On the Degenerate Pin Groups

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Abstract. We define a double covering homomorphism between a degenerate pin group and a degenerate orthogonal group which are given as semi-direct products. We also show that both of them are decomposed into four disjoint sets consisting of connected components. Then, using the components of pseudo-orthogonal group, we investigate the components of degenerate pin group and degenerate orthogonal group.

1. Introduction

In physics, the square root of Laplacian operator is known as Dirac operator. It is in some equations with physical applications and this makes it important. It was calculated easily for low dimensional Euclidean spaces and its existence in higher Euclidean spaces was revealed by Paul Dirac [7] in 1928 during the study of spin $\frac{1}{2}$ particles. By using these results, which is obtained in the n-dimensional Euclidean space, it is desired to define on the manifolds. However, since the tangent bundle is insufficient to define it on general Riemannian manifolds, it has remained as open problem for many years, and after that, it has been expressed with the development of the principal and associated bundles. So, the Dirac operator has attracted attention and many researchs have been carried out about it. One of these was accomplished by Brooke [4] who has obtained the spin group associated with the Galilei group by generalizing spin group for the degenerate orthogonal space and called it as Galilei spin group. Later, he was interested in matrix representations of Galilei spin group and the de Sitter spin group representations. Furthermore, Alagia and Sanchez [2] gave that the spin structures in the Riemannian manifold are generalizable to manifolds with nondegenerate metric by using signature (p, q). They stated (p, q)-orientability for those manifolds and occured the spin group $Spin(p, q)$. Then, they obtained the spin structure by using principal bundles on the manifolds which satisfy some conditions. Also, Ablamowicz [1] defined Clifford, pin and spin groups for degeneracy degree 1. As an example, he considered Galilei-Clifford algebra on the Galilei space-time. Then, Dereli et al. [6] showed that degenerate spin group, which was defined by Crumeyrolle [5], is isomorphic to the semi-direct product of the nondegenerate spin group and the matrix group.

For the expression of the Dirac operator on the manifolds, the principal bundles, the associated bundles and of course the spin manifolds are needed. This situation makes the spin manifolds essential. For this, Dirac operator was first described on Riemannian spin manifolds. Then, this operator was defined

on Lorentzian and pseudo-Riemannian manifolds. So, the question of how to define this operator on degenerate spin manifolds has been raised. Answering this question has been the motivation of our study. In order to define the degenerate spin manifolds, the degenerate pin groups are investigated firstly. For this, we generalize the obtained results by Brooke [4] and Ablamowicz [1] to $\mathbb{R}^{r,p,q}$, $r + p + q = n$ where r, p, q are the numbers of lightlike, spacelike, timelike basis vectors of $\mathbb{R}^n$, respectively. Also, to connect the degenerate pin groups and the degenerate orthogonal groups, we show that there exists a double covering homomorphism between the degenerate pin group and the degenerate orthogonal group. To obtain this, we decompose the pseudo-orthogonal group to the connected components since it is not connected. So, we get the components of the degenerate orthogonal group and the degenerate pin group. Then, we define the double covering homomorphism from each component of the degenerate pin group to each component of the degenerate orthogonal group. Thus, the degenerate spin manifolds could be described by using the obtained results for degenerate pin groups.

2. Preliminaries

Let V be a vector space over a commutative field k and Q be a quadratic form on V. Let $T(V) = \sum_{i=0}^{\infty} \otimes^i V$ denotes the tensor algebra of V and $I_Q(V)$ defines the ideal in $T(V)$ generated by all elements of the form $v \otimes v + Q(v) 1$ for $v \in V$ where $\otimes$ is tensor product. Then, a Clifford algebra is defined as the quotient

$$C\ell(V, Q) \equiv T(V) / I_Q(V). \tag{1}$$

Then, there is a decomposition

$$C\ell(V, Q) = C\ell^0(V, Q) \oplus C\ell^1(V, Q), \tag{2}$$

where

$$C\ell^i(V, Q) = \left\{ v \in C\ell(V, Q) : \alpha(v) = (-1)^i v \right\}. \tag{3}$$

Here, $\alpha^2 = I$ for identity map I and α is an automorphism on Clifford algebra $C\ell(V, Q)$. The subalgebra $C\ell^0(V, Q)$ of $C\ell(V, Q)$ is called the even part of $C\ell(V, Q)$ and the subspace $C\ell^1(V, Q)$ is called the odd part [10].

If the Clifford algebra is generated by n-dimensional real vector space $\mathbb{R}^n$ and Euclidean bilinear form g, then it is denoted by $C\ell_n$. Let g be a scalar product on $\mathbb{R}^n$ and $e_1, e_2, ..., e_n$ be standard basis vectors on $\mathbb{R}^n$. If the symmetric bilinear form g satisfies the condition

$$g(e_i, e_j) = \varepsilon_i \delta_{ij}, \varepsilon_i = \begin{cases} -1, & 1 \le i \le q \\ 1, & q \le i \le n \end{cases}, \delta_{ij} = \begin{cases} 0, & i \ne j \\ 1, & i = j \end{cases} \tag{4}$$

then it is called the pseudo-Euclidean bilinear form on $\mathbb{R}^n$ and the vector space $\mathbb{R}^n$ with pseudo-Euclidean bilinear form g is called the pseudo-Euclidean space [3].

Now, let us choose index q, $0 < q < n$ and $p = n - q$. In this situation, the set of all linear isometries $\psi : \mathbb{R}^{p,q} \to \mathbb{R}^{p,q}$ is the set of all matrices $\Psi \in GL(n, \mathbb{R})$ which preserve scalar product on $\mathbb{R}^{p,q}$. Then, this group is called the pseudo-orthogonal group and denoted by $O(p, q)$ for $p = n - q$ [11].

To define the pin group associated with pseudo-Euclidean space, we need a pseudo-orthogonal Clifford algebra. We assume that Q is a quadratic form for pseudo-Euclidean bilinear form g on $(\mathbb{R}^n, g)$. Then, the Clifford algebra $C\ell_{p,q} := C\ell(\mathbb{R}^n, Q)$ is called pseudo-orthogonal Clifford algebra. It has the following properties.

$$e_i^2 = -\varepsilon_i, \text{ for } i = 1, ..., n \tag{5}$$
$$e_i e_j + e_j e_i = 0, \text{ for } i \ne j, \text{ and } i, j = 1, ..., n.$$

Also, $(1, e_{i_1} \cdot ... \cdot e_{i_s}, 1 \le i_1 < ... < i_s \le n, 1 \le s \le n)$ is the basis of $C\ell_{p,q}$. Morever, the degree of element $a \in C\ell_{p,q}$ is defined by

$$\deg(a) = \begin{cases} 0, & \text{if } a \in C\ell_{p,q}^0 \\ 1, & \text{if } a \in C\ell_{p,q}^1. \end{cases} \tag{6}$$

Therefore, we use

$$n(a) = \begin{cases} 0, & \text{if the number of } a_i \text{ in } S_q^{n-1} \text{ is even for } i = 1, 2, ..., l \\ 1, & \text{if in other cases} \end{cases} \tag{7}$$

for the inverse of $a = a_1 \cdot ... \cdot a_l \in S_q^{n-1} \cup H_{q-1}^{n-1}$ where $S_q^{n-1} = \{v \in \mathbb{R}^{p,q} : g(v,v) = 1\}$ and $H_{q-1}^{n-1} = \{v \in \mathbb{R}^{p,q} : g(v,v) = -1\}$. Thus, a^{-1} is inverse of a such that $a^{-1} = a_l^{-1} \cdot ... \cdot a_1^{-1} = (-1)^{n(a)} a_l \cdot ... \cdot a_1$.

Pseudo-orthogonal pin group is defined as

$$Pin(p,q) := \left\{ a_1 \cdot ... \cdot a_l : a_i \in S_q^{n-1} \cup H_{q-1}^{n-1} \right\} \tag{8}$$

which consists of the inverse elements of Clifford algebra $C\ell_{p,q}$. If l is even, then the pseudo-orthogonal pin group is a pseudo-orthogonal spin group [3].

Proposition 2.1. *Let $O(p,q)$ be a pseudo-orthogonal group and $Pin(p,q)$ be a pseudo-orthogonal pin group. So, ρ is a double covering homomorphism such that $\rho : Pin(p,q) \to O(p,q)$, $\rho(a)v = a \cdot v \cdot a^{-1}(-1)^{dega}$ for $a \in Pin(p,q)$, $v \in \mathbb{R}^{p,q}$ [3].*

The Clifford algebras, which are formed by pseudo-Euclidean spaces, can be expanded to the degenerate spaces with similar concept. One of the important differences between the pseudo-Euclidean spaces and the degenerate spaces is that while the pseudo-orthogonal Clifford algebras could be represented by real, complex or quaternion matrices, the others could not because they have a nontrivial two-sided nilpotent ideal.

Let V be a real vector space with a symmetric bilinear form g and a quadratic form Q. It is called degenerate if there exists a vector $v \ne 0$ of V such that $g(v,w) = 0$ for $w \in V$ [8]. Also, the set of $v \in V$, which satisfies the condition $g(v,w) = 0$ for any $w \in V$, is a subspace of V and it is called a radical space of V. It is shown by $rad\, V$. If (V, Q) has a radical subspace, we have $V = V_1 \oplus rad\, V$ where $dim\, V_1 = n_1$ and Q induces a quadratic form Q_1 of $rank\, n_1$ on V_1. So, Clifford algebra, which is formed by vector space V with radical space, is called a degenerate Clifford algebra and this degenerate Clifford algebra is isomorphic to the graded tensor product of $C\ell(V_1, Q_1)$ and $\wedge rad\, V$ where $\wedge$ is an exterior product. If we set $V = \mathbb{R}^{r,p,q}$, then the degenerate Clifford algebra is written as $C\ell_{r,p,q}$ where r is the dimension of radical space in $\mathbb{R}^{r,p,q}$ [5].

We obtain the degenerate pin group with the following proposition.

Proposition 2.2. *There exists a homomorphism σ that it is defined from degenerate Clifford group Γ onto the group $\overline{T}$ of isometries of (V, Q) where the restriction to $rad\, V$ is the identity [5].*

The degenerate pin group of (V, Q) is denoted by $Pin(Q)$ and degenerate spin group of (V, Q) is denoted by $Spin(Q)$. For $\gamma_k = 1 + \sum_j c^{kj} e_k f_j$ where e_k is an orthogonal basis of (V_1, Q_1), f_j is an arbitrary basis of $Rad\, V$; every element of $Pin(Q)$ is a product

$$a_1 \cdot ... \cdot a_k \cdot \gamma_1 \cdot \gamma_2 \cdot ... \cdot \gamma_{n_1} \tag{9}$$

where $V = V_1 \oplus rad\, V$, Q_1 is a quadratic form on V_1 and $Q(a_i) = \pm 1$ for $a_i \in V_1$. If k is even, it is an element of $Spin(Q)$. If we set $V = \mathbb{R}^{r,p,q}$, then the degenerate pin group is written as $Pin(r, p, q)$ [5].

In the next section, since the covering homomorphism between the degenerate pin group and the degenerate orthogonal group will be obtained by the semi-direct product, it will be appropriate to give the definition of this product here.

Let B and B' be Lie groups, χ be a smooth map defined by

$$\chi : B \to Aut(B')$$
$$b \mapsto \chi(b)(b') \tag{10}$$

where $Aut(B')$ is automorphisms of B'. If χ is a homomorphism, we say that B acts on B'. In this situation, the group $B \times B'$ given by the multiplication

$$\left(b_1, b_1'\right)\left(b_2, b_2'\right) = \left(b_1 b_2, \chi\left(b_2^{-1}\right)\left(b_1'\right) b_2'\right) \tag{11}$$

is called a semi-direct product and denoted by $B \ltimes_\chi B'$ [9].

3. Degenerate Pin Groups

Thoughout this section, we assume that $p + q \geq 3$. By using Proposition 2.2 which has been given by Crumeyrolle [5], we write the following proposition.

Proposition 3.1. *Let L and L' be the groups defined by*

$$L = \left\{ \begin{bmatrix} I_{r\times r} & 0 \\ 0 & R \end{bmatrix} : R \in O(p,q) \right\}, \tag{12}$$

$$L' = \left\{ \begin{bmatrix} I_{r\times r} & \widetilde{A} \\ 0 & I_{(p+q)\times(p+q)} \end{bmatrix} : \widetilde{A} \in M_{r\times(p+q)}(\mathbb{R}) \right\}, \tag{13}$$

where $I_{r\times r}$ is an identity matrix, $O(p,q)$ is a pseudo-orthogonal group and $M_{r\times(p+q)}(\mathbb{R})$ is a matrix group with type $r \times (p+q)$ whose elements are real. Then, the semi-direct product $L \ltimes_{\chi'} L'$ gives a degenerate orthogonal group $O(r,p,q)$ which consists of the linear maps preserving quadratic form on $\mathbb{R}^{r,p,q}$. Also, r,p,q are the numbers of lightlike, spacelike, timelike basis vectors on $\mathbb{R}^{r,p,q}$, respectively and χ' is a map such that $\chi' : L \to Aut(L')$.

Proof. We assume that $\overline{L} = \begin{bmatrix} I_{r\times r} & 0 \\ 0 & R \end{bmatrix} \in L, \overline{L}' = \begin{bmatrix} I_{r\times r} & \widetilde{A} \\ 0 & I_{(p+q)\times(p+q)} \end{bmatrix} \in L'$. Then, we obtain that

$$\overline{L}\,\overline{L}'\,\overline{L}^{-1} = \begin{bmatrix} I_{r\times r} & \widetilde{A}R^T \\ 0 & I_{(p+q)\times(p+q)} \end{bmatrix} \in L'.$$

Here, T is the transpose of the matrix and $R^{-1} = R^T$ from $R \in O(p,q)$. Thus, χ' could be defined. To show that χ' is a homomorphism, we find

$$\chi'\left(\overline{L}_1\overline{L}_2\right)\left(\overline{L}'\right) = \begin{bmatrix} I_{r\times r} & \widetilde{A}R_2^T R_1^T \\ 0 & I_{(p+q)\times(p+q)} \end{bmatrix},$$

$$\chi'_{\overline{L}_1}\left(\chi'_{\overline{L}_2}\left(\overline{L}'\right)\right) = \begin{bmatrix} I_{r\times r} & \widetilde{A}R_2^T R_1^T \\ 0 & I_{(p+q)\times(p+q)} \end{bmatrix},$$

for $\overline{L}_1 = \begin{bmatrix} I_{r\times r} & 0 \\ 0 & R_1 \end{bmatrix}, \overline{L}_2 = \begin{bmatrix} I_{r\times r} & 0 \\ 0 & R_2 \end{bmatrix}, \overline{L}' = \begin{bmatrix} I_{r\times r} & \widetilde{A} \\ 0 & I_{(p+q)\times(p+q)} \end{bmatrix}$. Hence, χ' is a homomorphism. In this situation, let the multiplication on $L \times L'$ be

$$\cdot : (L \times L') \times (L \times L') \to (L \times L')$$

$$\left(\left(\overline{L}_1, \overline{L}_1'\right), \left(\overline{L}_2, \overline{L}_2'\right)\right) \mapsto \left(\overline{L}_1\overline{L}_2, \chi'\left(\overline{L}_2^{-1}\right)\left(\overline{L}_1'\right)\overline{L}_2'\right).$$

Then, the semi-direct product $L \ltimes_{\chi'} L'$ exists since $L \times L'$ is a group with this multiplication. $\quad\square$

Morever, we find that the degenerate pin group could be written as semi-direct product.

Proposition 3.2. *Let H and H′ be the groups defined by*

$$H = \left\{ a : a = a_1^{\alpha^1} \cdot ... \cdot a_{p+q}^{\alpha^{p+q}}, Q(a_i) = \pm 1, \alpha_i = 0, 1 \text{ for all } a_i \in \mathbb{R}^{p,q} \right\}, \tag{14}$$

$$H' = \left\{ \gamma : \gamma = \gamma_1 \cdot ... \cdot \gamma_{p+q} \cdot (1 + \Xi), \gamma_k = 1 + \sum_{j=1}^{r} c^{kj} e_k f_j, \Xi \in \wedge \mathbb{R}^r \right\}, \tag{15}$$

where $e_k, f_j \in \mathbb{R}^{r,p,q}$. Then, there exists the semi-direct product $H \ltimes_\chi H'$ where χ is a map such that $\chi : H \to Aut(H')$.

Proof. We assume that $a \in H, \gamma \in H'$. So, $\gamma = \gamma_1 \cdot ... \cdot \gamma_{p+q} \cdot (1 + \Xi)$ where $\gamma_k = 1 + \sum_{j=1}^{r} c^{kj} e_k f_j$ and $a = c_1^{\alpha^1} ... c_{p+q}^{\alpha^{p+q}} e_1^{\alpha^1} \cdot ... \cdot e_{p+q}^{\alpha^{p+q}}$. Here, $c_i^{\alpha^i} \in \mathbb{R}$ is a constant, e_i is a basis vector of $\mathbb{R}^{p,q}$ and $\alpha^i \in \{0, 1\}$. So, χ can be defined as

$$\chi : H \to Aut(H')$$
$$a \mapsto \chi(a)(\gamma) = a \cdot \gamma \cdot a^{-1}$$

for $a \in H, \gamma \in H'$ where $Aut(H')$ is the automorphisms of H'. Then, we write that $a \cdot \gamma \cdot a^{-1} \in H'$. Also, we have

$$\chi(a \cdot a')(\gamma) = a \cdot \left[a' \cdot \gamma \cdot (a')^{-1} \right] \cdot a^{-1} = a \cdot [\chi_{a'}(\gamma)] \cdot a^{-1} = \chi_a(\chi_{a'}(\gamma))$$

for $a, a' \in H, \gamma \in H'$. So, χ is a group homomorphism. Morever, the multiplication on $H \times H'$ is defined as

$$\cdot : (H \times H') \times (H \times H') \to (H \times H')$$
$$((a, \gamma), (a', \gamma')) \mapsto \left(aa', \chi \left((a')^{-1} \right) (\gamma) \gamma' \right).$$

$H \times H'$ is a group with respect to this multiplication. Thus, $H \ltimes_\chi H'$ is semi-direct product of the groups H and H' by χ. $\square$

$H \ltimes_\chi H'/N$ is called a degenerate pin group where N contains the elements $(1 + \Xi)$ for $\Xi \in \wedge \mathbb{R}^r$ and is a normal subgroup of $H \ltimes_\chi H'$. So, a covering homomorphism between the degenerate pin group and the degenerate orthogonal group, which are obtained with the semi-direct products, is introduced by the following theorem.

Theorem 3.3. *Let L and L′ be groups as in (12) and (13), H and H′ be the groups as in (14) and (15). Then, $\overline{\rho}$ is a double covering homomorphism such that*

$$\overline{\rho} : H \ltimes_\chi H'/N \to L \ltimes_{\chi'} L' \tag{16}$$

and its kernel is $\{1, -1\}$.

Proof. (1) Firstly, let us show that $\overline{\rho}$ is a homomorphism. Then, $H \ltimes_\chi H'$ and $L \ltimes_{\chi'} L'$ are semi-direct products and the multiplication on $H \ltimes_\chi H'$ is defined by

$$(a, \gamma)(a', \gamma') = \left(aa', \chi \left((a')^{-1} \right) (\gamma) \gamma' \right) = \left(aa', \chi_{(a')^{-1}}(\gamma) \gamma' \right)$$

for $\gamma, \gamma' \in H', a, a' \in H$. We assume that $\gamma, \gamma' \in H'/N$ and $\left(aa', \chi_{(a')^{-1}}(\gamma) \gamma' \right) \in H \ltimes_\chi H'/N$. In this situation, we assume that $aa' \chi_{(a')^{-1}}(\gamma) \gamma' = \overline{h}$ and $\chi_{(a')^{-1}}(\gamma) \gamma' = (a')^{-1} \gamma a' \gamma'$. Also, let us define $\overline{\rho}$ by

$$\overline{\rho}(\overline{h})(v) = (\overline{h}) \cdot v \cdot (\overline{h})^{-1} (-1)^{\deg a + \deg a'}$$

for $v \in \mathbb{R}^{r,p,q}$. Thus, $\overline{\rho}$ is a group homomorphism since we find

$$
\begin{aligned}
\overline{\rho}\left(\overline{h}\right)(v) &= \left(aa'a'^{-1}\gamma a'\gamma'\right) \cdot v \cdot \left(aa'a'^{-1}\gamma a'\gamma'\right)^{-1} (-1)^{\deg a + \deg a'} \\
&= (-1)^{\deg a} a\gamma \left[\left((-1)^{\deg a'} a'\gamma'\right) \cdot v \cdot \left((\gamma')^{-1} (a')^{-1}\right)\right] \gamma^{-1} a^{-1} \\
&= (-1)^{\deg a} a\gamma \left[\overline{\rho}\left(a'\gamma'\right)(v)\right] \gamma^{-1} a^{-1} \\
&= \overline{\rho}\left(a\gamma\right) \overline{\rho}\left(a'\gamma'\right)(v).
\end{aligned}
$$

(2) We assume that $u = a\gamma$ for $u \in H \ltimes_\chi H'/N$. If $\overline{\rho}$ could be written as a product of reflections such that

$$
\overline{\rho}_u(v) = \overline{\rho}_{a\gamma}(v) = \overline{\rho}_a\left(\overline{\rho}_\gamma(v)\right),
$$

then it is a surjective. To show this, we need to examine $\overline{\rho}_u(v)$ according to the elements v and u in degenerate or nondegenerate part. When u is equal to γ or a, $\overline{\rho}$ is examined in the following cases.

CASE 1 : We assume that $u = \gamma$ and $v = v' + \sum_{j=1}^{r} \lambda_j f_j$ for $v \in \mathbb{R}^{r,p,q}, v' \in \mathbb{R}^{p,q}$. We also have $\gamma = 1 + \sum_{i=1}^{r} \overline{b} f_i$

since γ is given by the product of the elements γ_i . Here, $\overline{b}$ consists of nondegenerate vectors e_k, degenerate vectors f_t and arbitrary constants. So, for $f_i \neq f_j$, we find that

$$
\begin{aligned}
\overline{\rho}_\gamma(v) &= \gamma \cdot v \cdot \gamma^{-1} \\
&= (v' + \sum_{j=1}^{r} \lambda_j f_j + \sum_{i=1}^{r} \overline{b} f_i v' + \sum_{i=1}^{r} \sum_{j=1}^{r} \lambda_j \overline{b} f_i f_j - \sum_{i=1}^{r} v' \overline{b} f_i - \sum_{j=1}^{r} \sum_{i=1}^{r} \lambda_j f_j \overline{b} f_i) \\
&= v' + \sum_{j=1}^{r} \left(\lambda_j - \overline{b} v' - v' \overline{b}\right) f_j.
\end{aligned}
$$

Also, for $f_i = f_j$, we obtain that

$$
\overline{\rho}_\gamma(v) = \gamma \cdot v \cdot \gamma^{-1} = v' + \sum_{j=1}^{r} \left(\lambda_j - \overline{b} v' - v' \overline{b}\right) f_j.
$$

In that case, when we consider the basis vectors of $\mathbb{R}^{r,p,q}$ instead of v, the matrix associated with $\overline{\rho}$ is given by $\begin{bmatrix} I_{r \times r} & \widetilde{A}_{r \times (p+q)} \\ 0 & I_{(p+q) \times (p+q)} \end{bmatrix}$. So, this matrix covers the group L'.

CASE 2 : We assume that $u = a$ for $a \in H$. Then, we have

$$
\begin{aligned}
\overline{\rho}_a(v) &= a \cdot v \cdot a^{-1} (-1)^{\deg a} \\
&= \sum_{j=1}^{r} \lambda_j f_j a a^{-1} (-1)^{\deg a} (-1)^{\deg a} + a v' a^{-1} (-1)^{\deg a} \\
&= \sum_{j=1}^{r} \lambda_j f_j + a v' a^{-1} (-1)^{\deg a}.
\end{aligned}
$$

When we consider the basis vectors f_i of $\mathbb{R}^{r,p,q}$ instead of v, the submatrix associated with $\overline{\rho}$ is given by $\begin{bmatrix} I_{r \times r} \\ 0 \end{bmatrix}$.

Also, when we consider the basis vectors e_i of $\mathbb{R}^{r,p,q}$ instead of v, the following situations are examined to obtain the matrix associated with $\overline{\rho}$.

(*i*) For $v \in \mathbb{R}^{r,p,q}$, we define a projection map

$$\pi : \mathbb{R}^{r,p,q} \to \mathbb{R}^{p,q}$$
$$v \mapsto v'.$$

So, we obtain the equation $\pi\left(\overline{\rho}_a(v)\right) = a \cdot v' \cdot a^{-1}(-1)^{\deg a}$. This means that the projection onto nondegenerate part provides a reflection when v has degenerate and nondegenerate part.

(*ii*) For $v \in \mathbb{R}^{p,q}$, $\overline{\rho}_a(v)$ is given by

$$\overline{\rho}_a(v) = a \cdot v' \cdot a^{-1}(-1)^{\deg a}$$

and it denotes the reflection.

(*iii*) Let $w = w' + \sum_{j=1}^{r} c_j f_j = w' + w''$, $v = v' + \sum_{j=1}^{r} \lambda_j f_j = v' + v''$ be for $w, v \in \mathbb{R}^{r,p,q}$. w'' and w' is the degenerate and the nondegenerate part of w, respectively. Then, we find

$$\overline{\rho}_w(v) = v - 2\frac{g(v,w)}{Q(w)}w = v'' + v' - 2\frac{g(v',w')}{Q(w')}(w'' + w').$$

Morever, we have

$$\pi\left(\overline{\rho}_w(v)\right) = v' - 2\frac{g(v',w')}{Q(w')}w'$$

for the projection map π. From there, we say that $\pi \circ \overline{\rho}_w$ is a reflection with respect to the hyperplane $(w')^{\perp}$.

Thus, when we consider the basis vectors of $\mathbb{R}^{r,p,q}$, the matrix associated with $\overline{\rho}$ is given by $\begin{bmatrix} I_{r\times r} & 0 \\ 0 & R \end{bmatrix}$, $R \in O(p,q)$. So, this matrix covers the group L.

Therefore, $\overline{\rho}$ is the subjective.

(3) Let Ker $\overline{\rho}$ be a kernel of $\overline{\rho}$ and $u = c_1^{\alpha^1}...c_{p+q}^{\alpha^{p+q}} e_1^{\alpha^1} \cdot ... \cdot e_{p+q}^{\alpha^{p+q}} \gamma_1 \cdot ... \cdot \gamma_{p+q}$ for $\alpha^i = 0,1$ and $i = 1,...,p+q$. In that case, if $u \in$ Ker $\overline{\rho}$, then there exist $\overline{\rho}(u)(e_i) = e_i$ and $\overline{\rho}(u)(f_i) = f_i$. We assume that $a = c_1^{\alpha^1}...c_{p+q}^{\alpha^{p+q}} e_1^{\alpha^1}...e_{p+q}^{\alpha^{p+q}}$. Then, we have

$$u \in \text{Ker}\overline{\rho} \Leftrightarrow \overline{\rho}(u)(e_i) = e_i$$
$$\Leftrightarrow \overline{\rho}_{a\gamma_1\cdot...\cdot\gamma_{p+q}}(e_i) = e_i$$
$$\Leftrightarrow \left(\overline{\rho}_a \circ \overline{\rho}_{\gamma_1} \circ ... \circ \overline{\rho}_{\gamma_{p+q}}\right)(e_i) = e_i.$$

Considering the homomorphism $\overline{\rho}$, if we use $\overline{\rho}_{\gamma_k}(e_i) = \gamma_k \cdot e_i \cdot \gamma_k^{-1}$, we obtain $\overline{\rho}_{\gamma_k}(e_k) = e_k + 2\varepsilon_k c^{kj} f_j$ for $i = k$ and $\overline{\rho}_{\gamma_k}(e_i) = e_i$ for $i \neq k$. Since there is $k = i$ at least one from γ_k for $\left(\overline{\rho}_{\gamma_1} \circ ... \circ \overline{\rho}_{\gamma_{p+q}}\right)(e_i)$, we have

$$\left(\overline{\rho}_{\gamma_1} \circ ... \circ \overline{\rho}_{\gamma_{p+q}}\right)(e_i) = e_i + 2\varepsilon_i c^{ij} f_j.$$

Thus, we write

$$\left(\overline{\rho}_a \circ \overline{\rho}_{\gamma_1} \circ ... \circ \overline{\rho}_{\gamma_{p+q}}\right)(e_i) = \overline{\rho}_a\left(e_i + 2\varepsilon_i c^{ij} f_j\right) = a \cdot \left(e_i + 2\varepsilon_i c^{ij} f_j\right) \cdot a^{-1}.$$

From $u \in$ Ker $\overline{\rho}$, we have

$$a \cdot \left(e_i + 2\varepsilon_i c^{ij} f_j\right) \cdot a^{-1} = e_i.$$

If we multiply both sides of equality by a and e_i, we get

$$e_i \cdot a \cdot \left(e_i + 2\varepsilon_i c^{ij} f_j \right) = -\varepsilon_i a \Rightarrow -\varepsilon_i (-1)^{\overset{\sum \alpha^j}{i \neq j}} a + 2\varepsilon_i c^{ij} e_i \cdot a \cdot f_j = -\varepsilon_i a.$$

For $\overline{\rho}(u)(e_i) = e_i$, there is one of the following two situations.

(a) $c^{ij} = 0$ and $c_1^{\alpha^1} ... c_{p+q}^{\alpha^{p+q}} = (-1)^{\overset{\sum \alpha^j}{i \neq j}} c_1^{\alpha^1} ... c_{p+q}^{\alpha^{p+q}}$,

(b) $c^{ij} = 0$ and $c_1^{\alpha^1} ... c_{p+q}^{\alpha^{p+q}} = 0$ for $\sum\limits_{i \neq j} \alpha^j \neq 0 \, (\mathrm{mod} \, 2)$.

Morever, we write

$$\left(\overline{\rho}_a \circ \overline{\rho}_{\gamma_1} \circ ... \circ \overline{\rho}_{\gamma_{p+q}} \right)(f_i) = \overline{\rho}_a (f_i) = a \cdot f_i \cdot a^{-1}.$$

From $u \in \mathrm{Ker}\, \overline{\rho}$, we find $a \cdot f_i = f_i \cdot a$ and

$$\left(c_1^{\alpha^1} ... c_{p+q}^{\alpha^{p+q}} e_1^{\alpha^1} \cdot ... \cdot e_{p+q}^{\alpha^{p+q}} \right) \cdot f_i = f_i \cdot \left(c_1^{\alpha^1} ... c_{p+q}^{\alpha^{p+q}} e_1^{\alpha^1} \cdot ... \cdot e_{p+q}^{\alpha^{p+q}} \right)$$

$$c_1^{\alpha^1} ... c_{p+q}^{\alpha^{p+q}} (-1)^{\overset{\sum \alpha^j}{j}} f_i \cdot e_1^{\alpha^1} \cdot ... \cdot e_{p+q}^{\alpha^{p+q}} = f_i \cdot c_1^{\alpha^1} ... c_{p+q}^{\alpha^{p+q}} e_1^{\alpha^1} \cdot ... \cdot e_{p+q}^{\alpha^{p+q}}.$$

For $\overline{\rho}(u)(f_i) = f_i$, we have $c_1^{\alpha^1} ... c_{p+q}^{\alpha^{p+q}} = (-1)^{\overset{\sum \alpha^j}{j}} c_1^{\alpha^1} ... c_{p+q}^{\alpha^{p+q}}$ or $c_1^{\alpha^1} ... c_{p+q}^{\alpha^{p+q}} = 0$ for $\sum\limits_{j} \alpha^j \neq 0 \, (\mathrm{mod} 2)$. Therefore,

the proof is completed since the kernel of $\overline{\rho}$ is $\{1, -1\}$. $\square$

If Ψ is the element of $O(r, p, q)$, we can write in block form as

$$\Psi = \begin{bmatrix} I_{r \times r} & \widetilde{A}_{r \times (p+q)} \\ 0_{(p+q) \times r} & R \end{bmatrix} \tag{17}$$

where $R = \begin{bmatrix} \Psi_T & \overline{C} \\ C & \Psi_S \end{bmatrix}$, $R \in O(p, q)$ and Ψ_T, Ψ_S is obtained by orthogonal projection of orthogonal map $\psi : \mathbb{R}^{p,q} \to \mathbb{R}^{p,q}$ on $\mathbb{R}^q, \mathbb{R}^p$, respectively. Thus, according to Ψ_T, Ψ_S, the degenerate orthogonal group $O(r, p, q)$ is decomposed into four disjoint sets

$$\begin{aligned} O^{++}(r, p, q) &= \{ \Psi \in O(r, p, q) : \det(\Psi_T) > 0, \det(\Psi_S) > 0 \}, \\ O^{--}(r, p, q) &= \{ \Psi \in O(r, p, q) : \det(\Psi_T) < 0, \det(\Psi_S) < 0 \}, \\ O^{+-}(r, p, q) &= \{ \Psi \in O(r, p, q) : \det(\Psi_T) > 0, \det(\Psi_S) < 0 \}, \\ O^{-+}(r, p, q) &= \{ \Psi \in O(r, p, q) : \det(\Psi_T) < 0, \det(\Psi_S) > 0 \}. \end{aligned} \tag{18}$$

Connectedness of the degenerate pin group is closely related to the covering homomorphism. So, this property is investigated by the following definition and theorem.

Definition 3.4. *Let u be an element of $Pin(r, p, q)$ such that*

$$u = c_1^{\alpha^1} ... c_{p+q}^{\alpha^{p+q}} e_1^{\alpha^1} \cdot ... \cdot e_{p+q}^{\alpha^{p+q}} \gamma_1 \cdot ... \cdot \gamma_{p+q} \tag{19}$$

for $c_i^{\alpha^i} \in \mathbb{R}$, $\alpha^i = 0, 1$ and $i = 1, ..., p + q$ where $\gamma_k = 1 + \sum_{j=1}^{r} c^{kj} e_k f_j$. We assume that $a = c_1^{\alpha^1} ... c_{p+q}^{\alpha^{p+q}} e_1^{\alpha^1} \cdot ... \cdot e_{p+q}^{\alpha^{p+q}}$ and

$\gamma = \gamma_1 \cdot ... \cdot \gamma_{p+q}$. Then, the degenerate pin group $Pin\,(r, p, q)$ is given by the components

$$Pin^{++}\,(r, p, q) = \{u \in Pin\,(r, p, q) : u = a\gamma, \deg(a) = 0, n(a) = 0\},$$
$$Pin^{--}\,(r, p, q) = \{u \in Pin\,(r, p, q) : u = a\gamma, \deg(a) = 0, n(a) = 1\}, \qquad (20)$$
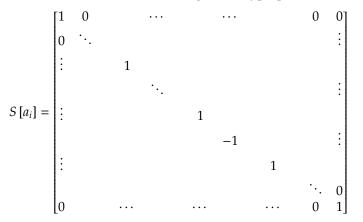$$Pin^{+-}\,(r, p, q) = \{u \in Pin\,(r, p, q) : u = a\gamma, \deg(a) = 1, n(a) = 1\},$$
$$Pin^{-+}\,(r, p, q) = \{u \in Pin\,(r, p, q) : u = a\gamma, \deg(a) = 1, n(a) = 0\}.$$

Theorem 3.5. *Degenerate pin group* $Pin\,(r, p, q)$ *is a Lie group having connected components* $Pin^{++}\,(r, p, q)$, $Pin^{--}\,(r, p, q)$, $Pin^{+-}\,(r, p, q)$, $Pin^{-+}\,(r, p, q)$ *associated with pseudo-orhogonal group* $O\,(p, q)$. *Also,* $\overline{\rho}$ *is a double covering homomorphism from each component of the degenerate pin group* $Pin\,(r, p, q)$ *to each component of the degenerate orthogonal group* $O\,(r, p, q)$.
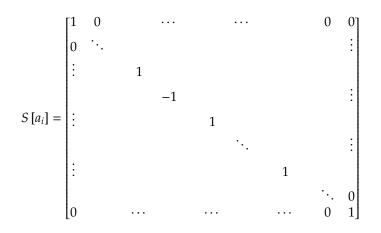
Proof. Let $S\,[u]$ be a reflection with regard to the hyperplane $\{u\}^{\perp}$ for $u \in Pin(r, p, q)$. Since $\overline{\rho}\,(u)$ could be written as the product of reflections, we could find the matrix $S\,[u]$ for $u = a\gamma$. We assume that $a_i = c_i^{\alpha^i} e_i^{\alpha^i}$ for $\alpha^i \neq 0$.

   CASE 1. $\langle a_i, a_i \rangle = 1$. While the reflection with regard to a_i of the vector a_i is $-a_i$, the reflection with regard to a_i of the other vectors are themselves. In this situation, when we consider the spacelike part of matrix which denotes the reflection according to the hyperplane $\{a_i\}^{\perp}$, we obtain that

$$S\,[a_i] = \begin{bmatrix} 1 & 0 & & \cdots & & \cdots & & 0 & 0 \\ 0 & \ddots & & & & & & & \vdots \\ \vdots & & 1 & & & & & & \\ & & & \ddots & & & & & \vdots \\ \vdots & & & & 1 & & & & \\ & & & & & -1 & & & \vdots \\ & & & & & & 1 & & \\ \vdots & & & & & & & \ddots & 0 \\ 0 & \cdots & & \cdots & & \cdots & & 0 & 1 \end{bmatrix}$$

where the matrix $S\,[a_i]$ is the matrix in the type of $n \times n$ ($n = r + p + q$) and its elements except the diagonal are zero. Also, the elements of diagonal of this matrix are 1 except the $(r + q + 1)(r + q + 1)$-element which is -1.

   CASE 2. $\langle a_i, a_i \rangle = -1$. Then, when we consider the timelike part of matrix, we have

$$S\,[a_i] = \begin{bmatrix} 1 & 0 & & \cdots & & \cdots & & 0 & 0 \\ 0 & \ddots & & & & & & & \vdots \\ \vdots & & 1 & & & & & & \\ & & & -1 & & & & & \vdots \\ \vdots & & & & 1 & & & & \\ & & & & & \ddots & & & \vdots \\ \vdots & & & & & & 1 & & \\ & & & & & & & \ddots & 0 \\ 0 & \cdots & & \cdots & & \cdots & & 0 & 1 \end{bmatrix}$$

where the matrix $S[a_i]$ is the matrix in the type of $n \times n$ ($n = r + p + q$) and its elements except the diagonal are zero. Also, the elements of diagonal are 1 except the $(r + 1)(r + 1)$-element which is -1.

CASE 3. With similar idea, the matrix corresponding to the reflection of the vector γ_i is in the form

$$S[a_i] = \begin{bmatrix} 1 & 0 & \cdots & \widetilde{A}_{r \times 1} & 0 & \cdots & & 0 & 0 \\ 0 & \ddots & & & & & & & \vdots \\ \vdots & & 1 & & & & & & \vdots \\ & & & \ddots & & & & & \\ \vdots & & & & 1 & & & & \vdots \\ & & & & & & \ddots & 0 & \\ 0 & & \cdots & & & \cdots & & 0 & 1 \end{bmatrix}$$

where the matrix $S[a_i]$ is the matrix in the type of $n \times n$ ($n = r + p + q$) and the elements of diagonal are 1. Also, $(r + 1)$-th column in the matrix is

$$\begin{bmatrix} \widetilde{A}_{r \times 1} \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Then, we get

$$\overline{\rho}\left(a_1 a_2 \cdot \ldots \cdot a_{2l} \gamma_1 \cdot \ldots \cdot \gamma_{p+q}\right) = S[a_1] \ldots S[a_{2l}] S[\gamma_1] \ldots S[\gamma_{p+q}]$$

since $\overline{\rho}(u)$ can be written as product of reflections. If $\deg(a) = 0$, $n(a) = 0$, then the determinant of the obtained matrix $S[a_1] \ldots S[a_{2l}] S[\gamma_1] \ldots S[\gamma_{p+q}]$ is one. So, we have $\overline{\rho}\left(a_1 a_2 \cdot \ldots \cdot a_{2l} \gamma_1 \cdot \ldots \cdot \gamma_{p+q}\right) \in O^{++}(r, p, q)$. When we do the similar operations for the other components, we have the following results.

$$\overline{\rho}\left(a_1 \cdot \ldots \cdot a_{2l} \gamma_1 \cdot \ldots \cdot \gamma_{p+q}\right) = S[a_1] \ldots S[a_{2l}] S[\gamma_1] \ldots S[\gamma_{p+q}] \in O^{--}(r, p, q),$$

$$\overline{\rho}\left(a_1 \cdot \ldots \cdot a_{2l+1} \gamma_1 \cdot \ldots \cdot \gamma_{p+q}\right) = S[a_1] \ldots S[a_{2l+1}] S[\gamma_1] \ldots S[\gamma_{p+q}] \in O^{+-}(r, p, q),$$

$$\overline{\rho}\left(a_1 \cdot \ldots \cdot a_{2l+1} \gamma_1 \cdot \ldots \cdot \gamma_{p+q}\right) = S[a_1] \ldots S[a_{2l+1}] S[\gamma_1] \ldots S[\gamma_{p+q}] \in O^{-+}(r, p, q).$$

Finally, we assume that $\overline{\rho}(u)$, $\overline{\rho}(v) \in O^{ab}(r, p, q)$ for $u, v \in Pin^{ab}(r, p, q)$ where ab shows the components of group, that is, ab can be $++, +-, -+, --$. In that case, there is a path ζ since $O^{ab}(r, p, q)$ is path connected and we write $\zeta(0) = \overline{\rho}(u)$, $\zeta(1) = \overline{\rho}(v)$. Let $\widetilde{\zeta}$ be a lift of the path ζ on $Pin^{ab}(r, p, q)$ at initial point u. Then, the end point of lift $\widetilde{\zeta}$ is $\pm v$, that is, $\widetilde{\zeta}(1) = \pm v$. So, we should show that v is connected to $-v$ with a path on $Pin^{ab}(r, p, q)$. Let us choose e_n and e_{n-1} which have the same causal characters by using the rearrangement of the basis vectors $e_i \in \mathbb{R}^{r,p,q}$, $p + q \geq 3$. Considering this, we assume that $\delta(t)$ is a curve such that

$$\delta(t) = v\left(1 + \sin t e_i f_j\right)\left(e_n \cos \frac{t}{2} + e_{n-1} \sin \frac{t}{2}\right)\left(e_n \cos \frac{t}{2} - e_{n-1} \sin \frac{t}{2}\right).$$

If we write $\delta_1(t) = 1 + \sin t e_i f_j$, $\delta_2(t) = e_n \cos \frac{t}{2} + e_{n-1} \sin \frac{t}{2}$, $\delta_3(t) = e_n \cos \frac{t}{2} - e_{n-1} \sin \frac{t}{2}$, we get $Q(\delta_2(t)) = Q(\delta_3(t)) = \mp 1$ where $Q(\delta_i(t)) = -\delta_i(t) \cdot \delta_i(t)$, $i = 2, 3$. So, according to the definition of $Pin^{ab}(r, p, q)$, we have $\delta(t) \in Pin^{ab}(r, p, q)$. Finally, the curve δ connects v and $-v$ on $Pin^{ab}(r, p, q)$ where $\delta(0) = -\varepsilon_n v$, $\delta(\pi) = \varepsilon_{n-1} v$. Hence, the proof is completed. $\square$

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