Filomat 34:6 (2020), 1915–1925
https://doi.org/10.2298/FIL2006915B

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Twisted Product *CR*-Submanifolds in a Locally Conformal Kähler Space Forms

Vittoria Bonanzinga[a], Koji Matsumoto[b]

[a] *University of Reggio Calabria*
[b] *University of Yamagata*

Abstract. Certain twisted product *CR*-submanifolds in a Kähler manifold and some inequalities of the second fundamental form of these submanifolds are presented ([14]). Then the length of the second fundamental form of a twisted product *CR*-submanifold in a locally conformal Kähler manifold is considered (2013), ([15]).

In this paper, we consider the relation of the mean curvature and the length of the second fundamental form in two twisted product *CR*-submanifolds in a locally conformal Kähler space forms.

Introduction

The study of twisted product submanifolds was initiated in 2000 by B. Y. Chen, ([10]). Twisted products $M_1 \times_f M_2$ are natural generalizations of warped products, namely, the function may depend on both factors, when f depends only on M_1 the twisted product becomes a warped product, ([7]). For a survey on geometry of warped product submanifolds in various ambient manifolds see [11]. During the last few years a broad scientific production has appeared on warped product submanifolds and in [11]. B. Y Chen has divided more than 100 published papers into 16 categories of warped product submanifolds. The length of the second fundamental form and the mean curvature in certain submanifolds of a Riemannian manifold are both interesting and important features in submanifold theory. In this paper, we consider these problems in twisted *CR*-submanifolds in locally conformal Kähler space forms.

In §1, we recall a twisted product manifold and give the Riemannian curvature tensor, the Ricci tensor and the scalar curvature. In §2 and §3, we consider a locally conformal Kähler manifold with a constant holomorphic sectional curvature (an l.c.K.-space forms) and its *CR*-submanifolds. In §4, we define two kinds of twisted product *CR*-submanifolds in a locally conformal Kähler manifold and give some essential properties of these submanifolds. In §5, we consider the length of the second fundamental form and the mean curvature of the above submanifolds in an l.c.K.-space forms (See Theorems 5.1, 5.2 and 5.3).

2010 *Mathematics Subject Classification*. Primary 53C40
Keywords. Twisted product manifold, *CR*-submanifold, twisted product *CR*-submanifold, locally conformal Kähler space forms, mean curvature
Received: 12 February 2019; Accepted: 22 December 2019
Communicated by Mića Stanković
Research supported by INDAM
Email addresses: vittoria.bonanzinga@unirc.it (Vittoria Bonanzinga), tokiko_matsumoto@yahoo.com (Koji Matsumoto)

1. Twisted Product manifolds

Let (M_1, g_1) and (M_2, g_2) be Riemannian manifolds and M be a (topological) product manifold of M_1 and M_2. We define a Riemannian metric g of M as

$$g(U,V) = e^{f^2} g_1(\pi_{1*}U, \pi_{1*}V) + g_2(\pi_{2*}U, \pi_{2*}V) \tag{1}$$

for any $U, V \in TM$, where, f denotes a positive differentiable function on M, TM the tangent bundle of M, π_1 (resp. π_2) a projection operator of M to M_1 (resp. M_2) and π_{1*} (resp. π_{2*}) the differential of π_1 (resp. π_2). Then, the manifold M is called a *twisted product manifold with an associated function or a warping function* f, and we write it $M = M_1 \times_f M_2$ ([10]). In particular, if the associated function f is in M_2, then the manifold M is a warped product ([16]).

Let $M = M_1 \times_f M_2$ be a twisted product manifold with the associated function f and let $\dim M_1 = n_1$, $\dim M_2 = n_2$ and $\dim M = n = n_1 + n_2$. Moreover, let $(x^1, x^2, ..., x^{n_1})$, $(x^{n_1+1}, ..., x^{n_1+n_2})$ be local coordinate systems of M_1 and M_2, respectively. Then $(x^1, x^2, ..., x^n)$ is a local coordinate system of M.

Using the above local coordinate systems, we can write

$$(g_{\mu\lambda}) = \begin{pmatrix} g_{ji} & 0 \\ 0 & g_{ba} \end{pmatrix} = \begin{pmatrix} e^{f^2} g_{1ji} & 0 \\ 0 & g_{2ba} \end{pmatrix}, \tag{2}$$

where the indices $(j, i, ..., h)$, $(d, c, ..., a)$ and $(\nu, \mu, ..., \lambda)$ vary in the ranges $(1, 2, ..., n_1)$, $(n_1+1, n_1+2, ..., n_1+n_2)$ and $(1, 2, ..., n_1 + n_2 = n)$, respectively.

Then, by the straightforward calculation, the covariant differentiation $\nabla_V U$ with respect to g is given by

$$\begin{cases} \nabla_Y X = \nabla_{1Y} X + f^2\{(Y \log f)X + (X \log f)Y\} \\ \qquad\quad -f^2\{\triangle_1 \log f + e^{f^2} \triangle_2 \log f\} g_1(Y, X), \\ \nabla_X Z = \nabla_Z X = f^2 (Z \log f) X, \quad \nabla_Z W = \nabla_{2Z} W \end{cases} \tag{3}$$

for any $Y, X \in TM_1$ and $Z, W \in TM_2$, where ∇_1 (resp. ∇_2) denotes the covariant differentiation with respect to g_1 (resp. g_2) and we put $\triangle_1 \log f = {g_1}^{lj} \partial_j \log f \partial_l$ (resp. $\triangle_2 \log f = {g_2}^{cb} \partial_b \log f \partial_c$).

We have from the above equation, the Riemannian curvature tensor ${R_{\omega\nu\mu}}^{\lambda}$, the Ricci tensor $\rho_{\mu\lambda}$ and the scalar curvature τ with respect to g are respectively given by

$$\begin{cases} {R_{kji}}^h = {R^1}_{kji}{}^h + (2 - f^2) f^2 [(\partial_i \log f)\{(\partial_k \log f)\delta_j{}^h - (\partial_j \log f)\delta_k{}^h\} \\ \qquad -(\partial_1{}^h \log f)\{(\partial_k \log f) g_{1ji} - (\partial_j \log f) g_{1ki}\}] \\ \qquad +f^2\{(\nabla_{1k}\partial_i \log f)\delta_j{}^h - (\nabla_{1j}\partial_i \log f)\delta_k{}^h \\ \qquad -(\nabla_{1k}\partial_1{}^h \log f) g_{1ji} + (\nabla_{1j}\partial_1{}^h \log f) g_{1ki}\} + f^4(\|\nabla_1 \log f\|^2 \\ \qquad +e^{f^2}\|\nabla_2 \log f\|^2)(g_{1ji}\delta_k{}^h - g_{1ki}\delta_j{}^h), \\ {R_{kji}}^a = f^2 e^{f^2} [2(\partial_2{}^a \log f)\{(\partial_j \log f) g_{1ki} - (\partial_k \log f) g_{1ji}\} \\ \qquad +(\partial_j \partial_2{}^a \log f) g_{1ki} - (\partial_k \partial_2{}^a \log f) g_{1ji}], \\ {R_{kjb}}^h = 2f^2(\partial_b \log f)\{(\partial_k \log f)\delta_j{}^h - (\partial_j \log f)\delta_k{}^h\} \\ \qquad +f^2\{(\partial_k \partial_b \log f)\delta_j{}^h - (\partial_j \partial_b \log f)\delta_k{}^h\}, \\ {R_{kbi}}^h = -f^2(\partial_b \log f)\{(\partial_i \log f)\delta_k{}^h - (\partial_1{}^h \log f) g_{1ki}\} \\ \qquad -f^2\{(\partial_k \partial_b \log f)\delta_i{}^h - (\partial_b \partial_1{}^h \log f) g_{1ki}\}, \\ {R_{kbi}}^a = f^2 e^{f^2}\{(2 + f^2)(\partial_b \log f)(\partial_2{}^a \log f) + (\nabla_{2b}\partial_2{}^a \log f)\} g_{1ki}, \\ {R_{kba}}^h = -f^2\{(2 + f^2)(\partial_b \log f)(\partial_a \log f) + \nabla_{2b}\partial_a \log f\}\delta_k{}^h, \\ {R_{kjb}}^a = 0, \quad {R_{kcb}}^a = 0, \quad {R_{dcb}}^h = 0, \quad {R_{dcb}}^a = {R^2}_{dcb}{}^a, \end{cases} \tag{4}$$

$$\begin{cases} \rho_{ji} = \rho^1{}_{ji} - (n_1-2)f^2\{(2-f^2)(\partial_j \log f)(\partial_i \log f) - \nabla_{1j}\partial_i \log f\} \\ \quad -f^2\{(2-2f^2+n_1f^2)\|\nabla_1 \log f\|^2 + \nabla_{1l}\partial_1{}^l \log f \\ \quad +(2+n_1f^2)e^{f^2}\|\nabla_2 \log f\|^2 + e^{f^2}(\nabla_{2e}\partial_2{}^e \log f\}g_{1ji}, \\ \rho_{ja} = -(n_1-1)f^2\{2(\partial_j \log f)(\partial_a \log f) + \partial_j\partial_a \log f\}, \\ \rho_{ba} = \rho^2{}_{ba} - n_1f^2\{(2+f^2)(\partial_b \log f)(\partial_a \log f) + \nabla_{2b}\partial_a \log f\}, \end{cases} \tag{5}$$

and

$$\tau = e^{-f^2}\tau^1 + \tau^2 - (n_1-1)f^2e^{-f^2}\{(4-2f^2+n_1f^2)\|\nabla_1 \log f\|^2 \tag{6}$$

$$+2\nabla_{1l}\partial_1{}^l \log f\} - n_1f^2\{(4+f^2+n_1f^2)\|\nabla_2 \log f\|^2 + 2\nabla_{2e}\partial_2{}^e \log f\},$$

where $R^1{}_{kji}{}^h$ (resp. $R^2{}_{dcb}{}^a$), $\rho^1{}_{ji}$ (resp. $\rho^2{}_{ba}$) and τ^1 (resp. τ^2) mean the curvature tensor, the Ricci tensor and the scalar curvature with respect to g_1 (resp. g_2).

2. Locally conformal Kaehler manifolds

A Hermitian manifold $\tilde{M}$ with structure $(J, \tilde{g})$ is called a *locally conformal Kaehler* (*an l.c.K.*) *manifold* if each point $x \in \tilde{M}$ has an open neighbourhood U with a positive differentiable function $\rho : U \to \mathcal{R}$ such that $\tilde{g}^* = e^{-2\rho}\tilde{g}_{|U}$ is a Kaehlerian metric on U, that is, $\nabla^* J = 0$, where J is the almost complex structure, $\tilde{g}$ is the Hermitian metric, ∇^* is the covariant differentiation with respect to $\tilde{g}^*$ and $\mathcal{R}$ is a real number space ([17]). Then, we know ([12])

Proposition 2.1. *A Hermitian manifold $\tilde{M}$ with structure $(J, \tilde{g})$ is an l.c.K.-manifold if and only if there exists a global 1-form α which is called Lee form satisfying*

$$d\alpha = 0 \quad (\alpha : \textit{closed}), \tag{7}$$

$$(\tilde{\nabla}_V J)U = -\tilde{g}(\alpha^\sharp, U)JV + \tilde{g}(V, U)\beta^\sharp + \tilde{g}(JV, U)\alpha^\sharp - \tilde{g}(\beta^\sharp, U)V \tag{8}$$

for any $V, U \in T\tilde{M}$, where $\tilde{\nabla}$ denotes the covariant differentiation with respect to $\tilde{g}$, $\alpha^\sharp$ is the dual vector field of α, the 1–form β is defined by $\beta(X) = -\alpha(JX)$, $\beta^\sharp$ is the dual vector field of β and $T\tilde{M}$ indicates the tangent bundle of $\tilde{M}$.

An l.c.K.-manifold $\tilde{M}(J, \tilde{g}, \alpha)$ is called an *l.c.K.-space form* if it has a constant holomorphic sectional curvature. We know that the Riemannian curvature tensor $\tilde{R}$ with respect to $\tilde{g}$ of an l.c.K.-space form with the constant holomorphic sectional curvature c is given by ([12]):

$$4\tilde{R}(X,Y,Z,W) = c\{\tilde{g}(X,W)\tilde{g}(Y,Z) - \tilde{g}(X,Z)\tilde{g}(Y,W) + \tilde{g}(JX,W)\tilde{g}(JY,Z) \tag{9}$$

$$-\tilde{g}(JX,Z)\tilde{g}(JY,W) - 2\tilde{g}(JX,Y)\tilde{g}(JZ,W)\} + 3\{P(X,W)\tilde{g}(Y,Z)$$

$$-P(X,Z)\tilde{g}(Y,W) + \tilde{g}(X,W)P(Y,Z) - \tilde{g}(X,Z)P(Y,W)\}$$

$$-\tilde{P}(X,W)\tilde{g}(JY,Z) + \tilde{P}(X,Z)\tilde{g}(JY,W) - \tilde{g}(JX,W)\tilde{P}(Y,Z)$$

$$+\tilde{g}(JX,Z)\tilde{P}(Y,W) + 2\{\tilde{P}(X,Y)\tilde{g}(JZ,W) + \tilde{g}(JX,Y)\tilde{P}(Z,W)\}$$

for any $X, Y, Z, W \in T\tilde{M}$, where P and $\tilde{P}$ are respectively defined by

$$P(X,Y) = -(\tilde{\nabla}_X\alpha)Y - \alpha(X)\alpha(Y) + \frac{1}{2}\,\|\,\alpha\,\|^2\,\tilde{g}(X,Y) \tag{10}$$

and

$$\tilde{P}(X,Y) = P(JX,Y) \tag{11}$$

for any $X, Y \in T\tilde{M}$, where $\|\,\alpha\,\|$ is the length of the Lee form α.

Remark 2.2. *To get (9), we have to assume that the symmetric (0,2)-tensor P is hybrid or, equivalently $\tilde{P}$ is skew-symmetric. This means that the Ricci tensor $\tilde{R}_1$ is hybrid ([12]).*

3. *CR*-submanifolds in an l.c.K.-manifold.

In general, for a Riemannian manifold $(\tilde{M}, \tilde{g})$ and its Riemannian submanifold, we know the Gauss and Weingarten formulas

$$\tilde{\nabla}_X Y = \nabla_X Y + \sigma(X, Y), \tag{12}$$

$$\tilde{\nabla}_X \xi = -A_\xi X + \nabla^\perp_X \xi \tag{13}$$

for any $X, Y \in TM$ and $\xi \in T^\perp M$, where σ is the second fundamental form and A_ξ is the shape operator with respect to ξ ([7]). The second fundamental form σ and the shape operator A are related by

$$\tilde{g}(A_\xi Y, X) = \tilde{g}(\sigma(Y, X), \xi)$$

for any $Y, X \in TM$ and $\xi \in T^\perp M$.

Moreover, we know the Gauss equation is given by

$$R(X, Y, Z, W) = \tilde{R}(X, Y, Z, W) + \tilde{g}(\sigma(X, W), \sigma(Y, Z)) \tag{14}$$

$$-\tilde{g}(\sigma(X, Z), \sigma(Y, W)),$$

for any $X, Y, Z, W \in TM$, where R is the curvature tensor with respect to g ([8]).

A submanifold M in an almost Hermitian manifold $\tilde{M}$ is called a *CR-submanifold* if there exists a differentiable distribution $\mathcal{D} : x \to \mathcal{D}_x \subset T_x M$ on M satisfying the following conditions:

(i) $\mathcal{D}$ is holomorphic, i.e., $J\mathcal{D}_x = D_x$ for each $x \in M$ and

(ii) the complementary orthogonal distribution $\mathcal{D}^\perp : x \to \mathcal{D}^\perp_x \subset T_x M$ is totally real, i.e., $J\mathcal{D}^\perp_x \subset T^\perp_x M$ for each $x \in M$, where $T_x M$ (resp. $T^\perp_x M$) denotes the tangent (resp. normal) vector space at x of M ([1],[2], [6]).

If $\dim \mathcal{D}^\perp_x = 0$ (resp. $\dim \mathcal{D}_x = 0$) for each $x \in M$, then the *CR*-submanifold is *holomorphic* (resp. *totally real*). A *CR*-submanifold M is said to be *anti-holomorphic* if $J\mathcal{D}^\perp_x = T^\perp_x M$ for any $x \in M$.

In [13], is proved the following

Proposition 3.1. *In a CR-submanifold M in an l.c.K.-manifold $\tilde{M}$, we have*

(i) the distribution $\mathcal{D}^\perp$ is integrable,

(ii) the distribution $\mathcal{D}$ is integrable if and only if

$$\tilde{g}(\sigma(X, JY) - \sigma(Y, JX) + 2\tilde{g}(JX, Y)\alpha^\sharp, JZ) = 0 \tag{15}$$

for any $X, Y \in \mathcal{D}$ and $Z \in \mathcal{D}^\perp$.

A *CR*-submanifold is said to be *proper* if it is neither holomorphic nor totally real.

A *CR*-submanifold is said to be *mixed geodesic* if the second fundamental form σ satisfies $\sigma(\mathcal{D}, \mathcal{D}^\perp) = \{0\}$, and to be $\mathcal{D}$ (resp.$\mathcal{D}^\perp$)-*geodesic* if the second fundamental form σ satisfies $\sigma(\mathcal{D}, \mathcal{D}) = \{0\}$ (resp. $\sigma(\mathcal{D}^\perp, \mathcal{D}^\perp) = \{0\}$).

In a *CR*-submanifold M of an almost Hermitian manifold $\tilde{M}$, we denote by ν the complementary orthogonal subbundle of $J\mathcal{D}^\perp$ in the normal bundle $T^\perp M$. Then we have the following direct sum decomposition

$$T^\perp M = J\mathcal{D}^\perp \oplus \nu, \qquad J\mathcal{D}^\perp \perp \nu. \tag{16}$$

For the next section, we define a twisted submanifold in a Riemannian manifold.

Definition 3.2. *Let $\tilde{M}$ be a Riemannian manifold with a metric tensor $\tilde{g}$. A submanifold M is said to be a* twisted product submanifold *of $\tilde{M}$ if it satisfies the following conditions:*

(i) M is a Riemannian submanifold of $\tilde{M}$,

(ii) M is a twisted product manifold of two submanifolds M_1 and M_2 of $\tilde{M}$,

(iii) for a certain Riemannian metric g_1 (resp. g_2) of M_1 (resp. M_2),

$$g(U, V) = e^{f^2} g_1(\pi_{1*}U, \pi_{1*}V) + g_2(\pi_{2*}U, \pi_{2*}V) \tag{17}$$

is an induced metric of $\tilde{g}$ for any $U, V \in TM$ and a positive differentiable function f on M, where π_1 (resp. π_2) is the projection operator of $\tilde{M}$ to M_1 (resp. M_2), and π_{1*} (resp. π_{2*}) is the differential of π_1 (resp. π_2).

(iv) the submanifolds M_1 and M_2 are orthogonal, that is, $\tilde{g}(X, Z) = 0$ for any $X \in TM_1$ and $Z \in TM_2$.

4. Twisted product *CR*-submanifolds in a locally conformal Kaehler manifold.

In this section, we consider a special twisted product submanifold in an l.c.K.-manifold.

Definition 4.1. *A submanifold M in an l.c.K.-manifold $\tilde{M}$ is said to be a* first *(resp.* second *)* kind twisted product CR-submanifold *in $\tilde{M}$ if it satisfies*

(i) M is a product manifold of a holomorphic submanifold $M_\top$ and a totally real submanifold $M_\perp$,

(ii) for a certain Riemannian metric tensor g_1 (resp. g_2) on $M_\top$ (resp. $M_\perp$) and a positive differentiable function f on M,

$$g(V,U) = e^{f^2} g_1(\pi_* V, \pi_* U) + g_2(\eta_* V, \eta_* U) \tag{18}$$

(resp.)

$$g(V,U) = g_1(\pi_* V, \pi_* U) + e^{f^2} g_2(\eta_* V, \eta_* U) \tag{19}$$

is a induced metric of $\tilde{g}$, that is, $\tilde{g}(V,U) = g(i_*V, i_*U)$, for any $V, U \in TM$, where π (resp. η) is a projection operator of M to $M_\top$ (resp. $M_\perp$) and i is an identity map of M to $\tilde{M}$.

Remark 4.2. *We write $\mathcal{D}$ (resp. $\mathcal{D}^\perp$) instead of $TM_\top$ (resp. $TM_\perp$).*

Remark 4.3. *In our submanifold, since the holomorphic distribution $\mathcal{D}$ is integrable, we have to assume that the second fundamental form σ satisfies (15).*

Remark 4.4. *Warped product and doubly warped product CR-submanifolds in an l.c.K.-manifold, can be found in [3], [4] and [5].*

In a *CR*-submanifold M of an l.c.K.-manifold $\tilde{M}$, let $\dim \mathcal{D} = 2p$, $\dim \mathcal{D}^\perp = q$, $\dim M = n$, $\dim \nu = 2s$ and $\dim \tilde{M} = m$.

Now we recall an adapted frame on $\tilde{M}$. We take a following local orthonormal frame on $\tilde{M}$,

(i) $\{e_1, e_2, ..., e_p, e^*{}_1, e^*{}_2, ..., e^*{}_p\}$ is an orthonormal frame of $\mathcal{D}$,

(ii) $\{e_{2p+1}, e_{2p+2}, ..., e_{2p+q}\}$ is an orthonormal frame of $\mathcal{D}^\perp$,

(iii) $\{e_{n+q+1}, e_{n+q+2}, ...e_{n+q+s}, e^*{}_{n+q+1}, e^*{}_{n+q+2}, ..., e^*{}_{n+q+s}\}$ is an orthonormal frame of ν. We call such a frame $\{e_1, e_2, ..., e^*{}_{n+q+s}\}$ an *adapted frame* of $\tilde{M}$.

First of all, we consider the first kind twisted product *CR*-submanifold M in an l.c.K.-manifold $\tilde{M}$. Then, by the definition, the induced metric g on M is defined by (18).

Then we have

$$\begin{cases} \nabla_Y X = \nabla_{1Y} X + f^2\{(Y \log f)X + (X \log f)Y\} \\ \qquad\quad -f^2\{\triangle_1 \log f + e^{f^2} \triangle_2 \log f\} g_1(Y,X), \\ \nabla_X Z = \nabla_Z X = f^2 (Z \log f) X, \quad \nabla_Z W = \nabla_{2Z} W \end{cases} \tag{20}$$

for any $Y, X \in \mathcal{D}$ and $Z, W \in \mathcal{D}^\perp$. Then we easily have, from (8) and (20)

Proposition 4.5. *For a proper first kind twisted product CR-submanifold $M = M_\top \times_f M_\perp$ in an l.c.K.-manifold $\tilde{M}$, we have*

(1) $\tilde{g}(\sigma(X, JY), JZ) = \tilde{g}(\alpha^\sharp, Z)\tilde{g}(X,Y) - \tilde{g}(\alpha^\sharp, JZ)\tilde{g}(X, JY)$
$\qquad\qquad -f^2(Z \log f)\tilde{g}(X,Y),$

(2) $\tilde{g}(\sigma(X,Y), JZ) = \tilde{g}(\alpha^\sharp, JZ)\tilde{g}(X,Y)$ *and* $\tilde{g}(\alpha^\sharp, Z) = f^2(Z \log f)$,

(3) $\tilde{g}(\sigma(JX, Z), JW) = -\tilde{g}(\alpha^\sharp, X)\tilde{g}(Z,W)$

for any $Y, X \in \mathcal{D}$ and $Z, W \in \mathcal{D}^\perp$.

By virtue of (2) in the above proposition, we can easily see

Proposition 4.6. *There does not exist a proper first kind of twisted product CR-submanifold in an l.c.K.-manifold whose the Lee vector field $\alpha^\sharp$ is normal to $\mathcal{D}^\perp$.*

Next, we consider the second kind twisted product CR-submanifold $M = M_\perp \times_f M_\top$ in an l.c.K.-manifold $\tilde{M}$. Then, (19) means

$$(g_{\mu\lambda}) = \begin{pmatrix} g_{ba} & 0 \\ 0 & g_{ji} \end{pmatrix} = \begin{pmatrix} e^{f^2} g_{2ba} & 0 \\ 0 & g_{1ji} \end{pmatrix} \quad . \tag{21}$$

In a similar way to a first kind case, we obtain

$$\begin{cases} \nabla_Z W = \nabla_{2Z} W + f^2\{(Z \log f)W + (W \log f)Z\} \\ \qquad -f^2 g_2(Z,W)\{(\triangle_2 \log f) + e^{f^2}(\triangle_1 \log f)\}, \\ \nabla_Z X = \nabla_X Z = f^2 (X \log f)Z, \quad \nabla_Y X = \nabla_{1Y} X \end{cases} \tag{22}$$

for any $Y, X \in \mathcal{D}$, $Z, W \in \mathcal{D}^\perp$.

Using (21) and (22), we obtain

$$\begin{cases} R_{kji}{}^h = R^1_{kji}{}^h, \qquad R_{dji}{}^h = 0, \quad R_{dci}{}^h = 0, \\ R_{bji}{}^a = -f^2\{(2+f^2)(\partial_j \log f)(\partial_i \log f) + (\nabla_{1j}\partial_i \log f)\}\delta_b{}^a, \\ R_{cib}{}^h = f^2 e^{f^2}\{(2+f^2)(\partial_i \log f)(\partial_1{}^h \log f) + (\nabla_{1i}\partial_1{}^h \log f)\}g_{2cb}, \\ R_{dib}{}^a = -f^2(\partial_i \log f)\{(\partial_b \log f)\delta_b{}^a - (\partial_2{}^a \log f)g_{2db}\} \\ \qquad -f^2\{(\partial_d\partial_i \log f)\delta_b{}^a - \partial_i\partial_2{}^a)g_{2db}\}, \\ R_{dci}{}^a = 2f^2(\partial_i \log f)\{(\partial_d \log f)\delta_c{}^a - \partial_c \log f)\delta_d{}^a\} \\ \qquad +f^2\{(\partial_d\partial_i \log f)\delta_c{}^a - (\partial_c\partial_i \log f)\delta_D{}^a\}, \\ R_{dcb}{}^h = f^2 e f^{f^2}[2(\partial_1{}^h \log f)\{(\partial_c \log f)g_{2db} - (\partial_d \log f)g_{2cb} \\ \qquad +(\partial_c\partial_1{}^h)g_{2db} - \partial_d\partial_1{}^h \log f)g_{2cb}], \\ R_{dcb}{}^a = R^2_{dcb}{}^a + ((2-f^2)f^2[(\partial_b \log f)\{(\partial_d \log f)\delta_c{}^a - (\partial_c \log f)\delta_d{}^a\} \\ \qquad -(\partial_2{}^a \log f)\{(\partial_d \log f)g_{2cb} - \partial_c \log f)g_{2db}\}] \\ \qquad +f^2\{(\nabla_{2d}\partial_b \log f)\delta_c{}^a - (\nabla_{2c}\partial_b \log f)\delta_d{}^a - (\nabla_{2d}\partial_2{}^a \log f)g_{2cb} \\ \qquad +(\nabla_{2c}\partial_2{}^a \log f)g_{2bd}\} + f^4(\|\nabla_2 \log f\|^2 \\ \qquad +e^{f^2}\|\nabla_1 \log f\|^2(g_{2cb}\delta_d{}^a - g_{2db}\delta_c{}^a). \end{cases} \tag{23}$$

$$\begin{cases} \rho_{ji} = \rho^1{}_{ji} - n_2 f^2\{(2+f^2)(\partial_j \log f)(\partial_i \log f) + \nabla_{1j}\partial_i \log f\}, \\ \rho_{ja} = -(n_2-1)\{2(\partial_i \log f)(\partial_a \log f) + \partial_a\partial_j \log f\}, \\ \rho_{ba} = \rho^2{}_{ba} - (n_2-2)f^2\{(2-f^2)(\partial_b \log f)(\partial_a \log f) - \nabla_{2b}\partial_a \log f\} \\ \qquad -f^2\{(2-2f^2+n_2f^2)\|\nabla_2 \log f\|^2 + \nabla_{2e}\partial_2{}^e \log f \\ \qquad +(2+n_2f^2)e^{f^2}\|\nabla_1 \log f\|^2 + e^{f^2}(\nabla_{1l}\nabla_1{}^l \log f)\}g_{2ba}, \end{cases} \tag{24}$$

and

$$\tau = e^{-f^2}\tau^2 + \tau^1 - (n_2-1)f^2 e^{-f^2}\{(4-2f^2+n_2f^2)\|\nabla_2 \log f\|^2 \tag{25}$$

$$+2\nabla_{2e}\partial_2{}^e \log f\} - n_2 f^2\{(4+f^2+n_2f^2)\|\nabla_1 \log f\|^2 + 2\nabla_{1l}\partial_1{}^l \log f\}.$$

Now, by virtue of (8) and (23), we obtain

Proposition 4.7. *For the second kind twisted product CR-submanifold in an l.c.K.-manifold $\tilde{M}$, we have*

(1) $\tilde{g}(\sigma(Y,JX),JZ) = \tilde{g}(\alpha^\sharp, Z)\tilde{g}(X,Y) + \tilde{g}(\alpha^\sharp, JZ)\tilde{g}(X,JY)$,

(2) $\tilde{g}(\sigma(X,Y),JZ) = -\tilde{g}(\alpha^\sharp, JZ)\tilde{g}(X,Y)$ *and* $\tilde{g}(\alpha^\sharp, Z) = 0$,

(3) $\tilde{g}(\sigma(JX,Z),JW) = \{-\tilde{g}(\alpha^\sharp, X) + f^2 X \log f\}\tilde{g}(Z,W)$

for any $Y, X \in \mathcal{D}$ and $Z, W \in \mathcal{D}^\perp$.

5. The length of the second fundamental form and the mean curvature.

In this section, we consider the length of the second fundamental form and the mean curvature of two kind twisted product CR-submanifolds in an l.c.K.-space form.

Let $\tilde{M}(c)$ and $\{e_1, e_2, ..., e_m\}$ be an l.c.K.-space form with the constant holomorphic sectional curvature c and an adapted frame of $\tilde{M}(c)$, respectively.

By virtue of (9), the curvature tensor $\tilde{R}_{\omega\nu\mu\lambda}$ in $\tilde{M}(c)$ is written as

$$\begin{cases}
4\tilde{R}_{lkji} = 4\tilde{R}_{l^*k^*j^*i^*} = c(\delta_{li}\delta_{kj} - \delta_{lj}\delta_{ki}) + 3(\delta_{kj}P_{li} - \delta_{ki}P_{lj} \\
\qquad +\delta_{li}P_{kj} - \delta_{lj}P_{ki}), \\
4\tilde{R}_{lkji^*} = 3(\delta_{kj}P_{li^*} - \delta_{lj}P_{ki^*}) - \delta_{ki}P_{lj^*} + \delta_{li}P_{kj^*} - 2\delta_{ji}P_{lk^*}, \\
4\tilde{R}_{lkj^*i^*} = c(\delta_{li}\delta_{kj} - \delta_{lj}\delta_{ki}) - \delta_{kj}P_{li} + \delta_{ki}P_{lj} - \delta_{li}P_{kj} + \delta_{lj}P_{ki}, \\
4\tilde{R}_{lk^*j^*i} = c(\delta_{li}\delta_{kj} + \delta_{lj}\delta_{ki} + 2\delta_{lk}\delta_{ji}) + 3(\delta_{kj}P_{li} + \delta_{li}P_{kj}) \\
\qquad -\delta_{ki}P_{lj} - \delta_{lj}P_{ki} - 2(\delta_{lk}P_{ji} + \delta_{ji}P_{lk}), \\
4\tilde{R}_{lk^*j^*i^*} = 3(\delta_{kj}P_{li^*} - \delta_{ki}P_{lj^*}) - \delta_{li}P_{k^*j} + \delta_{lj}P_{k^*i} + 2\delta_{lk}P_{j^*i}, \\
4\tilde{R}_{lkj(2p+a)} = 3\{\delta_{kj}P_{l(2p+a)} - \delta_{lj}P_{k(2p+a)}\}, \\
4\tilde{R}_{lkj^*(2p+a)} = \delta_{lj}P_{k^*(2p+a)} - \delta_{kj}P_{l^*(2p+a)}, \\
4\tilde{R}_{lk^*j(2p+a)} = -3\delta_{lj}P_{k^*(2p+a)} + \delta_{kj}P_{l^*(2p+a)} + 2\delta_{lk}P_{j^*(2p+a)}, \\
4\tilde{R}_{lk^*j^*(2p+a)} = 3\delta_{kj}P_{l(2p+a)} - \delta_{lj}P_{k(2p+a)} - 2\delta_{lk}P_{j(2p+a)}, \\
4\tilde{R}_{l^*k^*j^*(2p+a)} = 3\{\delta_{kj}P_{l^*(2p+a)} - \delta_{lj}P_{k^*(2p+a)}\}, \\
\tilde{R}_{lk(2p+b)(2p+a)} = 0, \qquad 2\tilde{R}_{lk^*(2p+b)(2p+a)} = \delta_{lk}P_{(2p+b)^*(2p+a)}, \\
\tilde{R}_{l^*k^*(2p+b)(2p+a)} = 0, \\
4\tilde{R}_{l(2p+c)(2p+b)(2p+a)} = 3\{\delta_{cb}P_{l(2p+a)} - \delta_{ca}P_{l(2p+b)}\}, \\
4\tilde{R}_{l^*(2p+c)(2p+b)(2p+a)} = 3\{\delta_{cb}P_{l^*(2p+a)} - \delta_{ca}P_{l^*(2p+b)}\}, \\
4\tilde{R}_{(2p+d)(2p+c)(2p+b)(2p+a)} = c(\delta_{da}\delta_{cb} - \delta_{db}\delta_{ca}) + 3\{\delta_{cb}P_{(2p+d)(2p+a)} - \\
\qquad \delta_{ca}P_{(2p+d)(2p+b)} + \delta_{da}P_{(2p+c)(2p+b)} - \delta_{db}P_{(2p+c)(2p+a)}\}, \\
4\tilde{R}_{j(2p+b)(2p+a)i} = 4\tilde{R}_{j^*(2p+b)(2p+a)i^*} = c\delta_{ji}\delta_{ba} \\
\qquad +3\{P_{ji}\delta_{ba} + \delta_{ji}P_{(2p+b)(2p+a)}\}, \\
4\tilde{R}_{j(2p+b)(2p+a)i^*} = 3\delta_{ba}P_{ji^*} - \delta_{ji}P_{(2p+b)^*(2p+a)},
\end{cases} \tag{26}$$

where the indices $k, j, ..., i$ and $c, b, ..., a$ run over the range $1, 2, ..., p$ and $1, 2, ..., q$, respectively. And we write $\tilde{R}(e_\omega, e_\nu, e_\mu, e_\lambda) = \tilde{R}_{\omega\nu\mu\lambda}$, etc., for any $\omega, \nu, ..., \lambda \in \{1, 2, ..., n.\}$

Now, the mean curvature vector H and the mean curvature $\|H\|$ are respectively given by ([8])

$$H = \frac{1}{n}\sum_{\mu=1}^{n}\sigma_{\mu\mu}, \quad \|H\|^2 = \frac{1}{n^2}\sum_{\nu,\lambda=1}^{n}\tilde{g}(\sigma_{\nu\nu}, \sigma_{\lambda\lambda}). \tag{27}$$

The length $\|\sigma\|$ of the second fundamental form σ is given by

$$\|\sigma\|^2 = \sum_{\mu,\lambda=1}^{n}\tilde{g}(\sigma_{\mu\lambda}, \sigma_{\mu\lambda}) = \sum_{\mu,\lambda=1}^{n}\sum_{\tau=n+1}^{m}\tilde{g}(\sigma_{\mu\lambda}, e_\tau)^2. \tag{28}$$

By virtue of (27), (28) and the Gauss equation, we have

$$4\tau = 4\sum_{\nu,\mu=1}^{n}\tilde{R}_{\nu\mu\mu\nu} + 4n^2\|H\|^2 - 4\|\sigma\|^2, \tag{29}$$

where τ is the scalar curvature with respect to the induced metric g in M.

By virtue of (26), we can write

$$4\sum_{\nu,\mu=1}^{n}\tilde{R}_{\nu\mu\mu\nu}=8\sum_{j,i=1}^{p}(\tilde{R}_{jiij}+\tilde{R}_{ji^*i^*j})+8\sum_{j=1}^{p}\sum_{a=1}^{q}(\tilde{R}_{j(2p+a)(2p+a)j}$$

$$+\tilde{R}_{j^*(2p+a)(2p+a)j^*})+4\sum_{b,a=1}^{q}\tilde{R}_{(2p+b)(2p+a)(2p+a)(2p+b)}.$$

So, using (26) and the above equation, we get

$$4\sum_{\omega,\nu=1}^{n}\tilde{R}_{\omega\nu\nu\omega}=c(n^2+4p-q)+6(n-1)\sum_{\mu=1}^{n}P_{\mu\mu}-6\sum_{j=1}^{2p}P_{jj}. \tag{30}$$

Substitution of (30) into (29) gives us

$$4\tau=c(n^2+4p-q)+6(n-1)\sum_{\mu=1}^{n}P_{\mu\mu}-6\sum_{j=1}^{2p}P_{jj} \tag{31}$$

$$+4n^2||H||^2-4||\sigma||^2.$$

Thus we have

Proposition 5.1. *In a CR-submanifold M in an l.c.K.-space form $\tilde{M}(c)$, the length of the second fundamental form $||\sigma||$ and the mean curvature $||H||$ respectively satisfy the following inequalities*

$$4||\sigma||^2\geq c(n^2+4p-q)+6(n-1)\sum_{\mu=1}^{n}P_{\mu\mu}-6\sum_{j=1}^{2p}P_{jj}-4\tau, \tag{32}$$

and

$$4n^2||H||^2\geq 4\tau-c(n^2+4p-q)-6(n-1)\sum_{\mu=1}^{n}P_{\mu\mu}+6\sum_{j=1}^{2p}P_{jj}. \tag{33}$$

In particular, if the above first (*resp. second*) *inequality satisfies equality, then the submanifold is minimal* (*resp. totally geodesic*).

Now, we assume that our submanifold M is the first kind twisted product CR-submanifold in an l.c.K.-space form $\tilde{M}(c)$. Since we know ([14])

$$||\sigma||^2=2\{p||\alpha^{\sharp}_{\mathcal{D}^\perp}||^2+q||\alpha^{\sharp}_{\mathcal{D}}||^2\}+2p\sum_{a=1}^{q}\{\tilde{g}(\alpha^{\sharp},e_{2p+a}) \tag{34}$$

$$-f^2(e_{2p+a}\log f)\}^2+\sum_{c,b,a=1}^{q}\{\tilde{g}(\sigma(e_{2p+c},e_{2p+b}),e^*_{2p+a})\}^2$$

$$+\sum_{r=n+q+1}^{m}\sum_{\mu,\lambda=1}^{n}\{\tilde{g}(\sigma(e_\lambda,e_\mu),e_r)\}^2,$$

the mean curvature $||H||$ satisfies

$$4\tau=c(n^2+4p-q)+6(n-1)\sum_{\mu=1}^{n}P_{\mu\mu}-6\sum_{j=1}^{2p}P_{jj} \tag{35}$$

$$+4n^2\|H\|^2 - 4[2\{p\|\alpha^\sharp_{\mathcal{D}^\perp}\|^2 + q\|\alpha^\sharp_{\mathcal{D}}\|^2\}$$

$$+2p\sum_{a=1}^{q}\{\tilde{g}(\alpha^\sharp, e_{2p+a}) - f^2(e_{2p+a}\log f)\}^2 + \sum_{c,b,a=1}^{q}\{\tilde{g}(\sigma(e_{2p+c}, e_{2p+b}), e^*_{2p+a})\}^2$$

$$+\sum_{r=n+q+1}^{m}\sum_{\mu,\lambda=1}^{n}\{\tilde{g}(\sigma(e_\lambda, e_\mu), e_r)\}^2],$$

where $\|\alpha^\sharp_{\mathcal{D}^\perp}\|$ (resp. $\|\alpha^\sharp_{\mathcal{D}}\|$) denotes the length of $\alpha^\sharp$ in $\mathcal{D}^\perp$ (resp. $\mathcal{D}$)-part. Thus we have

Theorem 5.2. *In the first kind twisted product CR-submanifold in the l.c.K.-space form $\tilde{M}(c)$, the mean curvature $\|H\|$ satisfies the inequality*

$$4n^2\|H\|^2 \geq 4\tau - c(n^2+4p-q) - 6(n-1)\sum_{\mu=1}^{n}P_{\mu\mu} + 6\sum_{j=1}^{2p}P_{jj} \tag{36}$$

$$+4[2\{p\|\alpha^\sharp{}_{\mathcal{D}^\perp}\|^2 + q\|\alpha^\sharp{}_{\mathcal{D}}\|^2\} + 2p\{\sum_{a=1}^{q}\{\tilde{g}(\alpha^\sharp, e_{2p+a}) - f^2(e_{2p+a}\log f)\}^2].$$

In particular, the equality case is the second fundamental form satisfies $\sigma(TM, TM) \subset J\mathcal{D}^\perp$ and $\sigma(\mathcal{D}^\perp, \mathcal{D}^\perp) \subset \nu$. Moreover if the submanifold is anti-holomorphic, then it is $\mathcal{D}^\perp$-geodesic.

In our case, by virtue of (6), the scalar curvature τ is written as

$$\tau = e^{-f^2}\tau^1 + \tau^2 - 2(2p-1)f^2e^{-f^2}\{(2 - f^2 + pf^2)\|\nabla_1\log f\|^2 \tag{37}$$

$$+\nabla_{1l}\partial_1{}^l\log f\} - 2pf^2\{(4 + f^2 + 2pf^2)\|\nabla_2\log f\|^2 + 2\nabla_{2e}\partial_2{}^e\log f\},$$

where τ^1 (resp. τ^2) means the scalar curvature with respect to g_1 (resp. g_2). Thus, we have from Theorem 5.1,

Theorem 5.3. *In the first kind twisted product CR-submanifold in the l.c.K.-space form $\tilde{M}(c)$, the mean curvature $\|H\|$ satisfies the inequality*

$$n^2\|H\|^2 \geq P + Q, \tag{38}$$

where we put

$$P = e^{-f^2}\tau^1 + \tau^2 - 2(2p-1)f^2e^{-f^2}\nabla_{1l}\partial_1{}^l\log f \tag{39}$$

$$-4pf^2\nabla_{2e}\partial_2{}^e\log f - \frac{c(n^2+4p-q)}{4} - \frac{3(n-1)}{2}\sum_{\mu=1}^{n}P_{\mu\mu}$$

$$+\frac{3}{2}\sum_{j=1}^{2p}P_{jj} + 2\{p\|\alpha^\sharp{}_{\mathcal{D}^\perp}\|^2 + q\|\alpha^\sharp{}_{\mathcal{D}}\|^2\}$$

$$+2p\{\sum_{a=1}^{q}\tilde{g}(\alpha^\sharp, e_{2p+a}) - f^2(e_{2p+a}\log f)\}^2$$

and

$$Q = -2(2p-1)f^2e^{-f^2}(2 - f^2 + pf^2)\|\nabla_1\log f)\|^2$$

$$-2pf^2(4 + f^2 + 2pf^2)\|\nabla_2\log f)\|^2.$$

As a corollary of Theorem 5.1, we can easily obtain

Corollary 5.4. *In the first kind proper twisted product CR-submanifold in the l.c.K.-space form $\tilde{M}(c)$, the mean curvature $\|H\|$ satisfies the inequality*

$$4n^2\|H\|^2 \geq 4\tau - c(n^2+4p-q) - 6(n-1)\sum_{\mu=1}^{n} P_{\mu\mu} + 6\sum_{j=1}^{2p} P_{jj}. \tag{40}$$

Proof. In fact, if the above inequality satisfies equality, then by Proposition 4.2 follows that the submanifold is not proper. □

Next, we assume that M is the second kind twisted product CR-submanifold in the l.c.K.-space form $\tilde{M}(c)$. Then, using Proposition 4.3, the length $\|\sigma\|^2$ satisfies ([14])

$$\begin{aligned}\|\sigma\|^2 &= 2p\|\alpha^{\sharp}_{J\mathcal{D}^{\perp}}\|^2 + 2q\{\|\alpha^{\sharp}_{\mathcal{D}}\|^2 - f^2\sum_{i=1}^{2p}\tilde{g}(\alpha^{\sharp}, e_i)(e_i\log f)\\ &+f^4\sum_{i=1}^{2p}\{(e_i\log f)(e^*{}_i\log f)\}^2\} + \sum_{c,b,a=1}^{q}\{\tilde{g}(\sigma_{(2p+c)(2p+b)}, e^*{}_{2p+a})\}^2\\ &+\sum_{r=n+q+1}^{m}\sum_{\mu,\lambda=1}^{n}\{\tilde{g}(\sigma_{\mu\lambda}, e_r)\}^2].\end{aligned}$$

By virtue of the above equation and (31), the mean curvature $\|H\|$ satisfies

$$\begin{aligned}4\tau &= c(n^2+4p-q) + 6(n-1)\sum_{\mu=1}^{n} P_{\mu\mu} - 6\sum_{j=1}^{2p} P_{jj} \qquad (41)\\ &+4n^2\|H\|^2 - 4[2p\|\alpha^{\sharp}_{J\mathcal{D}^{\perp}}\|^2 + 2q\{\|\alpha^{\sharp}_{\mathcal{D}}\|^2 - f^2\sum_{i=1}^{2p}\tilde{g}(\alpha^{\sharp}, e_i)(e_i\log f)\\ &+f^4\sum_{i=1}^{2p}\{(e_i\log f)(e^*{}_i\log f)\}^2\} + \sum_{c,b,a=1}^{q}\{\tilde{g}(\sigma_{(2p+c)(2p+b)}, e^*{}_{2p+a})\}^2\\ &+\sum_{r=n+q+1}^{m}\sum_{\mu,\lambda=1}^{n}\{\tilde{g}(\sigma_{\mu\lambda}, e_r)\}^2].\end{aligned}$$

Thus we have

Theorem 5.5. *In the second kind twisted product CR-submanifold in the l.c.K.-space form $\tilde{M}(c)$, the mean curvature $\|H\|$ satisfies the inequality*

$$\begin{aligned}4n^2\|H\|^2 &\geq 4\tau - c(n^2+4p-q) - 6(n-1)\sum_{\mu=1}^{n} P_{\mu\mu} + 6\sum_{j=1}^{2p} P_{jj} \qquad (42)\\ &+8\{p\|\alpha^{\sharp}_{J\mathcal{D}^{\perp}}\|^2 + q\|\alpha^{\sharp}_{\mathcal{D}}\|^2 - qf^2\sum_{i=1}^{2p}\tilde{g}(\alpha^{\sharp}, e_i)(e_i\log f) + qf^4\sum_{i=1}^{2p}(e_i\log f)^2(e^*{}_i\log f)^2\}.\end{aligned}$$

In particular, the equality case is the second fundamental form satisfies $\sigma(TM, TM)$ $\perp \nu$ and $\sigma(\mathcal{D}^{\perp}, \mathcal{D}^{\perp}) \perp J\mathcal{D}^{\perp}$.

Acknowledgments. This work was supported by the National Group for Algebraic and Geometric Structures, and their Applications (GNSAGA-INDAM). Part of the work has been carried out while Koji Matsumoto was visiting the Department of Information Engineering, Infrastructures and Sustainable Energy of the Mediterranean University of Reggio Calabria. Koji Matsumoto wishes to thank the Department of Information Engineering, Infrastructures and Sustainable Energy for the kind hospitality and the INdAM for the financial support.

References

[1] A. Bejancu, *CR*-submanifolds of a Kaehler manifold I, II, Proc. Amer. Math. Soc., 69 (1978), 134 –142. and Trans. Amer. Math. Soc., 250 (1979), 333–345.
[2] A. Bejancu, Geomerty of *CR*-submanifolds, D. Reidel Publishing Company, 1986.
[3] V. Bonanzinga, K. Matsumoto, Warped product *CR*-submanifolds in locally conformal Kaehler manifolds, Periodica Math. Hungarica, 48, (1-2), (2004), 207–221.
[4] V. Bonanzinga, K. Matsumoto, On doubly warped product *CR*-submanifolds in a locally conformal Kaehler manifold, Tensor (N.S.), 69, (2008), 76–82.
[5] V. Bonanzinga, K. Matsumoto, Doubly warped product *CR*-submanifolds in a locally conformal Kaehler space form, Acta Math. Acad. Paedagog. Nyházi. (N.S.) 24, (2008) 93–102.
[6] B. Y. Chen , *CR*-submanifolds of a Kaehler manifold I, II, J. of Differential Geometry, 16, (1981) 305–322 and 493–509.
[7] B. Y. Chen, Geometry of Submanifolds and Its Applications, Sci. Univ. Tokyo, 1981.
[8] . Y. Chen, Geometry of Submanifolds, Marcel Dekker, New York, 1973.
[9] B. Y. Chen, Totally umbilical submanifolds, Soochow J. Math., 5, (1979) 9–37.
[10] B. Y. Chen, Twisted product *CR*-submanifolds in Kaehler Manifolds, Tamusui Oxford J. Math. Sci., 16, (2), (2000) 105–121.
[11] B. Y. Chen, A survey on geometry of warped product submanifolds, arXiv:1307.0236v1, 2013.
[12] T. Kashiwada, Some properties of locally conformal Kähler manifolds, Hokkaido Math, J., 8, (1979) 191–198.
[13] K. Matsumoto, On *CR*-submanifolds of locally conformal Kähler manifolds I, II, J. Korean Math., 21, (1984), 49–61 and Tensor N, S. 45 (1987), 144–150.
[14] K. Matsumoto, Z. Şentürk, Certain twisted product *CR*-submanifolds in a Kaehler manifold, Proc. of the conference RIGA 2011, Riemannian Geometry and Applications, 2011.
[15] K. Matsumoto, Z. Şentürk, Twisted product *CR*-submanifolds in a locally conformal Kaehler manifold, Publications de l'Institut Mathematique, 94, (2013) 131–140.
[16] B. O'Neill, Semi-Riemannian Geometry, Academic Press, New York, 1981.
[17] I. Vaisman, Locally conformal almost Kähler manifolds, Israel J, Math., 24, (1976) 338–351.