szász–Mirakjan–Kantorovich operators and the Baskakov–Kantorovich operators are used in approximation theory. In this paper we prove that these operators are subject to the control of the Hardy–Littlewood maximal operator. What is important here is that the constant is 1 or 2 and that our bound is sharp in the case of the Szász–Mirakjan–Kantorovich operators.

Density of the continuous functions in $L^1([0, 1])$ plays a key role in many fields of mathematics. There is a constructive proof which uses the Baskakov–Kantorovich operators. Recall that the Baskakov–Kantorovich operator V_n of order $n \in \mathbb{N}$ on the interval $[0, 1]$ is defined for $f \in L^1([0, 1])$ by

$$V_n(f, x) := n \sum_{k=0}^{n-1} m_{n,k}(x) \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t) dt, \quad x \in [0, 1],$$

where

$$m_{n,k}(x) := \binom{n-1}{k} \frac{x^k}{(1+x)^{n+k}}, \quad \binom{n}{k} := \frac{n!}{k!(n-k)!}.$$

Let $1 \leq p < \infty$. We know that $\lim_{n \rightarrow \infty} V_n(f, x) = f(x)$ in $L^p([0, 1])$ for $f \in L^p([0, 1])$. This well-known fact is also a direct consequence of the estimate we will prove in this paper. We will show that each V_n is subject to the

2010 *Mathematics Subject Classification.* 41A10; 41A25

Keywords. variable exponent, Morrey space, Baskakov–Kantorovich operator, Szász–Mirakjan–Kantorovich operator, Hardy–Littlewood maximal operator

Received: 19 May 2019; Revised: 31 December 2019; Accepted: 14 January 2020

Communicated by Hari M. Srivastava

The corresponding author Jingshi Xu is supported by the Hainan Province Natural Science Foundation of China (2018CXTD338), National Natural Science Foundation of China (Grant No. 11761026 and 11761027) and Guangxi Natural Science Foundation (Grant No. 2020GXNSFAA159085).

Email addresses: ysawano@tmu.ac.jp (Yoshihiro Sawano), 875666986@qq.com (Xinxin Tian), jingshi.xu@126.com (Jingshi Xu)

control by the Hardy–Littlewood maximal operator M is given by

$$Mf(x) := \sup_{0 < r_1 < r_2 < 1} \frac{\chi_{(r_1, r_2)}(x)}{r_2 - r_1} \int_{r_1}^{r_2} |f(y)| dy$$

for $f \in L^1([0, 1])$. Here and in what follows χ_S is the characteristic function of the set S .

Theorem 1.1. *Let V_n be the Baskakov–Kantorovich operator of order $n \in \mathbb{N}$. Then*

$$|V_n(f, x)| \leq 2Mf(x), \quad x \in [0, 1],$$

where f is a locally integrable function on $[0, 1]$.

When we consider the approximation of defined on $[0, \infty)$, we can use the Szász–Mirakjan–Kantorovich operators $\{T_n\}_{n=1}^\infty$ defined for $f \in L^1_{\text{loc}}([0, \infty))$ by

$$T_n(f, x) := \sum_{k=0}^\infty np_{n,k}(x) \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t) dt, \quad x \in [0, \infty),$$

where $p_{n,k}(x) := e^{-nx} \frac{(nx)^k}{k!}$, $n \in \mathbb{N}$. We know that $\lim_{n \rightarrow \infty} T_n(f, x) = f(x)$ almost everywhere in $(0, \infty)$ for $f \in L^1(0, \infty)$ such that $f(x) = \frac{d}{dx} \int_0^x f(s) ds$; see [4, Theorem 2].

Accordingly, we consider the Hardy–Littlewood maximal operator M given by

$$Mf(x) := \sup_{r_2 > r_1 > 0} \frac{\chi_{(r_1, r_2)}(x)}{r_2 - r_1} \int_{r_1}^{r_2} |f(y)| dy$$

for $f \in L^1_{\text{loc}}([0, \infty))$.

We have a counterpart of Theorem 1.1.

Theorem 1.2. *Let T_n be the Szász–Mirakjan–Kantorovich operator of order $n \in \mathbb{N}$. Then $|T_n(f, x)| \leq Mf(x)$, for any $x \in [0, \infty)$ and $f \in L^1_{\text{loc}}([0, \infty))$.*

As the example of $f \equiv 1$ shows, we can not replace $Mf(x)$ with $aMf(x)$ for any $a \in (0, 1)$. Theorem 1.2 improves [4, Theorem 3], which asserts that $|T_n(f, x)| \leq 3Mf(x)$.

Theorems 1.1 and 1.2 have many applications. Among others we consider the uniform boundedness of V_n and T_n acting on variable Lebesgue spaces and variable Morrey spaces.

We recall definitions and fix notation. We call a measurable function $p(\cdot)$ a variable exponent. For a variable exponent $p(\cdot) : [0, 1] \rightarrow \mathbb{R}$, we denote

$$p_- := \text{essinf}_{x \in [0, 1]} p(x) \quad \text{and} \quad p_+ := \text{esssup}_{x \in [0, 1]} p(x).$$

The set $\mathcal{P}([0, 1])$ consists of all variable exponents $p(\cdot)$ satisfying $p_- > 1$ and $p_+ < \infty$. Let $p(\cdot) \in \mathcal{P}([0, 1])$. The Lebesgue space $L^{p(\cdot)}([0, 1])$ with variable exponent $p(\cdot)$ is defined by

$$L^{p(\cdot)}([0, 1]) := \left\{ f : [0, 1] \rightarrow \mathbb{C} : f \text{ is measurable, } \int_0^1 \left(\frac{|f(x)|}{\lambda} \right)^{p(x)} dx < \infty \text{ for some } \lambda > 0 \right\}.$$

The norm is given by

$$\|f\|_{L^{p(\cdot)}([0, 1])} := \inf \left\{ \lambda > 0 : \int_0^1 \left(\frac{|f(x)|}{\lambda} \right)^{p(x)} dx \leq 1 \right\}.$$

Variable function spaces go back to [17]. Starting from a seminal paper [16], the theory of variable function spaces has been developed quickly; see the books [6, 8, 10, 14, 15, 19].

Theorem 1.1 yields the uniform boundedness of the Baskakov–Kantorovich operators $\{V_n\}_{n=1}^\infty$.

Theorem 1.3. *If M is bounded in $L^{p(\cdot)}([0, 1])$, then there exists a constant C such that*

$$\|V_n(f)\|_{L^{p(\cdot)}([0,1])} \leq C\|f\|_{L^{p(\cdot)}([0,1])}$$

for all $f \in L^{p(\cdot)}([0, 1])$.

We do not prove Theorem 1.3 since Theorem 1.1 immediately reduces the matters to the boundedness of M on $L^{p(\cdot)}([0, 1])$.

The tools we need to prove Theorems 1.1, 1.2 and 4.2 are the Abel transformation and the following fundamental pointwise estimate for intervals I and measurable functions f :

$$\frac{1}{|I|} \int_I |f(x)| dx \leq Mf(y) \quad (y \in I)$$

or equivalently

$$\int_I |f(x)| dx \leq \int_I Mf(y) dx \quad (y \in I). \tag{1}$$

See also Lemma 4.1 to follow, which is a useful tool. Lemma 4.1 seems interesting in itself.

We can also investigate variable Morrey spaces. Here we recall the definition due to Almeida, Hasanov and Samko [1]. Let $p(\cdot), q(\cdot) \in \mathcal{P}([0, 1])$ satisfy $1 < q_- \leq q(\cdot) \leq p(\cdot) \leq p_+ < \infty$. Then the Morrey space $\mathcal{M}_{q(\cdot)}^{p(\cdot)}([0, 1])$ with variable exponents $p(\cdot)$ and $q(\cdot)$ is the set of all measurable functions f for which

$$\|f\|_{\mathcal{M}_{q(\cdot)}^{p(\cdot)}([0,1])} := \sup_{0 \leq \alpha < \beta \leq 1} |\beta - \alpha|^{\frac{1}{p(\frac{\alpha+\beta}{2})} - \frac{1}{q(\frac{\alpha+\beta}{2})}} \|f\|_{L^{q(\cdot)}([\alpha,\beta])} < \infty.$$

To guarantee that M is bounded on $\mathcal{M}_{q(\cdot)}^{p(\cdot)}([0, 1])$, we postulate the following conditions on the variable exponent $q(\cdot)$.

Definition 1.4. *Let $q(\cdot)$ be a real-valued measurable function on $[0, 1]$. If there exists a constant C_1 such that*

$$|q(x) - q(y)| \leq \frac{C_1}{-\log(|x - y|)} \quad \left(x, y \in [0, 1], |x - y| < \frac{1}{2}\right),$$

then $q(\cdot)$ is called locally log-Hölder continuous. In this case write $q(\cdot) \in C^{\log}([0, 1])$.

Actually, concerning the class $C^{\log}([0, 1])$, we have the following boundedness properties:

Lemma 1.5. [8, Theorem 4.3.8] *Let $p(\cdot) \in C^{\log}([0, 1])$ and satisfy $p_- > 1$. Then M is bounded on $L^{p(\cdot)}([0, 1])$.*

Lemma 1.6. [1, Theorem 2] *Let $p(\cdot) \in \mathcal{P}([0, 1])$ and $q(\cdot) \in C^{\log}([0, 1])$, and suppose $1 < q_- \leq q(\cdot) \leq p(\cdot) \leq p_+ < \infty$. Then M is bounded on $\mathcal{M}_{q(\cdot)}^{p(\cdot)}([0, 1])$.*

Let V_n be the Baskakov–Kantorovich operator of order $n \in \mathbb{N}$ as above. The following result is a consequence of Theorem 1.3 and Lemma 1.5.

Corollary 1.7. *If $p(\cdot) \in C^{\log}([0, 1])$ and $p_- > 1$, then $\{V_n\}_{n=1}^\infty$ is uniformly bounded on $L^{p(\cdot)}([0, 1])$.*

The following corollary follows from Theorem 1.3 and Lemma 1.6.

Theorem 1.8. *If $p(\cdot) \in \mathcal{P}([0, 1])$, $q(\cdot) \in C^{\log}([0, 1])$ and $1 < q_- \leq q(\cdot) \leq p(\cdot) \leq p_+ < \infty$, then $\{V_n\}_{n=1}^\infty$ is uniformly bounded on $\mathcal{M}_{q(\cdot)}^{p(\cdot)}([0, 1])$.*

We move on to function spaces on $[0, \infty)$. For a variable exponent $p(\cdot) : [0, \infty) \rightarrow \mathbb{R}$, we denote

$$p_- := \operatorname{ess\,inf}_{x \in [0, \infty)} p(x) \quad \text{and} \quad p_+ := \operatorname{ess\,sup}_{x \in [0, \infty)} p(x).$$

The set $\mathcal{P}([0, \infty))$ consists of all variable exponents $p(\cdot)$ satisfying $p_- > 1$ and $p_+ < \infty$. Let $p(\cdot) \in \mathcal{P}([0, \infty))$. The Lebesgue space $L^{p(\cdot)}([0, \infty))$ with variable exponent $p(\cdot)$ is defined by

$$L^{p(\cdot)}([0, \infty)) := \left\{ f \text{ is measurable} : \int_0^\infty \left(\frac{|f(x)|}{\lambda} \right)^{p(x)} dx < \infty \text{ for some } \lambda > 0 \right\}.$$

The norm is given by

$$\|f\|_{L^{p(\cdot)}([0, \infty))} := \inf \left\{ \lambda > 0 : \int_0^\infty \left(\frac{|f(x)|}{\lambda} \right)^{p(x)} dx \leq 1 \right\}.$$

Theorem 1.2 yields the uniform boundedness of the Szász–Mirakjan–Kantorovich operator $T_n, n \in \mathbb{N}$.

Theorem 1.9. *Let $p(\cdot) \in \mathcal{P}([0, \infty))$. If M is bounded on $L^{p(\cdot)}([0, \infty))$, then there exists a constant C independent of n such that*

$$\|T_n(f)\|_{L^{p(\cdot)}([0, \infty))} \leq C \|f\|_{L^{p(\cdot)}([0, \infty))}$$

for all $f \in L^{p(\cdot)}([0, \infty))$.

To guarantee that M is bounded on $L^{q(\cdot)}([0, \infty))$ we postulate the following conditions on the variable exponent $q(\cdot)$ or its reciprocal $\frac{1}{q(\cdot)}$.

Definition 1.10. *Let $r(\cdot)$ be a real-valued measurable function on $[0, \infty)$.*

(i) *If there exists a constant C_1 such that*

$$|r(x) - r(y)| \leq \frac{C_1}{-\log(|x - y|)} \quad \left(x, y \in [0, \infty), |x - y| < \frac{1}{2} \right),$$

then $r(\cdot)$ is called locally log-Hölder continuous. In this case write $r(\cdot) \in C_0^{\log}([0, \infty))$.

(ii) *If there exists $r_\infty \in \mathbb{R}$ satisfying*

$$|r(x) - r_\infty| \leq \frac{C_2}{\log(e + x)} \quad (x \in [0, \infty)),$$

then $r(\cdot)$ is called log-Hölder continuous at infinity. In this case write $r(\cdot) \in C_\infty^{\log}([0, \infty))$.

(iii) *If $r(\cdot)$ is both locally log-Hölder continuous and log-Hölder continuous at infinity, then $r(\cdot)$ is called global log-Hölder continuous. In this case write $r(\cdot) \in C^{\log}([0, \infty))$.*

Following [11], we define Morrey spaces with variable exponent on $[0, \infty)$. Let $p(\cdot), q(\cdot) \in \mathcal{P}([0, \infty))$ satisfy $1 < q_- \leq q(\cdot) \leq p(\cdot) \leq p_+ < \infty$. Assume that $r(\cdot) \in C^{\log}([0, \infty))$, where $r(\cdot) := q(\cdot)/p(\cdot) - 1$. Then the Morrey space $\mathcal{M}_{q(\cdot)}^{p(\cdot)}([0, \infty))$ with variable exponents $p(\cdot)$ and $q(\cdot)$ is the set of all measurable functions f for which

$$\|f\|_{\mathcal{M}_{q(\cdot)}^{p(\cdot)}([0, \infty))} := \sup_{0 \leq \alpha < \beta < \alpha + 1 < \infty} |\beta - \alpha|^{\frac{1}{p(\frac{\alpha+\beta}{2})} - \frac{1}{q(\frac{\alpha+\beta}{2})}} \|f\|_{L^{q(\cdot)}([\alpha, \beta])} + \sup_{0 \leq \alpha < \alpha + 1 \leq \beta < \infty} |\beta - \alpha|^{\frac{r_\infty}{q(\frac{\alpha+\beta}{2})}} \|f\|_{L^{q(\cdot)}([\alpha, \beta])} < \infty.$$

Actually, concerning the class $C^{\log}([0, \infty))$, we have the following boundedness properties:

Lemma 1.11. *Let $p(\cdot) \in \mathcal{P}([0, \infty)) \cap C^{\log}([0, \infty))$. Then M is bounded on $L^{p(\cdot)}([0, \infty))$.*

Lemma 1.12. *Let $p(\cdot) \in \mathcal{P}([0, \infty))$, $q(\cdot) \in C^{\log}([0, \infty))$, and let $1 < q_- \leq q(\cdot) \leq p(\cdot) \leq p_+ < \infty$. Assume in addition that $r(\cdot) \in C^{\log}([0, \infty))$, where $r(\cdot) := q(\cdot)/p(\cdot) - 1$. Then M is bounded on $\mathcal{M}_{q(\cdot)}^{p(\cdot)}([0, \infty))$.*

It seems that the proof of Lemma 1.12 is missing. However, by considering exponents $P(\cdot)$ and $Q(\cdot)$ obtained by extending $p(\cdot)$, $q(\cdot)$ to even functions respectively, we are in the position of using [11, Theorem 3.5]. So, we omit the details here.

Recall that T_n is the Szász–Mirakjan–Kantorovich operator of order $n \in \mathbb{N}$. The following corollary follows from Lemma 1.11 and Theorem 1.9.

Corollary 1.13. *If $p(\cdot) \in \mathcal{P}([0, \infty))$ satisfy $p(\cdot) \in C^{\log}([0, \infty))$, then $\{T_n\}_{n=1}^\infty$ is uniformly bounded on $L^{p(\cdot)}([0, \infty))$.*

Theorem 1.2 and Lemma 1.12 yield the following conclusion.

Theorem 1.14. *Let $p(\cdot), q(\cdot) \in C^{\log}([0, \infty))$ satisfy $1 < q_- \leq q(\cdot) \leq p(\cdot) \leq p_+ < \infty$. Assume in addition that $r(\cdot) \in C_0^{\log}([0, \infty))$, where $r(\cdot) := q(\cdot)/p(\cdot) - 1$. Then $\{T_n\}_{n=1}^\infty$ is uniformly bounded on $\mathcal{M}_{q(\cdot)}^{p(\cdot)}([0, \infty))$.*

Let $p(\cdot) \in \mathcal{P}([0, \infty))$. Denote by $\widetilde{\mathcal{M}}_{q(\cdot)}^{p(\cdot)}(I)$ the closure of $C_c^\infty(\text{Int}(I))$ in $\mathcal{M}_{q(\cdot)}^{p(\cdot)}(I)$. If we use Theorems 1.3, 1.8, 1.9 and 1.14, then we have the following conclusion as a byproduct of these theorems.

Theorem 1.15. *Under the assumptions in Theorem 1.3, $\lim_{n \rightarrow \infty} V_n(f) = f$ in $L^{p(\cdot)}([0, 1])$ for all $f \in L^{p(\cdot)}([0, 1])$.*

Theorem 1.16. *Under the assumptions in Theorem 1.8, $\lim_{n \rightarrow \infty} V_n(f) = f$ in $\mathcal{M}_{q(\cdot)}^{p(\cdot)}([0, 1])$ for all $f \in \widetilde{\mathcal{M}}_{q(\cdot)}^{p(\cdot)}([0, 1])$.*

Theorem 1.17. *Under the assumptions in Theorem 1.9, $\lim_{n \rightarrow \infty} T_n(f) = f$ in $L^{p(\cdot)}([0, \infty))$ for all $f \in L^{p(\cdot)}([0, \infty))$.*

Theorem 1.18. *Under the assumptions in Theorem 1.14, $\lim_{n \rightarrow \infty} T_n(f) = f$ in $\mathcal{M}_{q(\cdot)}^{p(\cdot)}([0, \infty))$ for all $f \in \widetilde{\mathcal{M}}_{q(\cdot)}^{p(\cdot)}([0, \infty))$.*

The proofs of Theorems 1.15–1.18 are based on the fact that the operators are uniformly bounded and that the operators approximate the smooth functions nicely. We omit the details.

We make a historical remark on these operators. We remark that the original Baskakov–Kantorovich operator \tilde{V}_n is considered in $(0, \infty)$ and given by

$$\tilde{V}_n(f, x) := n \sum_{k=0}^\infty m_{n,k}(x) \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t) dt \quad x \in [0, \infty),$$

see [9, Page 115]. We can say that the idea of a constructive approximation of the functions by polynomials goes back to the original functions as we considered in [2]. In [2], Bernstein introduced the operator B_n for each $n \in \mathbb{N}$. For f in $C[0, 1]$ the Bernstein operator B_n is defined by

$$B_n(f, x) := \sum_{k=0}^n b_{n,k}(x) f\left(\frac{k}{n}\right), \quad x \in [0, 1],$$

Theorem 1.1 substitutes for the estimate obtained in [3] for the functions in $L^1([0, 1])$. In [13] Kantorovich considered the approximation of the functions in $L^p[0, 1]$, $p \in [1, \infty)$. Write

$$b_{n,k}(x) := \binom{n}{k} x^k (1-x)^{n-k}, \quad x \in [0, 1]$$

for $x \in [0, 1]$, $k, n \in \mathbb{Z}$ satisfying $0 \leq k \leq n$. Kantorovich in [13] introduced the operator K_n , $n \in \mathbb{N}$ defined for $f \in L^1([0, 1])$ by

$$K_n(f; x) := (n+1) \sum_{k=0}^n b_{n,k}(x) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t) dt, \quad x \in [0, 1].$$

The operators V_n and K_n overcome the problem of B_n . In fact, we can not define $B_n(f, x)$ for $f \in L^1([0, 1])$. There are many variants and generalizations of the Kantorovich operators. Indeed, Srivastava and Zeng in [22] investigated a class of approximation operators (namely, the Szász-Bézier integral operators) which contain the modified Szász-Mirakjan operators as their special case. In [18], Özarlan, Dumanb and Srivastava considered a general sequence of Kantorovich-type operators associated with some special polynomials. In [21] the authors introduced a family of q -Szász-Mirakjan-Kantorovich type positive linear operators. In [20] the authors gave approximation properties of an extended family of the Szász-Mirakjan Beta-type operators. Recently, in [3] Burenkov, Ghorbanalizadeh and the first author of the paper obtained the uniform boundedness of Kantorovich operators in Morrey spaces. In [24], it was showed the uniform boundedness of Kantorovich operators in variable Morrey spaces. In [27], Zhou considered approximation by means of positive linear operators on variable Lebesgue spaces.

In Section 2, we prove Theorem 1.1 and consider Baskakov-Kantorovich operators, which are rational expressions. Section 3 is the proof of Theorem 1.2. In Section 4, we obtain that the conjugate operator of K_n is also controlled by the Hardy-Littlewood maximal operator. And we show that the Kantorovich-Stancu type of Szász-Mirakjan operators are controlled by the Hardy-Littlewood maximal operator in a certain interval. We will employ the method in [3] for the proof of Theorems 1.1, 1.2 and 4.2.

2. Baskakov-Kantorovich operators – Proof of Theorem 1.1

Let $x \in [0, 1]$ and $n \geq 2$. Write $n_x := [nx]$ here and below in Section 2, so that $n_x \leq nx < n_x + 1$.

Lemma 2.1.

1. The difference $m_{n,k}(x) - m_{n,k-1}(x)$ is non-negative for $k \in \{1, \dots, n_x - 1\}$.
2. The difference $m_{n,k-1}(x) - m_{n,k}(x)$ is non-negative for $k \in \{n_x + 1, \dots, n\}$.

Proof. Arithmetic shows

$$m_{n,k}(x) - m_{n,k-1}(x) = \frac{(n+k-2)!x^{k-1}(nx-x-k)}{k!(n-1)!(1+x)^{n+k}}.$$

This equality clearly yields the desired result. \square

We suppose f is a nonnegative measurable function to prove Theorem 1.1; otherwise, we replace f by $|f|$. Let $x \in [0, 1]$. We write

$$I := n \sum_{k=0}^{n_x-1} m_{n,k}(x) \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)dt, \quad II := nm_{n,n_x}(x) \int_{\frac{n_x}{n}}^{\frac{n_x+1}{n}} f(t)dt, \quad III := n \sum_{k=n_x+1}^{n-1} m_{n,k}(x) \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)dt,$$

so that $V_n(f, x) = I + II + III$. First, keeping in mind that $k + 1 \leq nx$ for $k \leq n_x - 1$, we have

$$I = n \sum_{k=0}^{n_x-1} m_{n,k}(x) \left(\int_{\frac{k}{n}}^x f(t)dt - \int_{\frac{k+1}{n}}^x f(t)dt \right).$$

We further decompose I to have

$$\begin{aligned} I &= nm_{n,0}(x) \int_0^x f(t)dt - nm_{n,n_x-1}(x) \int_{\frac{n_x}{n}}^x f(t)dt + n \sum_{k=1}^{n_x-1} m_{n,k}(x) \int_{\frac{k}{n}}^x f(t)dt - n \sum_{k=1}^{n_x-1} m_{n,k-1}(x) \int_{\frac{k}{n}}^x f(t)dt \\ &= nm_{n,0}(x) \int_0^x f(t)dt - nm_{n,n_x-1}(x) \int_{\frac{n_x}{n}}^x f(t)dt + n \sum_{k=1}^{n_x-1} (m_{n,k}(x) - m_{n,k-1}(x)) \int_{\frac{k}{n}}^x f(t)dt. \end{aligned}$$

Secondly, keeping in mind that $n_x \leq nx < n_x + 1$ we decompose

$$\text{II} = nm_{n,n_x}(x) \int_x^{\frac{n_x+1}{n}} f(t)dt + nm_{n,n_x}(x) \int_{\frac{n_x}{n}}^x f(t)dt.$$

Similar to I, we obtain that

$$\text{III} = nm_{n,n-1}(x) \int_x^1 f(t)dt - nm_{n,n_x+1}(x) \int_x^{\frac{n_x+1}{n}} f(t)dt + n \sum_{k=n_x+2}^{n-1} (m_{n,k-1}(x) - m_{n,k}(x)) \int_x^{\frac{k}{n}} f(t)dt.$$

Consequently we have

$$\begin{aligned} V_n(f, x) &= nm_{n,0}(x) \int_0^x f(t)dt + n \sum_{k=1}^{n_x-1} (m_{n,k}(x) - m_{n,k-1}(x)) \int_{\frac{k}{n}}^x f(t)dt \\ &\quad - nm_{n,n_x-1}(x) \int_{\frac{n_x}{n}}^x f(t)dt + nm_{n,n_x}(x) \int_{\frac{n_x}{n}}^x f(t)dt + nm_{n,n-1}(x) \int_x^1 f(t)dt \\ &\quad + n \sum_{k=n_x+1}^{n-1} (m_{n,k-1}(x) - m_{n,k}(x)) \int_x^{\frac{k}{n}} f(t)dt. \end{aligned} \tag{2}$$

Since $-nm_{n,n_x-1}(x) \int_{\frac{n_x}{n}}^x f(t)dt \leq 0$, we obtain

$$\begin{aligned} V_n(f, x) &\leq nm_{n,0}(x) \int_0^x f(t)dt + n \sum_{k=1}^{n_x-1} (m_{n,k}(x) - m_{n,k-1}(x)) \int_{\frac{k}{n}}^x f(t)dt \\ &\quad + nm_{n,n_x}(x) \int_{\frac{n_x}{n}}^x f(t)dt + nm_{n,n-1}(x) \int_x^1 f(t)dt + n \sum_{k=n_x+1}^{n-1} (m_{n,k-1}(x) - m_{n,k}(x)) \int_x^{\frac{k}{n}} f(t)dt. \end{aligned} \tag{3}$$

Using (1), we can replace the function f by the constant function $Mf(x)$ and obtain that

$$\begin{aligned} V_n(f, x) &\leq nm_{n,0}(x) \int_0^x Mf(x)dt + n \sum_{k=1}^{n_x-1} (m_{n,k}(x) - m_{n,k-1}(x)) \int_{\frac{k}{n}}^x Mf(x)dt \\ &\quad + nm_{n,n_x}(x) \int_{\frac{n_x}{n}}^x Mf(x)dt + nm_{n,n-1}(x) \int_x^1 Mf(x)dt \\ &\quad + n \sum_{k=n_x+1}^{n-1} (m_{n,k-1}(x) - m_{n,k}(x)) \int_x^{\frac{k}{n}} Mf(x)dt. \end{aligned} \tag{4}$$

Furthermore, in (2), replace the function f by the constant function $Mf(x)$, we have

$$\begin{aligned} Mf(x) &= V_n(Mf(x), x) \\ &= nm_{n,0}(x) \int_0^x Mf(x)dt + n \sum_{k=1}^{n_x-1} (m_{n,k}(x) - m_{n,k-1}(x)) \int_{\frac{k}{n}}^x Mf(x)dt \\ &\quad + nm_{n,n_x}(x) \int_{\frac{n_x}{n}}^x Mf(x)dt - nm_{n,n_x-1} \int_{\frac{n_x}{n}}^x Mf(x)dt + nm_{n,n-1}(x) \int_x^1 Mf(x)dt \\ &\quad + n \sum_{k=n_x+1}^{n-1} (m_{n,k-1}(x) - m_{n,k}(x)) \int_x^{\frac{k}{n}} Mf(x)dt + nm_{n,n_x-1} \int_{\frac{n_x}{n}}^x Mf(x)dt. \end{aligned}$$

Now we have

$$\begin{aligned}
 V_n(f, x) &\leq nm_{n,0}(x) \int_0^x Mf(x)dt + n \sum_{k=1}^{n_x-1} (m_{n,k}(x) - m_{n,k-1}(x)) \int_{\frac{k}{n}}^x Mf(x)dt \\
 &\quad + nm_{n,n_x}(x) \int_{\frac{n_x}{n}}^x Mf(x)dt + nm_{n,n-1}(x) \int_x^1 Mf(x)dt \\
 &\quad + n \sum_{k=n_x+1}^{n-1} (m_{n,k-1}(x) - m_{n,k}(x)) \int_x^{\frac{k}{n}} Mf(x)dt \\
 &= nm_{n,0}(x) \int_0^x Mf(x)dt + n \sum_{k=1}^{n_x-1} (m_{n,k}(x) - m_{n,k-1}(x)) \int_{\frac{k}{n}}^x Mf(x)dt \\
 &\quad + nm_{n,n_x}(x) \int_{\frac{n_x}{n}}^x Mf(x)dt - nm_{n,n_x-1} \int_{\frac{n_x}{n}}^x Mf(x)dt + nm_{n,n-1}(x) \int_x^1 Mf(x)dt \\
 &\quad + n \sum_{k=n_x+1}^{n-1} (m_{n,k-1}(x) - m_{n,k}(x)) \int_x^{\frac{k}{n}} Mf(x)dt + nm_{n,n_x-1} \int_{\frac{n_x}{n}}^x Mf(x)dt \\
 &\quad + nm_{n,n_x-1} \int_{\frac{n_x}{n}}^x Mf(x)dt \\
 &= Mf(x) + nm_{n,n_x-1} \int_{\frac{n_x}{n}}^x Mf(x)dt \\
 &\leq 2Mf(x).
 \end{aligned}$$

Consequently, we obtain the desired result.

3. Szász–Mirakjan–Kantorovich operators – Proof of Theorem 1.2

Let $x \geq 0$. We write $n_x := [nx]$, so that $n_x \leq nx < n_x + 1$. We set

$$\begin{aligned}
 \text{I} &= n \sum_{k=0}^{n_x-1} e^{-nx} \frac{(nx)^k}{k!} \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)dt, \\
 \text{II} &= ne^{-nx} \frac{(nx)^{n_x}}{n_x!} \int_{\frac{n_x}{n}}^{\frac{n_x+1}{n}} f(t)dt, \\
 \text{III} &= n \sum_{k=n_x+1}^{\infty} e^{-nx} \frac{(nx)^k}{k!} \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)dt.
 \end{aligned}$$

Then we have $T_n(f, x) = \text{I} + \text{II} + \text{III}$. First, we have

$$\text{I} = ne^{-nx} \int_0^x f(t)dt - ne^{-nx} \frac{(nx)^{n_x-1}}{(n_x - 1)!} \int_{\frac{n_x}{n}}^x f(t)dt + n \sum_{k=1}^{n_x-1} e^{-nx} \left(\frac{(nx)^k}{k!} - \frac{(nx)^{k-1}}{(k-1)!} \right) \int_{\frac{k}{n}}^x f(t)dt.$$

Next, keeping in mind that $n_x \leq nx < n_x + 1$ we decompose

$$\text{II} = ne^{-nx} \frac{(nx)^{n_x}}{(n_x)!} \int_x^{\frac{n_x+1}{n}} f(t)dt + ne^{-nx} \frac{(nx)^{n_x}}{(n_x)!} \int_{\frac{n_x}{n}}^x f(t)dt$$

Finally, as for III, we obtain that

$$\text{III} = n \sum_{k=n_x+2}^{\infty} e^{-nx} \left(\frac{(nx)^{k-1}}{(k-1)!} - \frac{(nx)^k}{k!} \right) \int_x^{\frac{k}{n}} f(t) dt - ne^{-nx} \frac{(nx)^{n_x+1}}{(n_x+1)!} \int_x^{\frac{n_x+1}{n}} f(t) dt,$$

Putting together all these decompositions, we obtain that

$$\begin{aligned} T_n(f, x) &= \text{I} + \text{II} + \text{III} \\ &= ne^{-nx} \int_0^x f(t) dt + n \sum_{k=1}^{n_x} e^{-nx} \left(\frac{(nx)^k}{k!} - \frac{(nx)^{k-1}}{(k-1)!} \right) \int_{\frac{k}{n}}^x f(t) dt \\ &\quad + n \sum_{k=n_x+1}^{\infty} e^{-nx} \left(\frac{(nx)^{k-1}}{(k-1)!} - \frac{(nx)^k}{k!} \right) \int_x^{\frac{k}{n}} f(t) dt. \end{aligned}$$

If we have $k \in [0, n_x]$, then $k \leq nx$. Hence,

$$\frac{(nx)^k}{k!} - \frac{(nx)^{k-1}}{(k-1)!} = \frac{(nx)^{k-1}}{(k-1)!} \left(\frac{nx}{k} - 1 \right) > 0.$$

By the same way, if $k \in [n_x + 1, \infty)$, then

$$\frac{(nx)^{k-1}}{(k-1)!} - \frac{(nx)^k}{k!} = \frac{(nx)^{k-1}}{(k-1)!} \left(1 - \frac{nx}{k} \right) > 0.$$

Consequently

$$\begin{aligned} |T_n(f, x)| &\leq ne^{-nx} \int_0^x |f(t)| dt + n \sum_{k=1}^{n_x} e^{-nx} \left(\frac{(nx)^k}{k!} - \frac{(nx)^{k-1}}{(k-1)!} \right) \int_{\frac{k}{n}}^x |f(t)| dt \\ &\quad + n \sum_{k=n_x+1}^{\infty} e^{-nx} \left(\frac{(nx)^{k-1}}{(k-1)!} - \frac{(nx)^k}{k!} \right) \int_x^{\frac{k}{n}} |f(t)| dt. \end{aligned}$$

Using (1), we have

$$\begin{aligned} |T_n(f, x)| &\leq ne^{-nx} \int_0^x Mf(x) dt + n \sum_{k=1}^{n_x-1} e^{-nx} \left(\frac{(nx)^k}{k!} - \frac{(nx)^{k-1}}{(k-1)!} \right) \int_{\frac{k}{n}}^x Mf(x) dt \\ &\quad + n \sum_{k=n_x+2}^{\infty} e^{-nx} \left(\frac{(nx)^{k-1}}{(k-1)!} - \frac{(nx)^k}{k!} \right) \int_x^{\frac{k}{n}} Mf(x) dt \\ &\quad + ne^{-nx} \frac{(nx)^{n_x-1}}{(n_x-1)!} \left(\frac{nx}{n_x} - 1 \right) \int_{\frac{n_x}{n}}^x Mf(x) dt + ne^{-nx} \frac{(nx)^{n_x}}{(n_x)!} \left(1 - \frac{nx}{n_x+1} \right) \int_x^{\frac{n_x+1}{n}} Mf(x) dt \\ &= T_n(Mf(x), x). \end{aligned}$$

So, we have

$$|T_n(f, x)| \leq n \sum_{k=0}^{\infty} e^{-nx} \frac{(nx)^k}{k!} \int_{\frac{k}{n}}^{\frac{k+1}{n}} Mf(x) dt = \sum_{k=0}^{\infty} e^{-nx} \frac{(nx)^k}{k!} Mf(x) = Mf(x).$$

4. Appendix

4.1. The conjugate of K_n

Then the conjugate operator of K_n is given by

$$K_n^*(g, t) = \sum_{k=0}^n \chi_{(\frac{k}{n+1}, \frac{k+1}{n+1})}(t) \int_0^1 (n+1)b_{n,k}(x)g(x)dx, \quad t \in [0, 1].$$

To handle the operator of this type, we will use the following lemma:

Lemma 4.1. *Let $g : [0, 1] \rightarrow [0, \infty)$ be a function increasing on $[0, a]$ and decreasing on $[a, 1]$. Then for any measurable function f ,*

$$\int_0^1 g(t)|f(t)|dt \leq Mf(a) \int_0^1 g(t)dt.$$

Proof. By approximating g by a function of the form $\sum_{j=0}^N \lambda_j \chi_{[a_j, b_j]}$, where $a_j \leq a \leq b_j$ and $\lambda_j \geq 0$, we may assume that g itself is such a function. In this case, we can resort to (1). \square

If we set

$$\tilde{b}_{n,k}(t) := \max\left(b_{n,k}\left(\frac{k}{n}\right)\chi_{(\frac{k}{n+1}, \frac{k+1}{n+1})}(t), b_{n,k}(t)\right),$$

then $\tilde{b}_{n,k}(t)$ attains its maximum at any point in $(\frac{k}{n+1}, \frac{k+1}{n+1})$, since $b_{n,k}$ increases in $[0, \frac{k}{n}]$ and decreases in $[\frac{k}{n}, 1]$. Furthermore,

$$\int_0^1 (n+1)\tilde{b}_{n,k}(t)dt \leq \int_0^1 (n+1)b_{n,k}(t)dt + \frac{1}{n+1}b_{n,k}\left(\frac{k}{n}\right) = 1 + \binom{n}{k} \frac{k^k(n-k)^{n-k}}{n^n(n+1)}.$$

Consequently

$$|K_n^*(g, t)| \leq \left(1 + \binom{n}{k} \frac{k^k(n-k)^{n-k}}{n^n(n+1)}\right)Mg(t) \leq \left(1 + \frac{1}{n+1}\right)Mg(t) \leq 2Mg(t)$$

for all $t \in [\frac{k}{n+1}, \frac{k+1}{n+1}]$.

According to [3, Theorem 1.1] we have $|K_n(f; x)| \leq Mf(x)$, $x \in [0, 1]$. Thus K_n are uniformly bounded on $L^{p(\cdot)}([0, 1])$ which the Hardy-Littlewood maximal operator is bounded on. However the necessity of the Hardy-Littlewood maximal operator bounds on $L^{p(\cdot)}([0, 1])$ is $p_- > 1$. Using the above estimate of K_n^* , we learn that the uniform boundedness of K_n on $L^{p(\cdot)}([0, 1])$ is possible for $p_- \geq 1$ as long as $p_+ < \infty$, in other words, if the Hardy-Littlewood maximal operator is bounded on $L^{p(\cdot)}([0, 1])$, then K_n are also uniformly bounded on $L^{p(\cdot)}([0, 1])$.

4.2. Kantorovich-Stancu type of Szasz-Mirakyan operators

The Kantorovich-Stancu type of Szasz-Mirakyan operators are defined as follows:

$$T_n^{(\alpha, \beta)}(f, x) = \sum_{k=0}^{\infty} (n+\beta)p_{n,k}(x) \int_{\frac{k+\alpha}{n+\beta}}^{\frac{k+\alpha+1}{n+\beta}} f(t)dt, \quad x \in [0, \infty),$$

where $p_{n,k}(x) := e^{-nx} \frac{(nx)^k}{k!}$, $n \in \mathbb{N}$, $k \in \mathbb{N} \cup \{0\}$ and $\alpha, \beta \geq 0$ are parameters. We have the following result:

Theorem 4.2. Let $T_n^{(\alpha,\beta)}$ be the Kantorovich-Stancu type for every $n \in \mathbb{N}$, M be the Hardy-Littlewood maximal function. Then

$$|T_n^{(\alpha,\beta)}(f, x)| \leq Mf(x),$$

whenever f is a locally integrable function on $[0, \infty)$ and

$$\max\left(\frac{\alpha - 1}{\beta}, \frac{\alpha}{n + \beta}\right) \leq x \leq \frac{\alpha}{\beta}.$$

Proof. By the triangle inequality, we may assume that f is non-negative. Let $n_x := [(n + \beta)x - \alpha]$ for $x \in [0, 1]$, so that $n_x \leq (n + \beta)x - \alpha < n_x + 1$. We set

$$\begin{aligned} \text{I} &:= \sum_{k=0}^{n_x-1} e^{-nx} \frac{(nx)^k}{k!} \int_{\frac{k+\alpha}{n+\beta}}^{\frac{k+\alpha+1}{n+\beta}} f(t)dt = \sum_{k=0}^{n_x-1} e^{-nx} \frac{(nx)^k}{k!} \left(\int_{\frac{k+\alpha}{n+\beta}}^x f(t)dt - \int_{\frac{k+\alpha+1}{n+\beta}}^x f(t)dt \right), \\ \text{II} &:= e^{-nx} \frac{(nx)^{n_x}}{n_x!} \int_{\frac{n_x+\alpha}{n+\beta}}^{\frac{n_x+\alpha+1}{n+\beta}} f(t)dt, \\ \text{III} &:= \sum_{k=n_x+1}^{\infty} e^{-nx} \frac{(nx)^k}{k!} \int_{\frac{k+\alpha}{n+\beta}}^{\frac{k+\alpha+1}{n+\beta}} f(t)dt = \sum_{k=n_x+1}^{\infty} e^{-nx} \frac{(nx)^k}{k!} \left(\int_x^{\frac{k+\alpha+1}{n+\beta}} f(t)dt - \int_x^{\frac{k+\alpha}{n+\beta}} f(t)dt \right). \end{aligned}$$

Then we have $T_n^{(\alpha,\beta)}(f, x) = (n + \beta)(\text{I} + \text{II} + \text{III})$.

We consider the Abel transform of I to have

$$\begin{aligned} \text{I} &= \sum_{k=0}^{n_x-1} e^{-nx} \frac{(nx)^k}{k!} \int_{\frac{k+\alpha}{n+\beta}}^x f(t)dt - \sum_{k=1}^{n_x} e^{-nx} \frac{(nx)^{k-1}}{(k-1)!} \int_{\frac{k+\alpha}{n+\beta}}^x f(t)dt \\ &= e^{-nx} \int_{\frac{\alpha}{n+\beta}}^x f(t)dt + \sum_{k=1}^{n_x-1} e^{-nx} \left(\frac{(nx)^k}{k!} - \frac{(nx)^{k-1}}{(k-1)!} \right) \int_{\frac{k+\alpha}{n+\beta}}^x f(t)dt - e^{-nx} \frac{(nx)^{n_x-1}}{(n_x-1)!} \int_{\frac{n_x+\alpha}{n+\beta}}^x f(t)dt \end{aligned}$$

Secondly, keeping in mind that $n_x \leq (n + \beta)x - \alpha < n_x + 1$, we decompose

$$\text{II} = e^{-nx} \frac{(nx)^{n_x}}{(n_x)!} \int_x^{\frac{n_x+\alpha+1}{n+\beta}} f(t)dt + e^{-nx} \frac{(nx)^{n_x}}{(n_x)!} \int_{\frac{n_x+\alpha}{n+\beta}}^x f(t)dt$$

We consider the Abel transform of III to have

$$\begin{aligned} \text{III} &= \sum_{k=n_x+2}^{\infty} e^{-nx} \frac{(nx)^{k-1}}{(k-1)!} \int_x^{\frac{k+\alpha}{n+\beta}} f(t)dt - \sum_{k=n_x+1}^{n-1} e^{-nx} \frac{(nx)^k}{k!} \int_x^{\frac{k+\alpha}{n+\beta}} f(t)dt \\ &= \sum_{k=n_x+2}^{\infty} e^{-nx} \left(\frac{(nx)^{k-1}}{(k-1)!} - \frac{(nx)^k}{k!} \right) \int_x^{\frac{k+\alpha}{n+\beta}} f(t)dt - e^{-nx} \frac{(nx)^{n_x+1}}{(n_x+1)!} \int_x^{\frac{n_x+\alpha+1}{n+\beta}} f(t)dt. \end{aligned}$$

So, we have

$$\begin{aligned} &\text{I} + \text{II} + \text{III} \\ &= e^{-nx} \int_{\frac{\alpha}{n+\beta}}^x f(t)dt + \sum_{k=1}^{n_x} e^{-nx} \left(\frac{(nx)^k}{k!} - \frac{(nx)^{k-1}}{(k-1)!} \right) \int_{\frac{k+\alpha}{n+\beta}}^x f(t)dt + \sum_{k=n_x+1}^{\infty} e^{-nx} \left(\frac{(nx)^{k-1}}{(k-1)!} - \frac{(nx)^k}{k!} \right) \int_x^{\frac{k+\alpha}{n+\beta}} f(t)dt. \end{aligned}$$

From the definition of n_x , $n_x \leq (n + \beta)x - \alpha < n_x + 1$. If $k \in [0, n_x]$, then $nx - k \geq nx - n_x \geq \alpha - \beta x \geq 0$. Hence, the following conclusion is established.

$$\frac{(nx)^k}{k!} - \frac{(nx)^{k-1}}{(k-1)!} = \frac{(nx)^{k-1}}{(k-1)!} \left(\frac{nx}{k} - 1 \right) > 0.$$

In the same way, if $k \in [n_x + 1, \infty)$, we have $k - nx \geq n_x + 2 - nx \geq \beta x + 1 - \alpha \geq 0$. Thus

$$\frac{(nx)^{k-1}}{(k-1)!} - \frac{(nx)^k}{k!} = \frac{(nx)^{k-1}}{(k-1)!} \left(1 - \frac{nx}{k} \right) > 0.$$

Using (1), we have

$$\begin{aligned} \text{I} + \text{II} + \text{III} &\leq e^{-nx} \int_{\frac{\alpha}{n+\beta}}^x Mf(x) dt + \sum_{k=1}^{n_x} e^{-nx} \left(\frac{(nx)^k}{k!} - \frac{(nx)^{k-1}}{(k-1)!} \right) \int_{\frac{k+\alpha}{n+\beta}}^x Mf(x) dt \\ &+ \sum_{k=n_x+1}^{\infty} e^{-nx} \left(\frac{(nx)^{k-1}}{(k-1)!} - \frac{(nx)^k}{k!} \right) \int_x^{\frac{k+\alpha}{n+\beta}} Mf(x) dt = Mf(x). \end{aligned}$$

□

References

- [1] A. Almeida, J. Hasanov and S. Samko, Maximal and potential operators in variable exponent Morrey spaces. *Georgian Mathematical Journal* 15 (2008) 198–208.
- [2] S. N. Bernstein, Démonstration du théorème de Weierstrass, fondée sur le calcul des probabilités, *Communications Society Mathematical Kharkow* 13 (1912–13) 1–2.
- [3] V. Burenkov, A. Ghorbanalizadeh, Y. Sawano, Uniform boundedness of Kantorovich operators in Morrey spaces, *Positivity* 22 (2018) 1097–1107.
- [4] P. L. Butzer, On the extensions of Bernstein polynomials to the infinite interval, *Proceedings of the American Mathematical Society* 5 (1954) 547–553.
- [5] I. Chlodowsky, Sur le developpement des fonctions dans un intervalle infini en series de polynomes de M.S. Bernstein, *Compositio Mathematica* 4 (1937) 380–393.
- [6] D. Cruz-Uribe and A. Fiorenza, *Variable Lebesgue Spaces*, Basel, Birkhäuser, 2015.
- [7] L. Diening, Maximal function on generalized Lebesgue spaces $L^{p(\cdot)}$, *Mathematical Inequalities and Applications* 7 (2004) 245–253.
- [8] L. Diening, P. Harjulehto, P. Hästö and M. Růžička, *Lebesgue and Sobolev spaces with variable exponents*, *Lecture Notes in Mathematics* 2017, Berlin, Springer, 2011.
- [9] Z. Ditzian, V. Totik, *Moduli of Smoothness*, Berlin, Springer, 1987.
- [10] D. Edmunds, J. Lang and O. Mendez, *Differential Operators on Spaces of Variable Integrability*, Singapore, World Scientific Publishing Company, 2014.
- [11] V. S. Guliyev and S. G. Samko, Maximal potential, and singular operators in the generalized variable exponent Morrey spaces on unbounded sets, *Journal of Mathematical Sciences* 193 (2013) 228–248.
- [12] P. Hästö, Local-to-global results in variable exponent spaces, *Mathematical Research Letters* 16 (2009) 263–278.
- [13] L. V. Kantorovich, Sur certaines developments suivant les polynômes de la forme de S. Bernstein I– II, *Comptes Rendus de l'Académie des Sciences de l'URSS A* (1930) 563–568; 595–600.
- [14] V. Kokilashvili, A. Meskhi, H. Rafeiro and S. Samko, *Integral Operators in Non-Standard Function Spaces, Volume 1: Variable Exponent Lebesgue and Amalgam Spaces*, Switzerland, Springer International Publishing, 2016.
- [15] V. Kokilashvili, A. Meskhi, H. Rafeiro and S. Samko, *Integral Operators in Non-Standard Function Spaces, Volume 2: Variable Exponent Hölder, Morrey-Campanato and Grand Spaces*, Switzerland, Springer International Publishing, 2016.
- [16] O. Kováčik and J. Rákosník, On spaces $L^{p(x)}$ and $W^{k,p(x)}$, *Czechoslovak Mathematical Journal* 41 (1991) 592–618.
- [17] W. Orlicz, Über konjugierte Exponentenfolgen, *Studia Mathematica* 3 (1931) 200–212.
- [18] M. A. Özarıslan, O. Dumanb, H. M. Srivastava, Statistical approximation results for Kantorovich-type operators involving some special polynomials, *Mathematical and Computer Modelling* 48 (2008), 388–401.
- [19] V. Rădulescu and D. Repovš, *Partial Differential Equations with Variable Exponents: Variational Methods and Qualitative Analysis*, New York, CRC Press, 2015.
- [20] H. M. Srivastava, G. İçöz and B. Çekim, Approximation properties of an extended family of the Szász-Mirakjan Beta-type operators, *Axioms* 8 (2019), Article ID 111, 1–13.
- [21] H. M. Srivastava, M. Mursaleenc, A. M. Alotaibi, Md. Nasiruzzamanc and A. A. H. Al-Abied, Some approximation results involving the q -Szász-Mirakjan-Kantorovich type operators via Dunkls generalization, *Mathematical Methods in the Applied Sciences* 40 (2017) 5437–5452.

- [22] H. M. Srivastava and X.-M. Zeng, Approximation by means of the Szász-Bézier integral operators, *International Journal of Pure and Applied Mathematics* 14 (2004) 283–294.
- [23] O. Szász, Generalization of Bernstein's polynomials to the infinite interval, *Journal of Research of the National Bureau of Standards* 45 (1950) 239–245.
- [24] X. Tian and J. Xu, Uniform boundedness of Kantorovich operators in Morrey spaces with variable exponents, *Journal of Hainan Normal University (Nat. Sci.)* 31 (2018) 387–390.
- [25] V. Totik, Approximation by Szász-Mirakjan type operators, *Acta Mathematica Hungarica* 41(1983) 291–307.
- [26] V. Totik, Approximation by Szász-Mirakjan-Kantorovich operators in L^p ($p > 1$), *Analysis Mathematica* 9 (1983) 147–167.
- [27] D X. Zhou, Approximation by positive linear operators on variable $L^p(\cdot)$ space, *Journal of Applied Functional Analysis* 9 (2014) 379–391.