On Uniform Perfectness in Quasimetric Spaces

Qingshan Zhou[a], Yaxiang Li[b], Ailing Xiao[c]

[a]School of Mathematics and Big Data, Foshan University, Foshan, Guangdong 528000, People's Republic of China
[b]Department of Mathematics, Hunan First Normal University, Changsha, Hunan 410205, People's Republic of China
[c]School of Finance, Guangdong University of Foreign Studies, Guangzhou 510006, People's Republic of China

Abstract. The aim of this paper is to investigate the equivalence conditions for uniform perfectness of quasimetric spaces. We also obtain the invariance property of uniform perfectness under quasimöbius mappings in quasimetric spaces. In the end, two applications are given.

1. Introduction and Main Results

In this paper, we study the equivalence conditions for uniform perfectness of quasimetric spaces. The first version of this paper was first finished in 2016 (an early version of this paper see [23]), there were several applications in the area of geometric function theory based on that version [1, 19–21]. We start with the definition of quasimetric spaces.

Definition 1.1. For a given set Z and a constant $K \geq 1$,

1. a function $\rho : Z \times Z \to [0, +\infty)$ is said to be *K-quasimetric*
   (a) if for all x and y in Z, $\rho(x, y) \geq 0$, and $\rho(x, y) = 0$ if and only if $x = y$;
   (b) $\rho(x, y) = \rho(y, x)$ for all $x, y \in Z$;
   (c) $\rho(x, z) \leq K(\rho(x, y) \vee \rho(y, z))$ for all $x, y, z \in Z$.
2. the pair (Z, ρ) is said to be a *K-quasimetric space* if the function $\rho : Z \times Z \to [0, +\infty)$ is K-quasimetric with $K \geq 1$. In the following, we always say that K is the quasimetric coefficient of (Z, ρ).

Clearly, if (Z, ρ) is K_1-quasimetric, it must be K_2-quasimetric for any $K_2 \geq K_1$. Hence, in the following, the quasimetric coefficients of all quasimetric spaces are always denoted by K with $K > 1$. We also assume that the quasimetric spaces considered in this paper contains at least three points.

Every quasimetric ρ defines a uniform structure on Z. The balls $\mathbb{B}_\rho(x, r) = \{y \in Z : \rho(x, y) < r\}$ $(r > 0)$ form a basis of neighborhoods of x for the topology induced by the uniformity on Z. We shall refer to this topology as the ρ-topology of Z (cf. [13]). If there is no risk of confusion, we will omit the subscript ρ in the symbol.

We know that every metric is a 2-quasimetric, but a quasimetric need not be a metric. For example, $Z = \mathbb{R}^2, \rho((x_1, y_2), (x_2, y_2)) = |x_1 - y_1|^{\alpha_1} + |x_2 - y_2|^{\alpha_2}$, where $\alpha_1 \neq \alpha_2$ are positive numbers, not all equal. In general, one can easily see that the nonnegative symmetric function is not necessary a metric but a quasimetric. This example follows from Coifman and Weiss [5, §2(2)] and they have referred to such property as *nonisotropy*. For more properties and examples concerning quasimetric spaces, see [6–9, 13] etc.

The following useful result on the relationship between quasimetric spaces and metric spaces follows from [4, Proposition 2.2.5].

Lemma 1.2. *Let (X, ρ) be a K-quasimetric space. For a constant $0 < \varepsilon \leq 1$, if $K^\varepsilon \leq 2$, then there is a metric d_ε on X such that*

$$\frac{1}{4}\rho^\varepsilon(z_1, z_2) \leq d_\varepsilon(z_1, z_2) \leq \rho^\varepsilon(z_1, z_2)$$

for all $z_1, z_2 \in X$.

To state our results, we need the definition of uniform perfectness in quasimetric spaces.

Definition 1.3. A quasimetric space (Z, ρ) is called *uniformly perfect* if there is a constant $\mu \in (0, 1)$ such that for each $x \in Z$ and every $r > 0$, the set $\mathbb{B}(x, r) \setminus \mathbb{B}(x, \mu r) \neq \emptyset$ provided that $Z \setminus \mathbb{B}(x, r) \neq \emptyset$.

Uniform perfectness is a weaker condition than connectedness. Connected spaces are uniformly perfect, and those with isolated points are not. Many disconnected fractals such as the Cantor ternary set, Julia sets and the limit set of a nonelementary, finitely generated Kleinian group of $\overline{\mathbb{R}}^n$ are uniformly perfect [10]. In particular, uniform perfectness has provided a useful tool in modern research of geometric functional theory, harmonic analysis and asymptotic geometry.

For example, it is worth to mention that Buyalo and Schroeder established the quasisymmetric and quasimöbius extension theorems for visual geodesic hyperbolic spaces which possess uniformly perfect boundaries at infinity [4, Chapter 7]. In [20], the first author and Wang found several conditions under which a weakly quasimöbius map is quasimöbius in uniformly perfect quasimetric spaces. The authors in [22] investigated the invariance of doubling property under sphericalization and flattening transformations in uniformly perfect spaces. Recently, Vellis [19] proved the classical quasisymmetric Schoenflies theorem for planar uniform domains with uniformly prefect boundaries.

As the first goal of this paper, we discuss the relationship among uniform perfectness, homogeneous density, σ-density etc in the setting of quasimetric spaces. We show that all these conditions are equivalent. (Note that other notions appearing in the following results will be introduced in the body of this paper.)

Theorem 1.4. *Suppose (Z, ρ) is a K-quasimetric space contains at least three points. Then the following are quantitatively equivalent.*

1. *(Z, ρ) is μ-uniformly perfect;*
2. *(Z, ρ) is (λ_1, λ_2)-homogeneous dense;*
3. *(Z, ρ) is σ-dense;*
4. *There are numbers μ_1 and μ_2 such that $0 < \mu_1 \leq \mu_2 < 1$ and for any triple (a, c, d) of distinct points in Z, there is a point $x \in Z$ such that the cross ratio $r(a, x, c, d)$ satisfying*

$$\mu_1 \leq r(a, x, c, d) \leq \mu_2.$$

The constants $\mu, \lambda_1, \lambda_2, \sigma, \mu_1$ and μ_2 depend only on each other and K.

Remark 1.5. In metric spaces, the equivalence between (1) and (4) coincides with [7, Lemma 11.7], the equivalence between (2) and (3) follows from [9, Lemma 3.1], and the equivalence between (3) and (4)) follows from [2, Remark 3.3].

In [18], Väisälä introduced the concept of quasimöbius mappings in metric spaces and obtained the close connections among quasimöbius mappings, quasiconformal mappings and quasisymmetric mappings. See [2–4, 7, 16, 17] for more background materials in this line. Further, in [16], Tukia and Väisälä proved that every quasisymmetric mapping in uniformly prefect spaces is power quasisymmetric. As the second goal of this paper, we shall investigate the relationships of the uniform perfectness with (power) quasisymmetric mappings and (power) quasimöbius mappings, respectively. The next of our results reads as follows.

Theorem 1.6. *Suppose* $f : (Z_1, \rho_1) \to (Z_2, \rho_2)$ *is a quasimöbius mapping between K-quasimetric spaces* (Z_1, ρ_1) *and* (Z_2, ρ_2). *Then* (Z_1, ρ_1) *is* μ_1-*uniformly perfect if and only if* (Z_2, ρ_2) *is* μ_2-*uniformly perfect, where* μ_1 *and* μ_2 *are constants depending only on each other and the constant K.*

Remark 1.7. We remark that Theorem 1.6 coincides with [10, Corollary 4.6] in the setting of $\mathbb{R}^n$. Our result is new in metric space as far as we know. Moreover, as we mentioned in the beginning of this paper, there were already several applications based on Theorems 1.4 and 1.6. By using Theorems 1.4 and 1.6, the first author and Wang investigated the relations between quasimöbius mappings and weakly quasimöbius mappings in uniform perfect quasimetric spaces [20]. By using the equivalence of weakly quasisymmetric mappings and quasisymmetric mappings on uniform perfect quasimetric spaces as established in [20], Vellis further studied the quasisymmetric and bilipschitz extension properties of planer uniform domains with uniform perfect boundaries [19]. Moreover, Assev [1] applied the main results in [20] and an early version of this paper [23] to prove that in uniform perfect Ptolemaic Möbius structures, a single-valued mapping is of the BAD class if and only if it is quasimöbius. In a recent work [21], the authors studied deformations on ultrametric spaces and demonstrated the quasimöbius uniformization of symbolic Cantor sets by means of the invariance of uniform perfect sets under sphericalization and flattening transformations.

By using Theorem 1.6, we further obtain the following equivalent conditions for uniform perfectness concerning quasisymmetric and quasimöbius mappings in quasimetric spaces.

Theorem 1.8. *Suppose* (Z, ρ) *is a quasimetric space contains at least three points. If* (Z, ρ) *has no isolated point, then*

1. (Z, ρ) *is uniformly perfect if and only if every quasisymmetric mapping of* (Z, ρ) *to a quasimetric space is power quasisymmetric;*
2. (Z, ρ) *is uniformly perfect if and only if every quasimöbius mapping of* (Z, ρ) *to a quasimetric space is power quasimöbius.*

Remark 1.9. In the context of metric spaces, Theorem 1.8(1) was proved in [15, Theorems 4.13 and 6.20]. In $\mathbb{R}^n$, Theorem 1.8(2) follows from [2, Theorem 4.1]. In fact, Aseev and Trotsenko proved that if (Z, ρ) is σ-dense, then every quasimöbius mapping of (Z, ρ) is power quasimöbius by applying the conformal moduli of families of curves (see [2, Theorem 4.1]). Because the general quasimetric spaces may even have no rectifiable curve, one finds that the method of proof in [2] is no longer valid in quasimetric spaces. So we establish a new method to prove 1.8(2).

Recently, Meyer studied the relationship between Gromov hyperbolic spaces and their boundaries at infinity. He proved the invariance property of the uniform perfectness with respect to the inversions in quasimetric spaces (see [14, Theorem 7.1]). This result is one of the main results in [14], whose proof is lengthy. As an application of Theorem 1.6, we shall give a different proof to [14, Theorem 7.1] (see Theorem 6.2 below). Also, we shall discuss the uniform perfectness of a complete quasimetric space and the corresponding boundary of its hyperbolic approximation by applying Theorem 1.6 (see Theorem 6.3 below).

The organization of this paper is as follows. In the second section, we shall introduce some necessary concepts and discuss the condition in quasimetric spaces under which quasimöbius mappings and quasisymmetric mappings are the same. In the third section, some concepts will be introduced and the equivalence of uniform perfectness with homogeneous density, σ-density etc will be proved. The invariance property of uniform perfectness with respect to quasimöbius mappings will be shown in the forth section, and in the fifth section, relationships among (power) quasisymmetric mappings, (power) quasimöbius mappings and uniform perfectness will be established. In the last section, some applications of Theorem 1.6 will be given.

2. Quasimöbius Mappings and Quasisymmetric Mappings in Quasimetric Spaces

In this section, we shall introduce certain necessary notations and concepts, and prove several basic results. The main result in this section is Theorem 2.6, which concerns the condition under which power quasisymmetric mappings and power quasimöbius mappings are the same. In the following, we use the notations: $r \vee s$ and $r \wedge s$ for numbers r, s in $\mathbb{R}$, where

$$r \vee s = \max\{r, s\} \quad \text{and} \quad r \wedge s = \min\{r, s\}.$$

2.1. Cross Ratios

For four points a, b, c, d in a quasimetric space (Z, ρ), its *cross ratio* is defined as

$$r(a, b, c, d) = \frac{\rho(a, c)\rho(b, d)}{\rho(a, b)\rho(c, d)}.$$

Then we have

Proposition 2.1. (1) *For a, b, c and d in (Z, ρ),*

$$r(a, b, c, d) = \frac{1}{r(b, d, a, c)};$$

(2) *For a, b, c, d and z in (Z, ρ),*

$$r(a, b, c, d) = r(a, b, z, d)r(a, z, c, d).$$

In [3], Bonk and Kleiner introduced the following useful notation:

$$\langle a, b, c, d \rangle = \frac{\rho(a, c) \wedge \rho(b, d)}{\rho(a, b) \wedge \rho(c, d)}.$$

Moreover, they established a relation between $r(a, b, c, d)$ and $\langle a, b, c, d \rangle$ in the setting of metric spaces (see [3, Lemma 3.3]). The following result shows that this relation also holds in quasimetric spaces.

Lemma 2.2. *For any a, b, c, d in (Z, ρ), we have*

1. $\frac{1}{\theta_K(1/r(a,b,c,d))} \leq \langle a, b, c, d \rangle \leq \theta_K(r(a, b, c, d));$
2. $\theta_K^{-1}(\langle a, b, c, d \rangle) \leq r(a, b, c, d) \leq \frac{1}{\theta_K^{-1}(1/\langle a,b,c,d \rangle)},$

where $\theta_K(t) = K^2(t \vee \sqrt{t})$. (Here, we recall that K denotes the coefficient of the quasimetric space (Z, ρ).)

Proof. Obviously, we only need to prove the inequality (1) in the lemma. Let

$$\langle a, b, c, d \rangle = s \quad \text{and} \quad r(a, b, c, d) = t.$$

Without loss of generality, we may assume that $\rho(a, c) \leq \rho(b, d)$. Then we have

$$
\begin{aligned}
\rho(a, b) &\leq K(\rho(a, c) \vee \rho(c, b)) \leq K^2\big(\rho(a, c) \vee \rho(c, d) \vee \rho(d, b)\big) \\
&= K^2\big(\rho(c, d) \vee \rho(d, b)\big),
\end{aligned}
$$

and similarly,

$$\rho(c, d) \leq K^2\big(\rho(a, b)) \vee \rho(d, b)\big).$$

The combination of these two estimates leads to

$$\rho(a, b) \vee \rho(c, d) \leq K^2\big((\rho(a, b) \wedge \rho(c, d)) \vee \rho(b, d)\big) \leq K^2(1 \vee \frac{1}{s})\rho(d, b),$$

and so we get

$$t = r(a,b,c,d) = \frac{\rho(a,c)\rho(b,d)}{\left(\rho(a,b) \wedge \rho(c,d)\right)\left(\rho(a,b) \vee \rho(c,d)\right)} \geq \frac{s}{K^2(1 \vee \frac{1}{s})},$$

which implies

$$s \leq \theta_K(t) = K^2(t \vee \sqrt{t}).$$

Hence the right side inequality in (1) holds.

By Proposition 2.1, we see that the left side inequality in (1) easily follows from the right side one, and so the proof of the lemma is complete. $\square$

2.2. Quasisymmetric Mappings and Quasimöbius Mappings

Definition 2.3. Suppose η and θ are homeomorphisms from $[0, \infty)$ to $[0, \infty)$. A homeomorphism $f : (Z_1, \rho_1) \to (Z_2, \rho_2)$ is said to be

1. (a) *η-quasisymmetric* if $\rho_1(x, a) \leq t\rho_1(x, b)$ implies

$$\rho_2(x', a') \leq \eta(t)\rho_2(x', b')$$

   for all a, b, x in (Z_1, ρ_1) and $t \geq 0$, where primes mean the images of points under f, for example, $x' = f(x)$ etc;
   (b) *power quasisymmetric* if it is η-quasisymmetric, where η has the form

$$\eta(t) = M_1(t^{1/\alpha} \vee t^\alpha)$$

   for some constants $\alpha \geq 1$ and $M_1 \geq 1$.

2. (a) *θ-quasimöbius* if $r(a, b, c, d) \leq t$ implies

$$r(a', b', c', d') \leq \theta(t)$$

   for all a, b, c, d in (Z_1, ρ_1) and $t \geq 0$;
   (b) *power quasimöbius* if it is θ-quasimöbius, where θ has the form

$$\theta(t) = M_2(t^{1/\beta} \vee t^\beta)$$

   for some constants $\beta \geq 1$ and $M_2 \geq 1$.

As a direct consequence of Lemma 2.2, we have the following two results.

Lemma 2.4. *Suppose $f : (Z_1, \rho_1) \to (Z_2, \rho_2)$ is a homeomorphism between two quasimetric spaces.*

1. *If f is η-quasisymmetric, then it is θ-quasimöbius, where $\theta(t) = \frac{1}{\theta_K^{-1}(\frac{1}{\eta \circ \theta_K(t)})}$ and θ_K is from Lemma 2.2.*

2. *If f is a power quasisymmetric mapping with its control function $\eta(t) = M(t^\alpha \vee t^{\frac{1}{\alpha}})$, where $M \geq 1$ and $\alpha \geq 1$, then it is power quasimöbius with its control function $\theta(t) = M^2 K^{4(1+\alpha)}(t^{2\alpha} \vee t^{\frac{1}{2\alpha}})$.*

We remark that Lemma 2.4(1) is a generalization of [18, Theorem 3.2] in the setting of quasimetric spaces.

Next, we consider the converse of Lemma 2.4(2) in the setting of bounded quasimetric spaces. To this end, we introduce the following condition.

Definition 2.5. Suppose both (Z_1, ρ_1) and (Z_2, ρ_2) are bounded quasimetric spaces. Let $\lambda \geq 1$ be a constant. A homeomorphism $f : (Z_1, \rho_1) \to (Z_2, \rho_2)$ is said to satisfy the *λ-three-point condition* if there are points z_1, z_2, z_3 in (Z_1, ρ_1) such that

$$\rho_1(z_i, z_j) \geq \frac{1}{\lambda}\text{diam}(Z_1) \quad \text{and} \quad \rho_2(z_i', z_j') \geq \frac{1}{\lambda}\text{diam}(Z_2)$$

for all $i \neq j \in \{1, 2, 3\}$, where "diam" means "diameter".

Theorem 2.6. *Suppose that both (Z_1, ρ_1) and (Z_2, ρ_2) are bounded quasimetric spaces and that $f : (Z_1, \rho_1) \to (Z_2, \rho_2)$ satisfies the λ-three-point condition. Then f is power quasisymmetric if and only if it is power quasimöbius.*

Proof. The necessity of the theorem obviously follows from Lemma 2.4(2). In the following, we prove the sufficiency. Let $f : (Z_1, \rho_1) \to (Z_2, \rho_2)$ be a power quasimöbius mapping between two bounded quasimetric spaces, which satisfies the λ-three-point condition for some constant $\lambda \geq 1$ and points $z_1, z_2, z_3 \in Z_1$. We assume that the control function of f is

$$\theta(t) = M(t^{1/\beta} \vee t^{\beta})$$

for some constants $M \geq 1$ and $\beta \geq 1$.

To prove the power quasisymmetry of f, let x, a, b be any three points in (Z_1, ρ_1) with $\rho_1(x, a) = t\rho_1(x, b)$ with $t \geq 0$. Then we shall show that

$$\rho_2(x', a') \leq \eta(t)\rho_2(x', b'),$$

where $\eta(t) = K^{3+6\beta}M(2\lambda)^{1+2\beta}(t^{1/(2\beta)} \vee t^{2\beta})$.

It follows from the λ-three-point condition that for any $w \in Z_1$, there are $i \neq j \in \{1, 2, 3\}$ such that

$$\rho_1(w, z_i) \wedge \rho_1(w, z_j) \geq \frac{\text{diam}(Z_1)}{2K\lambda}.$$

Similarly, for any $u' \in Z_2$, there exist $m \neq n \in \{1, 2, 3\}$ such that

$$\rho_2(u', z'_m) \wedge \rho_2(u', z'_n) \geq \frac{\text{diam}(Z_2)}{2K\lambda}.$$

Therefore, there must exist $z_i \in \{z_1, z_2, z_3\}$ such that

$$\rho_1(a, z_i) \geq \frac{\text{diam}(Z_1)}{2K\lambda} \quad \text{and} \quad \rho_2(b', z'_i) \geq \frac{\text{diam}(Z_2)}{2K\lambda}.$$

Thus

$$\rho_1(a, z_i) \wedge \rho_1(x, b) \geq \frac{\rho_1(x, b)}{2K\lambda} \quad \text{and} \quad \rho_2(b', z'_i) \wedge \rho_2(x', a') \geq \frac{\rho_2(x', a')}{2K\lambda}, \tag{1}$$

from which we deduce that

$$\langle x, b, a, z_i \rangle \leq 2K\lambda\frac{\rho_1(x, a)}{\rho_1(x, b)} \quad \text{and} \quad \langle x', b', a', z'_i \rangle \geq \frac{\rho_2(x', a')}{2K\lambda\rho_2(x', b')}. \tag{2}$$

On the other hand, since f is power quasimöbius with its control function θ, we see from Lemma 2.2 that

$$\langle x', b', a', z'_i \rangle \leq \theta'(\langle x, b, a, z_i \rangle),$$

where $\theta'(t) = \theta_K \circ \theta\left(\frac{1}{\theta_K^{-1}(1/t)}\right)$ and θ_K is from Lemma 2.2. Then we deduce from (2) that

$$\frac{\rho_2(x', a')}{\rho_2(x', b')} \leq 2K\lambda\theta'(\langle x, b, a, z_i \rangle) \leq 2K\lambda\theta'\left(2K\lambda\frac{\rho_1(x, a)}{\rho_1(x, b)}\right).$$

By taking $\eta(t) = K^{3+6\beta}M(2\lambda)^{1+2\beta}(t^{1/(2\beta)} \vee t^{2\beta})$, we see from elementary computations that

$$\frac{\rho_2(x', a')}{\rho_2(x', b')} \leq \eta\left(\frac{\rho_1(x, a)}{\rho_1(x, b)}\right).$$

Hence the proof of the theorem is complete. $\square$

Lemma 2.7. *Suppose $f : (Z_1, \rho_1) \to (Z_2, \rho_2)$ and $g : (Z_2, \rho_2) \to (Z_3, \rho_3)$ are homeomorphisms.*

1. *If f is θ_1-quasimöbius and g is θ_2-quasimöbius, then $g \circ f$ is θ-quasimöbius, where $\theta = \theta_2 \circ \theta_1$;*
2. *If f is θ-quasimöbius and g is η-quasisymmetric, then $g \circ f$ is θ_1-quasimöbius, where $\theta_1(t) = \dfrac{1}{\theta_K^{-1}\left(\frac{1}{\eta \circ \theta_K \circ \theta(t)}\right)}$;*

3. *If f is power quasimöbius and g is power quasisymmetric (or power quasimöbius), then $g \circ f$ is power quasimöbius, quantitatively.*

3. Uniform Perfectness, Homogeneous Density and σ-Density

We start this section with several definitions, and then establish the invariance property of uniform perfectness with respect to quasisymmetric mappings in quasimetric spaces (Lemma 3.4 below). Based on this result, Theorem 1.4 will be proved.

3.1. Homogeneous Density and σ-Density

Definition 3.1. Suppose $\{x_i\}_{i\in\mathbb{Z}}$ denotes a sequence of points in a quasimetric space (Z, ρ) with $a \neq x_i \neq b$.

(i) If $x_i \to a$ as $i \to -\infty$ and $x_i \to b$ as $i \to +\infty$, then $\{x_i\}$ is called a *chain* joining a and b; further, if there is a constant $\sigma > 1$ such that for all i,

$$|\log r(a, x_i, x_{i+1}, b)| \leq \log \sigma,$$

then $\{x_i\}$ is called a σ-*chain*.

(ii) (Z, ρ) is said to be σ-*dense* $(\sigma > 1)$ if any pair of points in (Z, ρ) can be joined by a σ-chain.

We remark that a σ-dense space does not contain any isolated point, and also, every σ-dense space must be σ'-dense for any $\sigma' \geq \sigma$.

Definition 3.2. A quasimetric space (Z, ρ) is said to be *homogeneously dense*, if there are constants λ_1 are λ_2 with $0 < \lambda_1 \leq \lambda_2 < 1$ such that for each pair of points $a, b \in Z$, there is $x \in Z$ satisfying

$$\lambda_1 \rho(a, b) \leq \rho(a, x) \leq \lambda_2 \rho(a, b).$$

To emphasize the parameters, we also say that (Z, ρ) is (λ_1, λ_2)-homogeneously dense.

Lemma 3.3. (1) *If a quasimetric space is (λ_1, λ_2)-homogeneously dense, then it is $(\lambda_1^n, \lambda_2^n)$-homogeneously dense for any $n \in \mathbb{N}^+ = \{1, 2, \ldots\}$.*
(2) *Suppose that both (Z_1, ρ_1) and (Z_2, ρ_2) are quasimetric spaces and that $f : (Z_1, \rho_1) \to (Z_2, \rho_2)$ is η-quasisymmetric. If (Z_1, ρ_1) is (λ_1, λ_2)-homogeneously dense, then (Z_2, ρ_2) is (μ_1, μ_2)-homogeneously dense, where both μ_1 and μ_2 depend only on λ_1, λ_2 and η.*

We remark that, in the setting of metric spaces, Lemma 3.3 coincides with [16, Lemma 3.9]. Also the proof of Lemma 3.3 is similar to that of [16, Lemma 3.9]. We omit it here.

3.2. The Invariance Property of Uniform Perfectness with Respect to Quasisymmetric Mappings

It is known that uniform perfectness is an invariant with respect to quasisymmetric mappings in metric spaces (cf. [7, Exercise 11.2]). In the following, we prove that this fact is still valid in quasimetric spaces.

Lemma 3.4. *Let $f : (Z_1, \rho_1) \to (Z_2, \rho_2)$ be η-quasisymmetric, where both (Z_i, ρ_i) $(i = 1, 2)$ are K-quasimetric. Then (Z_1, ρ_1) is μ_1-uniformly perfect if and only if (Z_2, ρ_2) is μ_2-uniformly perfect, where μ_1 and μ_2 depend only each other and K, η.*

Proof. Since the inverse of a quasisymmetric mapping is also quasisymmetric, to prove the lemma, it suffices to show that the uniform perfectness of (Z_1, ρ_1) implies the uniform perfectness of (Z_2, ρ_2).

Now, we assume that (Z_1, ρ_1) is μ-uniformly perfect for some $\mu \in (0, 1)$. Then we shall show that (Z_2, ρ_2) is uniformly perfect too. To reach this goal, it suffices to find a constant $\mu' \in (0, 1)$ such that for any $z' \in Z_2$ and $r > 0$, if $Z_2 \setminus \mathbb{B}(z', r) \neq \emptyset$, then there is a point u' in (Z_2, ρ_2) such that

$$\mu' r \leq \rho_2(z', u') < r.$$

By the assumption $Z_2 \setminus \mathbb{B}(z', r) \neq \emptyset$, we see that there is a point $u'_0 \in Z_2$ such that

$$\rho_2(z', u'_0) \geq r. \tag{3}$$

Choose $0 < \alpha < 1$ small enough such that $\eta(\alpha) < 1$. Then there exists an integer k such that

$$\eta(\alpha)^k \rho_2(z', u_0') < r \leq \eta(\alpha)^{k-1} \rho_2(z', u_0'). \tag{4}$$

Since (Z_1, ρ_1) is μ-uniformly perfect and $u_0 \in Z_1 \backslash \mathbb{B}(z, \alpha\rho_1(z, u_0))$, we see that $\mathbb{B}(z, \alpha\rho_1(z, u_0)) \backslash \mathbb{B}(z, \mu\alpha\rho_1(z, u_0)) \neq \emptyset$. So there is a point $u_1 \in Z_1$ such that

$$\mu\alpha\rho_1(z, u_0)) \leq \rho_1(z, u_1) < \alpha\rho_1(z, u_0).$$

Hence

$$\mu'\rho_2(z', u_0') \leq \rho_2(z', u_1') < \eta(\alpha)\rho_2(z', u_0'), \tag{5}$$

where $\mu' = \frac{1}{\eta(\frac{1}{\mu\alpha})}$.

If $\rho_2(z', u_1') < r$, then (3) and (5) lead to

$$\mu'r \leq \rho_2(z', u_1') < r.$$

At present, we can take $u' = u_1'$.

Now, we consider the case:

$$\rho_2(z', u_1') \geq r. \tag{6}$$

Since (Z_1, ρ_1) is μ-uniformly perfect and $u_1 \in Z_1 \backslash \mathbb{B}(z, \alpha\rho_1(z, u_1))$, we see that $\mathbb{B}(z, \alpha\rho_1(z, u_1)) \backslash \mathbb{B}(z, \mu\alpha\rho_1(z, u_1)) \neq \emptyset$. So there is a point $u_2 \in Z_1$ such that

$$\mu\alpha\rho_1(z, u_1) \leq \rho_1(z, u_2) < \alpha\rho_1(z, u_1).$$

Hence

$$\mu'\rho_2(z', u_1') \leq \rho_2(z', u_2') < \eta(\alpha)\rho_2(z', u_1') < \eta(\alpha)^2\rho_2(z', u_0').$$

If $\rho_2(z', u_2') < r$, then (6) leads to

$$\mu'r \leq \rho_2(z', u_2') < r.$$

Hence, we can take $u' = u_2'$.

Next, we consider the case:

$$\rho_2(z', u_2') \geq r.$$

By repeating this procedure, we can reach the following conclusion: There is $u_k' \in Z_2$ such that

1. For any $i \in \{1, \ldots, k-1\}$, $\rho_2(z', u_i') \geq r$;
2. $\mu'\rho_2(z', u_{k-1}') \leq \rho_2(z', u_k') < \eta(\alpha)^k\rho_2(z', u_0')$.

Then (4) guarantees that

$$\mu'r \leq \rho_2(z', u_k') < r.$$

By taking $u' = u_k'$, we finish the proof. $\square$

3.3. The Proof of Theorem 1.4

By applying Lemmas 1.2, 3.3 and 3.4, together with Lemma 4.1 below, we see that the equivalence between (1) and (2) easily follows from [7, Lemma 11.7], and the equivalence between (2) and (3)) follows from [9, Lemma 3.1]. Hence, to finish the proof, it remains to show the equivalence between (3) and (4), whose proof is as follows.

$(3) \Rightarrow (4)$ Assume that (Z, ρ) is σ-dense. Let a, c, d be three distinct points in (Z, ρ). Then there is a σ-chain $\{x_i\}_{i\in\mathbb{Z}}$ in (Z, ρ) joining a and d such that

$$\frac{1}{\sigma} \leq r(a, x_i, x_{i+1}, d) \leq \sigma. \tag{7}$$

To prove this implication, it suffices to show that there is an integer k such that

$$\frac{1}{2\sigma^2} \le r(a, x_{k-1}, c, d) \le \frac{1}{2\sigma}. \tag{8}$$

For the proof, we let

$$k = \inf\{i \in \mathbb{Z} : r(a, x_i, c, d) < \frac{1}{2\sigma^2}\}.$$

Since $\lim_{i \to +\infty} r(a, x_i, c, d) = 0$ and $\lim_{i \to -\infty} r(a, x_i, c, d) = +\infty$, we see that k is finite, and so

$$r(a, x_k, c, d) < \frac{1}{2\sigma^2} \quad \text{and} \quad r(a, x_{k-1}, c, d) \ge \frac{1}{2\sigma^2}.$$

Then (7) implies

$$r(a, x_{k-1}, c, d) = r(a, x_k, c, d) r(a, x_{k-1}, x_k, d) < \frac{1}{2\sigma}.$$

Hence (8) is true, and thus the implication from (3) to (4) is proved.

$(4) \Rightarrow (3)$ For any two distinct points a and $d \in Z$, let c be a fixed point in (Z, ρ), which is different from a and d. Then there is a point $x_0 \in Z$ such that

$$\mu_1 \le r(a, x_0, c, d) \le \mu_2,$$

where $0 < \mu_1 \le \mu_2 < 1$.

By repeating this procedure, we can find a sequence $\{x_i\}_{i \in \mathbb{N}^+}$ in (Z, ρ) such that

$$\mu_1 \le r(a, x_i, x_{i-1}, d) \le \mu_2.$$

Then

$$\mu_1 \le \frac{r(a, x_i, c, d)}{r(a, x_{i-1}, c, d)} = r(a, x_i, x_{i-1}, d) \le \mu_2,$$

which implies that

$$\mu_1^{i+1} \le r(a, x_i, c, d) \le \mu_2^{i+1},$$

and so $x_i \to d$ as $i \to +\infty$.

Similarly, we know that there exists $\{x_{-i}\}_{i \in \mathbb{N}_+}$ in (Z, ρ) such that

$$\mu_1 \le r(d, x_{-i}, x_{1-i}, a) = r(a, x_{1-i}, x_{-i}, d) \le \mu_2$$

and

$$\mu_1 \mu_2^{1-i} \le r(a, x_{-i}, c, d) \le \mu_1^{1-i} \mu_2.$$

Then $x_{-i} \to a$ as $i \to +\infty$, and hence we have proved that (Z, ρ) is $\frac{1}{\mu_1}$-dense. $\square$

4. The Invariance Property of Uniform Perfectness with Respect to Quasimöbius Mappings

The aim of this section is to prove Theorem 1.6. To this end, by Theorem 1.4, it suffices to show the following lemma.

Lemma 4.1. *Let* $f : (Z_1, \rho_1) \to (Z_2, \rho_2)$ *be* θ-*quasimöbius, where both* (Z_i, ρ_i) *(i = 1, 2) are quasimetric. Then* (Z_1, ρ_1) *is* σ-*dense if and only if* (Z_2, ρ_2) *is* σ'-*dense with* σ *and* σ' *depend on each other and* θ.

Proof. Since the inverse of a θ-quasimöbius mapping is θ'-quasimöbius with $\theta'(t) = \frac{1}{\theta^{-1}(1/t)}$, to prove this lemma, it suffices to show that (Z_2, ρ_2) is σ'-dense under the assumption "(Z_1, ρ_1) being σ-dense", where $\sigma > 1$ and σ' depends only on σ and θ. For this, we only need to check that for each pair of points a', b' in (Z_2, ρ_2), there is a σ'-chain in (Z_2, ρ_2) joining them. Now, we assume that $\{x_i\}_{i\in\mathbb{Z}}$ is a σ-chain in (Z_1, ρ_1) joining the points a and b with

$$\frac{1}{\sigma} \leq r(a, x_i, x_{i+1}, b) \leq \sigma.$$

Then for all i, we have

$$\frac{1}{\theta(\sigma) + 1} \leq r(a', x_i', x_{i+1}', b') \leq \theta(\sigma) + 1,$$

which shows that $\{x_i'\}_{i\in\mathbb{Z}}$ is a σ'-chain in (Z_2, ρ_2) joining a' and b' with $\sigma' = \theta(\sigma) + 1$. $\quad\square$

5. Uniform Perfectness, (Power) Quasisymmetric Mappings and (Power) Quasimöbius Mappings

This section is devoted to the proof of Theorem 1.8 concerning the relationships among uniform perfectness, (power) quasisymmetric mappings and (power) quasimöbius mappings in quasimetric spaces. It consists of two subsections. In the first subsection, we shall prove a relationship among uniform perfectness, quasisymmetric mappings and power quasisymmetric mappings, i.e. Theorem 1.8(1), and in the second subsection, the proof of a relationship among uniform perfectness, quasimöbius mappings and power quasimöbius mappings, i.e. Theorem 1.8(2), will be presented.

5.1. The Proof of Theorem 1.8(1)

Let $\varepsilon \in (0, 1)$ be a constant such that $K^\varepsilon \leq 2$. Then it follows from Lemma 1.2 that there exists a metric d_ε (briefly d in the following) in Z such that

$$\frac{1}{4}\rho^\varepsilon(z_1, z_2) \leq d(z_1, z_2) \leq \rho^\varepsilon(z_1, z_2)$$

for all $z_1, z_2 \in Z$. Let id denote the identity mapping from (Z, ρ) to (Z, d), i.e.,

$$\text{id} : (Z, \rho) \rightarrow (Z, d).$$

Obviously, id is power quasisymmetric with its control function $\eta(t) = 4(t^\varepsilon \vee t^{\frac{1}{\varepsilon}})$.

We first assume that (Z, ρ) is uniformly perfect, and consider a quasisymmetric mapping f defined in (Z, ρ). It follows from the power quasisymmetry of id and Lemma 3.4 that (Z, d) is uniformly perfect, and so Theorem 1.4 implies that (Z, d) is (λ_1, λ_2)-homogeneously dense for constants λ_1 and λ_2 with $0 < \lambda_1 \leq \lambda_2 < 1$. Since the composition of two quasisymmetric mappings is still a quasisymmetric mapping, we have $f \circ \text{id}^{-1}$ is quasisymmetric in (Z, d). Moreover, we see from [16, Corollary 3.12] that every quasisymmetric mapping defined in a homogeneously dense space is a power quasisymmetric mapping, so we get that $f \circ \text{id}^{-1}$ is power quasisymmetric, which implies that f itself is power quasisymmetric.

Next, we assume that every quasisymmetric mapping of (Z, ρ) is power quasisymmetric. Then we see that for any quasisymmetric mapping g in (Z, d), $g \circ \text{id}$ is power quasisymmetric in (Z, ρ), and so g itself is power quasisymmetric. Hence, by [15, Theorems 4.13 and 6.20], (Z, d) is uniformly perfect. Since id is power quasisymmetric, it follows from Lemma 3.4 that (Z, ρ) is uniformly perfect. $\quad\square$

5.2. The Proof of Theorem 1.8(2)

We start this subsection with the following result in metric spaces.

Lemma 5.1. *Suppose (Z, d) is a metric space with no isolated points. Then the following statements are quantitatively equivalent.*

1. *(Z, d) is uniformly perfect;*

2. *every quasimöbius mapping of* (Z, d) *is power quasimöbius.*

Proof. By [2, Theorem 3.2], we only need to prove the implication from (2) to (1). Assume that every quasimöbius mapping in (Z, d) is a power quasimöbius mapping. To prove the uniform perfectness of (Z, d), we divide the proof into two cases.

Case 5.2. (Z, d) *is unbounded.*

We shall apply Theorem 1.8(1) to finish the proof in this case. For this, we assume that f is a quasisymmetric mapping in (Z, d). Then Lemma 2.4 implies that f is quasimöbius, so it follows from condition (2) that f is power quasimöbius. And further, [18, Theorem 3.10] guarantees that $f(z) \to \infty$ as $z \to \infty$. Again, it follows from [18, Theorem 3.10] that f is power quasisymmetric, and so Theorem 1.8(1) ensures that (Z, d) is uniformly perfect. Hence the lemma is true in this case.

Case 5.3. (Z, d) *is bounded.*

By the Kuratowski embedding theorem [12], we may assume that Z is a subset of a Banach space E. By performing an auxiliary translation, further, we assume that $0 \in Z$. Let

$$u(x) = \frac{x}{|x|^2}$$

be the inversion in $\dot{E} = E \cup \{\infty\}$. Then, clearly, $u(Z)$ is unbounded. By [18, §1.6], u is θ-quasimöbius, where $\theta(t) = 81t$, and obviously, it is power quasimöbius. To prove that (Z, d) is uniformly perfect, by Theorem 1.6, it suffices to show that $u(Z)$ is uniformly perfect. Again, we shall apply Theorem 1.8(1) to reach this goal. For this, we assume that g is quasisymmetric in $u(Z)$. Once more, by Lemma 2.4, g is quasimöbius. Then $g \circ u$ is quasimöbius in (Z, d), which implies that $g \circ u$ is power quasimöbius, and thus we deduce from Lemma 2.7(3) that g itself is power quasimöbius. So we infer from [18, Theorem 3.10] that g is power quasisymmetric. Then it follows from Theorem 1.8(1) that $u(Z)$ is uniformly perfect. Hence the proof of the lemma is complete. $\square$

The proof of Theorem 1.8(2). Let id : $(Z, \rho) \to (Z, d)$ be the same as that in the proof of Theorem 1.8(1). Then id is power quasisymmetric with its control function $\eta(t) = 4(t^\epsilon \vee t^{\frac{1}{\epsilon}})$, where $\epsilon \in (0, 1)$.

Assume now that (Z, ρ) is uniformly perfect, and so is (Z, d) by Lemma 3.4. For any quasimöbius mapping f in (Z, ρ), it follows from Lemma 5.1 that $f \circ$ id is power quasimöbius, and so is f itself by Lemma 2.7. This shows that the necessity in Theorem 1.8(2) is true.

To prove the sufficiency in Theorem 1.8(2), it suffices to prove the uniform perfectness of (Z, d) under the assumption that every quasisymmetric mapping in (Z, ρ) is power quasimöbius. By Lemma 5.1, we only need to show the power quasisymmetry of each quasisymmetric mapping in (Z, d). This fact easily follows from Lemma 2.7. Hence the proof of Theorem 1.8(2) is complete. $\square$

6. Applications

The aim of this section is twofold. First, as an application of Theorem 1.6, we will give a different proof to [14, Theorem 7.1]. Second, we shall apply Theorem 1.6 to discuss the uniform perfectness of a complete quasimetric space and the corresponding boundary of its hyperbolic approximation.

6.1. Application I

We begin this subsection with a definition.

Definition 6.1. ([4, Page 57]) For $p \in (Z, \rho)$, let

$$\rho_p(x, y) = \frac{r^2 \rho(x, y)}{\rho(x, p) \rho(y, p)}$$

for all $x, y \in Z \setminus \{p\}$. Then ρ_p is said to be *the inversion* with respect to ρ centered at p with radius $r > 0$.

Theorem 6.2. ([14, Theorem 7.1]) *For any $p \in Z$, if $(Z \setminus \{p\}, \rho)$ is a uniformly perfect quasimetric space, then $(Z \setminus \{p\}, \rho_p)$ is a uniformly perfect quasimetric space.*

Proof. First, if $(Z \setminus \{p\}, \rho)$ is a K-quasimetric space, by [4, Proposition 5.3.6], we know that $(Z \setminus \{p\}, \rho_p)$ is a K^2-quasimetric space. Then a direct computation gives that the identity mapping from $(Z \setminus \{p\}, \rho)$ to $(Z \setminus \{p\}, \rho_p)$ is θ-quasimöbius with $\theta(t) = t$. Hence the proof of the theorem easily follows from Theorem 1.6. $\square$

6.2. Application II

Let $\mathrm{Hyp}_r(Z, \rho)$ denote the hyperbolic approximation of (Z, ρ) with parameter r, $\partial_\infty^{a,o} \mathrm{Hyp}_r(Z, \rho)$ the boundary at infinity of $\mathrm{Hyp}_r(Z, \rho)$ with respect to the quasimetric $a^{-(\cdot|\cdot)_o}$ based at $o \in \mathrm{Hyp}_r(Z, \rho)$ with $a > 1$, and $\partial_\infty^{a',b} \mathrm{Hyp}_r(Z, \rho)$ the boundary at infinity of $\mathrm{Hyp}_r(Z, \rho)$ with respect to the quasimetric $a'^{-(\cdot|\cdot)_\omega}$ based at ω with $a' > 1$, where b is a Busemann function based at ω. See [11, §3] for their precise definitions.

Theorem 6.3. *Suppose (Z, ρ) is a complete quasimetric space and $r \in (0, 1)$. Then the following are quantitatively equivalent.*

(a) *(Z, ρ) is uniformly perfect;*
(b) *$\partial_\infty^{a,o} \mathrm{Hyp}_r(Z, \rho)$ is uniformly perfect;*
(c) *$\partial_\infty^{a',b} \mathrm{Hyp}_r(Z, \rho)$ is uniformly perfect.*

Proof. First, by [4, Proposition 2.2.9 and 5.2.8], we know that the identity mapping from $\partial_\infty^{a',b} \mathrm{Hyp}_r(Z, \rho)$ to $\partial_\infty^{a,o} \mathrm{Hyp}_r(Z, \rho)$ is quasimöbius, and so Theorem 1.6 implies the quantitative equivalence of (b) and (c).

To finish the proof of the theorem, we divide the discussions into two cases. The first case is that (Z, ρ) is unbounded. By [11, Theorem 3], we know that for any Busemann function $b \in B(\omega)$, the identity mapping from $\partial_\infty^{a',b} \mathrm{Hyp}_r(Z, \rho)$ to (Z, ρ) is bi-Hölder, and thus Theorem 1.6 guarantees the quantitative equivalence of (a) and (c). For the remainder case, that is, (Z, ρ) is bounded, again, by [11, Theorem 3], we see that the identity mapping from $\partial_\infty^{a,o} \mathrm{Hyp}_r(Z, \rho)$ to (Z, ρ) is bi-Hölder. Once more, it follows from Theorem 1.6 that (a) and (b) are quantitatively equivalent. Hence the proof of this theorem is complete. $\square$

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