Filomat 34:6 (2020), 1959–1973
https://doi.org/10.2298/FIL2006959C

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Painlevé σ-Equations s_1, s_2, s_4 and Their Value Distribution

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Abstract. Local solutions of Painlevé equations P_1, P_2, P_4, as well as of the equations S_1, S_2, S_4 satisfied by their Hamiltonians, can be extended to functions meromorphic in $\mathbb{C}$. This way they become a point of interest for value distribution theory. Distribution of values of solutions of P_1, P_2 and P_4 is already well described. In the paper we discuss mostly S_1, S_2 and S_4 in this context. In particular, we pay attention to deficient, asymptotic and ramified values of solutions of these equations.

1. Introduction

Value distribution and growth properties of meromorphic solutions of the main Painlevé equations are quite well recognized. The Painlevé σ-equations are the second order differential equations fulfilled by the Hamiltonians of Painlevé equations [6, 19]. In this paper we concentrate on the equations S_1, S_2, and S_4 related to the first, second and fourth Painlevé equations, and compare their properties with those of P_1, P_2 and P_4.

One of the prominent features of the Painlevé equations is the possibility to represent them as a Hamiltonian system

$$\frac{dq}{dz} = \frac{\partial H}{\partial p}, \quad \frac{dp}{dz} = -\frac{\partial H}{\partial q}$$

with polynomial or rational Hamiltonians [15, 18]. The first Painlevé equation

$$\mathbf{P_1}: \qquad f'' = 6f^2 + z$$

can be represented as a Hamiltonian system with the Hamiltonian

$$H_1(p, q, z) = \frac{1}{2}p^2 - 2q^3 - zq$$

and

$$\begin{cases} q' = p \\ p' = 6q^2 + z \end{cases}, \tag{1}$$

2010 *Mathematics Subject Classification*. Primary 30D35; Secondary 34M05, 34M55
Keywords. Painlevé equation, meromorphic function, value distribution
Received: 19 February 2019; Revised: 14 January 2020; Accepted: 18 April 2020
Communicated by Miodrag Mateljević
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where q fulfills P_1. If we put

$$\sigma_1(z) := H_1(p, q, z), \tag{2}$$

then

$$\begin{cases} q &= -\sigma_1' \\ p &= -\sigma_1'' \end{cases},$$

and the function σ_1 fulfills the second order equation

$$\mathbf{S_1}: \qquad (\sigma'')^2 + 4(\sigma')^3 + 2z\sigma' - 2\sigma = 0.$$

Conversely, if σ solves S_1, then $q = -\sigma',\ p = -\sigma''$ solve (1).

The second Painlevé equation

$$\mathbf{P_2}(\alpha): \qquad f'' = 2f^3 + zf + \alpha,$$

where α is a fixed complex parameter, can be represented with the Hamiltonian

$$H_2(p, q, z) := \frac{1}{2}p^2 - (q^2 + \frac{1}{2}z)p - (\alpha + \frac{1}{2})q.$$

Here

$$\begin{cases} q' = p - q^2 - \dfrac{1}{2}z \\ p' = 2qp + \alpha + \dfrac{1}{2} \end{cases}.$$

Then q satisfies P_2 and p satisfies the equation

$$ff'' = \frac{1}{2}(f')^2 + 2f^3 - zf^2 - \frac{1}{2}(\alpha + \frac{1}{2})^2,$$

known as P_{34}. The function defined as

$$\sigma_2(z) := H_2(p, q, z). \tag{3}$$

is then a solution of the equation

$$\mathbf{S_2}(\alpha): \qquad (\sigma'')^2 + 4(\sigma')^3 + 2\sigma'(z\sigma' - \sigma) = \frac{1}{4}(\alpha + \frac{1}{2})^2,$$

where α is a complex parameter. Conversely, if σ is a solution of S_2, then

$$\begin{cases} q = \dfrac{4\sigma'' + 2\alpha + 1}{8\sigma'}, \\ p = -2\sigma', \end{cases}$$

are solutions of P_2 and P_{34}, respectively [19].

The fourth Painlevé equation is

$$\mathbf{P_4}(\alpha, \beta): \qquad f'' = \frac{f'^2}{2f} + \frac{3f^3}{2} + 4zf^2 + 2(z^2 - \alpha)f + \frac{\beta}{f},$$

where $\alpha,\ \beta$ are arbitrary complex parameters. If we represent P_4 as a Hamiltonian system with the Hamiltonian

$$H_4(p, q, z) := 2qp^2 - (q^2 + 2zq + 2\theta_0)p + \theta_\infty q,$$

we get

$$\begin{cases} q' = 4qp - q^2 - 2zq - 2\theta_0, \\ p' = -2p^2 + 2qp + 2zp - \theta_\infty \end{cases}.$$

Thus, eliminating p in the system we obtain that q satisfies $P_4(\alpha,\beta)$ with $\alpha = 1 - \theta_0 + 2\theta_\infty$, $\beta = -2\theta_0^2$, and by eliminating q we obtain, that $-2p$ satisfies $P_4(\alpha,\beta)$ with $\alpha = 2\theta_0 - \theta_\infty - 1$, $\beta = -2\theta_\infty^2$. The function

$$\sigma_4(z) := H_4(p,q,z) \tag{4}$$

satisfies the equation

$$\mathbf{S_4}(\theta_0,\theta_\infty): \qquad (\sigma'')^2 - 4(z\sigma' - \sigma)^2 + 4\sigma'(\sigma' + 2\theta_0)(\sigma' + 2\theta_\infty) = 0,$$

and conversely, if σ is a solution of S_4, then

$$q = \frac{\sigma'' - 2z\sigma' + 2\sigma}{2(\sigma' + 2\theta_\infty)}, \qquad p = \frac{\sigma'' + 2z\sigma' - 2\sigma}{4(\sigma' + 2\theta_0)}$$

are solutions of the Hamiltonian system for P_4 [19].

2. Value distribution and σ-equations

We apply the following notations standard in value distribution theory [10, 14]. By $N(r,f)$ and $\overline{N}(r,f)$ we denote Nevanlinna's integrated functions counting poles of a meromorphic function f. Then

$$N(r,a,f) := N(r,\frac{1}{f-a}), \qquad \overline{N}(r,a,f) := \overline{N}(r,\frac{1}{f-a})$$

count a-points with, and without multiplicity, respectively. We also put $N_1(r,a,f) = N(r,a,f) - \overline{N}(r,a,f)$. Next,

$$m(r,f) = m(r,\infty,f) \qquad \text{and} \qquad m(r,a,f) := m(r,\frac{1}{f-a})$$

denote mean proximity functions, $T(r,f)$ the characteristic function.

We define $\delta(a,f)$, the deficiency of f at a value $a \in \overline{\mathbb{C}}$, by the formula

$$\delta(a,f) = \liminf_{r\to\infty} \frac{m(r,a,f)}{T(r,f)} = 1 - \limsup_{r\to\infty} \frac{N(r,a,f)}{T(r,f)}$$

and $\vartheta(a,f)$, the index of multiplicity of a value a, by the formula

$$\vartheta(a,f) = \liminf_{r\to\infty} \frac{N_1(r,a,f)}{T(r,f)}.$$

If $\delta(a,f) > 0$, then we say that the value a is deficient (in the sense of Nevanlinna), and if $\vartheta(a,f) > 0$ we call a a ramified value of f. Let us remind that it follows from the first and the second main theorems of Nevanlinna that the set of deficient values of a meromorphic function f is at most countable and the following relations are true:

$$0 \le \delta(a,f) + \vartheta(a,f) \le 1,$$
$$\sum_{a\in\overline{\mathbb{C}}} (\delta(a,f) + \vartheta(a,f)) \le 2.$$

The order and the lower order of a meromorphic function f are defined by

$$\varrho(f) := \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r}, \qquad \mu(f) := \liminf_{r\to\infty} \frac{\log T(r,f)}{\log r}.$$

If $\varrho(f) = \mu(f)$ then f is called a function of regular growth.

2.1. Main results

Since 1913–14 and the papers of Boutroux it has been known that the solutions of P_1 are transcendental meromorphic functions of regular growth, of finite growth order $\varrho(f) = 5/2$. The rigorous proofs were given by Steinmetz in 2002 [29] and, independently, by Shimomura in 2001 [23] and in 2003 [24]. Thus P_1 does not admit rational solutions. Moreover, the solutions do not have values deficient in the sense of Nevanlinna [22] or even in the sense of Petrenko [3].

The following result holds for solutions of S_1.

Theorem 2.1. *The solutions of S_1 are transcendental meromorphic functions of finite order $\varrho(\sigma) = \frac{5}{2}$. Moreover,*

$$m(r, \sigma) = O(\log r) \qquad (r \to \infty)$$

and for all $a \in \mathbb{C}$,

$$m(r, a, \sigma) = O(\log r) \qquad (r \to \infty).$$

Thus for all $a \in \overline{\mathbb{C}}$, the deficiency $\delta(a, \sigma) = 0$ and no value is deficient.

Theorem 2.1 means, in particular, that S_1 does not admit entire solutions.

Solutions of P_2 and P_4 are meromorphic functions, for recent proofs see: Hinkkanen and Laine in [11] or Steinmetz in [28]. Steinmetz in [29] and Shimomura in [23, 24] also proved that solutions of P_2 are of order $\varrho(f) \leq 3$ and solutions of P_4, of $\varrho(f) \leq 4$, while the respective lower bounds for the order of growth of transcendental solutions $\varrho(f) \geq 3/2$ and $\varrho(f) \geq 2$ were shown by Hinkkanen and Laine in [12] and independently, by Shimomura in [25] and by Steinmetz in [30] (for an overview see also: [32]). It follows from the relationship with P_2 that the solutions of P_{34} are also meromorphic and of finite order of growth $\varrho(f) \leq 3$ [2]. Apart from solutions of order 3, equation P_2 may admit rational solutions and solutions of order 3/2 for some values of the parameter of the equation, hence the same applies to solutions of P_{34}. The estimates of deficiencies for transcendental solutions of P_2 were given by Schubart in 1956 [21] and by Schubart and Wittich in 1957 [22]. For a solution f of $P_2(\alpha)$ with $\alpha \neq 0$, we have $\delta(a, f) = 0$ for every value a. In case of $P_2(0)$, the same holds both for all non-zero values and for the value zero, which has been recently shown by Steinmetz [31]. Estimates of deficiencies for P_{34} were shown by Filipuk and Ciechanowicz in [1]. Now we can formulate the result concerning the σ-function of P_2.

Theorem 2.2. *The solutions of $S_2(\alpha)$ for any fixed $\alpha \in \mathbb{C}$ are meromorphic functions of finite order $\varrho(\sigma) \leq 3$ fulfilling the condition*

$$m(r, \sigma) = O(\log r) \qquad (r \to \infty).$$

If $\alpha \neq -\frac{1}{2}$, then the condition

$$m(r, a, \sigma) = O(\log r) \qquad (r \to \infty)$$

holds for all complex values a. If $\alpha = -\frac{1}{2}$ then

$$m(r, a, \sigma) = O(\log r) \qquad (r \to \infty)$$

holds for all $a \neq 0$. Thus for a transcendental solution σ we have $\delta(a, \sigma) = 0$ for all $a \in \overline{\mathbb{C}}$ if $\alpha \neq -\frac{1}{2}$, and for all $a \in \overline{\mathbb{C}} \setminus \{0\}$ if $\alpha = -\frac{1}{2}$.

It should be mentioned that in [31] Steinmetz proved that $\delta(0, f) = 0$ for transcendental solutions of $P_2(0)$, which means that zero is not a deficient value of a solution of $P_2(\alpha)$, regardless of the choice of parameter α. It seems plausible that the same is true both for solutions of P_{34} and S_2.

Apart from solutions of order 4, equation P_4 may admit rational solutions and solutions of order 2 for some values of the parameters. The estimates of deficiencies for transcendental solutions of P_4 were originally given by Steinmetz in 1982 [27]. The following result concerns the σ-form of P_4.

Theorem 2.3. *The solutions of $S_4(\theta_0, \theta_\infty)$ are meromorphic functions of finite order $\varrho(\sigma) \leq 4$ fulfilling the condition*

$$m(r, \sigma) = O(\log r) \qquad \text{and} \qquad m(r, a, \sigma) = O(\log r) \qquad (r \to \infty)$$

for all values $a \in \mathbb{C} \setminus \{0\}$. Thus if σ is a transcendental solution of $S_4(\theta_0, \theta_\infty)$, then for all non-zero values $\delta(a, \sigma) = 0$.

Zero may actually be a deficient value in case of certain solutions of S_4, which the following example shows.

Example 2.4. *If we consider $S_4(1, 0)$ or $S_4(0, 1)$ we get the equation*

$$(\sigma'')^2 - 4(z\sigma' - \sigma)^2 + 4\sigma'^2(\sigma' + 1) = 0.$$

This equation is satisfied by solutions of the Riccati equation

$$\sigma' = -(\sigma^2 + 2z\sigma).$$

An example of such a solution is $\sigma_0(z) = \left(e^{z^2} \int_0^z e^{-t^2} dt\right)^{-1}$. Here the order $\varrho(\sigma_0) = 2$. Therefore, similarly as in the case of P_4, solutions of S_4 of order 2 are possible. Let us also notice that $\delta(0, \sigma_0) = 1$ [14, 27].

The quantity

$$\beta(a, f) = \liminf_{r \to \infty} \frac{\mathcal{L}(r, a, f)}{T(r, f)}$$

is called *deviation of a meromorphic function with respect to $a \in \overline{\mathbb{C}}$*, where

$$\mathcal{L}(r, a, f) := \begin{cases} \max\limits_{|z|=r} \log^+ |f(z)| & \text{for} \quad a = \infty, \\ \max\limits_{|z|=r} \log^+ \left|\frac{1}{f(z)-a}\right| & \text{for} \quad a \neq \infty. \end{cases}$$

If $\beta(a, f) > 0$ we say, that a is *deficient in the sense of Petrenko.* For all $a \in \overline{\mathbb{C}}$ the inequality

$$\delta(a, f) \leq \beta(a, f)$$

follows easily from the respective definitions. Thus for each value deficient in the sense of Nevanlinna also $\beta(a, f) > 0$. For meromorphic functions of finite lower order we have upper bounds of deviations similar to the deficiency relations following from the first and second main theorems of Nevanlinna. Namely, it was proved by Petrenko in [20] that for a function f of finite lower order μ,

$$\beta(a, f) \leq B(\mu) := \begin{cases} \frac{\pi\mu}{\sin \pi\mu} & \text{if} \quad \mu \leq 0.5\,, \\ \pi\mu & \text{if} \quad \mu > 0.5 \end{cases}$$

and the set of deficient values in this sense is at most countable. Marchenko and Shcherba in [17] proved that

$$\sum_{a \in \overline{\mathbb{C}}} \beta(a, f) \leq 2B(\mu).$$

Both estimates are sharp. In general, the sets of deficient values in the sense of Nevanlinna and in the sense of Petrenko may differ even in case of functions of finite order and of regular growth.

A value $a \in \overline{\mathbb{C}}$ is *an asymptotic value* of a meromorphic function f if there exists a continuous curve $\Gamma \subset \mathbb{C}$,

$$\Gamma : \ z = z(t), \ 0 \leq t < \infty, \qquad z(t) \to \infty \ \text{ for } \ t \to \infty,$$

such that

$$\lim_{z\to\infty, z\in\Gamma} f(z) = \lim_{t\to\infty} f(z(t)) = a.$$

We call such a pair $\{a, \Gamma\}$ *an asymptotic spot of* f. Two asymptotic spots $\{a_1, \Gamma_1\}$ and $\{a_2, \Gamma_2\}$ are considered equal if $a_1 = a_2 = a$ and there exists a sequence of continuous curves γ_k with one end of each γ_k belonging to Γ_1 and the other to Γ_2, and

$$\lim_{k\to\infty} \min_{z\in\gamma_k} |z| = \infty, \quad \lim_{z\to\infty, z\in\bigcup_k \gamma_k} f(z) = a.$$

By the Denjoy-Carleman-Ahlfors theorem [8], an entire function of finite lower order λ cannot have more than $\max\{[2\lambda], 1\}$ different asymptotic spots. Similar estimates have been formulated with respect to asymptotic functions, in attempt to prove so-called Denjoy conjecture (see: [7, 26]). The number of asymptotic values of an entire function of infinite lower order or a meromorphic function even of finite order may be infinite.

Among asymptotic values we can distinguish those, for which the function moves along the curve towards the value a with a fixed higher speed, for instance comparable with characteristic $T(r, f)$. We say that $a \in \overline{\mathbb{C}}$ is *an* α_0*−strong asymptotic value* of a meromorphic function f, if there exists a continuous curve $\Gamma: \ z = z(t)$, $0 \le t < \infty$, $z(t) \to \infty$ as $t \to \infty$, such, that

$$\liminf_{t\to\infty} \frac{\log |f(z(t)) - a|^{-1}}{T(|z(t)|, f)} = \alpha(a) \ge \alpha_0 > 0, \qquad \text{if} \quad a \ne \infty;$$

$$\liminf_{t\to\infty} \frac{\log |f(z(t))|}{T(|z(t)|, f)} \ge \alpha_0 > 0, \qquad \text{if} \quad a = \infty.$$

If a is an α_0-strong asymptotic value of f, then an asymptotic spot $\{a, \Gamma\}$ is called *an* α_0*- strong asymptotic spot* [16]. It is easy to notice that, if a is an α_0-strong asymptotic value of f, then the magnitude of Petrenko's deviation $\beta(a, f) \ge \alpha_0$. It means that a is also a deficient value in the sense of Petrenko. Marchenko proved that the number k of distinct α_0-strong asymptotic spots of a meromorphic function of finite lower order λ is finite and the inequality

$$k \le \left[\frac{2B(\lambda)}{\alpha_0} \right]$$

holds (see: [16]). Similar results concerning strong asymptotic functions can be found, for example, in [4, 5].

The next result concerns asymptotic values in case of solutions of P_1, P_2, P_{34} and P_4.

Theorem 2.5. *If f is a solution of Painlevé equation P_1 then f does not have any strong asymptotic values. Transcendental solutions of the second Painlevé equation $P_2(\alpha)$ do not have strong asymptotic values if $\alpha \ne 0$. If $\alpha = 0$ the solutions of the equation do not have non-zero strong asymptotic values. Transcendental solutions of Painlevé equation $P_{34}(\alpha)$ do not have strong asymptotic values if $\alpha \ne -\frac{1}{2}$. If $\alpha = -\frac{1}{2}$ the solutions of the equation do not have non-zero strong asymptotic values. Transcendental solutions of $P_4(\alpha, \beta)$ do not have strong asymptotic values other that, possibly, zero.*

Now we move on to discuss deficient values in the sense of Petrenko and asymptotic values of the σ-equations. We say that $\phi : (0, +\infty) \to \mathbb{R}$ is $S(r, f)$ if

$$\phi(r) = o(T(r, f)) \quad (r \to \infty, \ r \notin E),$$

where E is a set of finite linear measure.

Theorem 2.6. *Let σ be a solution of S_1. For all values a in $\overline{\mathbb{C}}$ we have*

$$\mathcal{L}(r, a, \sigma) = S(r, \sigma) \qquad (r \to \infty)$$

and $\beta(a, \sigma) = 0$.

Corollary 2.7. *The solutions of Painlevé equation S_1 do not have strong asymptotic values.*

Theorem 2.8. *Transcendental solutions of $S_2(\alpha)$ fulfill the condition*

$$\mathcal{L}(r,\sigma) = S(r,f) \qquad (r \to \infty).$$

Thus $\beta(\infty,\sigma) = 0$. Moreover, if $\alpha \neq -\frac{1}{2}$, for all complex values a, and if $\alpha = -\frac{1}{2}$ for all $a \in \mathbb{C} \setminus \{0\}$,

$$\mathcal{L}(r,a,\sigma) = S(r,f) \qquad \text{and} \qquad \beta(a,\sigma) = 0.$$

Theorem 2.8 leads to the following conclusion.

Corollary 2.9. *Transcendental solutions of Painlevé equation $S_2(\alpha)$ do not have strong asymptotic values apart from, possibly, the value zero when $\alpha = -\frac{1}{2}$.*

Similar results can be formulated with respect to S_4.

Theorem 2.10. *Transcendental solutions of $S_4(\theta_0,\theta_\infty)$ fulfill the condition*

$$\mathcal{L}(r,\sigma) = S(r,f) \qquad (r \to \infty).$$

Thus $\beta(\infty,\sigma) = 0$. Moreover, for all non-zero complex values a,

$$\mathcal{L}(r,a,\sigma) = S(r,f) \qquad \text{and} \qquad \beta(a,\sigma) = 0.$$

Theorem 2.10 leads to the following conclusion.

Corollary 2.11. *Transcendental solutions of Painlevé equation $S_4(\theta_0,\theta_\infty)$ do not have strong asymptotic values apart from, possibly, the value zero.*

Zero may actually be a strong asymptotic value of solution of S_4, which the following example shows.

Example 2.12. *Let us again consider $S_4(\theta_0,\theta_\infty)$ with $(\theta_0,\theta_\infty) = (0,1)$ or $(\theta_0,\theta_\infty) = (1,0)$, that is the equation*

$$(\sigma'')^2 - 4(z\sigma' - \sigma)^2 + 4\sigma'^2(\sigma' - 2) = 0.$$

We can see that in case of value zero a positive deviation is possible as $\beta(0,\sigma_0) \geq \delta(0,\sigma_0) = 1$ for the solution $\sigma_0(z) = \left(e^{z^2}\int_0^z e^{-t^2}dt\right)^{-1}$ of this equation. Applying the asymptotics of the function $\int_0^z e^{-t^2}dt$ (see: [10, §2.5]*), we can make more accurate computatons and find out that*

$$T(r,\sigma_0) = \frac{r^2}{\pi} + O(1) \qquad (r \to \infty)$$

$$\mathcal{L}(r,0,\sigma_0) \sim r^2 \qquad (r \to \infty), \qquad \beta(0,\sigma_0) = \pi$$

$$\mathcal{L}(r,\infty,\sigma_0) \sim \log r \qquad (r \to \infty), \qquad \beta(\infty,\sigma_0) = 0.$$

It should be mentioned that in case of functions of order 2, the highest possible value of deviation is 2π. It follows from Theorem 2.10 that $\sum_{a\in\overline{\mathbb{C}}}\beta(a,\sigma_0) = \pi$, while the extremal value of the sum of deviations for a function of order 2 is 4π. Moreover, the function has two asymptotic values: the value zero with two separate asymptotic spots $\{0,\Gamma_1\}$, $\{0,\Gamma_2\}$, where

$$\Gamma_1 : \; z(t) = t, \; t \in [0,\infty), \qquad \Gamma_2 : \; z(t) = -t, \; t \in [0,\infty),$$

and the value ∞ with two asymptotic spots $\{\infty,\Gamma_3\}$, $\{\infty,\Gamma_4\}$, where

$$\Gamma_3 : \; z(t) = it, \; t \in [0,\infty), \qquad \Gamma_4 : \; z(t) = -it, \; t \in [0,\infty).$$

Here 0 is a strong asymptotic value and $\{0,\Gamma_1\}$, $\{0,\Gamma_2\}$ are strong asymptotic spots, while ∞ is not strongly asymptotic.

2.2. Ramification of Hamiltonians

For the estimates of multiplicity indices for P_2 see [13], for P_{34} see [1], and for P_4 see [27]. The following results concerning ramification of values hold for solutions of S_1, S_2 and S_4.

Theorem 2.13. *Let σ be a solution of S_1. The a-points of σ for $a \neq 0$ are at most double and $\vartheta(a,\sigma) \leq \frac{1}{2}$. The non-zero zeros of σ are of multiplicity at most 3 and $\vartheta(0,\sigma) \leq \frac{2}{3}$. The poles of σ are simple with residuum 1 and $\vartheta(\infty,\sigma) = 0$.*

Theorem 2.14. *Let σ be a solution of $S_2(\alpha)$. If $\alpha \neq -\frac{1}{2}$, the a-points of σ are at most double and $\vartheta(a,\sigma) \leq \frac{1}{2}$. If $\alpha = -\frac{1}{2}$, then the multiplicity of non-zero a-points is at most 3 and $\vartheta(a,\sigma) \leq \frac{2}{3}$. The zeros are simple, so $\vartheta(0,\sigma) = 0$. For any choice of the parameter α, the poles of a solution of $S_2(\alpha)$ are simple and $\vartheta(\infty,\sigma) = 0$.*

Theorem 2.15. *Let σ be a solution of $S_4(\theta_0,\theta_\infty)$. The non-zero a-points of σ are at most double and $\vartheta(a,\sigma) \leq \frac{1}{2}$. If $\theta_0\theta_\infty = 0$ the zeros of σ are simple, so $\vartheta(0,\sigma) = 0$. If $\theta_0\theta_\infty \neq 0$, the multiple zeros are of multiplicity 3 and $\vartheta(0,\sigma) \leq \frac{2}{3}$. For any choice of the parameters θ_0, θ_∞ the poles of a solution of $S_4(\theta_0,\theta_\infty)$ are simple and $\vartheta(\infty,\sigma) = 0$.*

3. Proofs of the main results

In order to prove the theorems presented in the previous section we need a few auxiliary results.

3.1. Auxiliaries

We first recall the well-known lemmas of Clunie and of A.Z. Mohon'ko and V.D. Mohon'ko as formulated in [9, App. B].

Lemma 3.1. *Let f be a transcendental meromorphic function of finite order such that*

$$f^{p+1} = Q(z,f),\ p \in \mathbb{N},$$

where $Q(z,u)$ is a polynomial in u and its derivatives with meromorphic coefficients b_μ ($\mu \in M$). If the total degree of $Q(z,u)$ as a polynomial in u and its derivatives does not exceed p, then

$$m(r,f) = O(\sum_{\mu\in M} m(r,b_\mu)) + O(\log r) \qquad (r \to \infty).$$

Lemma 3.2. *Let $F(z,u)$ be a polynomial in u and its derivatives with meromorphic coefficients b_μ ($\mu \in M$). Suppose that f is a transcendental meromorphic function of finite order such that $F(z,f) = 0$ and let $c \in \mathbb{C}$. If $F(z,c) \not\equiv 0$, then*

$$m(r,c,f) = O(\sum_{\mu\in M} T(r,b_\mu)) + O(\log r) \qquad (r \to \infty).$$

The following results concerning deviation are modified versions of the (generalized) lemma on the logarithmic derivative, Clunie lemma and Mohon'ko-Mohon'ko lemma.

Lemma 3.3. [1] *Let f be a meromorphic function. Then, possibly except for r in a set of finite linear measure, for $k = 1,2,...$ we have*

$$\mathcal{L}\left(r,\infty,\frac{f^{(k)}}{f}\right) = O(\log(rT(r,f)))\ , \qquad (r \to \infty),$$

where $f^{(k)}$ means the k-th derivative of f.

Lemma 3.4. [1] *Let f be a transcendental meromorphic solution of*

$$f^n P(z, f) = Q(z, f), \tag{5}$$

where n is a positive integer, $P(z, f)$, $Q(z, f)$ are polynomials in f and its derivatives with meromorphic coefficients a_ν, b_ν, respectively, which are small with respect to f in the sense that

$$\mathcal{L}(r, \infty, a_\nu) = S(r, f), \qquad \mathcal{L}(r, \infty, b_\nu) = S(r, f).$$

If the total degree d of $Q(z, f)$ as a polynomial in f and its derivatives is $d \le n$, then

$$\mathcal{L}(r, \infty, P(z, f)) = S(r, f).$$

Lemma 3.5. [1] *Let*

$$P(z, f, f', ..., f^{(n)}) = 0 \tag{6}$$

be an algebraic differential equation ($P(z, u_0, u_1, ..., u_n)$ is a polynomial in all arguments) and let f be its transcendental meromorphic solution. If a constant a does not solve the equation (6)*, then $\mathcal{L}(r, a, f) = S(r, f)$ and $\beta(a, f) = 0$.*

3.2. Proof of Theorem 2.1

Let σ be a solution of S_1. Then $q = -\sigma'$, $p = -\sigma''$ fulfill (1), so q is a solution of P_1. Thus q is meromorphic, which means that σ' and σ'' are also meromorphic functions. It follows from S_1, that

$$\sigma = \frac{1}{2}(\sigma'')^2 - 2(\sigma')^3 + z\sigma', \tag{7}$$

so σ is a sum of meromorphic functions and thus is also meromorphic in $\mathbb{C}$. We know that solutions of P_1 are transcendental meormorphic functions of order $\varrho(f) = \frac{5}{2}$. If f is a solution of P_1 then $f = -\sigma'$ for a certain solution σ of the equation S_1. It follows that σ' has to be a transcendental meromorphic function of order $\varrho(\sigma') = \frac{5}{2}$. Since for meromorphic functions the order of the derivative is the same as of the function itself, we get $\varrho(\sigma) = \frac{5}{2}$.

Next, from (7) and by basic properties of the mean proximity function,

$$m(r, \sigma) \le 2m(r, \sigma'') + 4m(r, \sigma') + O(\log r) \le 2m(r, \frac{\sigma''}{\sigma'}) + 6m(r, \sigma') + O(\log r).$$

As $m(r, \sigma') = m(r, f)$ for a certain solution f of P_1, it follows that $m(r, \sigma') = O(\log r)$. Applying the lemma on the logarithmic derivative to σ', we get

$$m(r, \sigma) = O(\log r) \qquad (r \to \infty).$$

Let us notice now that if $a \in \mathbb{C} \setminus \{0\}$, then a constant a does not satisfy the equation S_1. Thus, by Lemma 3.2,

$$m(r, a, \sigma) = O(\log r) \qquad (r \to \infty). \tag{8}$$

Next,

$$m(r, \frac{1}{\sigma}) \le m(r, \frac{\sigma'}{\sigma}) + m(r, \frac{1}{\sigma'}). \tag{9}$$

Moreover, $-\sigma'$ solves P_1 and for a solution f of P_1,

$$m(r, \frac{1}{f}) = O(\log r) \qquad (r \to \infty),$$

which means that also

$$m(r, \frac{1}{\sigma'}) = O(\log r) \qquad (r \to \infty).$$

Applying the lemma on the logarithmic derivative to (9), we now obtain

$$m(r, 0, \sigma) = m(r, \frac{1}{\sigma}) = O(\log r) \qquad (r \to \infty). \tag{10}$$

It follows from (8) and (10) that

$$m(r, a, \sigma) = O(\log r) \qquad (r \to \infty)$$

for all $a \in \mathbb{C}$, which completes the proof.

3.3. Proof of Theorem 2.2

By the definition (3), each solution of the equation $S_2(\alpha)$ can be expressed as a polynomial with rational coefficients and in variables p, q, where q satisfies P_2 and p satisfies P_{34}. Since p, q are meromorphic, σ is also a meromorphic function. Moreover, by the relationship $p = -2\sigma'$ between a solution p of the equation P_{34} and the derivative of σ we get

$$\varrho(\sigma) = \varrho(\sigma') = \varrho(p).$$

We have $\varrho(p) \leq 3$ [2], which implies $\varrho(\sigma) \leq 3$.

Next, by (3) again,

$$m(r, \sigma) = m(r, \frac{1}{2}p^2 - (q^2 + \frac{1}{2}z)p - (\alpha + \frac{1}{2})q),$$

where p, q are the solutions of P_{34} and P_2, respectively. It follows that

$$m(r, \sigma) \leq 3m(r, q) + 3m(r, p) + O(\log r) \qquad (r \to \infty).$$

Since for solutions of P_2 we have $m(r, q) = O(\log r)$ and for solutions of P_{34} also $m(r, p) = O(\log r)$, we get

$$m(r, \sigma) = O(\log r) \qquad (r \to \infty).$$

Consider now the equation $S_2(\alpha)$ with $\alpha \neq -\frac{1}{2}$. In this case no constant fulfills the equation, so by Lemma 3.2, we get for all $a \in \mathbb{C}$

$$m(r, a, \sigma) = O(\log r) \qquad (r \to \infty). \tag{11}$$

If $\alpha = -\frac{1}{2}$ the equality (11) holds for all non-zero complex numbers, by the same argument, which completes the proof.

3.4. Proof of Theorem 2.3

By the definition (4), each solution σ of the equation $S_4(\theta_0, \theta_\infty)$ can be represented as

$$\sigma = 2qp^2 - (q^2 + 2zq + 2\theta_0)p + \theta_\infty q,$$

where q satisfies $P_4(\alpha, \beta)$ with $\alpha = 1 - \theta_0 + 2\theta_\infty$, $\beta = -2\theta_0^2$, and $-2p$ satisfies $P_4(\alpha, \beta)$ with $\alpha = 2\theta_0 - \theta_\infty - 1$, $\beta = -2\theta_\infty^2$. Since p, q are meromorphic, then σ is also a meromorphic function. Moreover,

$$\begin{aligned} T(r, \sigma) &= T(r, 2qp^2 - (q^2 + 2zq + 2\theta_0)p + \theta_\infty q) \\ &\leq 5T(r, q) + 3T(r, p) + O(\log r) \end{aligned}$$

As both $\varrho(q) \leq 4$ and $\varrho(p) \leq 4$, we get $\varrho(\sigma) \leq 4$. Similarly,

$$m(r,\sigma) \leq 5m(r,q) + 3m(r,p) + O(\log r).$$

By known estimates for solutions of P_4 we have

$$m(r,q) = O(\log r) \qquad \text{and} \qquad m(r,p) = O(\log r),$$

so we get $m(r,\sigma) = O(\log r)$ $(r \to \infty)$, and $\delta(\infty,\sigma) = 0$.

Let us now notice that, as a non-zero constant a does not solve $S_4(\theta_0,\theta_\infty)$, by Lemma 3.2 we get for all $a \in \mathbb{C} \setminus \{0\}$

$$m(r,a,\sigma) = O(\log r).$$

3.5. *Proof of Theorem 2.5*

Let f be solution of P_1. Then, by Theorem 3.16 in [3] we have $\beta(a,f) = 0$ for every value $a \in \overline{\mathbb{C}}$. If a were to be an α_0-strong asymptotic value of f, then $\beta(a,f) \geq \alpha_0 > 0$ - a contradiction. It follows that f does not have α_0- strong asymptotic values for any $\alpha_0 > 0$.

By Theorem 3.3 in [1] and a similar reasoning we obtain conclusions concerning strong asymptotic values of solutions P_2 and P_4.

3.6. *Proof of Theorem 2.6*

Let σ be a solution of S_1. By a similar argument as in the proof of Theorem 2.1, since σ fulfills the equation (7) and by the properties of the function of deviation, we get

$$\mathcal{L}(r,\sigma) \leq 2\mathcal{L}(r,\sigma'') + 4\mathcal{L}(r,\sigma') + O(\log r) \leq \mathcal{L}(r,\frac{\sigma''}{\sigma'}) + 6\mathcal{L}(r,\sigma') + O(\log r).$$

As $\mathcal{L}(r,\sigma') = \mathcal{L}(r,f)$ for a certain solution f of P_1, it follows from Theorem 3.16 in [3] that $\mathcal{L}(r,\sigma') = S(r,\sigma')$. Applying Lemma 3.3 to σ' and the fact that $S(r,\sigma') = S(r,\sigma)$, we obtain

$$\mathcal{L}(r,\sigma) = S(r,\sigma) \qquad (r \to \infty).$$

If $a \in \mathbb{C} \setminus \{0\}$, then a constant a does not fulfill the equation S_1. Thus, by Lemma 3.5,

$$\mathcal{L}(r,a,\sigma) = S(r,\sigma).$$

Next,

$$\mathcal{L}(r,\frac{1}{\sigma}) \leq \mathcal{L}(r,\frac{\sigma'}{\sigma}) + \mathcal{L}(r,\frac{1}{\sigma'}).$$

We apply again Lemma 3.3. Moreover, it follows from Theorem 3.16 in [3] in connection with the fact that $\sigma' = -f$ for a solution f of P_1, that

$$\mathcal{L}(r,\frac{1}{\sigma'}) = S(r,\sigma') = S(r,\sigma).$$

This way we obtain the statement.

3.7. Proof of Theorem 2.8

By a similar argument as in the proof of Theorem 2.2, since we consider transcendental solutions of the equation $S_2(\alpha)$ and by the properties of the function of deviation, we get

$$\mathcal{L}(r,\sigma) = \mathcal{L}(r, \frac{1}{2}p^2 - (q^2 + \frac{1}{2}z)p - (\alpha + \frac{1}{2})q),$$

where p, q are the solutions of P_{34} and P_2, respectively. It follows that

$$\mathcal{L}(r,\sigma) \leq 3\mathcal{L}(r,q) + 3\mathcal{L}(r,p) + O(\log r) \qquad (r \to \infty).$$

Applying Theorems 3.3 and 3.9 from [1], we get

$$\mathcal{L}(r,\sigma) = S(r,p) + S(r,q) \qquad (r \to \infty).$$

By the relationships of σ with p and q, we have

$$S(r,p) = S(r,\sigma) \qquad \text{and} \qquad S(r,q) = S(r,\sigma).$$

It follows that $\mathcal{L}(r,\sigma) = S(r,\sigma)$.

If $\alpha \neq -\frac{1}{2}$ then no constant $a \in \mathbb{C}$ fulfills the equation $S_2(\alpha)$. Thus, by Lemma 3.5,

$$\mathcal{L}(r,a,\sigma) = S(r,\sigma)$$

and $\beta(a,\sigma) = 0$. If, on the other hand $\alpha = -\frac{1}{2}$, then the same argument holds for non-zero values, so $\beta(a,\sigma) = 0$ for all $a \in \mathbb{C} \setminus \{0\}$, which completes the proof.

3.8. Proof of Theorem 2.10

By definition (4), a solution σ of $S_4(\theta_0, \theta_\infty)$ can be represented as

$$\sigma = 2qp^2 - (q^2 + 2zq + 2\theta_0)p + \theta_\infty q,$$

where q satisfies $P_4(\alpha,\beta)$ with $\alpha = 1 - \theta_0 + 2\theta_\infty$, $\beta = -2\theta_0^2$, and $-2p$ satisfies $P_4(\alpha,\beta)$ with $\alpha = 2\theta_0 - \theta_\infty - 1$, $\beta = -2\theta_\infty^2$. Applying the properties of the function of deviation, we get

$$\mathcal{L}(r,\sigma) \leq 5\mathcal{L}(r,q) + 3\mathcal{L}(r,p) + O(\log r).$$

As σ is transcendental, applying Theorem 3.3 from [1], we obtain

$$\mathcal{L}(r,\sigma) = S(r,p) + S(r,q).$$

It follows from the relationships between p and q with σ that $\mathcal{L}(r,\sigma) = S(r,\sigma)$.

Let us now notice that, as a non-zero constant a does not solve $S_4(\theta_0, \theta_\infty)$. By Lemma 3.5 we get for all $a \in \mathbb{C} \setminus \{0\}$

$$\mathcal{L}(r,a,\sigma) = S(r,\sigma).$$

3.9. Proof of Theorem 2.13

Let z_0 be an a-point of a solution σ of S_1 $(a \neq \infty)$. If $a \neq 0$, the assumption that $\sigma(z_0) = a$, $\sigma'(z_0) = 0$ leads to the conclusion that $(\sigma''(z_0))^2 = 2a$. Thus the non-zero $a-$points of σ are at most double. It follows that

$$N_1(r,a,\sigma) = N(r,a,\sigma) - \overline{N}(r,a,\sigma) \leq \frac{1}{2}N(r,a,\sigma)$$

and

$$\vartheta(a,\sigma) = \liminf_{r\to\infty} \frac{N_1(r,a,\sigma)}{T(r,\sigma)} \leq \frac{1}{2}.$$

Assume now that z_0 is a zero of σ. Standard computations lead to the Taylor series around z_0 of the form

$$\sigma(z) = a_1(z-z_0) + \sqrt{-a_1(a_1^2+\frac{z_0}{2})}(z-z_0)^2 + (-a_1^2 - \frac{z_0}{6})(z-z_0)^3$$

$$+(-a_1\sqrt{-a_1(a_1^2+\frac{z_0}{2})} - \frac{1}{24})(z-z_0)^4 + \sum_{k=5}^{\infty} a_k(z-z_0)^k$$

with a_k depending on z_0 and a_1.

If we assume that z_0 is a multiple zero we get the series around z_0 of the form

$$\sigma(z) = -\frac{z_0}{6}(z-z_0)^3 - \frac{1}{24}(z-z_0)^4 - \frac{z_0^2}{20}(z-z_0)^7 + \ldots.$$

It follows that, apart from possibly a zero point at zero, all the zeros of σ are of multiplicity at most 3. so

$$N_1(r,\frac{1}{\sigma}) \leq \frac{2}{3}N(r,\frac{1}{\sigma}) + O(\log r) \qquad \text{and} \qquad \vartheta(0,\sigma) \leq \frac{2}{3}.$$

Let now z_0 be a pole of the solution of S_1. The expansion around z_0 is

$$\sigma(z) = \frac{1}{z-z_0} + a_0 + \frac{z_0}{30}(z-z_0)^3 + \frac{1}{24}(z-z_0)^4 + a_5(z-z_0)^5 + \ldots,$$

so z_0 is a simple pole with residuum 1. It follows that $\vartheta(\infty,\sigma) = 0$.

3.10. *Proof of Theorem 2.14*

Let σ be a solution of the equation $S_2(\alpha)$. The assumption that z_0 is an a-point of σ ($a \neq \infty$) leads to the system of equations binding the initial coefficients of the expansion around z_0 :

$$\begin{cases}
4a_2^2 + 4a_1^3 + 2z_0a_1^2 - 2aa_1 & = \frac{1}{4}(\alpha+\frac{1}{2})^2 \\
6a_2a_3 + 6a_1^2a_2 + 2z_0a_1a_2 - aa_2 & = 0 \\
18a_3^2 + 24a_2a_4 + 24a_1a_2^2 + 18a_1^2a_3 + a_1a_2 + 4z_0a_2^2 + 6z_0a_1a_3 - 3aa_3 & = 0 \\
36a_3a_4 + 20a_2a_5 + 8a_2^3 + 36a_1a_2a_3 + 12a_1^2a_4 + a_2^2 + a_1a_3 & \\
+6z_0a_2a_3 + 4z_0a_1a_4 - 2aa_4 & = 0 \\
72a_4^2 + 120a_3a_5 + 60a_2a_6 + 72a_2^2a_3 + 54a_1a_3^2 + 96a_1a_2a_4 + 30a_1^2a_5 & \\
+7a_2a_3 + 3a_1a_4 + 9z_0a_3^2 + 16z_0a_2a_4 + 10z_0a_1a_5 - 5aa_5 & = 0 \\
\ldots\ldots &
\end{cases}$$

If $\alpha \neq -\frac{1}{2}$ the multiple a-points of σ are at most double. Indeed, the assumption that $\sigma(z_0) = a$, $\sigma'(z_0) = 0$ leads to the equality $(\sigma''(z_0))^2 = \frac{1}{4}(\alpha+\frac{1}{2})^2$. Thus we have $\vartheta(a,\sigma) \leq \frac{1}{2}$. If $\alpha = -\frac{1}{2}$, the assumption that z_0 is a multiple a-point of σ with $a \neq 0$ leads to the conclusion that the multiplicity is 3 and $\vartheta(a,\sigma) \leq \frac{2}{3}$. The zeros in this case are simple with the expansion

$$\sigma(z) = a_1(z-z_0) + \sqrt{-a_1^2(a_1+\frac{1}{2}z_0)}(z-z_0)^2 + (-a_1^2 - \frac{z_0}{3}a_1)(z-z_0)^3 + \ldots,$$

so $\vartheta(0,\sigma) = 0$.

The initial part of the Laurent expansion around a pole z_0 is

$$\sigma(z) = \frac{1}{z-z_0} + a_0 - \frac{z_0}{6}(z-z_0) - \frac{1}{8}(z-z_0)^2 - \frac{1}{30}(a_0 + \frac{1}{6}z_0)(z-z_0)^2 + \ldots.$$

Thus the poles of σ are simple with residuum 1, so $\vartheta(\infty,\sigma) = 0$.

3.11. Proof of Theorem 2.15

Let σ be a solution of $S_4(\theta_0, \theta_\infty)$. If $\sigma(z_0) = a$ ($a \neq 0, \infty$) and assuming that $\sigma'(z_0) = 0$, we get $\sigma''(z_0) = \pm 2a$. Thus multiple non-zero a-points are at most double and $\vartheta(a, \sigma) \leq \frac{1}{2}$.

Around an a-point z_0 of σ ($a \neq \infty$) we have a Taylor expansion with the initial coefficients bound by the system

$$\begin{cases} a_2^2 + a_1^3 + (2\theta_\infty + 2\theta_0 - z_0^2)a_1^2 + 2z_0 a a_1 + 4\theta_0\theta_\infty a_1 - a^2 & = & 0 \\ 3a_3 + 3a_1^2 + (4\theta_\infty + 4\theta_0 - 2z_0^2)a_1 + 2z_0 a + 4\theta_0\theta_\infty & = & 0 \\ 6a_4 + 6a_1a_2 + (4\theta_\infty + 4\theta_0 - 2z_0^2)a_2 - z_0a_1 + a & = & 0 \\ 5a_5 + 3a_1a_3 + (2\theta_\infty + 2\theta_0 - z_0^2)a_3 + 2a_2^2 - z_0a_2 & = & 0 \\ a_6 + 12a_1a_4 + (8\theta_\infty + 8\theta_0 - 4z_0^2)a_4 + 18a_2a_3 - 5z_0a_3 - a_2 & = & 0 \\ 105a_7 + 30a_1a_5 + 10(2\theta_\infty + 2\theta_0 - z_0^2)a_5 + 48a_2a_4 - 14z_0a_4 + 27a_3^2 - 4a_3 & = & 0 \\ \ldots\ldots \end{cases}$$

Let now z_0 be a zero of σ. The system above leads to the following expansion around z_0

$$\begin{aligned} \sigma(z) &= a_1(z - z_0) + \sqrt{-a_1^3 - (2\theta_\infty + 2\theta_0 - z_0^2)a_1^2 - 4\theta_\infty\theta_0 a_1}(z - z_0)^2 \\ &\quad -(a_1^2 + \tfrac{2}{3}(2\theta_\infty + 2\theta_0 - z_0^2)a_1 + \tfrac{4}{3}\theta_\infty\theta_0)(z - z_0)^3 + \textstyle\sum_{k=4}^{\infty} a_k(z - z_0)^k \end{aligned}$$

with a_k depending on z_0, a_1 and parameters θ_∞, θ_0.

If $\theta_\infty\theta_0 = 0$ the zeros are simple and thus $\vartheta(0, \sigma) = 0$. If $\theta_\infty\theta_0 \neq 0$, the multiple zeros are of multiplicity 3 with the expansion

$$\sigma(z) = -\frac{4}{3}\theta_\infty\theta_0(z - z_0)^3 + \frac{4}{15}\theta_\infty\theta_0(2\theta_\infty + 2\theta_0 - z_0^2)(z - z_0)^5 - \frac{20}{3}z_0\theta_\infty\theta_0(z - z_0)^6 + \ldots.$$

It follows that $\vartheta(0, \sigma) \leq \frac{2}{3}$ in this case.

Let z_0 be a pole of a solution σ of S_4. The expansion around z_0 is

$$\sigma(z) = \frac{1}{z - z_0} + a_0 + \frac{z_0^2 - 2(\theta_\infty + \theta_0)}{3}(z - z_0) + \frac{z_0}{2}(z - z_0)^2 + \ldots,$$

so the pole is simple with residuum 1 and thus $\vartheta(\infty, \sigma) = 0$.

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