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Dual Wavelet Frames in Sobolev Spaces on Local Fields of Positive Characteristic

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Abstract. Wavelet frames have gained considerable popularity during the past decade, primarily due to their substantiated applications in diverse and widespread fields of engineering and science. In this article, we obtain the characterization of nonhomogeneous wavelet frames and nonhomogeneous dual wavelet frames in a Sobolev spaces on a local field of positive characteristic by means of a pair of equations.

1. Introduction

The notion of frame was first introduced by Duffin and Schaeffer [7] in connection with some deep problems in non-harmonic Fourier series. Frames are basis-like systems that span a vector space but allow for linear dependency, which can be used to reduce noise, find sparse representations, or obtain other desirable features unavailable with orthonormal bases. An important example about frame is wavelet frame, which is obtained by translating and dilating a finite family of functions. In recent years there has been a considerable interest in the problem of constructing periodic wavelet bases and frames in Hilbert spaces as most of the signals of practical interest are periodic in nature. Apart from signals that are inherently periodic, all signals resulting from experiments with a finite duration can in principle be modeled as periodic signals [11]. Since the setup of tight wavelet frames provides great flexibility in approximating and representing periodic functions. Nonhomogeneous wavelet dual frames in $L^2(\mathbb{R}^d)$ were first characterized by Han [13], and then further investigated by Bownik [5]. For the homogeneous wavelet dual frames, regularity and vanishing moments have been both required. But in case of nonhomogeneous wavelet dual frames in Sobolev space pairs $\left(H^s(\mathbb{R}^d), H^{-s}(\mathbb{R}^d)\right)$, they can be relaxed. Therefore it is easy to construct dual frames. Recently, the theory of nonhomogeneous wavelet frames have been studied by various researchers [17, 18, 25].

During the last decade, there is a tremendous interest in the problem of constructing wavelet bases and frames on various spaces other than $\mathbb{R}$, such as locally compact Abelian groups [10], Vilenkin groups [9], Cantor dyadic groups [20], zero-dimensional groups [22]. The local field K is a natural model for the structure of wavelet frame systems, as well as a domain upon which one can construct wavelet basis functions. There is a substantial body of work that has been concerned with the construction of wavelets and frames on local fields.

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R. L. Benedetto and J. J. Benedetto [4] developed a wavelet theory for Local fields and related groups. They did not develop the multiresolution analysis (MRA) approach, their method is based on the theory of wavelet sets and only allows the construction of wavelet functions whose Fourier transforms are characteristic functions of some sets. Jiang et al. [14] pointed out a method for constructing orthogonal wavelets on local field $\mathbb{K}$ with a constant generating sequence and derived necessary and sufficient conditions for a solution of the refinement equation to generate a multiresolution analysis of $L^2(\mathbb{K})$. Later on, Li and Jiang [16] have obtained a necessary condition and a set of sufficient conditions for the wavelet system $\left\{\psi_{j,k} =: q^{j/2}\psi\left(\mathfrak{p}^{-j}x - u(k)\right) : j,k \in \mathbb{N}_0\right\}$ to be a tight wavelet frame on local fields in the frequency domain. Ahmad and Sheikh [1] defined a new type of inner product on Local fields called *a-Inner product* and introduced the concept of *a-frames* in this concern. As far as the characterization of wavelet frames on local fields is concerned, Shah and Abdullah [24] have established a complete characterization of tight wavelet frames on local fields by virtue of two basic equations in the frequency domain and show how to construct an orthonormal wavelet basis for $L^2(K)$.

The paper is structured in the following manner. In Section 2, we discuss some preliminary Fourier analysis about local fields of positive characteristic and also some results which are required in the subsequent sections. Sections 3 and 4 state and prove our main results about the characterizations of nonhomogeneous wavelet frames in Sobolev spaces on local fields of positive characteristic. In the rest of this paper, we use the symbols $\mathbb{N}, \mathbb{N}_0$ and $\mathbb{Z}$ to denote the sets of natural, non-negative integers and integers, respectively.

2. Notations, Preliminaries and NonHomogeneous Wavelet Frames on Local Fields

Let K be a field and a topological space. Then K is called a *local field* if both K^+ and K^* are locally compact Abelian groups, where K^+ and K^* denote the additive and multiplicative groups of K, respectively. If K is any field and is endowed with the discrete topology, then K is a local field. Further, if K is connected, then K is either $\mathbb{R}$ or $\mathbb{C}$. If K is not connected, then it is totally disconnected. Hence by a local field, we mean a field K which is locally compact, non-discrete and totally disconnected.

For example the field of p-adic numbers Q_p is an example of a Local field. An extensive work has been done by many authors on p-adic fields [2–4], which are the extension of p-adic number system and extends the ordinary arithmetic of rational numbers obtained by an alternative interpretation of *"Closeness"* or *absolute value*. Formally, for a given prime p, the field Q_p of p-adic numbers is a completion of the rational numbers. The field Q_p is equipped with a topology derived from a metric obtained from p-adic order, an alternative valuation on rational numbers. This kind of metric is complete in the sense that every Cauchy sequence converges to a point in Q_p. More details are referred to [15, 19, 22].

Let K be a local field. Let dx be the Haar measure on the locally compact Abelian group K^+. If $\alpha \in K$ and $\alpha \neq 0$, then $d(\alpha x)$ is also a Haar measure. Let $d(\alpha x) = |\alpha|dx$. We call $|\alpha|$ the *absolute value* of α. Moreover, the map $x \to |x|$ has the following properties:

- $|x| = 0$ if and only if $x = 0$;
- $|xy| = |x||y|$ for all $x, y \in K$; and
- $|x + y| \leq \max\{|x|, |y|\}$ for all $x, y \in K$.

Property (c) is called the *ultrametric inequality.* The set $\mathfrak{D} = \{x \in K : |x| \leq 1\}$ is called the *ring of integers* in K. Define $\mathfrak{B} = \{x \in K : |x| < 1\}$. The set $\mathfrak{B}$ is called the *prime ideal* in K. The prime ideal in K is the unique maximal ideal in $\mathfrak{D}$ and hence as a result $\mathfrak{B}$ is both principal and prime. Since the local field K is totally disconnected, so there exist an element of $\mathfrak{B}$ of maximal absolute value. Let $\mathfrak{p}$ be a fixed element of maximum absolute value in $\mathfrak{B}$. Such an element is called a *prime element* of K. Therefore, for such an ideal $\mathfrak{B}$ in $\mathfrak{D}$, we have $\mathfrak{B} = \langle \mathfrak{p} \rangle = \mathfrak{p}\mathfrak{D}$. As it was proved in [22], the set $\mathfrak{D}$ is compact and open. Hence, $\mathfrak{B}$ is compact and open. Therefore, the residue space $\mathfrak{D}/\mathfrak{B}$ is isomorphic to a finite field $GF(q)$, where $q = p^k$ for some prime p and $k \in \mathbb{N}$.

Let $\mathfrak{D}^* = \mathfrak{D} \setminus \mathfrak{B} = \{x \in K : |x| = 1\}$. Then, it can be proved that $\mathfrak{D}^*$ is a group of units in K^* and if $x \neq 0$, then we may write $x = \mathfrak{p}^k x', x' \in \mathfrak{D}^*$. For a proof of this fact we refer to [18]. Moreover, each $\mathfrak{B}^k = \mathfrak{p}^k\mathfrak{D} = \left\{x \in K : |x| < q^{-k}\right\}$ is a compact subgroup of K^+ and usually known as the *fractional ideals* of K^+. Let $\mathcal{U} = \{c_i\}_{i=0}^{q-1}$ be any fixed full set of coset representatives of $\mathfrak{B}$ in $\mathfrak{D}$, then every element $x \in K$ can be expressed uniquely as $x = \sum_{\ell=k}^{\infty} c_\ell \mathfrak{p}^\ell$ with $c_\ell \in \mathcal{U}$. Let χ be a fixed character on K^+ that is trivial on $\mathfrak{D}$ but is non-trivial on $\mathfrak{B}^{-1}$. Therefore, χ is constant on cosets of $\mathfrak{D}$ so if $y \in \mathfrak{B}^k$, then $\chi_y(x) = \chi(yx), x \in K$. Suppose that χ_u is any character on K^+, then clearly the restriction $\chi_u|\mathfrak{D}$ is also a character on $\mathfrak{D}$. Therefore, if $\{u(n) : n \in \mathbb{N}_0\}$ is a complete list of distinct coset representative of $\mathfrak{D}$ in K^+, then, as it was proved in [22], the set $\left\{\chi_{u(n)} : n \in \mathbb{N}_0\right\}$ of distinct characters on $\mathfrak{D}$ is a complete orthonormal system on $\mathfrak{D}$.

The Fourier transform $\widehat{f}$ of a function $f \in L^1(K) \cap L^2(K)$ is defined by

$$\widehat{f}(\xi) = \int_K f(x)\overline{\chi_\xi(x)}\, dx.$$

It is to be noted that

$$\widehat{f}(\xi) = \int_K f(x)\, \overline{\chi_\xi(x)} dx = \int_K f(x)\chi(-\xi x)\, dx.$$

Furthermore, the properties of Fourier transform on local field K are much similar to those of on the real line. In particular Fourier transform is unitary on $L^2(K)$. Also, if $f \in L^2(\mathfrak{D})$, then we define the Fourier coefficients of f as

$$\widehat{f}\big(u(n)\big) = \int_{\mathfrak{D}} f(x)\overline{\chi_{u(n)}(x)}\, dx.$$

The series $\sum_{n\in\mathbb{N}_0} \widehat{f}\big(u(n)\big)\chi_{u(n)}(x)$ is called the Fourier series of f. From the standard L^2-theory for compact Abelian groups, we conclude that the Fourier series of f converges to f in $L^2(\mathfrak{D})$ and Parseval's identity holds:

$$\left\|f\right\|_2^2 = \int_{\mathfrak{D}} \left|f(x)\right|^2 dx = \sum_{n\in\mathbb{N}_0} \left|\widehat{f}\big(u(n)\big)\right|^2.$$

We now impose a natural order on the sequence $\{u(n)\}_{n=0}^{\infty}$. We have $\mathfrak{D}/\mathfrak{B} \cong GF(q)$ where $GF(q)$ is a c-dimensional vector space over the field $GF(p)$. We choose a set $\{1 = \zeta_0, \zeta_1, \zeta_2, \ldots, \zeta_{c-1}\} \subset \mathfrak{D}^*$ such that span $\left\{\zeta_j\right\}_{j=0}^{c-1} \cong GF(q)$. For $n \in \mathbb{N}_0$ satisfying

$$0 \leq n < q, \;\; n = a_0 + a_1 p + \cdots + a_{c-1}p^{c-1}, \;\; 0 \leq a_k < p, \;\text{ and } k = 0, 1, \ldots, c-1,$$

we define

$$u(n) = (a_0 + a_1\zeta_1 + \cdots + a_{c-1}\zeta_{c-1})\, \mathfrak{p}^{-1}.$$

Also, for $n = b_0 + b_1 q + b_2 q^2 + \cdots + b_s q^s,\ n \in \mathbb{N}_0,\ 0 \leq b_k < q, k = 0, 1, 2, \ldots, s$, we set

$$u(n) = u(b_0) + u(b_1)\mathfrak{p}^{-1} + \cdots + u(b_s)\mathfrak{p}^{-s}.$$

This defines $u(n)$ for all $n \in \mathbb{N}_0$. In general, it is not true that $u(m+n) = u(m) + u(n)$. But, if $r, k \in \mathbb{N}_0$ and $0 \leq s < q^k$, then $u(rq^k + s) = u(r)\mathfrak{p}^{-k} + u(s)$. Further, it is also easy to verify that $u(n) = 0$ if and only if $n = 0$ and $\{u(\ell) + u(k) : k \in \mathbb{N}_0\} = \{u(k) : k \in \mathbb{N}_0\}$ for a fixed $\ell \in \mathbb{N}_0$. Hereafter, we use the notation $\chi_n = \chi_{u(n)},\ n \geq 0$.

Let the local field K be of characteristic $p > 0$ and $\zeta_0, \zeta_1, \zeta_2, \ldots, \zeta_{c-1}$ be as above. We define a character χ on K as follows:

$$\chi(\zeta_\mu \mathfrak{p}^{-j}) = \begin{cases} \exp(2\pi i/p), & \mu = 0 \text{ and } j = 1, \\ 1, & \mu = 1, \ldots, c-1 \text{ or } j \neq 1. \end{cases}$$

We also denote the test function space on K by $\Omega(K)$, that is, each function f in $\Omega(K)$ is a finite linear combination of functions of the form $\mathbf{1}_k(x-h), h \in K, k \in \mathbb{Z}$, where $\mathbf{1}_k$ is the characteristic function of $\mathfrak{B}^k$. This class of functions can also be described in the following way. A function $g \in \Omega(K)$ if and only if there exist integers k, ℓ such that g is constant on cosets of $\mathfrak{B}^k$ and is supported on $\mathfrak{B}^\ell$. It follows that Ω is closed under Fourier transform and is an algebra of continuous functions with compact support, which is dense in $C_0(K)$ as well as in $L^p(K), 1 \le p < \infty$.

Definition 2.1 For $s \in \mathbb{K}$, we define the sobolev spaces $H^s(\mathbb{K})$ as the space of all tempered distributions f such that

$$\|f\|^2_{H^s(\mathbb{K})} = \int_{\mathbb{K}} |\hat{f}(\xi)|^2 \left(1 + \|\xi\|^2\right)^s d\xi < \infty,$$

where $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{K}$. The inner product in $H^s(\mathbb{K})$ is given by

$$\langle f, g\rangle_{H^s(\mathbb{K})} = \int_{\mathbb{K}} \hat{f}(\xi)\overline{\hat{g}(\xi)} \left(1 + \|\xi\|^2\right)^s d\xi, \ \ f, g \in H^s(\mathbb{K}).$$

Moreover, for each $f \in H^s(\mathbb{K}),\ g \in H^{-s}(\mathbb{K})$, we have

$$\langle f, g\rangle = \int_{\mathbb{K}} \hat{f}(\xi)\overline{\hat{g}(\xi)}\, d\xi,$$

is a linear continuous functional in $H^s(\mathbb{K})$. The spaces $H^s(\mathbb{K})$ and $H^{-s}(\mathbb{K})$ form pairs of dual spaces.

For functions $f, g : \mathbb{K}^d \to \mathbb{C}$, define

$$[f, g]_t(\xi) = \sum_{k \in \mathbb{N}^d} f\left(\xi + u(k)\right) \overline{g\left(\xi + u(k)\right)} \left(1 + \|\xi + u(k)\|^2\right)^t, \ \ t \in \mathbb{K}^d.$$

For a distribution $f,\ j \in \mathbb{Z},\ k \in \mathbb{N}^d,\ s \in \mathbb{K}$, we can write

$$f_{j,k} = q^{j/2} f\left(\mathfrak{p}^{-j}\xi - u(k)\right) \ \ and \ \ f^s_{j,k} = q^{j(1/2-s)} f\left(\mathfrak{p}^{-j}\xi - u(k)\right).$$

For a given $L \in \mathbb{N}$ and $s \in \mathbb{K}$, let $\phi, \psi_1, \psi_2, \cdots .\psi_L \in H^s(\mathbb{K})$ and $\widetilde{\phi}, \widetilde{\psi}_1, \widetilde{\psi}_2, \cdots .\widetilde{\psi}_L \in H^{-s}(\mathbb{K})$, we denote by $X^s\left(\phi; \psi_1, \psi_2, \cdots .\psi_L\right)$ and $X^{-s}\left(\widetilde{\phi}; \widetilde{\psi}_1, \widetilde{\psi}_2, \cdots .\widetilde{\psi}_L\right)$, the following two nonhomogeneous wavelet systems in $H^s(\mathbb{K})$ and $H^{-s}(\mathbb{K})$, respectively:

$$X^s\left(\phi; \psi_1, \psi_2, \cdots .\psi_L\right) = \left\{\phi_{0,k} : k \in \mathbb{N}^d\right\} \cup \left\{\psi^s_{\ell,j,k} : j \in \mathbb{N}_0, k \in \mathbb{N}^d, 1 \le \ell \le L\right\} \tag{1}$$

and

$$X^s\left(\widetilde{\phi}; \widetilde{\psi}_1, \widetilde{\psi}_2, \cdots \widetilde{\psi}_L\right) = \left\{\widetilde{\phi}_{0,k} : k \in \mathbb{N}^d\right\} \cup \left\{\widetilde{\psi}^{-s}_{\ell,j,k} : j \in \mathbb{N}_0, k \in \mathbb{N}^d, 1 \le \ell \le L\right\} \tag{2}$$

we say that $X^s\left(\phi; \psi_1, \psi_2, \cdots .\psi_L\right)$ is a *nonhomogeneous wavelet frame* in $H^s(\mathbb{K})$ if there exist two positive constants A, B such that

$$A\|f\|^2_{H^s(\mathbb{K})} \le \sum_{k \in \mathbb{N}^d} \left|\langle f, \phi_{0,k}\rangle_{H^s(\mathbb{K})}\right|^2 + \sum_{\ell=1}^{L} \sum_{j=0}^{\infty} \sum_{k \in \mathbb{N}^d} \left|\langle f, \psi^s_{\ell,j,k}\rangle_{H^s(\mathbb{K})}\right|^2 \le B\|f\|^2_{H^s(\mathbb{K})} \tag{3}$$

where A, B are called *frame bounds*; it is called a *nonhomogeneous wavelet Bessel sequence* in $H^s(K)$ if the right hand inequality in (3) holds, where B is called a *Bessel bound*. Furthermore, we say that $\left(X^s\left(\phi;\psi_1,\psi_2,\cdots.\psi_L\right), X^{-s}\left(\widetilde{\phi};\widetilde{\psi_1},\widetilde{\psi_2},\cdots.\widetilde{\psi_L}\right)\right)$ is a pair of *nonhomogeneous wavelet dual frame* in $(H^s(\mathbb{K}), H^{-s}(\mathbb{K}))$ if $X^s\left(\phi;\psi_1,\psi_2,\cdots.\psi_L\right)$ and $X^{-s}\left(\widetilde{\phi};\widetilde{\psi_1},\widetilde{\psi_2},\cdots.\widetilde{\psi_L}\right)$ are Bessel sequences in $H^s(\mathbb{K})$ and $H^{-s}(\mathbb{K})$ respectively, and

$$A\langle f,g\rangle = \sum_{k\in\mathbb{N}^d}\langle f,\widetilde{\phi}_{0,k}\rangle\langle\phi_{0,k},g\rangle + \sum_{\ell=1}^{L}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{N}^d}\langle f,\widetilde{\psi}^s_{\ell,j,k}\rangle\langle\psi^s_{\ell,j,k},g\rangle \tag{4}$$

holds for all $f\in H^s(\mathbb{K})$ and $g\in H^{-s}(\mathbb{K})$.

3. Main Results

Lemma 3.1. Let $s\in\mathbb{K}^d$, $j\in\mathbb{N}^d$ and $\psi\in H^{-s}(\mathbb{K}^d)$. Then for $f\in H^s(\mathbb{K}^d), k\in\mathbb{N}^d$, the k-th Fourier coefficient of $[q^i\widehat{f}(\mathfrak{p}^j.),\widehat{\psi}(.)]_0(\xi)$ is $\left\langle f,\psi_{_{j,k}}\right\rangle$. In particular

$$\left[q^{id}\widehat{f}(\mathfrak{p}^j\cdot),\widehat{\psi}(\cdot)\right]_0(\xi) = \sum_{k\in\mathbb{N}^d}\left\langle f,\psi_{_{j,k}}\right\rangle e^{ik\xi} \tag{5}$$

if $\{\psi_{_{j,k}}:k\in\mathbb{N}^d\}$ is a Bessel sequence in $H^{-s}(\mathbb{K}^d)$.

Proof. Since $f\in H^s(\mathbb{K}^d)$ and $\psi\in H^{-s}(\mathbb{K}^d)$, we have $\widehat{f}(\mathfrak{p}^j.)\overline{\widehat{\psi}(.)}\in L^1(\mathbb{K}^d)$, and thus

$$\begin{aligned}\int_{\mathfrak{D}}[q^i\widehat{f}(\mathfrak{p}^j.),\widehat{\psi}(.)]_0(\xi)e^{-i\langle k,\xi\rangle}d\xi &= q^i\int_{\mathfrak{D}}\sum_{\ell\in\mathbb{N}}\widehat{f}\Big(q^j(\xi+u(\ell))\Big)\overline{\widehat{\psi}\Big(\xi+u(\ell)\Big)}e^{-ik\xi}d\xi\\ &= q^i\int_{\mathbb{K}^d}\widehat{f}(\mathfrak{p}^j\xi)\overline{\widehat{\psi}(\xi)}e^{-i\langle k,\xi\rangle}d\xi\\ &= q^{-i}\int_{\mathbb{K}^d}\widehat{f}(\xi)\overline{\widehat{\psi}(\mathfrak{p}^{-j}\xi)}e^{-i\langle k,\mathfrak{p}^{-j}\xi\rangle}d\xi\\ &= \int_{\mathbb{K}^d}\widehat{f}(\xi)\overline{[\psi_{_{j,k}}(.)]^\wedge(\xi)}d\xi,\end{aligned}$$

by the Plancheral theorem and hence

$$\int_{\mathfrak{D}}[q^i\widehat{f}(\mathfrak{p}^j.),\widehat{\psi}(.)]_0(\xi)e^{-i\langle k,\xi\rangle}d\xi = \langle f,\psi_{_{j,k}}\rangle. \tag{6}$$

If $\{\psi_{_{j,k}}:k\in\mathbb{N}^d\}$ is a Bessel sequence in $H^{-s}(\mathbb{K}^d)$, then $\{\langle f,\psi_{_{j,k}}\rangle\}_{k\in\mathbb{N}^d}\in\ell^2(\mathbb{N}^d)$, and hence (5) follows by (6). □

Lemma 3.2. Let $s\in\mathbb{K}^d, f,\psi_1,\psi_2,...,\psi_L\in H^s(\mathbb{K}^d)$. Then $X^s(f;\psi_1,\psi_2,...,\psi_L)$ is a Bessel sequence in $H^s(\mathbb{K}^d)$ with Bessel bound B if and only if

$$\sum_{k\in\mathbb{N}^d}\left|\langle h,f_{0,k}\rangle\right|^2 + \sum_{\ell=1}^{L}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{N}^d}\left|\left\langle h,\psi^s_{_{\ell,j,k}}\right\rangle\right|^2 \le B\|h\|^2_{H^{-s}(\mathbb{K}^d)},\quad for\ all\ h\in H^{-s}(\mathbb{K}^d). \tag{7}$$

Lemma 3.3 Let $s\in\mathbb{K}^d, f,\psi_1,\psi_2,...,\psi_L\in H^s(\mathbb{K}^d)$. Suppose that $X^s(f;\psi_1,\psi_2,...,\psi_L)$ is a Bessel sequence in $H^s(\mathbb{K}^d)$ with Bessel bound B, then

$$|\widehat{f}(.)|^2 + \sum_{\ell=1}^{L}\sum_{j=0}^{\infty}q^{-js}|\widehat{\psi_\ell}(p^{-j}.)|^2 \le B\left(1+\|.\|^2\right)^{-s} \tag{8}$$

holds a.e on $\mathbb{K}^d$.

Proof. Since $X^s(f;\psi_1,\psi_2,...,\psi_L)$ is a Bessel sequence in $H^s(\mathbb{K}^d)$ with Bessel bound B, by Lemma 3.2, one has

$$\sum_{k\in\mathbb{N}^d}\left|\langle h,f_{0,k}\rangle\right|^2+\sum_{\ell=1}^{L}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{N}^d}|\left\langle h,\psi^s_{\ell,j,k}\right\rangle|^2\le B\|h\|^2_{H^{-s}(\mathbb{K}^d)}\ \ for\ all\ h\in H^{-s}(\mathbb{K}^d) \tag{9}$$

By Lemma 3.1, we get

$$\begin{aligned}\sum_{k\in\mathbb{N}^d}|\langle h,f_{0,k}\rangle|^2+\sum_{\ell=1}^{L}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{N}^d}|\langle h,\psi^s_{\ell,j,k}\rangle|^2\\ =\int_{\mathbb{K}^d}\widehat{f}(\xi)\overline{\widehat{h}(\xi)}\sum_{k\in\mathbb{N}^d}\widehat{h}(\xi+u(k))\overline{\widehat{f}(\xi+u(k))}d\xi\\ +\sum_{\ell=1}^{L}\sum_{j=0}^{\infty}q^{-js}\int_{\mathbb{K}^d}\widehat{\psi}_\ell(p^{-j}\xi)\overline{\widehat{f}(\xi)}\sum_{k\in\mathbb{N}^d}\widehat{h}(\xi+\mathfrak{p}^ju(k))\overline{\widehat{\psi}_\ell(p^{-j}\xi+u(k))}d\xi.\end{aligned}$$

or

$$\begin{aligned}\sum_{k\in\mathbb{N}^d}|\langle h,f_{0,k}\rangle|^2+\sum_{\ell=1}^{L}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{N}^d}|\langle h,\psi^s_{\ell,j,k}\rangle|^2\\ =\int_{\mathbb{K}^d}|\widehat{h}(\xi)|^2\left\{|\widehat{\psi}(\xi)|^2+\sum_{\ell=1}^{L}\sum_{j=0}^{\infty}q^{-js}|\widehat{\psi}_\ell(\mathfrak{p}^j\xi)|^2\right\}d\xi\\ +\int_{\mathbb{K}^d}\overline{\widehat{f}(\xi)}\sum_{0\neq k\in\mathbb{N}^d}\widehat{h}(\xi+u(k))\\ \times\left\{\widehat{f}(\xi)\overline{\widehat{f}(\xi+u(k))}+\sum_{\ell=1}^{L}\sum_{j=0}^{\infty}q^{-js}\widehat{\psi}_\ell\Big(\mathfrak{p}^{-j}(\xi+u(k))\Big)\right\}d\xi\end{aligned} \tag{10}$$

Suppose (8) does not hold. Then there exist $E\subset\mathbb{K}^d$ with $|E|>0$ such that

$$|\widehat{f}(.)|^2+\sum_{\ell=1}^{L}\sum_{j=0}^{\infty}q^{-js}|\widehat{\psi}_\ell(p^{-j}.)|^2>B(1+\|.\|^2)^{-s}\ \ on\ \ E,$$

and thus

$$|\widehat{f}(.)|^2+\sum_{\ell=1}^{L}\sum_{j=0}^{\infty}q^{-js}|\widehat{\psi}_\ell(p^{-j}.)|^2>B\Big(1+\|.\|^2\Big)^{-s}$$

on some $E'=E\cap(\mathfrak{D}+u(k_0))$ with $|E'|>0$ and $k_0\in\mathbb{N}^d$. Take h such that $\widehat{h}(.)=(1+\|.\|^2)^{s/2}\chi_{E'}$ in (10), we obtain

$$\sum_{k\in\mathbb{N}^d}|\langle h,f_{0,k}\rangle|^2+\sum_{\ell=1}^{L}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{N}^d}|\langle h,\psi^s_{\ell,j,k}\rangle|^2>B|E'|=B\|h\|^2_{H^{-s}(\mathbb{K}^d)},$$

contradicting (9). □

Theorem 3.1 Let $s\in\mathbb{K}^d$, $f,\psi_1,\psi_2,...,\psi_L\in H^s(\mathbb{K}^d)$ and $\tilde{f},\tilde{\psi}_1,\tilde{\psi}_2,...,\tilde{\psi}_L\in H^{-s}(\mathbb{K}^d)$. Consider the wavelet systems $X^s(f;\psi_1,\psi_2,...,\psi_L)$ and $X^{-s}(\tilde{f};\tilde{\psi}_1,\tilde{\psi}_2,...,\tilde{\psi}_L)$ as in (1) and (2) respectively. Suppose that $X^s(f;\psi_1,\psi_2,...,\psi_L)$

is a Bessel sequence in $H^s(\mathbb{K}^d)$ and $X^{-s}(\tilde{f}, \tilde{\psi}_1, ..., \tilde{\psi}_L)$ is a Bessel sequence in $H^{-s}(\mathbb{K}^d)$. Then $\Big(X^s(f; \psi_1, \psi_2, ..., \psi_L), X^{-s}(\tilde{f}; \tilde{\psi}_1, ..., \tilde{\psi}_L)\Big)$ is a pair of dual frames in $\Big(H^s(\mathbb{K}^d), H^{-s}(\mathbb{K}^d)\Big)$ if and only if, for every $k \in \mathbb{N}^d$

$$\hat{f}(\cdot)\overline{\widehat{\tilde{f}}(\cdot + u(k))} + \sum_{\ell=1}^{L}\sum_{j=0}^{\kappa(k)} \widehat{\psi_\ell}(\mathfrak{p}^{-j}\cdot)\overline{\widehat{\tilde{\psi}}_\ell(\mathfrak{p}^{-j}\cdot + u(k))} = \delta_{0,k} \ \ a.e. \ on \ \mathbb{K}. \tag{11}$$

Proof. Since by definition, $\Big(X^s(f; \psi_1, \psi_2, ..., \psi_L), X^{-s}(\tilde{f}; \tilde{\psi}_1, ..., \tilde{\psi}_L)\Big)$ is a pair of dual frames for $\Big(H^s(\mathbb{K}^d), H^{-s}(\mathbb{K}^d)\Big)$ if and only if

$$\sum_{k\in\mathbb{N}^d}\Big\langle \psi, \tilde{f}_{0,k}\Big\rangle + \sum_{\ell=1}^{L}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{N}^d}\Big\langle \psi, \tilde{\psi}_{\ell,j,k}\Big\rangle\Big\langle \psi^s_{\ell,j,k}, h\Big\rangle = \langle\psi, h\rangle, \ \ \psi\in H^s(\mathbb{K}^d), \ \ for\ all\ h\in H^{-s}(\mathbb{K}^d) \tag{12}$$

By the Plancheral Theorem and Lemma 3.1, we conclude that

$$\sum_{k\in\mathbb{N}^d}\Big\langle \psi, \tilde{\psi}(. - k)\Big\rangle\langle\psi(. - k), h\rangle + \sum_{\ell=1}^{L}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{N}^d}\Big\langle \psi, \tilde{\psi}^{-s}_{\ell,j,k}\Big\rangle\Big\langle \psi^s_{\ell,j,k}, h\Big\rangle$$

$$= \int_{\mathfrak{D}^d}\left\{\sum_{k\in\mathbb{N}^d}\widehat{\psi}\,(\xi + u(k))\,\overline{\widehat{\tilde{f}}\,(\xi + u(k))}\right\}\left\{\sum_{k\in\mathbb{N}^d}\widehat{f}(\xi + u(k))\overline{\widehat{h}(\xi + u(k))}\right\}d\xi$$

$$+ \sum_{\ell=1}^{L}\sum_{j=0}^{\infty} q^{id}\int_{\mathfrak{D}^d}\left\{\sum_{k\in\mathbb{N}^d}\widehat{\psi}\Big(\mathfrak{p}^j(\xi + u(k))\Big)\overline{\widehat{\tilde{\psi}}(\xi + u(k)}\right\}\times\left\{\sum_{k\in\mathbb{N}^d}\widehat{\psi_\ell}(\xi + u(k))\overline{\widehat{h}\Big(\mathfrak{p}^j(\xi + u(k))\Big)}\right\}d\xi$$

$$= \int_{\mathbb{K}^d}\sum_{k\in\mathbb{N}^d}\widehat{\psi}(\xi + u(k))\overline{\widehat{\tilde{g}}(\xi + u(k))}\widehat{f}(\xi)\overline{\widehat{h}(\xi)}d\xi$$

$$+ \sum_{\ell=1}^{L}\sum_{j=0}^{\infty} q^{id}\int_{\mathbb{K}^d}\sum_{\mathbb{N}^d}\widehat{\psi}\Big(\mathfrak{p}^j(\xi + u(k))\Big)\overline{\widehat{\tilde{\psi}}_{\xi+u(k)}}\widehat{\psi_\ell}(\xi)\overline{\widehat{h}(\mathfrak{p}^j\xi)}d\xi$$

$$= \int_{\mathbb{K}^d}\widehat{\psi}(\xi)\overline{\widehat{h}(\xi)}\left\{\widehat{g}(\xi)\overline{\widehat{\tilde{g}}(\xi)} + \sum_{\ell=1}^{L}\sum_{j=0}^{\infty}\widehat{\psi_\ell}(p^{-j}\xi)\overline{\widehat{\tilde{\psi}}_\ell(p^{-j}\xi)}\right\}d\xi$$

$$+ \int_{\mathbb{K}^d}\overline{\widehat{h}(\xi)}\Bigg(\sum_{0\neq k\in\mathbb{N}^d}\widehat{\psi}\Big(\xi + u(k)\Big)\widehat{g}(\xi)\overline{\widehat{\tilde{g}}(\xi + u(k))}\,d\xi$$

$$+ \sum_{\ell=1}^{L}\sum_{j=0}^{\infty}\sum_{0\neq k\in\mathbb{N}^d}\widehat{\psi}\Big(\xi + \mathfrak{p}^j u(k)\Big)\widehat{\psi_\ell}\Big(\overline{\widehat{\tilde{\psi}}_\ell}(p^{-j}\xi + u(k))\Big)\Bigg)d\xi$$

$$= \int_{\mathbb{K}^d}\widehat{\psi}(\xi)\overline{\widehat{h}(\xi)}\Big(\widehat{g}(\xi)\overline{\widehat{\tilde{\psi}}(\xi)} + \sum_{\ell=1}^{L}\sum_{j=0}^{\infty}\psi_\ell(p^{-j}\xi)\overline{\widehat{\tilde{\psi}}_\ell(p^{-j}\xi)}\Big)d\xi$$

$$+ \int_{\mathbb{K}^d}\overline{\widehat{h}(\xi)}\sum_{0\neq k\in\mathbb{N}^d}\widehat{\psi}(\xi + u(k))\Big(\widehat{g}(\xi)\overline{\widehat{\tilde{g}}(\xi + u(k))} + \sum_{\ell=1}^{L}\sum_{j=0}^{\infty}\widehat{\psi_\ell}(p^{-j}(\xi + u(k)))\Big)d\xi.$$

And hence (12) can be written as

$$\begin{aligned}&\int_{\mathbb{K}^d}\widehat{\psi}(\xi)\overline{\widehat{h}(\xi)}\Big(\widehat{g}(\xi)\overline{\widehat{\tilde{g}}(\xi)}+\sum_{\ell=1}^{L}\sum_{j=0}^{\infty}\widehat{\psi}_\ell(p^{-j}\xi)\overline{\widehat{\tilde{\psi}}(p^{-j}\xi)}\Big)d\xi\\&+\int_{\mathbb{K}^d}\overline{\widehat{h}(\xi)}\sum_{0\neq k\mathbb{N}^d}\widehat{\psi}(\xi+u(k))\Big(\widehat{g}(\xi)\overline{\widehat{\tilde{g}}(\xi+u(k))}+\sum_{\ell=1}^{L}\sum_{j=0}^{\infty}\widehat{\psi}_\ell(p^{-j}\xi)\overline{\widehat{\tilde{\psi}}(p^{-j}(\xi+u(k)))}\Big)d\xi\\&=\int_{\mathbb{K}^d}\widehat{\psi}(\xi)\overline{\widehat{h}(\xi)}d\xi.\end{aligned}\tag{13}$$

Clearly (11) implies (13). For the completion of the proof, we prove the converse implication. Suppose (13) holds. By Lemma 3.3 and Cauchy-Schwarz inequality, the series

$$\widehat{g}(.)\overline{\widehat{\tilde{g}}(.+k)}+\sum_{\ell=1}^{L}\sum_{j=0}^{\infty}\widehat{\psi}_\ell(p^{-j}.)\overline{\widehat{\tilde{\psi}}_\ell(p^{-j}(.+k))}$$

with $k\in\mathbb{N}^d$ converges absolutely a.e on K^d and belongs to $L^\infty(\mathbb{K}^d)$. Let ξ_0 be any point $\mathbb{K}^d$, for $\epsilon>0$, choose ψ, h such that $\widehat{\psi}=(1+\|.\|^2)^{-s/2}\mathbf{1}_{\mathcal{B}}$ and $\widehat{h}(.)=\Big(1+\|.\|^2\Big)^{s/2}\mathbf{1}_{\mathcal{B}}$ in (13), then

$$\int_{\mathcal{B}}\Big(\widehat{g}(\xi)\overline{\widehat{\tilde{g}}(\xi)}+\sum_{\ell=1}^{L}\sum_{j=0}^{\infty}\widehat{\psi}_\ell(p^{-j\xi})\overline{\widehat{\tilde{\psi}}}_\ell(p^{-j}\xi)\Big)d\xi=1.$$

By Lebesgue differentiation theorem, we have

$$\widehat{g}(\xi_0)\overline{\widehat{\tilde{g}}(\xi_0)}+\sum_{\ell=1}^{L}\sum_{j=0}^{\infty}\widehat{\psi}_\ell(p^{-j}\xi_0)\overline{\widehat{\tilde{\psi}}_\ell(p^{-j}\xi_0)}=1.$$

For $0\neq k_0\in\mathbb{N}^d$, we take ψ and h such that $\widehat{\psi}(.+k_0)\Big(1+\|.\|^2\Big)^{-s/2}\mathbf{1}_{\mathcal{B}}$ and $\widehat{h}(.)=\Big(1+\|.\|^2\Big)^{s/2}\mathbf{1}_{\mathcal{B}}$ in (13), then

$$\int_{\mathcal{B}}\Big(\widehat{f}(\xi)\overline{\widehat{\tilde{f}}(\xi+u(k_0)}+\sum_{\ell=1}^{L}\sum_{j=0}^{\infty}\widehat{\psi}_\ell(p^{-j}\xi)\overline{\widehat{\tilde{\psi}}\Big(p^{-j}(\xi+u(k_0))\Big)}\Big)d\xi=0.$$

On the application of Lebesgue differentiation theorem, we obtain

$$\widehat{f}(\xi_0)\overline{\widehat{\tilde{f}}(\xi_0+u(k_0))}\sum_{\ell=1}^{\infty}\sum_{j=0}^{\infty}\widehat{\psi}_\ell(p^{-j}\xi_0)\overline{\widehat{\tilde{\psi}}_\ell\Big(p^{-j}(\xi_0+u(k_0))\Big)}=0.$$

Since ξ_0 and k_0 are arbitrary, we obtain (11). □

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