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On the Fourth and Fifth Coefficients in the Carathéodory Class

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Abstract. In the paper, we give general parametric formulas for the fourth and fifth coefficients of functions in the Carathéodory class.

1. Introduction

One of the classical research topics in the class $\mathcal{A}$ of analytic functions f of the form

$$f(z) = \sum_{n=1}^{\infty} a_n z^n, \quad a_1 := 1,\ z \in \mathbb{D} := \{z \in \mathbb{C} : |z| < 1\},$$

is the study of coefficient functionals. These studies include both the whole class $\mathcal{A}$, as well as subfamilies, in particular the class $\mathcal{S}$ of univalent functions and its subclasses. Although normalized differently, an important class of analytic functions in $\mathbb{D}$ is the class $\mathcal{P}$ consisting of functions p of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, \quad z \in \mathbb{D}, \tag{1}$$

having positive real part in $\mathbb{D}$, generally known as the Carathéodory class.

For a great many subclasses of $\mathcal{A}$, it is possible express coefficient functionals in terms of the coefficients of functions in $\mathcal{P}$. Thus knowledge about the coefficients in $\mathcal{P}$ can form the basis of computational techniques when considering coefficient problems in these classes. In recent years, attention has focused on the study of e.g. inverse and logarithmic coefficients, the Fekete-Szegö functional, Hankel determinants, the Zalcman functional, and a number of other related functionals, (see e.g., [5], [2], [11] and [6] for further references). Parametric formulas for the initial coefficients of functions $\mathcal{P}$ are usually used to study these problems, and formulas for the first and second coefficients c_1 and c_2 derived by Carathéodory ([1], [10, p. 166]), and a formula for the third coefficient c_3 found by Libera and Złotkiewicz [8], [9] with the additional assumption

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that $c_1 \geq 0$, have been used extensively. This latter condition on c_1 restricts its use to rotationally invariant classes and functionals. In [7] a parametric formula for fourth coefficient c_4 was derived using a determinant method, but also with the restriction $c_1 \geq 0$. Due to computational difficulties, the determinant method does not seem to be effective in finding further coefficient formula for c_n, when $n \geq 5$.

As a result, in [3] an algorithmic method based on the Schwarz lemma was developed for deriving parametric formulas for all the coefficients c_n. In particular Lemma 2.4 in [3] gives a general formula for the third coefficient c_3, which was able to be used to study certain non-rotationally invariant functionals in $\mathcal{A}$.

In this paper we give general formulas for c_4 and c_5, together with extreme functions, without the restriction $c_1 \geq 0$, thus providing tools for further study into more difficult problems concerning functionals in $\mathcal{A}$.

We remark that the general method used will give formulas for c_n when $n \geq 6$, but the computations soon become too complicated to yield useful results.

2. Parametric formulas for coefficients of Carathéodory functions

Denote by $\mathcal{B}_0$, the class of all self-maps of $\mathbb{D}$ of the form

$$\omega(z) = \sum_{n=1}^{\infty} b_n z^n, \quad z \in \mathbb{D}, \tag{2}$$

Such functions are normally referred to as Schwarz functions.

Given $\alpha \in \mathbb{D}$, let

$$\psi_\alpha(z) := \frac{z-\alpha}{1-\overline{\alpha}z}, \quad z \in \overline{\mathbb{D}} := \{z \in \mathbb{C} : |z| \leq 1\}.$$

It is well known that ψ_α in a conformal automorphism of $\mathbb{D}$, $\psi_\alpha(\mathbb{D}) = \mathbb{D}$, $\psi_\alpha(\mathbb{T}) = \mathbb{T}$, and $\psi_\alpha^{-1} = \psi_{-\alpha}$, where $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$. Moreover for $n \in \mathbb{N}$,

$$\psi_\alpha^{(n)}(\alpha) = \frac{n!\overline{\alpha}^{n-1}}{(1-|\alpha|^2)^n}. \tag{3}$$

Let

$$L(z) := \frac{1+z}{1-z}, \quad z \in \mathbb{D}.$$

Now we prove the main result of this paper, noting that (4) is due to Carathéodory [1], (5) can be found in [10, p. 166], and (6) in [3]. We state our results as lemmas rather than theorems, since they provide tools for use in considering functions in $\mathcal{A}$.

Lemma 2.1. *If $p \in \mathcal{P}$ and is of the form* (1), *then*

$$c_1 = 2\zeta_1, \tag{4}$$

$$c_2 = 2\zeta_1^2 + 2(1-|\zeta_1|^2)\zeta_2, \tag{5}$$

$$c_3 = 2\zeta_1^3 + 2(1-|\zeta_1|^2)\left(2\zeta_1 - \overline{\zeta_1}\zeta_2\right)\zeta_2 + 2(1-|\zeta_1|^2)(1-|\zeta_2|^2)\zeta_3, \tag{6}$$

$$\begin{aligned} c_4 =& 2\zeta_1^4 + 2(1-|\zeta_1|^2)\left(3\zeta_1^2 + \overline{\zeta_1}^2\zeta_2^2 - 3|\zeta_1|^2\zeta_2 + \zeta_2\right)\zeta_2 \\ &+ 2(1-|\zeta_1|^2)(1-|\zeta_2|^2)\left(2\zeta_1 - 2\overline{\zeta_1}\zeta_2 - \overline{\zeta_2}\zeta_3\right)\zeta_3 \\ &+ 2(1-|\zeta_1|^2)(1-|\zeta_2|^2)(1-|\zeta_3|^2)\zeta_4, \end{aligned} \tag{7}$$

and

$$\begin{aligned} c_5 =& 2\zeta_1^5 + 2(1-|\zeta_1|^2)\Big(4\zeta_1^3 + 3\zeta_1\zeta_2 - 2\overline{\zeta_1}\zeta_2^2 - \overline{\zeta_1}^3\zeta_2^3 - 6\zeta_1|\zeta_1|^2\zeta_2 \\ & + 4\overline{\zeta_1}|\zeta_1|^2\zeta_2^2\Big)\zeta_2 + 2(1-|\zeta_1|^2)(1-|\zeta_2|^2)\Big(3\zeta_1^2 - 2\zeta_1\overline{\zeta_2}\zeta_3 + 3\overline{\zeta_1}^2\zeta_2^2 \\ & + 3\overline{\zeta_1}|\zeta_2|^2\zeta_3 - \overline{\zeta_1}\zeta_3 - 6|\zeta_1|^2\zeta_2 + 2\zeta_2 + \overline{\zeta_2}^2\zeta_3^2\Big)\zeta_3 \\ & + 2(1-|\zeta_1|^2)(1-|\zeta_2|^2)(1-|\zeta_3|^2)\Big(2\zeta_1 - 2\overline{\zeta_1}\zeta_2 - 2\overline{\zeta_2}\zeta_3 - \overline{\zeta_3}\zeta_4\Big)\zeta_4 \\ & + 2(1-|\zeta_1|^2)(1-|\zeta_2|^2)(1-|\zeta_3|^2)(1-|\zeta_4|^2)\zeta_5 \end{aligned} \tag{8}$$

for some $\zeta_i \in \overline{\mathbb{D}}$, $i \in \{1, \ldots, 5\}$.

For $\zeta_1 \in \mathbb{T}$, *there is a unique function* $p \in \mathcal{P}$ *with* c_1 *as in* (4), *namely,*

$$p(z) = \frac{1+\zeta_1 z}{1-\zeta_1 z}, \quad z \in \mathbb{D}. \tag{9}$$

For $\zeta_1 \in \mathbb{D}$ *and* $\zeta_2 \in \mathbb{T}$, *there is a unique function* $p = L \circ \omega \in \mathcal{P}$ *with* c_1 *and* c_2 *as in* (4) and (5), *where*

$$\omega(z) = z\psi_{-\zeta_1}(\zeta_2 z), \quad z \in \mathbb{D}, \tag{10}$$

i.e.,

$$p(z) = \frac{Q_2(z)}{R_2(z)}, \quad z \in \mathbb{D}, \tag{11}$$

where for $z \in \mathbb{D}$,

$$Q_2(z) := 1 + \Big(\overline{\zeta_1}\zeta_2 + \zeta_1\Big)z + \zeta_2 z^2,$$
$$R_2(z) := 1 + \Big(\overline{\zeta_1}\zeta_2 - \zeta_1\Big)z - \zeta_2 z^2.$$

For $\zeta_1, \zeta_2 \in \mathbb{D}$ *and* $\zeta_3 \in \mathbb{T}$, *there is a unique function* $p = L \circ \omega \in \mathcal{P}$ *with* c_1, c_2 *and* c_3 *as in* (4) to (6), *where*

$$\omega(z) = z\psi_{-\zeta_1}(z\psi_{-\zeta_2}(\zeta_3 z)), \quad z \in \mathbb{D}, \tag{12}$$

i.e.,

$$p(z) = \frac{Q_3(z)}{R_3(z)}, \quad z \in \mathbb{D}, \tag{13}$$

where for $z \in \mathbb{D}$,

$$Q_3(z) := 1 + \Big(\overline{\zeta_2}\zeta_3 + \overline{\zeta_1}\zeta_2 + \zeta_1\Big)z + \Big(\overline{\zeta_1}\zeta_3 + \zeta_1\overline{\zeta_2}\zeta_3 + \zeta_2\Big)z^2 + \zeta_3 z^3,$$
$$R_3(z) := 1 + \Big(\overline{\zeta_2}\zeta_3 + \overline{\zeta_1}\zeta_2 - \zeta_1\Big)z + \Big(\overline{\zeta_1}\zeta_3 - \zeta_1\overline{\zeta_2}\zeta_3 - \zeta_2\Big)z^2 - \zeta_3 z^3.$$

For $\zeta_1, \zeta_2, \zeta_3 \in \mathbb{D}$ *and* $\zeta_4 \in \mathbb{T}$, *there is a unique function* $p = L \circ \omega \in \mathcal{P}$ *with* c_1, c_2, c_3 *and* c_4 *as in* (4) to (7), *where*

$$\omega(z) = z\psi_{-\zeta_1}(z\psi_{-\zeta_2}(z\psi_{-\zeta_3}(\zeta_4 z))), \quad z \in \mathbb{D}, \tag{14}$$

i.e.,

$$p(z) = \frac{Q_4(z)}{R_4(z)}, \quad z \in \mathbb{D}, \tag{15}$$

where for $z \in \mathbb{D}$,

$$\begin{aligned} Q_4(z) :=& 1 + \left(\overline{\zeta_1}\zeta_2 + \overline{\zeta_2}\zeta_3 + \overline{\zeta_3}\zeta_4 + \zeta_1\right) z \\ &+ \left(\overline{\zeta_1}\zeta_3 + \overline{\zeta_2}\zeta_4 + \overline{\zeta_1}\zeta_2\overline{\zeta_3}\zeta_4 + \zeta_1\overline{\zeta_2}\zeta_3 + \zeta_1\overline{\zeta_3}\zeta_4 + \zeta_2\right) z^2 \\ &+ \left(\overline{\zeta_1}\zeta_4 + \zeta_1\overline{\zeta_2}\zeta_4 + \zeta_2\overline{\zeta_3}\zeta_4 + \zeta_3\right) z^3 + \zeta_4 z^4, \end{aligned}$$

$$\begin{aligned} R_4(z) :=& 1 + \left(\overline{\zeta_1}\zeta_2 + \overline{\zeta_2}\zeta_3 + \overline{\zeta_3}\zeta_4 - \zeta_1\right) z \\ &+ \left(\overline{\zeta_1}\zeta_3 + \overline{\zeta_2}\zeta_4 + \overline{\zeta_1}\zeta_2\overline{\zeta_3}\zeta_4 - \zeta_1\overline{\zeta_2}\zeta_3 - \zeta_1\overline{\zeta_3}\zeta_4 - \zeta_2\right) z^2 \\ &+ \left(\overline{\zeta_1}\zeta_4 - \zeta_1\overline{\zeta_2}\zeta_4 - \zeta_2\overline{\zeta_3}\zeta_4 - \zeta_3\right) z^3 - \zeta_4 z^4. \end{aligned}$$

For $\zeta_i \in \mathbb{D}$, $i = 1, \dots, 4$, *and* $\zeta_5 \in \mathbb{T}$, *there is a unique function* $p = L \circ \omega \in \mathcal{P}$ *with* c_i, $i = 1, \dots, 5$, *as in* (4) to (8), *where*

$$\omega(z) = z\psi_{-\zeta_1}(z\psi_{-\zeta_2}(z\psi_{-\zeta_3}(z\psi_{-\zeta_4}(\zeta_5 z)))), \quad z \in \mathbb{D}, \tag{16}$$

i.e.,

$$p(z) = \frac{Q_5(z)}{R_5(z)}, \quad z \in \mathbb{D}, \tag{17}$$

where for $z \in \mathbb{D}$,

$$\begin{aligned} Q_5(z) :=& 1 + \left(\overline{\zeta_1}\zeta_2 + \overline{\zeta_2}\zeta_3 + \overline{\zeta_3}\zeta_4 + \overline{\zeta_4}\zeta_5 + \zeta_1\right) z \\ &+ \Big(\overline{\zeta_1}\zeta_3 + \overline{\zeta_2}\zeta_4 + \overline{\zeta_3}\zeta_5 + \overline{\zeta_1}\zeta_2\overline{\zeta_3}\zeta_4 + \overline{\zeta_1}\zeta_2\overline{\zeta_4}\zeta_5 \\ &+ \overline{\zeta_2}\zeta_3\overline{\zeta_4}\zeta_5 + \zeta_1\overline{\zeta_2}\zeta_3 + \zeta_1\overline{\zeta_3}\zeta_4 + \zeta_1\overline{\zeta_4}\zeta_5 + \zeta_2\Big) z^2 \\ &+ \Big(\overline{\zeta_1}\zeta_4 + \overline{\zeta_2}\zeta_5 + \overline{\zeta_1}\zeta_3\overline{\zeta_4}\zeta_5 + \overline{\zeta_1}\zeta_2\overline{\zeta_3}\zeta_5 + \zeta_1\overline{\zeta_2}\zeta_4 \\ &+ \zeta_1\overline{\zeta_3}\zeta_5 + \zeta_2\overline{\zeta_3}\zeta_4 + \zeta_2\overline{\zeta_4}\zeta_5 + \zeta_1\overline{\zeta_2}\zeta_3\overline{\zeta_4}\zeta_5 + \zeta_3\Big) z^3 \\ &+ \left(\overline{\zeta_1}\zeta_5 + \zeta_1\overline{\zeta_2}\zeta_5 + \zeta_2\overline{\zeta_3}\zeta_5 + \zeta_3\overline{\zeta_4}\zeta_5 + \zeta_4\right) z^4 + \zeta_5 z^5, \end{aligned}$$

$$\begin{aligned} R_5(z) :=& 1 + \left(\overline{\zeta_1}\zeta_2 + \overline{\zeta_2}\zeta_3 + \overline{\zeta_3}\zeta_4 + \overline{\zeta_4}\zeta_5 - \zeta_1\right) z \\ &+ \Big(\overline{\zeta_1}\zeta_3 + \overline{\zeta_2}\zeta_4 + \overline{\zeta_3}\zeta_5 + \overline{\zeta_1}\zeta_2\overline{\zeta_3}\zeta_4 + \overline{\zeta_1}\zeta_2\overline{\zeta_4}\zeta_5 \\ &+ \overline{\zeta_2}\zeta_3\overline{\zeta_4}\zeta_5 - \zeta_1\overline{\zeta_2}\zeta_3 - \zeta_1\overline{\zeta_3}\zeta_4 - \zeta_1\overline{\zeta_4}\zeta_5 - \zeta_2\Big) z^2 \\ &+ \Big(\overline{\zeta_1}\zeta_4 + \overline{\zeta_2}\zeta_5 + \overline{\zeta_1}\zeta_3\overline{\zeta_4}\zeta_5 + \overline{\zeta_1}\zeta_2\overline{\zeta_3}\zeta_5 - \zeta_1\overline{\zeta_2}\zeta_4 \\ &- \zeta_1\overline{\zeta_3}\zeta_5 - \zeta_2\overline{\zeta_3}\zeta_4 - \zeta_2\overline{\zeta_4}\zeta_5 - \zeta_1\overline{\zeta_2}\zeta_3\overline{\zeta_4}\zeta_5 - \zeta_3\Big) z^3 \\ &+ \left(\overline{\zeta_1}\zeta_5 - \zeta_1\overline{\zeta_2}\zeta_5 - \zeta_2\overline{\zeta_3}\zeta_5 - \zeta_3\overline{\zeta_4}\zeta_5 - \zeta_4\right) z^4 - \zeta_5 z^5. \end{aligned}$$

Proof. Let $p \in \mathcal{P}$ and be of the form (1). Then there exists $\omega \in \mathcal{B}_0$ of the form (2) such that

$$(1 - \omega(z))p(z) = 1 + \omega(z), \quad z \in \mathbb{D}. \tag{18}$$

Substituting (1) and (2) into (18), and equating coefficients, we obtain for $n \in \mathbb{N}$

$$c_n = c_{n-1}b_1 + c_{n-2}b_2 + \cdots + c_1 b_{n-1} + 2b_n.$$

Thus

$$\begin{aligned}
c_1 &= 2b_1,\\
c_2 &= 2b_1^2 + 2b_2,\\
c_3 &= 2b_1^3 + 4b_1b_2 + 2b_3,\\
c_4 &= 2b_1^4 + 6b_1^2b_2 + 4b_1b_3 + 2b_2^2 + 2b_4,\\
c_5 &= 2b_1^5 + 8b_1^3b_2 + 6b_1^2b_3 + 6b_1b_2^2 + 4b_1b_4 + 4b_2b_3 + 2b_5.
\end{aligned} \tag{19}$$

1. By the Schwarz lemma (see e.g., [4, Vol. I, pp. 84-85]),

$$|b_1| = |\omega'(0)| \le 1, \tag{20}$$

i.e.,

$$b_1 = \zeta_1, \tag{21}$$

for some $\zeta_1 \in \overline{\mathbb{D}}$, and so from (19) we obtain (4).

Moreover equality is attained in (20), i.e., the case $\zeta_1 \in \mathbb{T}$ in (21) if, and only if,

$$\omega(z) = \zeta_1 z, \quad z \in \mathbb{D},$$

(e.g., [4, Vol. I, p. 85]). Thus from (18) it follows that p can be only as (9).

2. By Part 1 we can assume that $b_1 \in \mathbb{D}$, and so from (2) we have

$$\omega(z) = z\varphi_1(z), \quad z \in \mathbb{D}, \tag{22}$$

where

$$\varphi_1(z) := b_1 + b_2z + b_3z^2 + b_4z^3 + b_5z^4 + \ldots, \quad z \in \mathbb{D}. \tag{23}$$

From (22) and the maximum principle for analytic functions it follows that φ_1 is a self-map of $\mathbb{D}$. Thus

$$\omega_1(z) := \psi_{b_1}(\varphi_1(z)) = b_1^{(1)}z + b_2^{(1)}z^2 + b_3^{(1)}z^3 + b_4^{(1)}z^4 + \ldots, \quad z \in \mathbb{D}, \tag{24}$$

is a Schwarz function. By the Schwarz lemma we have

$$|b_1^{(1)}| = |\omega_1'(0)| \le 1, \tag{25}$$

i.e.,

$$b_1^{(1)} = \zeta_2, \tag{26}$$

for some $\zeta_2 \in \overline{\mathbb{D}}$. Using (24), (3) and (23) we obtain

$$b_1^{(1)} = \omega_1'(0) = \psi_{b_1}'(\varphi_1(0))\varphi_1'(0) = \frac{b_2}{1 - |b_1|^2},$$

which, using (26) and (21), gives

$$b_2 = (1 - |\zeta_1|^2)\zeta_2. \tag{27}$$

Substituting (21) and (27) into (19) gives (5).

Noting (26), equality in (25) holds in the case $\zeta_2 \in \mathbb{T}$ if, and only if, $\omega_1(z) = \zeta_2 z,\ z \in \mathbb{D}$. Thus by (22), (24) and (21) we have

$$\omega(z) = z\varphi_1(z) = z\psi_{-b_1}(\omega_1(z)) = z\psi_{-\zeta_1}(\zeta_2 z), \quad z \in \mathbb{D},$$

i.e., ω is as in (10). Hence and from (18) it follows that p can be only as in (11).

3. By Parts 1 and 2 we can assume that $b_1, b_1^{(1)} \in \mathbb{D}$. By (24) we have

$$\omega_1(z) = z\varphi_2(z), \quad z \in \mathbb{D}, \tag{28}$$

where

$$\varphi_2(z) := b_1^{(1)} + b_2^{(1)} z + b_3^{(1)} z^2 + b_4^{(1)} z^3 + \dots, \quad z \in \mathbb{D}. \tag{29}$$

From (28) and again by the maximum principle for analytic functions, it follows that φ_2 is a self-map of $\mathbb{D}$. Thus

$$\omega_2(z) := \psi_{b_1^{(1)}}(\varphi_2(z)) = b_1^{(2)} z + b_2^{(2)} z^2 + b_3^{(2)} z^3 + \dots, \quad z \in \mathbb{D}, \tag{30}$$

is a Schwarz function. By the Schwarz lemma we have

$$|b_1^{(2)}| = |\omega_2'(0)| \le 1, \tag{31}$$

i.e.,

$$b_1^{(2)} = \zeta_3, \tag{32}$$

for some $\zeta_3 \in \overline{\mathbb{D}}$. Using (30), (3) and (29) we obtain

$$b_1^{(2)} = \omega_2'(0) = \psi'_{b_1^{(1)}}(\varphi_2(0))\varphi_2'(0) = \frac{b_2^{(1)}}{1 - |b_1^{(1)}|^2},$$

which, taking into account (32) and (26), yields

$$b_2^{(1)} = (1 - |\zeta_2|^2)\zeta_3. \tag{33}$$

On the other hand, from (24) and (23), (3), (21) and (27) we get

$$\begin{aligned} b_2^{(1)} &= \frac{1}{2}\omega_1''(0) = \frac{1}{2}\left(\psi''_{b_1}(\varphi_1(0))(\varphi_1'(0))^2 + \psi'_{b_1}(\varphi_1(0))\varphi_1''(0)\right) \\ &= \frac{1}{2}\psi''_{b_1}(b_1) b_2^2 + \psi'_{b_1}(b_1) b_3 = \frac{\overline{b_1} b_2^2}{(1 - |b_1|^2)^2} + \frac{b_3}{1 - |b_1|^2} \\ &= \overline{\zeta_1}\zeta_2^2 + \frac{b_3}{1 - |\zeta_1|^2}. \end{aligned}$$

This, together with (33), gives

$$b_3 = -(1 - |\zeta_1|^2)\overline{\zeta_1}\zeta_2^2 + (1 - |\zeta_1|^2)(1 - |\zeta_2|^2)\zeta_3. \tag{34}$$

Substituting (21), (27) and (34) into (19) gives (6).

Moreover equality in (31), i.e., the case $\zeta_3 \in \mathbb{T}$ in (32) holds if, and only if, $\omega_2(z) = \zeta_3 z,\ z \in \mathbb{D}$. Thus by (28), (30) and (26) we have

$$\omega_1(z) = z\varphi_2(z) = z\psi_{-b_1^{(1)}}(\omega_2(z)) = z\psi_{-\zeta_2}(\zeta_3 z), \quad z \in \mathbb{D}.$$

Now (22) and (24) with (21) yields

$$\omega(z) = z\varphi_1(z) = z\psi_{-b_1}(\omega_1(z)) = z\psi_{-\zeta_1}(z\psi_{-\zeta_2}(\zeta_3 z), \quad z \in \mathbb{D},$$

i.e., ω is as in (12). Hence from (18) it follows that p can be only as (13).

4. Arguing as before, by Parts 1 to 3 we can assume that $b_1, b_1^{(1)}, b_1^{(2)} \in \mathbb{D}$, and so by (30) we have

$$\omega_2(z) = z\varphi_3(z), \quad z \in \mathbb{D}, \tag{35}$$

where

$$\varphi_3(z) := b_1^{(2)} + b_2^{(2)} z + b_3^{(2)} z^2 + \dots, \quad z \in \mathbb{D}. \tag{36}$$

From (35) and by the maximum principle for analytic functions it follows that φ_3 is a self-map of $\mathbb{D}$. Thus

$$\omega_3(z) := \psi_{b_1^{(2)}}(\varphi_3(z)) = b_1^{(3)} z + b_2^{(3)} z^2 + \dots, \quad z \in \mathbb{D}, \tag{37}$$

is a Schwarz function. By the Schwarz lemma we have

$$|b_1^{(3)}| = |\omega_3'(0)| \le 1, \tag{38}$$

i.e.,

$$b_1^{(3)} = \zeta_4, \tag{39}$$

for some $\zeta_4 \in \overline{\mathbb{D}}$. Using (37), (3) and (36) we obtain

$$b_1^{(3)} = \omega_3'(0) = \psi'_{b_1^{(2)}}(\varphi_3(0))\varphi_3'(0) = \frac{b_2^{(2)}}{1 - |b_1^{(2)}|^2},$$

which, taking into account (32) and (39), yields

$$b_2^{(2)} = (1 - |\zeta_3|^2)\zeta_4. \tag{40}$$

On the other hand from (30), applying (29), (3), (26) and (33) we obtain

$$\begin{aligned} b_2^{(2)} &= \frac{1}{2}\omega_2''(0) = \frac{1}{2}\left(\psi''_{b_1^{(1)}}(\varphi_2(0))(\varphi_2'(0))^2 + \psi'_{b_1^{(1)}}(\varphi_2(0))\varphi_2''(0)\right) \\ &= \frac{1}{2}\psi''_{b_1^{(1)}}(b_1^{(1)})(b_2^{(1)})^2 + \psi'_{b_1^{(1)}}(b_1^{(1)})b_3^{(1)} = \frac{\overline{b_1^{(1)}}(b_2^{(1)})^2}{(1 - |b_1^{(1)}|^2)^2} + \frac{b_3^{(1)}}{1 - |b_1^{(1)}|^2} \\ &= \overline{\zeta_2}\zeta_3^2 + \frac{b_3^{(1)}}{1 - |\zeta_2|^2}. \end{aligned}$$

This, together with (40), yields

$$b_3^{(1)} = -(1 - |\zeta_2|^2)\overline{\zeta_2}\zeta_3^2 + (1 - |\zeta_2|^2)(1 - |\zeta_3|^2)\zeta_4. \tag{41}$$

From (24), (23), (3), (21), (27) and (34) we have

$$\begin{aligned} b_3^{(1)} &= \frac{1}{6}\omega_1^{(3)}(0) \\ &= \frac{1}{6}\left(\psi_{b_1}^{(3)}(\varphi_1(0))(\varphi_1'(0))^3 + 3\psi_{b_1}''(\varphi_1(0))\varphi_1'(0)\varphi_1''(0) + \psi_{b_1}'(\varphi_1(0))\varphi_1^{(3)}(0)\right) \\ &= \frac{1}{6}\left(\psi_{b_1}^{(3)}(b_1)b_2^3 + 6\psi_{b_1}''(b_1)b_2b_3 + 6\psi_{b_1}'(b_1)b_4\right) \\ &= \frac{\overline{b_1}^2 b_2^3}{(1-|b_1|^2)^3} + \frac{2\overline{b_1}b_2b_3}{(1-|b_1|^2)^2} + \frac{b_4}{1-|b_1|^2} \\ &= -\overline{\zeta_1}^2\zeta_2^3 + 2(1-|\zeta_2|^2)\overline{\zeta_1}\zeta_2\zeta_3 + \frac{b_4}{1-|\zeta_1|^2}. \end{aligned} \tag{42}$$

Comparing (41) and (42) we obtain

$$\begin{aligned} b_4 =&(1-|\zeta_1|^2)\overline{\zeta_1}^2\zeta_2^3 - (1-|\zeta_1|^2)(1-|\zeta_2|^2)\left(\overline{\zeta_2}\zeta_3^2 + 2\overline{\zeta_1}\zeta_2\zeta_3\right) \\ &+ (1-|\zeta_1|^2)(1-|\zeta_2|^2)(1-|\zeta_3|^2)\zeta_4, \end{aligned} \tag{43}$$

and substituting (21), (27), (34) and (43) into (19) we obtain

$$\begin{aligned} c_4 =&2b_1^4 + 6b_1^2b_2 + 4b_1b_3 + 2b_2^2 + 2b_4 \\ =&2\zeta_1^4 + 6(1-|\zeta_1|^2)\zeta_1^2\zeta_2 + 4\zeta_1\left[-(1-|\zeta_1|^2)\overline{\zeta_1}\zeta_2^2 + (1-|\zeta_1|^2)(1-|\zeta_2|^2)\zeta_3\right] \\ &+ 2(1-|\zeta_1|^2)^2\zeta_2^2 + 2(1-|\zeta_1|^2)\overline{\zeta_1}^2\zeta_2^3 - 2(1-|\zeta_1|^2)(1-|\zeta_2|^2)\left(\overline{\zeta_2}\zeta_3^2 + 2\overline{\zeta_1}\zeta_2\zeta_3\right) \\ &+ 2(1-|\zeta_1|^2)(1-|\zeta_2|^2)(1-|\zeta_3|^2)\zeta_4, \end{aligned}$$

which gives (7).

For equality in (38), i.e., the case $\zeta_4 \in \mathbb{T}$ in (39), we argue as before, so that equality and holds if, and only if,

$$\omega_3(z) = \zeta_4 z, \quad z \in \mathbb{D}. \tag{44}$$

By (22) and (24) with (21) we have

$$\omega(z) = z\varphi_1(z) = z\psi_{-b_1}(\omega_1(z)) = z\psi_{-\zeta_1}(\omega_1(z)), \quad z \in \mathbb{D}. \tag{45}$$

Also by (28), (30) and (26) we have

$$\omega_1(z) = z\varphi_2(z) = z\psi_{-b_1^{(1)}}(\omega_2(z)) = z\psi_{-\zeta_2}(\omega_2(z)), \quad z \in \mathbb{D}. \tag{46}$$

Next by (35) and (37) with (32), and using (44), we have

$$\omega_2(z) = z\varphi_3(z) = z\psi_{-b_1^{(2)}}(\omega_3(z)) = z\psi_{-\zeta_3}(\zeta_4 z), \quad z \in \mathbb{D}.$$

Thus taking into account (45) and (46), it follows that ω is as in (14), hence from (18) it follows that p can be only as (15).

5. We proceed as before so that by Parts 1 to 4 we can assume that $b_1, b_1^{(1)}, b_1^{(2)}, b_1^{(3)} \in \mathbb{D}$, and so

$$\omega_3(z) = z\varphi_4(z), \quad z \in \mathbb{D}, \tag{47}$$

where

$$\varphi_4(z) := b_1^{(3)} + b_2^{(3)} z + b_3^{(3)} z^2 + \dots, \quad z \in \mathbb{D}. \tag{48}$$

From (47) and by the maximum principle for analytic functions it follows that φ_4 is a self-map of $\mathbb{D}$. Thus

$$\omega_4(z) := \psi_{b_1^{(3)}}(\varphi_4(z)) = b_1^{(4)} z + b_2^{(4)} z^2 + \dots, \quad z \in \mathbb{D}, \tag{49}$$

is a Schwarz function. By the Schwarz lemma we have

$$|b_1^{(4)}| = |\omega_4'(0)| \leq 1, \tag{50}$$

i.e.,

$$b_1^{(4)} = \zeta_5, \tag{51}$$

for some $\zeta_5 \in \overline{\mathbb{D}}$. Using (49), (3) and (48) we obtain

$$b_1^{(4)} = \omega_4'(0) = \psi'_{b_1^{(3)}}(\varphi_4(0))\varphi_4'(0) = \frac{b_2^{(3)}}{1 - |b_1^{(3)}|^2}, \tag{52}$$

which, taking into account (39) and (51) yields

$$b_2^{(3)} = (1 - |\zeta_4|^2)\zeta_5. \tag{53}$$

On the other hand, from (37), (36), (3), (32) and (40) we get

$$\begin{aligned} b_2^{(3)} &= \frac{1}{2}\omega_3''(0) = \frac{1}{2}\left(\psi''_{b_1^{(2)}}(\varphi_3(0))(\varphi_3'(0))^2 + \psi'_{b_1^{(2)}}(\varphi_3(0))\varphi_3''(0)\right) \\ &= \frac{1}{2}\psi''_{b_1^{(2)}}(b_1^{(2)})(b_2^{(2)})^2 + \psi'_{b_1^{(2)}}(b_1^{(2)})b_3^{(2)} = \frac{\overline{b_1^{(2)}}(b_2^{(2)})^2}{(1 - |b_1^{(2)}|^2)^2} + \frac{b_3^{(2)}}{1 - |b_1^{(2)}|^2} \\ &= \overline{\zeta_3}\zeta_4^2 + \frac{b_3^{(2)}}{1 - |\zeta_3|^2}, \end{aligned}$$

which, together with (53), yields

$$b_3^{(2)} = -(1 - |\zeta_3|^2)\overline{\zeta_3}\zeta_4^2 + (1 - |\zeta_3|^2)(1 - |\zeta_4|^2)\zeta_5. \tag{54}$$

From (30), (29), (3), (26), (33) and (41) we have

$$\begin{aligned} b_3^{(2)} &= \frac{1}{6}\omega_2^{(3)}(0) \\ &= \frac{1}{6}\left(\psi^{(3)}_{b_1^{(1)}}(\varphi_2(0))(\varphi_2'(0))^3 + 3\psi''_{b_1^{(1)}}(\varphi_2(0))\varphi_2'(0)\varphi_2''(0) + \psi'_{b_1^{(1)}}(\varphi_2(0))\varphi_2^{(3)}(0)\right) \\ &= \frac{1}{6}\left(\psi^{(3)}_{b_1^{(1)}}(b_1^{(1)})(b_2^{(1)})^3 + 6\psi''_{b_1^{(1)}}(b_1^{(1)})b_2^{(1)}b_3^{(1)} + 6\psi'_{b_1^{(1)}}(b_1^{(1)})b_4^{(1)}\right) \\ &= \frac{\left(\overline{b_1^{(1)}}\right)^2(b_2^{(1)})^3}{(1 - |b_1^{(1)}|^2)^3} + \frac{2\overline{b_1^{(1)}}b_2^{(1)}b_3^{(1)}}{(1 - |b_1^{(1)}|^2)^2} + \frac{b_4^{(1)}}{1 - |b_1^{(1)}|^2} \\ &= -\overline{\zeta_2}^2\zeta_3^3 + 2(1 - |\zeta_3|^2)\overline{\zeta_2}\zeta_3\zeta_4 + \frac{b_4^{(1)}}{1 - |\zeta_2|^2}, \end{aligned} \tag{55}$$

and so comparing (54) and (55) we obtain

$$\begin{aligned} b_4^{(1)} =&(1-|\zeta_2|^2)\overline{\zeta_2}^2\zeta_3^3 - (1-|\zeta_2|^2)(1-|\zeta_3|^2)\left(\overline{\zeta_3}\zeta_4^2 + 2\overline{\zeta_2}\zeta_3\zeta_4\right) \\ &+ (1-|\zeta_2|^2)(1-|\zeta_3|^2)(1-|\zeta_4|^2)\zeta_5. \end{aligned} \tag{56}$$

From (24), (23), (3), (21), (27), (34) and (43) we have

$$\begin{aligned} b_4^{(1)} =&\frac{1}{24}\omega_1^{(4)}(0) \\ =&\frac{1}{24}\left(\psi_{b_1}^{(4)}(\varphi_1(0))(\varphi_1'(0))^4 + 6\psi_{b_1}^{(3)}(\varphi_1(0))(\varphi_1'(0))^2\varphi_1''(0)\right. \\ &\left.+4\psi_{b_1}''(\varphi_1(0))\varphi_1'(0)\varphi_1^{(3)}(0) + 3\psi_{b_1}''(\varphi_1(0))(\varphi_1''(0))^2 + \psi_{b_1}'(\varphi_1(0))\varphi_1^{(4)}(0)\right) \\ =&\frac{1}{24}\left(\psi_{b_1}^{(4)}(b_1))b_2^4 + 12\psi_{b_1}^{(3)}(b_1)b_2^2b_3 + 24\psi_{b_1}''(b_1)b_2b_4\right. \\ &\left.+12\psi_{b_1}''(b_1)b_3^2 + 24\psi_{b_1}'(b_1)b_5\right) \\ =&\frac{\overline{b_1}^3b_2^4}{(1-|b_1|^2)^4} + \frac{3\overline{b_1}^2b_2^2b_3}{(1-|b_1|^2)^3} + \frac{2\overline{b_1}b_2b_4}{(1-|b_1|^2)^2} + \frac{\overline{b_1}b_3^2}{(1-|b_1|^2)^2} + \frac{b_5}{1-|b_1|^4} \\ =&\overline{\zeta_1}^3\zeta_2^4 + 3\overline{\zeta_1}^2\zeta_2^2\left[-\overline{\zeta_1}\zeta_2^2 + (1-|\zeta_2|^2)\zeta_3\right] \\ &+ 2\overline{\zeta_1}\zeta_2\left[\overline{\zeta_1}^2\zeta_2^3 - (1-|\zeta_2|^2)\left(\overline{\zeta_2}\zeta_3^2 + 2\overline{\zeta_1}\zeta_2\zeta_3\right) + (1-|\zeta_2|^2)(1-|\zeta_3|^2)\zeta_4\right] \\ &+ \overline{\zeta_1}\left[-\overline{\zeta_1}\zeta_2^2 + (1-|\zeta_2|^2)\zeta_3\right]^2 + \frac{b_5}{1-|\zeta_1|^2}. \end{aligned} \tag{57}$$

Finally comparing (56) and (57) we obtain

$$\begin{aligned} b_5 =& -(1-|\zeta_1|^2)\overline{\zeta_1}^3\zeta_2^4 \\ &-(1-|\zeta_1|^2)(1-|\zeta_2|^2)\left(\overline{\zeta_1}\zeta_3^2 - \overline{\zeta_2}^2\zeta_3^3 - 3\overline{\zeta_1}^2\zeta_2^2\zeta_3 - 3\overline{\zeta_1}|\zeta_2|^2\zeta_3^2\right) \\ &-(1-|\zeta_1|^2)(1-|\zeta_2|^2)(1-|\zeta_3|^2)\left(\overline{\zeta_3}\zeta_4^2 + 2\overline{\zeta_1}\zeta_2\zeta_4 + 2\overline{\zeta_2}\zeta_3\zeta_4\right) \\ &+(1-|\zeta_1|^2)(1-|\zeta_2|^2)(1-|\zeta_3|^2)(1-|\zeta_4|^2)\zeta_5. \end{aligned} \tag{58}$$

Substituting (21), (27), (34), (43) and (58) into (19) gives

$$\begin{aligned} c_5 =&2b_1^5 + 8b_1^3b_2 + 6b_1^2b_3 + 6b_1b_2^2 + 4b_1b_4 + 4b_2b_3 + 2b_5 \\ =&2\zeta_1^5 + 8\zeta_1^3\zeta_2(1-|\zeta_1|^2) + 6\zeta_1^2\left[-(1-|\zeta_1|^2)\overline{\zeta_1}\zeta_2^2 + (1-|\zeta_1|^2)(1-|\zeta_2|^2)\zeta_3\right] \\ &+ 6\zeta_1\zeta_2^2(1-|\zeta_1|^2)^2 \\ &+ 4\zeta_1\left[(1-|\zeta_1|^2)\overline{\zeta_1}^2\zeta_2^3 - (1-|\zeta_1|^2)(1-|\zeta_2|^2)\left(\overline{\zeta_2}\zeta_3^2 + 2\overline{\zeta_1}\zeta_2\zeta_3\right)\right. \\ &\left.+ (1-|\zeta_1|^2)(1-|\zeta_2|^2)(1-|\zeta_3|^2)\zeta_4\right] \\ &+ 4\zeta_2(1-|\zeta|_1|^2)\left[-(1-|\zeta_1|^2)\overline{\zeta_1}\zeta_2^2 + (1-|\zeta_1|^2)(1-|\zeta_2|^2)\zeta_3\right] \\ &+ 2\left[-(1-|\zeta_1|^2)\overline{\zeta_1}^3\zeta_2^4\right. \\ &-(1-|\zeta_1|^2)(1-|\zeta_2|^2)\left(\overline{\zeta_1}\zeta_3^2 - \overline{\zeta_2}^2\zeta_3^3 - 3\overline{\zeta_1}^2\zeta_2^2\zeta_3 - 3\overline{\zeta_1}|\zeta_2|^2\zeta_3^2\right) \\ &-(1-|\zeta_1|^2)(1-|\zeta_2|^2)(1-|\zeta_3|^2)\left(\overline{\zeta_3}\zeta_4^2 + 2\overline{\zeta_1}\zeta_2\zeta_4 + 2\overline{\zeta_2}\zeta_3\zeta_4\right) \\ &\left.+(1-|\zeta_1|^2)(1-|\zeta_2|^2)(1-|\zeta_3|^2)(1-|\zeta_4|^2)\zeta_5\right] \end{aligned}$$

which yields (8).

Moreover, equality in (50), i.e., the case $\zeta_4 \in \mathbb{T}$ in (51) holds if, and only if,

$$\omega_4(z) = \zeta_5 z, \quad z \in \mathbb{D}. \tag{59}$$

Note that ω and ω_1 are given by (45) and (46) respectively. From (35) and (37) with (32) it follows that

$$\omega_2(z) = z\varphi_3(z) = z\psi_{-b_1^{(2)}}(\omega_3(z)) = z\psi_{-\zeta_3}(\omega_3(z)), \quad z \in \mathbb{D}. \tag{60}$$

By (47), (49), (39) and (59) we have

$$\omega_3(z) = z\varphi_4(z) = z\psi_{-b_1^{(3)}}(\omega_4(z)) = z\psi_{-\zeta_4}(\zeta_5 z), \quad z \in \mathbb{D}.$$

Hence taking into account (45), (46) and (60) we see that ω is as in (16), and so from (18) it follows that p can be only as (17). □

Remark 2.2. When $c_1 \geq 0$, (6) was found in [8] and [9], and (7) in [7].

Directly from the proof of Lemma 2.1, using (21), (27), (34), (43) and (58) we can formulate the following.

Lemma 2.3. *If $\omega \in \mathcal{B}_0$ and is of the form* (2), *then*

$$b_1 = \zeta_1, \tag{61}$$

$$b_2 = (1 - |\zeta_1|^2)\zeta_2, \tag{62}$$

$$b_3 = -(1 - |\zeta_1|^2)\overline{\zeta_1}\zeta_2^2 + (1 - |\zeta_1|^2)(1 - |\zeta_2|^2)\zeta_3, \tag{63}$$

$$\begin{aligned} b_4 =&(1 - |\zeta_1|^2)\overline{\zeta_1}^2\zeta_2^3 - (1 - |\zeta_1|^2)(1 - |\zeta_2|^2)\left(\overline{\zeta_2}\zeta_3^2 + 2\overline{\zeta_1}\zeta_2\zeta_3\right) \\ &+ (1 - |\zeta_1|^2)(1 - |\zeta_2|^2)(1 - |\zeta_3|^2)\zeta_4 \end{aligned} \tag{64}$$

and

$$\begin{aligned} b_5 =& - (1 - |\zeta_1|^2)\overline{\zeta_1}^3\zeta_2^4 \\ &- (1 - |\zeta_1|^2)(1 - |\zeta_2|^2)\left(\overline{\zeta_1}\zeta_3^2 - \overline{\zeta_2}^2\zeta_3^3 - 3\overline{\zeta_1}^2\zeta_2^2\zeta_3 - 3\overline{\zeta_1}|\zeta_2|^2\zeta_3^2\right) \\ &- (1 - |\zeta_1|^2)(1 - |\zeta_2|^2)(1 - |\zeta_3|^2)\left(\overline{\zeta_3}\zeta_4^2 + 2\overline{\zeta_1}\zeta_2\zeta_4 + 2\overline{\zeta_2}\zeta_3\zeta_4\right) \\ &+ (1 - |\zeta_1|^2)(1 - |\zeta_2|^2)(1 - |\zeta_3|^2)(1 - |\zeta_4|^2)\zeta_5, \end{aligned} \tag{65}$$

for some $\zeta_i \in \overline{\mathbb{D}}$, $i \in \{1, \ldots, 5\}$.

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