Filomat 34:6 (2020), 2073–2089
https://doi.org/10.2298/FIL2006073K

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Approximate Optimality for Quasi Approximate Solutions in Nonsmooth Semi-Infinite Programming Problems, Using ε-Upper Semi-Regular Semi-Convexificators

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Abstract. In this paper, we study optimality conditions of quasi approximate solutions for nonsmooth semi-infinite programming problems (for short, (SIP)), in terms of ε-upper semi-regular semi- convexificator which is introduced here. Some classes of functions, namely $(\varepsilon - \partial^*_\varepsilon)$-pseudoconvex functions and $(\varepsilon - \partial^*_\varepsilon)$-quasiconvex functions with respect to a given ε-upper semi-regular semi-convexificator are introduced, respectively. By utilizing these new concepts, sufficient optimality conditions of approximate solutions for the nonsmooth (SIP) are established. Moreover, as an application, optimality conditions of quasi approximate weakly efficient solution for nonsmooth multi-objective semi-infinite programming problems (for short, (MOSIP)) are presented.

1. Introduction

It is well known that semi-infinite programming problems became an active research topic in mathematical programming due to its extensive applications in many fields such as reverse Chebyshev approximate, robust optimization, minimax problems, design centering and disjunctive programming; see ([12, 30, 33]). Recently, a great deal of results have appeared in the literature; see [3, 5, 9, 10, 15, 20, 21, 23, 24] and the references therein.

We note that the approximate solutions of optimization problems are very important from both the theoretical and practical points of view because they exist under very mild hypotheses and a lot of solution methods (for example, iterative algorithms or heuristic algorithms) obtain this kind of solutions. Thus, it is meaningful to consider various concepts of approximate solutions to optimization problems. The first concept of approximate solutions for optimization problems was introduced by Kutateladze [22]. We remark that, in recent years, many authors devoted their efforts to propose new notions of approximate solutions in connection with the optimization problems [26].

On the other hand, the idea of convexificators has been used to extend, unify, and sharpen various results in nonsmooth analysis and optimization (see, for instance [8, 17, 19]. They represent a weaker version of the

2010 *Mathematics Subject Classification*. 90C30; 90C46; 49J52
Keywords. Nonsmooth semi-infinite programming, Approximate optimality conditions, Quasi approximate solutions, Convexificator, Pseudoconvexity, Quasiconvexity
Received: 10 May 2019; Accepted: 28 June 2020
Communicated by Vasile Berinde
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notion of subdifferentials, and are more suitable to analysis and applications. These convexificators, which firstly were assumed to be compact [7]), produce both upper convex and lower concave approximations at a point for a function. In the paper [18], the notion of convexificator is introduced as a closed set, but is not necessarily bounded or convex. The significance of noncompact convexificators is that they allow applications of convexificators to continuous functions. For a locally Lipschitz function, most known subdifferentials, which are convex and compact sets, such as the subdifferential of Clarke [6] , Michel–Penot [28], Ioffe–Morduchovich [16, 29], and Treiman [35] are convexificators. Moreover, for locally Lipschitz functions, these known subdifferentials may strictly contain the convex hull of a convexificator (see, [36, Example 2.2]).

Finding the exact description of the solution in optimization problem, for instance, the description of weakly efficient solution, sometimes it turns out to be practically impossible or computationally too expensive. Thus, many researchers turn their attention on approximate solutions, and for various approximate solution concepts we refer the reader to [2, 13, 14, 27, 34, 37]. According to above paragraph, the description of optimality conditions for solutions and approximate solutions in terms of convexificators provides sharp results. Surely, such description in terms of ε-upper convexificators, which were introduced very recently in [4], also. These new concepts were used for obtaining the results on approximation of solutions in optimization problems. In the paper, with the continuous objective function, necessary and sufficient conditions for a point to be an ε-quasi solution of a scalar optimization problem via ε-convexificators are provided.

Motivated by these important problems and interesting concepts about convexificators and their genaralization, we aim to establish results on approximation of solutions in (SIP) via some tools related to convexificators. In order to reach our goals, we introduce the notion of ε-upper semi-regular semi-convexificator, and some classes of functions, namely $(\varepsilon - \partial^*_\varepsilon)$-pseudoconvex of type I, $(\varepsilon - \partial^*_\varepsilon)$-pseudoconvex of type II and $(\varepsilon - \partial^*_\varepsilon)$-quasiconvex functions with respect to a given upper semi-regular semi-convexificator. Then employed these notions for deriving necessary and sufficient optimality conditions for characterizing the quasi approximate solutions of our considered (SIP). Further, we then consider the optimality conditions of quasi approximate efficient solution for a nonsmooth (MOSIP) and obtain the desired results.

The rest of the paper is organized as follows. Section 2 contains preliminaries. The optimality conditions are investigated in Section 3. Then, an application of the results is presented in Section 4. Finally, the conclusion can be found in Section 5.

2. Preliminaries

For a set $A \subseteq \mathbb{R}^n$, we use the notations co$A$, int$A$ and cl$A$ to denote the convex hull, the interior and the closure of A, respectively. The considered norm $\|\cdot\|$ is the Euclidean norm, the notation $\langle \cdot, \cdot \rangle$ is utilized to denote inner product and the symbol B^* stands for a closed unit ball in $\mathbb{R}^n$.

A nonempty set $A \subseteq \mathbb{R}^n$ is called a cone if for each $x \in K$ and each scalar $\alpha \geq 0$, we have $\alpha x \in K$. A cone K is said to be pointed whenever $K \cap (-K) = \{0\}$.

Let $A \subseteq \mathbb{R}^n$ be a nonempty subset and $x \in A$. Denote by d_A the distance function of A, i.e., $d_A(x) := \inf\{\|x - y\| : y \in A\}$. A vector $v \in \mathbb{R}^n$ is tangent to A at x provided $d^\circ_A(x; v) = 0$. The set of all vectors tangent to A at x, namely the Clarke tangent cone to A at x is denoted by $T_C(A, x)$. The Clarke normal cone to A at x is defined by

$$N_C(A, x) := \{\xi \in \mathbb{R}^n : \langle \xi, v \rangle \leq 0, \forall v \in T_C(A, x)\}.$$

Let $A \subseteq \mathbb{R}^n$ be a nonempty closed convex subset. The contigent and the normal cone to A at $x \in A$ are respectively defined by

$$T(A, x_0) = \{d \in \mathbb{R}^n : \exists t_n \downarrow 0, \exists \{d_n\} \subseteq \mathbb{R}^n \text{ s. t. } d_n \to d, x_0 + t_n d_n \in A\},$$

and

$$N(A, x) := \{\xi \in \mathbb{R}^n : \langle \xi, y - x \rangle \leq 0, \forall y \in A\}.$$

Clearly, if a set A is closed and convex, then $N_C(A,x) = N(A,x)$.
The polar cone of a set $A \subseteq \mathbb{R}^n$ is defined by

$$A^\circ = \{d \in \mathbb{R}^n : \langle d, x\rangle \leq 0, \forall x \in K\}.$$

It is clear that $N_C(A,x) = (T(A,x))^\circ$ and $N_C(A,x) = (T(A,x))^\circ$.

The Hadamard Dini directional derivatives, defined as follows, play a vital role in this work. Hereafter, dom f stands for the effective domain of f.

Definition 2.1. Consider $f : \mathbb{R}^n \to \mathbb{R}$.

1. The *Hadamard Dini directional derivative* of f at $x \in \text{dom}f$ in direction $v \in \mathbb{R}^n$ is defined by

$$f^{HD}(x;v) := \limsup_{\substack{d\to v\\ t\downarrow 0}} \frac{f(x+td)-f(x)}{t}.$$

2. The *lower and upper Dini directional derivatives* of f at $x \in \text{dom}f$ in direction $d \in \mathbb{R}^n$ are, respectively, defined by

$$f^-(x;d) := \liminf_{t\downarrow 0} \frac{f(x+td)-f(x)}{t}$$
$$f^+(x;d) := \limsup_{t\downarrow 0} \frac{f(x+td)-f(x)}{t}.$$

3. The *directional derivative* of f at $x \in \mathbb{R}^n$ in direction $d \in \mathbb{R}^n$ (if exists), denoted by $f'(x;d)$, is defined as

$$f'(x;d) := \lim_{t\downarrow 0} \frac{f(x+td)-f(x)}{t}.$$

Remark 2.2. If f is locally Lipschitz, then its upper Dini directional derivative and Hadamard directional derivative are same.

Definition 2.3. [11, 18] Let $f : \mathbb{R}^n \to \mathbb{R}$ and $x_0 \in \text{dom}f$. The function f is said to have:

1. an *upper convexificator* $\partial^* f(x_0)$ at x_0 if $\partial^* f(x_0) \subseteq \mathbb{R}^n$ is closed and, for each $d \in \mathbb{R}^n$,

$$f^-(x_0;d) \leq \sup_{\eta\in\partial^* f(x_0)} \langle \eta, d\rangle. \tag{1}$$

2. a *lower convexificator* $\partial^* f(x_0)$ at x_0 if $\partial^* f(x_0) \subseteq \mathbb{R}^n$ is closed and, for each $d \in \mathbb{R}^n$,

$$f^+(x_0;d) \geq \inf_{\eta\in\partial^* f(x_0)} \langle \eta, d\rangle. \tag{2}$$

3. a *convexificator* $\partial^* f(x_0)$ at x_0 if $\partial^* f(x_0) \subseteq \mathbb{R}^n$ is both of upper and lower convexificator (i.e. $\partial^* f(x_0)$ is closed and both of (1) and (2) are fulfilled for each $d \in \mathbb{R}^n$.)
4. an *upper semi-regular convexificator* $\partial^* f(x_0)$ at x_0 if $\partial^* f(x_0) \subseteq \mathbb{R}^n$ is closed and, for each $d \in \mathbb{R}^n$,

$$f^+(x_0;d) \leq \sup_{\eta\in\partial^* f(x_0)} \langle \eta, d\rangle. \tag{3}$$

5. an *upper regular convexificator* of f at x_0 if inequality 3 holds as equality.

Definition 2.4. [4] Let $\varepsilon \geq 0$ be given. Let $f : \mathbb{R}^n \to \mathbb{R}$ and $x_0 \in \text{dom}f$. The function f is said to have:

1. a *ε-upper convexificator* $\overline{\partial}^*_\varepsilon f(x_0)$ at x_0 if $\overline{\partial}^*_\varepsilon f(x_0) \subseteq \mathbb{R}^n$ is closed and, for each $d \in \mathbb{R}^n$,
$$f^-(x_0; d) \leq \sup_{\eta \in \overline{\partial}^*_\varepsilon f(x_0)} \langle \eta, d \rangle + \varepsilon \|d\|. \tag{4}$$

2. a *ε-lower convexificator* $\underline{\partial}^*_\varepsilon f(x_0)$ at x_0 if $\underline{\partial}^*_\varepsilon f(x_0) \subseteq \mathbb{R}^n$ is closed and, for each $d \in \mathbb{R}^n$,
$$f^+(x_0; d) \geq \inf_{\eta \in \underline{\partial}^*_\varepsilon f(x_0)} \langle \eta, d \rangle - \varepsilon \|d\|. \tag{5}$$

3. a *ε-upper regular convexificator* $\overline{\partial}^*_\varepsilon f(x_0)$ at x_0 if $\overline{\partial}^*_\varepsilon f(x_0) \subseteq \mathbb{R}^n$ is closed and, for each $d \in \mathbb{R}^n$,
$$f^+(x_0; d) = \sup_{\eta \in \overline{\partial}^*_\varepsilon f(x_0)} \langle \eta, d \rangle + \varepsilon \|d\|. \tag{6}$$

4. a *ε-lower regular convexificator* $\underline{\partial}^*_\varepsilon f(x_0)$ at x_0 if $\underline{\partial}^*_\varepsilon f(x_0) \subseteq \mathbb{R}^n$ is closed and, for each $d \in \mathbb{R}^n$,
$$f^-(x_0; d) = \inf_{\eta \in \underline{\partial}^*_\varepsilon f(x_0)} \langle \eta, d \rangle + \varepsilon \|d\|. \tag{7}$$

Following the idea of definitions for upper(and lower) semi-regular convexificators, we introduce the following notions.

Definition 2.5. Let $\varepsilon \geq 0$ be given. Let $f : \mathbb{R}^n \to \mathbb{R}$ and $x_0 \in \mathrm{dom} f$. The function f is said to have an *ε- upper semi-regular semi-convexificator* $\overline{\partial}^*_\varepsilon f(x_0)$ at x_0 if $\partial^* f(x_0) \subseteq \mathbb{R}^n$ is closed and, for each $d \in \mathbb{R}^n$,
$$f^{HD}(x_0; d) \leq \sup_{\eta \in \overline{\partial}^*_\varepsilon f(x_0)} \langle \eta, d \rangle + \sqrt{\varepsilon} \|d\|. \tag{8}$$

Remark 2.6. If f is a locally Lipschitz function and $\varepsilon = 0$, then whenever f admits an ε upper semi-regular semi-convexificator at x_0 it also admits upper semi-regular convexificator at the point.

Example 2.7. Consider $f(x) = -|x|$, $x \in \mathbb{R}$. We know that $f'(x; v) = -|v|$, $v \in \mathbb{R}$, $f^\circ = |v|$, $v \in \mathbb{R}$, and $\partial_C f(0) = [-1, 1]$. It is not difficult to check that the closed set $\partial^*_\varepsilon f(0) = [-1, -1 + \varepsilon] \cup [1 - \varepsilon, 1]$ is an ε upper semi-regular semi-convexificator at $x = 0$ as follows:
Case I: $v \geq 0$;
$$f^{HD}(0; v) = -v \leq v + \sqrt{\varepsilon} v = \sup_{x^* \in [-1, -1+\varepsilon] \cup [1-\varepsilon, 1]} \langle x^*, v \rangle + \sqrt{\varepsilon} |v|,$$
Case II: $v < 0$;
$$f^{HD}(0; v) = v \leq -(1 + \sqrt{\varepsilon}) v \leq \sup_{x^* \in [-1, -1+\varepsilon] \cup [1-\varepsilon, 1]} \langle x^*, v \rangle + \sqrt{\varepsilon} |v|.$$
Notice that this ε semi-regular semi-convexificator is contained in the Clarke subdifferential $\partial_C f(0)$ of f at 0 and it is not equal to the convexificator $\partial^* f(0) = \{-1, 1\}$ of f at 0.

In locally Lipschitz optimization programming, in 2009, Son et al. [32] introduced the following generalized convexity which is a generalization of the convexity and the semiconvexity (in locally Lipschitz optimization programming).

Definition 2.8. [32] Let $A \subseteq \mathbb{R}^n$ be a nonempty subset and $\varepsilon \geq 0$. A locally Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be ε-semiconvex at $x \in K$ if f is regular at x and
$$f'(x; y - x) + \sqrt{\varepsilon} \|y - x\| \geq 0 \Rightarrow f(y) + \sqrt{\varepsilon} \|y - x\| \geq f(x), \ \forall y \in A.$$

Very recently, X.-J. Long et al.[25]introduced the generalized convex functions called ε-pseudoconvex of Type I, ε-pseudoconvex of Type II, and ε-quasiconvex function.

Definition 2.9. [25] Let $A \subseteq \mathbb{R}^n$ be a nonempty set and $\varepsilon \geq 0$. For a locally Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$, $\partial^\circ f(x)$ stands for its Clark subdifferential at a point $x \in \mathbb{R}^n$. The function f is said to be

1. ε-pseudoconvex of type I at $x \in A$ if, for all $y \in A$,

$$f(y) + \sqrt{\varepsilon}\|y - x\| < f(x) \Rightarrow \langle \xi, y - x \rangle + \sqrt{\varepsilon}\|y - x\| < 0, \forall \xi \in \partial^\circ f(x_0).$$

2. ε-pseudoconvex of type II at $x \in A$ if, for all $y \in A$,

$$f(y) + \sqrt{\varepsilon}\|y - x\| < f(x) \Rightarrow \langle \xi, y - x \rangle < 0, \forall \xi \in \partial^\circ f(x).$$

3. ε-quasiconvex at $x \in A$ if, for all $y \in A$,

$$f(y) \leq f(x) \Rightarrow \langle \xi, y - x \rangle + \sqrt{\varepsilon}\|y - x\| \leq 0, \forall \xi \in \partial^\circ f(x).$$

Recently, by going along the lines of Dutta and Chandra [11], Ahmad et.al [1] gave the definitions of ∂^*-convex, strict ∂^*-convex, ∂^*-pseudoconvex, strict ∂^*-pseudoconvex, and ∂^*-quasiconvex functions by using the concept of convexifactors.

Definition 2.10. [1] Let a function $f : \mathbb{R}^n \to \mathbb{R}$ admit a convexifactor $\partial^* f(x)$ at $x \in \mathbb{R}^n$. The function f is said to be

1. ∂^*-convex at $x \in \mathbb{R}^n$ if, for all $y \in \mathbb{R}^n$,

$$f(y) - f(x) \geq \langle \xi, y - x \rangle, \ \forall \xi \in \partial^* f(x);$$

and if above inequality holds strictly for all $y \in \mathbb{R}^n, y \neq x$, f is said to be strict ∂^*-convex at $x \in \mathbb{R}^n$.

2. ∂^*-pseudoconvex at $x \in \mathbb{R}^n$ if, for all $y \in \mathbb{R}^n$,

$$f(y) < f(x) \Rightarrow \langle \xi, y - x \rangle < 0, \ \forall \xi \in \partial^* f(x);$$

3. strict ∂^*-pseudoconvex at $x \in \mathbb{R}^n$ if, for all $y \in \mathbb{R}^n, y \neq x$,

$$f(y) \leq f(x) \Rightarrow \langle \xi, y - x \rangle < 0, \ \forall \xi \in \partial^* f(x);$$

4. ∂^*-quasiconvex at $x \in \mathbb{R}^n$ if, for all $y \in \mathbb{R}^n$,

$$f(y) \leq f(x) \Rightarrow \langle \xi, y - x \rangle \leq 0, \ \forall \xi \in \partial^* f(x).$$

Now, we shall introduce the following classes of functions dealing with pseudoconvexity and quasiconvexity.

Definition 2.11. Let $A \subseteq \mathbb{R}^n$ be a nonempty set and $\varepsilon \geq 0$ be given. Assume that $f : \mathbb{R}^n \to \mathbb{R}$ admits an ε upper (semi-regular) semi-convexifactor $\partial^*_\varepsilon f(x)$ at $x \in \mathbb{R}^n$. The function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be

1. $(\varepsilon - \partial^*_\varepsilon)$*-pseudoconvex of type I* at $x \in A$ if, for all $y \in A$,

$$f(y) + \sqrt{\varepsilon}\|y - x\| < f(x) \Rightarrow \langle \xi, y - x \rangle + \sqrt{\varepsilon}\|y - x\| < 0, \ \forall \xi \in \partial^*_\varepsilon f(x);$$

2. $(\varepsilon - \partial^*_\varepsilon)$*-pseudoconvex of type II* at $x \in A$ if, for all $y \in A$,

$$f(y) + \sqrt{\varepsilon}\|y - x\| < f(x) \Rightarrow \langle \xi, y - x \rangle < 0, \ \forall \xi \in \partial^*_\varepsilon f(x);$$

3. $(\varepsilon - \partial^*_\varepsilon)$*-quasiconvex* at $x \in A$ if, for all $y \in \mathbb{R}^n$,

$$f(y) \leq f(x) \Rightarrow \langle \xi, y - x \rangle + \sqrt{\varepsilon}\|y - x\| \leq 0, \ \forall \xi \in \partial^*_\varepsilon f(x).$$

In order to connect with approximation, we now recall some definitions of approximate solutions. Denote the feasible set of (23) by C, i.e., $C := \{x \in \Omega : g_i(x) \leq 0, \forall i \in I\}$.

Definition 2.12. [26] Let $\varepsilon \geq 0$. A feasible solution $x_0 \in C$ is called:

1. *an ε-minimum* of (9) if for each $x \in C$,

$$f(x_0) \leq f(x) + \varepsilon,$$

2. *an ε-quasi minimum* of (9) if for each $x \in C$

$$f(x_0) \leq f(x) + \sqrt{\varepsilon}\|x - x_0\|.$$

Example 2.13. Consider

$$f(x) := \begin{cases} x^{\frac{3}{2}} + x; & \text{if } x \geq 0, \\ 2x; \text{ if } x < 0, \end{cases}$$

By direct computation, we get that

$$f^{HD}(0; v) = f'(0; v) = \begin{cases} v; & \text{if } x \geq 0, \\ 2v \text{ if } x < 0, \end{cases}$$

and

$$f^{\circ}(0; v) = \begin{cases} 2v; & \text{if } x \geq 0, \\ v \text{ if } x < 0. \end{cases}$$

Let $\varepsilon = \dfrac{1}{4}$. It can be checked that $\partial^*_\varepsilon f(0) = [\frac{5}{4}, \frac{3}{2}]$ is an ε upper semi-convexificator of f at 0. Note that this ε upper semi-convexificator is contained in Clarke subdifferential $\partial_C f(0) = [1, 2]$ of f at 0. The $(\varepsilon - \partial^*_\varepsilon)$-pseudoconvexity of type I of f at 0 is sastisfied as follows:
for $y \geq 0$; consider $\xi = \frac{5}{4} \in [\frac{5}{4}, \frac{3}{2}]$ with $\frac{7}{4}y = \langle \frac{5}{4}, y \rangle \geq 0$. Clearly, $f(y) = y^3 + y > 0 = f(0)$. On the other hand, for $y < 0$; for any $\xi \in [\frac{5}{4}, \frac{3}{2}]$ we have $\langle \xi, y \rangle \leq \dfrac{3}{4}y < 0$. Also, the $(\varepsilon - \partial^*_\varepsilon)$-pseudoconvexity of type II of f is fulfilled at 0. However, f is not semiconvex at 0 since it is not regular at the point.

Let us denote by C_ε the ε-feasible set, which is nonempty and closed, is in the following form:

$$C_\varepsilon := \left\{x \in \Omega : g_i(x) \leq \sqrt{\varepsilon}, \, i \in I\right\}.$$

Definition 2.14. [24] Let $\varepsilon > 0$ be given. A point $x_0 \in X$ is said to be *an almost ε-quasi minimum)* for (23) if x_0 satisfies the following conditions:

1. $x_0 \in C_\varepsilon$;
2. $f(x_0) \leq f(x) + \sqrt{\varepsilon}\|x - x_0\|$, for all $x \in C$.

3. Approximate optimality for quasiapproximate solutions in (SIP)

In this section, we firstly investigate the following (SIP):

$$\begin{aligned} &\text{Minimize } f(x) \\ &\text{subject to } g_i(x) \leq 0, i \in I, \end{aligned} \tag{9}$$

where $f, g_i : \mathbb{R}^n \to \mathbb{R}$ for $i \in I$ are real-valued functions. Here, I is an arbitrary (possibly infinite) nonempty set. Unlike various related publications existing in the literature, in the present work, appearing functions are not locally Lipschitz or convex necessarily. The set of feasible solutions of Problem (9) is

$$K := \{x \in \mathbb{R}^n : g_i(x) \leq 0, i \in I\}.$$

We assume $K \neq \emptyset$. For a given $x_0 \in K$, set

$$I(x_0) := \{i \in I : g_i(x_0) = 0\}.$$

If for each $i \in I$, g_i is a convex or quasiconvex function, then C is a convex set.

Now, let us denote by $\mathbb{R}^{(I)}$ the following linear vector space [1]:

$$\mathbb{R}^{(I)} := \{r = (r_i)_{i\in I} : r_i = 0 \text{ for all } i \in I \text{ except for initely many } r_i \neq 0\}.$$

The nonnegative cone of $\mathbb{R}^{(I)}$ is denoted by

$$\mathbb{R}^{(I)}_+ := \{r = (r_i)_{i\in I} \in \mathbb{R}^{(I)} : r_i \geq 0, i \in I\}.$$

It is easy to see that $\mathbb{R}^{(I)}_+$ is a convex cone of $\mathbb{R}^{(I)}$. For $\alpha \in \mathbb{R}^{(I)}_+$, the supporting set corresponding to α is defined by

$$I(\alpha) := \{i \in I : \alpha_i > 0\},$$

which is a finite subset of I.

Let Z be a linear vector space. For $\alpha \in \mathbb{R}^{(I)}$ and $\{z_i\}_{i\in I} \subseteq Z$, we set

$$\sum_{i\in I} \alpha_i z_i := \begin{cases} \sum_{i\in I(\alpha)} \alpha_i z_i, & I(\alpha) \neq \emptyset, \\ 0, & I(\alpha) = \emptyset. \end{cases}$$

Now, we concentrate on consideration of the following (SIP):

$$\begin{aligned} &\text{Minimize } f(x) \\ &\text{subject to } g_i(x) \leq 0, i \in I, x \in \Omega \end{aligned} \tag{10}$$

where $f, g_i : \mathbb{R}^n \to \mathbb{R}$ for $i \in I$ are real-valued functions, while $\Omega \subseteq \mathbb{R}^n$ is a closed convex set. Here, I is an arbitrary (possibly infinite) nonempty set. The set of feasible solutions of Problem (10) is

$$C := \{x \in \Omega : g_i(x) \leq 0, i \in I\}, \tag{11}$$

which is assumed to be nonempty.

To obtain the necessary optimality condition, we consider the following constraint qualification condition:

$$N_C(C, x_0) \subseteq \bigcup_{\mu \in A(x_0)} \left[\sum_{i\in I} \mu_i \text{co}\,(\partial^* g_i(x_0)) \right] + N(\Omega, x_0), \tag{12}$$

where $A(x_0) := \{\mu \in \mathbb{R}^{(I)}_+ : \mu_i g_i(x_0) = 0, \forall i \in I\}$ and $x_0 \in C$.

Theorem 3.1. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a fuction and let C be a closed subset of $\mathbb{R}^n$. Consider the problem:

$$\text{Minimize } f(x) \text{ subject to } x \in C. \tag{13}$$

Let $x_0 \in C$. Assume that

$$\text{int}\,(T_C(C, x_0)) \cap \text{dom } g \neq \emptyset, \tag{14}$$

where $g(v) = \sup_{x^* \in \overline{\partial}^*_\varepsilon f(x_0)} \langle x^*, v\rangle + \sqrt{\varepsilon}\|v\|$. If x_0 is an ε-quasi solution of the problem (13), then $0 \le f^{HD}(x_0; v) + \sqrt{\varepsilon}\|v\|$, $\forall v \in T(C, x_0)$. In addition, if f admits an ε upper semiregular semi-convexificator $\overline{\partial}^*_\varepsilon f(x_0)$ at x_0, i.e., $\overline{\partial}^*_\varepsilon f(x_0)$ is closed and $f^{HD}(x_0; v) \le \sup_{x^* \in \overline{\partial}^*_\varepsilon f(x_0)} \langle x^*, v\rangle + \sqrt{\varepsilon}\|v\|$, then $0 \in \mathrm{clco}\overline{\partial}^*_\varepsilon f(x_0) + \sqrt{\varepsilon}B^* + N_C(C, x_0)$.

Proof. Let $v \in T(C, x_0)$. Then, there exist $\lambda_n > 0$ and $x_n \in C$ such that $x_n \to x_0$ and $v = \lim_{n\to+\infty} \lambda_n(x_n - x_0)$. Putting $v_n = \lambda_n(x_n - x_0)$ implies

$$v_n \to v \text{ and } x_n = x_0 + \frac{1}{\lambda_n} v_n.$$

Let $\frac{1}{\lambda_n} = t_n$, then

$$x_n = x_0 + t_n v_n \in C.$$

We may assume that $v \neq 0$. Then $\lambda_n \to +\infty$ and so $t_n \downarrow 0$. Thus, we have that

$$\begin{aligned} f^{HD}(x_0; v) &= \limsup_{\substack{d\to v\\ t\downarrow 0}} \frac{f(x_0 + td) - f(x_0)}{t} \\ &\ge \limsup_{n\to\infty} \frac{f(x_0 + t_n v_n) - f(x_0)}{t_n} \\ &= \limsup_{n\to\infty} \frac{f(x_0 + t_n v_n) - f(x_0) + \sqrt{\varepsilon}\|t_n v\|}{t_n} - \sqrt{\varepsilon}\|v\| \\ &\ge -\sqrt{\varepsilon}\|v\|. \end{aligned}$$

Hence, we obtain

$$0 \le f^{HD}(x_0; v) + \sqrt{\varepsilon}\|v\|, \ \forall v \in T(C, x_0).$$

Next, we assume that $f^{HD}(x_0; v) \le \sup_{x^* \in \partial^* f(x_0)} \langle x^*, v\rangle + \sqrt{\varepsilon}\|v\|$ for all $v \in \mathbb{R}^n$. It follows that

$$\begin{aligned} 0 &\le \sup_{x^* \in \overline{\partial}^*_\varepsilon f(x_0)} \langle x^*, v\rangle + \sqrt{\varepsilon}\|v\|, \ \forall v \in T(c, x_0) \\ &= \sup_{x^* \in \mathrm{clco}\overline{\partial}^*_\varepsilon f(x_0) + \sqrt{\varepsilon}B^*} \langle x^*, v\rangle, \ \forall v \in T(c, x_0). \end{aligned}$$

Let $g(v) = \sup_{x^* \in \mathrm{clco}\overline{\partial}^*_\varepsilon f(x_0)} \langle x^*, v\rangle$. Then $g(0) = 0$ and so g is a proper convex function. Since $T_C(C, x_0) \subseteq T(C, x_0)$, we have

$$0 \le g(v), \ \forall v \in T_C(C, x_0),$$

and so

$$0 \le g(v) + \delta_{T_C(C,x_0)}(v), \ \forall v \in \mathbb{R}^n.$$

Hence, we get that

$$\begin{aligned} 0 &\in \partial\left(g + \delta_{T_C(C,x_0)}\right)(0) \\ &= \partial g(0) + \partial\delta_{T_C(C,x_0)}(0) \\ &= \partial g(0) + N_C(C, x_0). \end{aligned} \tag{15}$$

Note that

$$g(v) = \sup_{x^* \in \mathrm{clco}\overline{\partial}^*_\varepsilon f(x_0) + \sqrt{\varepsilon}B^*} \geq g(0) + \langle x^*, v\rangle.$$

so, we obtain $\mathrm{clco}\overline{\partial}^*_\varepsilon f(x_0) + \varepsilon B \subseteq \partial g(0)$. On the other hand, assume that there exists $\xi \in \mathrm{clco}\overline{\partial}^*_\varepsilon f(x_0) + \sqrt{\varepsilon}B^*$ such that $\xi \notin \partial g(0)$. Thus, there is $\hat{v} \in \mathbb{R}^n$ such that

$$g(\hat{v}) < g(0) + \langle \xi, \hat{v}\rangle.$$

So,

$$\sup_{x^* \in \mathrm{clco}\overline{\partial}^*_\varepsilon f(x_0) + \sqrt{\varepsilon}B^*} \langle x^*, v\rangle < \langle \xi, \hat{v}\rangle,$$

a contradiction. From (15), $0 \in \mathrm{clco}\overline{\partial}^*_\varepsilon f(x_0) + \sqrt{\varepsilon}B^* + N_C(C, x_0)$. □

Theorem 3.2. Let $\varepsilon \geq 0$ be given and x_0 be an ε-quasi-minimizer for (10). Suppose that f admits an ε-upper semi-regular semi-convexificator $\overline{\partial}^*_\varepsilon f(x_0)$ at x_0, each $g_i, i \in I$ admits an upper convexificator $\overline{\partial}^* g_i(x_0)$ at x_0, assume the assumption (14) and the constraint qualification condition (12) hold at x_0. Then, there exist $\mu_i \geq 0, \forall i \in I$ and $\mu \in \mathbb{R}^{(I)}_+$ such that

$$0 \in \mathrm{clco}\overline{\partial}^*_\varepsilon f(x_0) + \sqrt{\varepsilon}B^* + \sum_{i \in I} \mu_i(\mathrm{co}\overline{\partial}^* g_i(x_0)) + N(\Omega, x_0),\ g_i(x_0) = 0, \forall i \in I(\mu). \tag{16}$$

Proof. Since f admits ε-semi regular convexificator at x_0, we have

$$f^{HD}(x_0; v) \leq \sup_{x^* \in \partial^*_\varepsilon f(x_0)} \langle x^*, v\rangle + \sqrt{\varepsilon}\|v\|,\ \forall v \in \mathbb{R}^n.$$

Due to assumptionas and Theorem 3.1, we obtain that

$$0 \in \mathrm{clco}\overline{\partial}^*_\varepsilon f(x_0) + \sqrt{\varepsilon}B^* + N_C(C, x_0).$$

Since (12) is satisfied at x_0, we obtain that

$$0 \in \mathrm{clco}\overline{\partial}^*_\varepsilon f(x_0) + \sqrt{\varepsilon}B^* + \bigcup_{\mu \in A(x_0)} \left[\sum_{i \in I} \mu_i \mathrm{co}\,(\partial^* g_i(x_0))\right] + N(\Omega, x_0),$$

where $A(x_0) := \{\mu \in \mathbb{R}^{(I)}_+ : \mu_i g_i(x_0) = 0, \forall i \in I\}$. Therefore, there exists $\mu_i \geq 0,\ \forall i \in I$ such that

$$0 \in \mathrm{clco}\overline{\partial}^*_\varepsilon f(x_0) + \sqrt{\varepsilon}B^* + \sum_{i \in I} \mu_i \left(\mathrm{co}\overline{\partial}^* g_i(x_0)\right) + N(\Omega, x_0),$$
$$g_i(x_0) = 0, i \in I(\mu).$$

Therefore (16) is verified. □

We next formulate some sufficient conditions for an almost ε-quasi minimizer for the problem (10).

Theorem 3.3. $(x_0, \mu) \in C_\varepsilon \times \mathbb{R}^{(I)}_+$ be given. Suppose that f admits ε-upper semi-regular semi-convexificator $\overline{\partial}^*_\varepsilon f(x_0)$ at x_0, $g_i, i \in I$ admit upper convexificators $\overline{\partial}^* g_i(x_0)$, respectively, at x_0, and (x_0, μ) is such that

$$0 \in \mathrm{co}\overline{\partial}^*_\varepsilon f(x_0) + \sum_{i \in I} \mu_i \left(\mathrm{co}\overline{\partial}^* g_i(x_0)\right) + N(\Omega, x_0) + \sqrt{\varepsilon}B^*, \tag{17}$$

$$g_i(x_0) \geq 0, \forall i \in I(\mu). \tag{18}$$

Assume that for each $i \in I$, the function g_i is quasiconvex at x_0.

1. If f is $\overline{\partial}^*_\varepsilon$-pseudoconvex of type I at x_0, then x_0 is an almost ε-quasi minimizer for (10).
2. If f is $\overline{\partial}^*_\varepsilon$-pseudoconvex of type II at x_0, then x_0 is an almost ε-quasi minimizer for (10).

Proof. (i) Let $(x_0, \mu) \in C_\varepsilon \times \mathbb{R}^{(I)}_+$ be such that (17) holds. Then, there exist $a \in \mathrm{co}\overline{\partial}^*_\varepsilon f(x_0)$, $b_i \in \mathrm{co}(\overline{\partial}^* g_i(x_0))$ with $\mu_i \in \mathbb{R}_+, \forall i \in I, c \in N(\Omega, x_0)$ and $d \in B^*$, such that $g_i(x_0) \geq 0$ for all $i \in I(\mu)$ and

$$a + \sum_{i\in I} \mu_i b_i + c + d = 0. \tag{19}$$

Since $c \in N(\Omega, x_0)$, and $d \in B^*$, one has

$$\langle c, x - x_0\rangle \leq 0,\ d(x - x_0) \leq \|x - x_0\|,\ \forall x \in \Omega.$$

Because of these inequalities and (19), we obtain that

$$\left\langle a + \sum_{i\in I} \mu_i b_i, x - x_0 \right\rangle + \sqrt{\varepsilon}\|x - x_0\| \geq 0.$$

Therefore, it follows that

$$\left\langle a + \sum_{i\in I(\mu)} \mu_i b_i, x - x_0 \right\rangle + \sqrt{\varepsilon}\|x - x_0\| \geq 0,$$

which is equivalent to

$$\langle a, x - x_0\rangle + \sqrt{\varepsilon}\|x - x_0\| \geq -\left\langle \sum_{i\in I(\mu)} \mu_i b_i, x - x_0 \right\rangle. \tag{20}$$

Due to the hypothesis and property of any feasible points, respectively, we have $g_i(x_0) \geq 0$ for all $i \in I(\mu)$ and $g_i(x) \leq 0$ for all $i \in I, x \in C$. Thus, for any $x \in C$ and $i \in I(\mu)$,

$$g_i(x) \leq g_i(x_0).$$

By the $\overline{\partial}^*_\varepsilon$-quasiconvexity of $g_i, i \in I(\mu)$ at x_0, above inequality implies

$$\langle b_i, x - x_0\rangle \leq 0,\ \forall b_i \in \partial^* g_i(x_0),\ \forall i \in I(\mu).$$

Thus, one has

$$\left\langle \sum_{i\in I(\mu)} \mu_i b_i, x - x_0 \right\rangle \leq 0. \tag{21}$$

Then, we obtain by combining (20) and (21) that

$$\langle a, x - x_0\rangle + \sqrt{\varepsilon}\|x - x_0\| \geq 0. \tag{22}$$

Immediately, the $\overline{\partial}^*_\varepsilon$- pseudoconvexity of type I of f at x_0 yields

$$f(x_0) \leq f(x) + \sqrt{\varepsilon}\|x - x_0\|,$$

which means x_0 is an almost ε-quasi minimizer of (10) as desired.

(ii) From (20) and the fact that $-\sqrt{\varepsilon}\|x - x_0\| \leq 0$, it is not hard to see that

$$\langle a, x - x_0 \rangle \geq -\left\langle \sum_{i \in I(\mu)} \mu_i b_i, x - x_0 \right\rangle,$$

which together with (21) yields

$$\langle a, x - x_0 \rangle \geq 0.$$

Therefore, it follows from the $\overline{\partial}^*_\varepsilon$-pseudoconvexity of type II of the function f that x_0 is an almost ε-quasi minimizer of (10). This completes the proof. □ □

Example 3.4. Consider the following problem (9):

$$\begin{aligned} &\min f(x) \\ &\text{subject to } g_i(x) \leq 0, i \in I = [0, 1], \\ &\qquad x \in \Omega = [-1, 1], \end{aligned}$$

where

$$f(x) := \begin{cases} 0; & \text{if } x \geq 0, \\ -\frac{x}{4}; & \text{if } x < 0, \end{cases}$$

and $g_i(x) := x^3 i$, for $x \in \mathbb{R}$ and $i \in I$. Simple calculations provide

$$f^{HD}(x; d) = f^+(x; d) = \begin{cases} 0; & \text{if } d \geq 0, \\ -\frac{d}{4}; & \text{if } d < 0. \end{cases}$$

Let $\varepsilon = \dfrac{1}{4}$. By direct computation, we can see that $\overline{\partial}^*_{\frac{1}{4}} = [\frac{1}{2}, 1]$ is an $\dfrac{1}{4}$- upper convexificator of f at $x_0 = 0$. Moreover, we can check that f is $\overline{\partial}^*_{\frac{1}{4}}$-pseudoconvex of type I (obviously, also of type II) of the function f at 0, and $g_i, i \in I$, is quasiconvex at 0. The feasible set of the considered problem is $K = [1, 0]$. It is clear that, $N(\Omega; 0) = \{0\}$, and $\overline{\partial}^*_0 g_i(0) = \partial_C g_i(0) = \{3x^2 i\}$ for all $i \in I$, are respectively convexificators of $g_i, i \in I$ at 0. Let μ be such that $\mu_0 = 1$ and $\mu_i = 0$ for all $i \in I \setminus \{0\}$. We can check that the optimality condition (17) corresponding $(0, \mu)$ holds. In this case we obtain $I(\mu) = \{0\}$. By Theorem 4.3, $x_0 = 0$ is an almost $\frac{1}{4}$-quasi minimum for (9).

4. Approximate optimality for quasiapproximate weakly efficiency in (MOSIP)

This section is an attempt to investigate constraint qualifications (CQs) and to characterize quasi efficient solutions of the following (MOSIP):

$$\begin{aligned} &\text{Minimize } f(x) := (f_1(x), \ldots, f_k(x)) \\ &\text{subject to } g_i(x) \leq 0, i \in I, \end{aligned} \tag{23}$$

where $f_j, g_i : \mathbb{R}^n \to \mathbb{R}$ for $j = 1, 2, \ldots, k$ and $i \in I$ are real-valued functions. Here, I is an arbitrary (possibly infinite) nonempty set. Unlike various related publications existing in the literature, in the present work, appearing functions are not locally Lipschitz or convex necessarily. The set of feasible solutions of Problem (23) is

$$K := \{x \in \mathbb{R}^n : g_i(x) \leq 0, i \in I\}.$$

We assume $K \neq \emptyset$. Set $I := \{1, \dots, k\}$. Furthermore, for a given $x_0 \in K$, set

$$I(x_0) := \{i \in I : g_i(x_0) = 0\}.$$

Now, we concentrate on consideration of the following (MOSIP):

$$\begin{aligned} &\text{Minimize } f(x) := (f_1(x), \dots, f_k(x)) \\ &\text{subject to } g_i(x) \leq 0, i \in I, x \in \Omega \end{aligned} \tag{24}$$

where $f_j, g_i : \mathbb{R}^n \to \mathbb{R}$ for $j = 1, 2, \dots, k$ and $i \in I$ are real-valued functions, while $\Omega \subseteq \mathbb{R}^n$ is a closed convex set. Here, I is an arbitrary (possibly infinite) nonempty set. The set of feasible solutions of Problem (24) is

$$C := \{x \in \Omega : g_i(x) \leq 0, i \in I\}, \tag{25}$$

which is assumed to be nonempty.

Now, we recall the notions of approximate quasi efficiency in (23).

Definition 4.1. [14] Let $\varepsilon \geq 0$. A point $x_0 \in C$ is said to be

1. an ε-quasi efficient solution of (23) if

$$f(x) - f(x_0) + \varepsilon\|x - x_0\|e_k \notin -\mathbb{R}^k_+ \setminus \{0\}, \ \forall x \in C,$$

2. an ε-quasi weakly efficient solution of (23) if

$$f(x) - f(x_0) + \varepsilon\|x - x_0\|e_k \notin -\mathrm{int}\mathbb{R}^k_+ \setminus \{0\}, \ \forall x \in C.$$

Similarly, we denote by C_ε the ε-feasible set, which is nonempty and closed, is in the following form:

$$C_\varepsilon := \left\{x \in \Omega : g_i(x) \leq \sqrt{\varepsilon}, \ i \in I\right\}.$$

Next, we introduce the following concepts of almost (approximate) quasi efficient solution.

Definition 4.2. Let $\varepsilon > 0$ be given. A point $x_0 \in X$ is said to be *an almost ε-quasi weakly efficient solution* for (23) if $x_0 \in C_\varepsilon$ and for any $x \in C$

$$f(x) - f(x_0) + \sqrt{\varepsilon}\|x - x_0\|e_k \notin -\mathrm{int}\mathbb{R}^k_+ \setminus \{0\}.$$

Theorem 4.3. Let $\varepsilon \geq 0$ be given and x_0 be an ε-quasi weakly efficient solution for (24). Suppose that each continuous function $f_j, j \in J$ and $g_i, i \in I$ admit ε-upper semi-regular semi-convexificator $\overline{\partial}^*_\varepsilon f_j(x_0)$ and upper convexificator $\overline{\partial}^* g_i(x_0)$, respectively, at x_0. Assume the assumption (14) and let constraint qualification (12) be satisfied at x_0. Then, there exist $(\lambda, \mu) \in \mathbb{R}^k_+ \times \mathbb{R}^{(I)}_+$ such that $\sum_{j \in J} \lambda_j = 1$ and

$$\begin{aligned} 0 \in \mathrm{clco}\left(\bigcup_{j \in J(x_0)} \overline{\partial}^*_\varepsilon f_j(x_0)\right) + \sqrt{\varepsilon}B^* + \sum_{i \in I} \mu_i \mathrm{co}(\overline{\partial}^* g_i(x_0)) + N(\Omega, x_0), \\ g_i(x_0) = 0, \forall i \in I(\mu). \end{aligned} \tag{26}$$

where $J(x_0) = \left\{j \in J : f_j(x_0) = \Phi(x_0)\right\}$.

Proof. By the definition of an ε-quasi weakly efficient solution and feasible solution for (MOSIP), we have that for all $x \in C$,

$$f(x) - f(x_0) + \sqrt{\varepsilon}\|x - x_0\|e_k \notin -\mathrm{int}\mathbb{R}^k_+ \setminus \{0\},$$

where $e_k := (1, \dots, k) \in \mathbb{R}^k_+$, and $g_i(x) \leq 0, \ \forall i \in I$. In other words, there is no $x \in \mathbb{R}^n$ such that $f_j(x) - f_j(x_0) + \sqrt{\varepsilon}\|x - x_0\| \leq 0, \forall j \in J$, $f_l(x) - f(x_0) + \sqrt{\varepsilon}\|x - x_0\| < 0, \exists l \in J$, and $g_i(x) \leq 0, \forall i \in I$. So, for any $x \in C$, we have that

$$\max_{j\in J}\{f_j(x) + \sqrt{\varepsilon}\|x - x_0\|\} \geq \max_{j\in J}\{f_j(x_0)\}, \text{ and } g_i(x) \leq 0, \forall i \in I.$$

This implies

$$\max_{j\in J}\{f_j(x)\} + \sqrt{\varepsilon}\|x - x_0\| \geq \max_{j\in J}\{f_j(x_0)\}, \text{ and } g_i(x) \leq 0, \forall i \in I.$$

Let

$$\Phi(x) := \max_{j\in J} f_j(x), \ \forall x \in C.$$

Clearly, x_0 is an ε-quasi minimizer of Φ over C. By Theorem 3.1 and definition of Φ, we have

$$\begin{aligned} 0 &\in \operatorname{clco}\overrightarrow{\partial}^*_\varepsilon \Phi(x_0) + \sqrt{\varepsilon}B^* + N_C(C, x_0) \\ &= \operatorname{clco}\overrightarrow{\partial}^*_\varepsilon\left(\max_{j\in J} f_j(x_0)\right) + \sqrt{\varepsilon}B^* + N_C(C, x_0). \end{aligned}$$

Since the constraint qualification (12) is satisfied at x_0, one has

$$0 \in \operatorname{clco}\overrightarrow{\partial}^*_\varepsilon\left(\max_{j\in J} f_j(x_0)\right) + \sqrt{\varepsilon}B^* + \bigcup_{\mu\in A(x_0)}\left[\sum_{i\in I}\mu_i \operatorname{co}\left(\overrightarrow{\partial}^* g_i(x_0)\right)\right] + N(\Omega, x_0).$$

Due to the same reasons as in the proof of Theorem 3.2, there exists $\mu = (\mu_1, \mu_2, \dots) \in \mathbb{R}^{(I)}_+$ such that

$$0 \in \operatorname{clco}\overrightarrow{\partial}^*_\varepsilon\left(\max_{j\in J} f_j(x_0)\right) + \sqrt{\varepsilon}B^* + \sum_{i\in I}\mu_i \operatorname{co}\left(\overrightarrow{\partial}^* g_i(x_0)\right) + N(\Omega, x_0), \ g_i(x_0) = 0, \forall i \in I(\mu).$$

Using the Proposition 2 in [4] yields

$$\begin{aligned} 0 \in \operatorname{clco}\left(\bigcup_{j\in J(x_0)} \overrightarrow{\partial}^*_\varepsilon f_j(x_0)\right) + \sqrt{\varepsilon}B^* + \sum_{i\in I}\mu_i \operatorname{co}\left(\overrightarrow{\partial}^* g_i(x_0)\right) + N(\Omega, x_0), \\ g_i(x_0) = 0, \ \forall i \in I(\mu). \end{aligned}$$

□

Theorem 4.4. Let $\varepsilon \geq 0$ be given and x_0 be an ε-quasi efficient solution for (24). Suppose all assumptions of Theorem 4.3. In addition, assume that $\operatorname{co}\left(\bigcup_{j\in J(x_0)} \overrightarrow{\partial}^*_\varepsilon f_j(x_0)\right) + \sqrt{\varepsilon}B^*$ is closed. Then, there exist $(\lambda, \mu) \in \mathbb{R}^n \times \mathbb{R}^{(I)}_+$, $\lambda_j \geq 0, \forall j \in J$, $\sum_{j\in J}\lambda_j = 1$, and $\mu_i \geq 0, \forall i \in I$ such that

$$0 \in \sum_{j\in J(x_0)} \lambda_j \operatorname{co}(\overrightarrow{\partial}^*_\varepsilon f_j(x_0)) + \sum_{i\in I}\mu_i \operatorname{co}(\overrightarrow{\partial}^* g_i(x_0)) + N(\Omega, x_0) + \sqrt{\varepsilon}B^*, \ g_i(x_0) = 0, \forall i \in I(\mu). \tag{27}$$

where $J(x_0) = \left\{j \in J : f_j(x_0) = \Phi(x_0)\right\}$.

Proof. The proof is completed by following above proof of Theorem 4.3 and using a well-known result from [11]. □

We next formulate some sufficient conditions for an almost ε-quasi weakly efficient solution for MOSIP (24).

Theorem 4.5. Let $(x_0, \lambda, \mu) \in C_\varepsilon \times \mathbb{R}^k_+ \times \mathbb{R}^{(I)}_+$ be given. Suppose that $f_j, \forall j \in J$ and $g_i, \forall i \in I$ admit ε-upper semi-regular semi-convexificators $\overline{\partial}^*_\varepsilon f_j(x_0)$ and upper convexificators $\overline{\partial}^* g_i(x_0)$, respectively, at x_0 and assume that (x_0, λ, μ) is such that

$$0 \in \sum_{j \in J(x_0)} \lambda_j \mathrm{co}(\overline{\partial}^*_\varepsilon f_j(x_0)) + \sum_{i \in I} \mu_i \mathrm{co}(\overline{\partial}^* g_i(x_0)) + N(\Omega, x_0),$$
$$g_i(x_0) \geq 0, \forall i \in I(\mu). \tag{28}$$

Assume that $g_i, i \in I$ is quasiconvex at x_0.

1. If for each $j \in J, f_j$ is $\overline{\partial}^*_\varepsilon$-pseudoconvex of type I at x_0, then x_0 is an almost ε-quasi weakly efficient solution for (24).;
2. If for each $j \in J, f_j$ is $\overline{\partial}^*_\varepsilon$-pseudoconvex of type II at x_0, then there exists $\varepsilon \geq 0$ such that x_0 is an almost ε-quasi weakly efficient solution for (24).;
3. for each $j \in J, f_j$ is quasiconvex at x_0, then x_0 is an almost ε-quasi weakly efficient solution for (24).

Proof. (i) Let $(x_0, \lambda, \mu) \in C_\varepsilon \times \mathbb{R}^k_+ \times \mathbb{R}^{(I)}_+$ be such that (28) holds. Then, there exist $a_j \in \mathrm{co}(\overline{\partial}^*_\varepsilon f_j(x_0))$ with $\lambda_j \in \mathbb{R}_+, \forall j \in J, b_i \in \mathrm{co}(\overline{\partial}^* g_i(x_0))$ with $\mu_i \in \mathbb{R}_+, \forall i \in I, c \in N(\Omega, x_0)$ and $d \in B^*$ such that $g_i(x_0) \geq 0$ for all $i \in I(\mu)$ and

$$\sum_{j \in J(x_0)} \lambda_j a_j + \sum_{i \in I} \mu_i b_i + c + d = 0. \tag{29}$$

Therefore, we obtain

$$\begin{aligned} 0 &= \left\langle \sum_{j \in J(x_0)} \lambda_j a_j + \sum_{i \in I} \mu_i b_i + c + d, x - x_0 \right\rangle \\ &= \left\langle \sum_{j \in J} \lambda_j a_j + \sum_{i \in I} \mu_i b_i, x - x_0 \right\rangle + \langle c, x - x_0 \rangle + \langle d, x - x_0 \rangle. \end{aligned}$$

Since $c \in N(\Omega, x_0)$, and $d \in B^*$, one has

$$\langle c, x - x_0 \rangle \leq 0, \ d(x - x_0) \leq \|x - x_0\|, \ \forall x \in \Omega$$

and so

$$\left\langle \sum_{j \in J} \lambda_j a_j + \sum_{i \in I} \mu_i b_i, x - x_0 \right\rangle + \sqrt{\varepsilon}\|x - x_0\| \geq 0. \tag{30}$$

Hence, we obtain that

$$\left\langle \sum_{j \in J} \lambda_i a_i, x - x_0 \right\rangle + \sqrt{\varepsilon}\|x - x_0\| \geq -\left\langle \sum_{i \in I(\mu)} \mu_i b_i, x - x_0 \right\rangle. \tag{31}$$

Again, the hypothesis and property of any feasible points imply, respectively, that $g_i(x_0) \geq 0$ for all $i \in I(\mu)$ and $g_i(x) \leq 0$ for all $i \in I, x \in C$. Thus, for any $x \in C$ and $i \in I(\mu)$,

$$g_i(x) \leq g_i(x_0).$$

By the quasiconvexity of $g_i, i \in J(\mu)$ at x_0, above inequality implies

$$\left\langle \sum_{i\in I(\mu)} \mu_i b_i, x - x_0 \right\rangle \leq 0, \ \forall x \in C. \tag{32}$$

Then, we obtain by using (31) and combining this new inequality with (32) that

$$\left\langle \sum_{i\in I} \lambda_j a_j, x - x_0 \right\rangle + \sqrt{\varepsilon_0}\|x - x_0\| \geq 0. \tag{33}$$

Therefore, there exists $\bar{a} = \sum_{j\in J} \bar{\lambda}_j a_j = \sum_{j\in J}(\lambda_j/\lambda)a_j \in \mathrm{co}(\overline{\partial}^*_\varepsilon \Phi(x_0))$ such that

$$\langle \bar{a}, x - x_0 \rangle + \sqrt{\varepsilon_0}\|x - x_0\| \geq 0, \tag{34}$$

Immediately, the $\overline{\partial}^*_\varepsilon$- pseudoconvexity of type I of Φ at x_0 yields

$$\max_{j\in J} f_j(x_0) \leq \max_{j\in J} f_j(x) + \sqrt{\varepsilon_0}\|x - x_0\|,$$

which means for any $x \in C$

$$f(x) - f(x_0) + \sqrt{\varepsilon}\|x - x_0\|e_k \notin -\mathrm{int}\mathbb{R}^k_+ \setminus \{0\}.$$

Hence, x_0 is an almost ε-quasi weakly efficient solution of (24) as desired.

(ii) It is clear that inequality (30) implies

$$\left\langle \sum_{i\in I} \lambda_i a_i, x - x_0 \right\rangle \geq -\left\langle \sum_{i\in I(\mu)} \mu_i b_i, x - x_0 \right\rangle, \tag{35}$$

which together with (33) yields

$$\left\langle \sum_{i\in I} \lambda_i a_i, x - x_0 \right\rangle \geq 0.$$

Thus, there exists $\bar{a} \in \mathrm{co}\left(\overline{\partial}^*_\varepsilon \Phi(x_0)\right)$ such that

$$\langle \bar{a}, x - x_0 \rangle \geq 0. \tag{36}$$

By the $\overline{\partial}^*_\varepsilon$-pseudoconvexity of type II of Φ at x_0,

$$\max_{j\in J} f_j(x_0) < \max_{j\in J} f_j(x) + \sqrt{\varepsilon}\|x - x_0\|$$

and we obtain the desired result.

(iii) In order to obtain (36), we follow the same way in proof of statement (ii). Then, since $\varepsilon > 0$ and $x \neq x_0$, the strictly inequality $\langle \bar{a}, x - x_0 \rangle + \sqrt{\varepsilon}\|x - x_0\| > 0$ is satisfied and so the quasiconvexity of Φ at x_0 provides

$$\max_{j\in J} f_j(x_0) < \max_{j\in J} f_j(x),$$

thereby

$$\max_{j\in J} f_j(x_0) < \max_{j\in J} f_j(x) + \sqrt{\varepsilon}\|x - x_0\|, \ \forall x \in C.$$

Thus, there is no $x \in C$ such that $f(x) - f(x_0) + \sqrt{\varepsilon}\|x - x_0\|e_j \in -\mathrm{int}\mathbb{R}^k_+ \setminus \{0\}$ and hence x_0 is an almost ε-quasi weakly efficient solution of (24). This completes the proof. □

5. Conclusions

Some classes of functions, namely $(\varepsilon - \partial^*_\varepsilon)$-pseudoconvex function and $(\varepsilon - \partial^*_\varepsilon)$-quasiconvex functions with respect to a given ε upper semi-regular semi-convexificator are introduced, respectively. By utilizing these new concepts, sufficient optimality conditions of approximate solutions for the nonsmooth (SIP), in terms of ε-upper semi-regular semi-convexificator, are established. Moreover, as an application, optimality conditions of quasi approximate weakly efficient solution for nonsmooth multi-objective semi-infinite programming problems (MOSIP) are presented.

Acknowledgments

The authors would like to thank the referees for their careful reading and constructive comments. This research is partially supported by the Science Achievement Scholarship of Thailand and Naresuan University. The second author was supported by ICST-HCM City Grant Number 03/2019/HD-KHCNTT. The third author was supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MSIT) (NRF-2017R1E1A1A03069931).

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