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Fixed Point Theorems via w-Distance in Relational Metric Spaces with an Application

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Abstract. In this paper, we establish fixed point theorems for generalized nonlinear contractive mappings using the concept of w-distance on metric spaces endowed with an arbitrary binary relation. Our fixed point theorems generalize recent results of Senapati and Dey [J. Fixed Point Theory Appl., 19, 2945-2961, (2017)] and many other important results of the existing literature. Moreover, in order to revel the usability of our findings an example and an application to first order periodic boundary value problem are given.

1. Introduction

The classical Banach contraction principle (Bcp) has many inferences and huge applicability in mathematical theory and because of this, Bcp has been improved and generalized in various metric settings. One such interesting and important setting is to establish fixed point results in metric spaces equipped with an arbitrary binary relation. Utilizing the notions of various kind of binary relations such as partial order, strict order, near order, transitive and tolerance etc. on metric spaces, many researcher are doing their research during several years (see, for instance [1-4], [6], [9], [11-16], [18-21], [25-26]) and attempting to obtain new extensions of the celebrated Bcp. However, the first try to investigate the analogous version of the Banach contraction principle in this setting was performed by Turinici ([22]-[24]) in 1986. Among these extensions, we must quote the one due to Alam and Imdad [3], where some relation theoretic analogues of standard metric notions such as continuity and completeness were used. Further, Alam and Imdad [2] extended the above setting for nonlinear contractions by using T-transitivity of the ambient relation $\mathcal{R}$, and obtained an extension of the Boyd-Wong [5] fixed point theorem to such spaces.

On the other hand, most recently Senapati and Dey [21] improved and refined the main result of Alam and Imdad [3] and many other relevant results of the existing literature, by utilizing the notion of w-distance in relational metric spaces, that is, the metric spaces endowed with an arbitrary binary relation. In this way the investigations of the respective authors unveil another direction for the relation-theoretic metrical fixed point results. Our aim in this paper, is to give an extension of these results to nonlinear φ-contraction using the comparison function and also explore the possibility of their application by solving the first order periodic boundary value problem.

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2. Preliminaries

Throughout this paper, $\mathcal{R}$ stands for a non-empty binary relation, $\mathbb{N}_0$ stands for the the set of whole numbers, i.e., $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\mathbb{R}$ for the set of all real numbers.

Definition 2.1. [3]. *Let R be a binary relation on a non-empty set X and $x, y \in X$. We say that x and y are $\mathcal{R}$-comparative if either $(x, y) \in \mathcal{R}$ or $(y, x) \in \mathcal{R}$. We denote it by $[x, y] \in \mathcal{R}$.*

Definition 2.2. [2]. *Let X be a non-empty set and $\mathcal{R}$ a binary relation on X.*
(1) The inverse or transpose or dual relation of $\mathcal{R}$, denoted by $\mathcal{R}^{-1}$ is defined by $\mathcal{R}^{-1} = \{(x, y) \in X^2 : (y, x) \in \mathcal{R}\}$.
(2) The symmetric closure of $\mathcal{R}$, denoted by $\mathcal{R}^s$ is defined to be the set $\mathcal{R} \cup R^{-1}$ i.e. $\mathcal{R}^s := R \cup \mathcal{R}^{-1}$. Indeed $\mathcal{R}^s$ is the smallest symmetric relation on X containing $\mathcal{R}$.

Theorem 2.3. [3]. *For a binary relation $\mathcal{R}$ defined on a non-empty set $X, (x, y) \in \mathcal{R}^s$ if and only if $[x, y] \in \mathcal{R}$.*

Definition 2.4. [3]. *Let X be a non-empty set and $\mathcal{R}$ a binary relation on X. A sequence $\{x_n\} \subset X$ is called $\mathcal{R}$-preserving if $(x_n, x_{n+1}) \in \mathcal{R}$ for all $n \in \mathbb{N}_0$.*

Definition 2.5. [3]. *Let X be a non-empty set and T be a self-mapping on X. A binary relation $\mathcal{R}$ on X is called T-closed if for any $x, y \in X, (x, y) \in \mathcal{R}$ implies $(Tx, Ty) \in \mathcal{R}$.*

Theorem 2.6. [3]. *Let X be a non-empty set, $\mathcal{R}$ a binary relation on X and T a self-mapping on X. If $\mathcal{R}$ is T-closed, then $\mathcal{R}^s$ is also T-closed.*

Theorem 2.7. [3]. *Let X be a non-empty set, $\mathcal{R}$ a binary relation on X and T a self-mapping on X. If $\mathcal{R}$ is T-closed, then for all $n \in N_0$, $\mathcal{R}$ is also T^n-closed where T^n denotes n th iterate of T.*

Definition 2.8. [2]. *Let (X, d) be a metric space and $\mathcal{R}$ a binary relation on X. We say that (X, d) is $\mathcal{R}$-complete if every $\mathcal{R}$-preserving Cauchy sequence in X converges.*

The following notion is a generalization of d-self-closedness of a partial order relation $(\preceq)$ (defined by Turinici [26]).

Definition 2.9. [26]. *Let (X, d) be a metric space. A binary relation $\mathcal{R}$ on X is called d-self-closed if for any $\mathcal{R}$-preserving sequence $\{x_n\}$ such that $x_n \xrightarrow{d} x$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ with $[x_{n_k}, x] \in \mathcal{R}$ for all $k \in \mathbb{N}_0$.*

Definition 2.10. [2]. *Let X be a non-empty set and $\mathcal{R}$ a binary relation on X. A subset E of X is called $\mathcal{R}$-connected if for each pair $x, y \in E$, there exists a path in $\mathcal{R}$ from x to y.*

Definition 2.11. [20]. *Let X be a non-empty set and $\mathcal{R}$ a binary relation on X. A subset E of X is called $\mathcal{R}$-directed if for each pair $x, y \in E$, there exists $z \in X$ such that $(x, z) \in \mathcal{R}$ and $(y, z) \in \mathcal{R}$.*

Definition 2.12. [2]. *Let X be a non-empty set and T be a self-mapping on X. A binary relation $\mathcal{R}$ on X is called T-transitive if for any $x, y, z \in X, (Tx, Ty), (Ty, Tz) \in \mathcal{R}$ implies $(Tx, Tz) \in \mathcal{R}$.*

Inspired by Turinici [23] Alam and Imdad [2] introduce the following notion by localizing the notion of transitivity.

Definition 2.13. [2]. *A binary relation $\mathcal{R}$ on a non-empty set X is called locally transitive if for each (effectively) $\mathcal{R}$-preserving sequence $\{x_n\} \subset X$ (with range $E := \{x_n : n \in \mathbb{N}_0\}$), the binary relation $\mathcal{R}|_E$ is transitive. Where $\mathcal{R}|_E$ denote the restriction of $\mathcal{R}$ to E.*

Definition 2.14. [10]. *Let X be a non-empty set and $\mathcal{R}$ a binary relation on X. For $x, y \in X$, a path of length k (where k is a natural number) in $\mathcal{R}$ from x to y is a finite sequence $\{z_0, z_1, z_2, ..., z_k\} \subset X$ satisfying the following conditions:*
(i) $z_0 = x$ and $z_k = y$,
(ii) $(z_i, z_{i+1}) \in \mathcal{R}$ for each $i (0 \leq i \leq k - 1)$.
Notice that a path of length k involves $k + 1$ elements of X, although they are not necessarily distinct.

Definition 2.15. [2]. *Let (X,d) be a metric space, $\mathcal{R}$ a binary relation on X and $x \in X$. A self-mapping T on X is called $\mathcal{R}$-continuous at x if for any $\mathcal{R}$-preserving sequence $\{x_n\}$ such that $x_n \xrightarrow{d} x$, we have $T(x_n) \xrightarrow{d} T(x)$. Moreover, T is called $\mathcal{R}$-continuous if it is $\mathcal{R}$-continuous at each point of X.*

The notion of $\mathcal{R}$-lower semi-continuity (briefly, $\mathcal{R}$-LSC) of a function is defined by Senapati and Dey [21] as follows:

Definition 2.16. [21]*Let (X,d) be a metric space and R be a binary relation defined on X. A function $f : X \to R \cup \{-\infty, +\infty\}$ is said to be $\mathcal{R}$-LSC at x if for every $\mathcal{R}$-preserving sequence x_n converging to x, we have* $\liminf_{n\to+\infty} f(x_n) \geq f(x)$.

By presenting examples the respective authors explained that the $\mathcal{R}$-LSC is weaker than $\mathcal{R}$-continuity as well as lower semi-continuity (see for details [21]) and modify the definition of w-distance (Definition 2.8) and the corresponding Lemma 1 presented in [8] in the context of metric spaces endowed with an arbitrary binary relation $\mathcal{R}$ as follows:

Definition 2.17. *Let (X,d) be a metric space and $\mathcal{R}$ be a binary relation on X. A function $p : X \times X \to [0,+\infty)$ is said to be a w-distance on X if*
(w_1) $p(x,z) \leq p(x,y) + p(y,z)$ for any $x,y,z \in X$;
(w_2) for any $x \in X$, $p(x,.) : X \to [0,+\infty)$ is $\mathcal{R}$-lower semi-continuous;
(w_3) for any $\epsilon > 0$, there exists $\delta > 0$, such that $p(z,x) \leq \delta$ and $p(z,y) \leq \delta$ imply $d(x,y) \leq \epsilon$.

The following two lemmas are required in our subsequent discussion.

Lemma 2.18. [21]. *Let (X,d) be a metric space endowed with binary relation $\mathcal{R}$ and $p : X \times X \to [0,+\infty)$ be a w-distance. Suppose (x_n) and (y_n) are two $\mathcal{R}$-preserving sequences in X and $x,y,z \in X$. Let (u_n) and (v_n) be sequences of positive real numbers converging to 0. Then, we have the followings:*
(L_1) If $p(x_n,y) \leq u_n$ and $p(x_n,z) \leq v_n$ for all $n \in \mathbb{N}$, then $y = z$. Moreover, if $p(x,y) = 0$ and $p(x,z) = 0$, then $y = z$.
(L_2) If $p(x_n,y_n) \leq u_n$ and $p(x_n,z) \leq v_n$ for all $n \in \mathbb{N}$, then $y_n \to z$.
(L_3) If $p(x_n,x_m) \leq u_n$ for all $m > n$, then (x_n) is an $\mathcal{R}$-preserving Cauchy sequence in X.
(L_4) If $p(x_n,y) \leq u_n$ for all $n \in N$, then (x_n) is an $\mathcal{R}$-preserving Cauchy sequence in X.

The following family of control functions is due to Lakshmikantham and Ciric [12].
$\Phi = \{\varphi : [0,+\infty) \to [0,+\infty) : \varphi(t) < t$ for each $t > 0$ and $\lim_{r\to t^+} \varphi(t) < t$ for each $t > 0$ $\}$.
The following family of control functions is indicated by Boyd and Wong [5] but was later used in Jotic [7].
$\Omega = \{\varphi : [0,+\infty) \to [0,+\infty) : \varphi(t) < t$ for each $t > 0$ and $\limsup_{r\to t^+} \varphi(t) < t$ for each $t > 0$ $\}$.
Note that the class Ω enlarges the class Φ, i.e., $\Phi \subset \Omega$.

Lemma 2.19. [4]. *Let $\varphi \in \Omega$. If $\{x_n\} \subset (0,+\infty)$ is a sequence such that $a_{n+1} \leq \varphi(a_n)$ for all $n \in \mathbb{N}$, then* $\lim_{n\to+\infty} a_n = 0$.

Lemma 2.20. [17, 23]. *Let (X,d) be a metric space, p be a w-distance and $\{x_n\}$ a sequence in X. If $\{x_n\}$ is not a Cauchy, then there exist $\epsilon > 0$ and two subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$ such that*
(i) $k \leq m_k \leq n_k$ for all $k \in \mathbb{N}$,
(ii) $p(x_{m_k}, x_{n_k}) > \epsilon$ for all $k \in \mathbb{N}$,
(iii) $p(x_{m_k}, x_{n_{k-1}}) \leq \epsilon$ for all $k \in \mathbb{N}$.
Moreover, suppose that $\lim\limits_{k\to+\infty} p(x_{m_k}, x_{n_k}) = 0$, then
(iv) $\lim\limits_{k\to+\infty} p(x_{m_k}, x_{n_k}) = \epsilon$,
(v) $\lim\limits_{k\to+\infty} p(x_{m_{k+1}}, x_{n_{k+1}}) = \epsilon$.

Given a binary relation $\mathcal{R}$ and a self-mapping T on a nonempty set X, we use the following notations.
(i) $F(T)$:= the set of all fixed points of T,
(ii) $X(T,\mathcal{R}) := \{x \in X : (x,Tx) \in \mathcal{R}\}$.

3. Main Results

In this section, firstly we prove the existence of fixed points for mappings in relational metric spaces.

Theorem 3.1. *Let (X, d) be a metric space with a w-distance p and $\mathcal{R}$ be any arbitrary binary relation on X. Suppose T is a self-mapping on X with following conditions :*
(a) there exists $Y \subseteq X$ with $T(X) \subseteq Y$ such that (Y, d) is $\mathcal{R}$-complete,
(b) $\mathcal{R}$ is T-closed and locally T-transitive,
(c) either T is $\mathcal{R}$-continuous or $\mathcal{R}_{|Y}$ is d-self-closed,
(d) $X(T, \mathcal{R})$ is non-empty,
(e) there exists $\varphi \in \Omega$ such that

$$p(Tx, Ty) \leq \varphi(p(x, y)),$$

for all $x, y \in X$ with $(x, y) \in \mathcal{R}$. Then T has a fixed point.

Proof. In light of assumption (d), suppose that x_0 be an arbitrary element of $X(T, \mathcal{R})$. Define the sequence $\{x_n\}$ of Picard iterates with initial point x_0, i.e

$$x_n = T^n(x_0) \text{ for all } n \in \mathbb{N}_0. \tag{1}$$

Since $(x_0, Tx_0) \in \mathcal{R}$ and $\mathcal{R}$ is T-closed, we have

$$(Tx_0, T^2x_0), (T^2x_0, T^3x_0)..., (T^nx_0, T^{n+1}x_0), ... \in \mathcal{R}.$$

so that

$$(x_n, x_{n+1}) \in \mathcal{R} \text{ for all } n \in \mathbb{N}_0. \tag{2}$$

Thus the sequence $\{x_n\}$ is $\mathcal{R}$-preserving. Applying the contractive condition (e), we have

$$p(x_n, x_{n+1}) = p(Tx_{n-1}, Tx_n) \leq \varphi(p(x_{n-1}, x_n)) \text{ for all } n \in \mathbb{N}_0.$$

Hence by Lemma 2.17, we have

$$\lim_{n \to +\infty} p(x_n, x_{n+1}) = 0. \tag{3}$$

Now, we shall show that $\{x_n\}$ is a Cauchy sequence. On contrary, suppose that $\{x_n\}$ be not Cauchy sequence. Therefore, by Lemma 2.18, there exist $\epsilon > 0$ and two subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$ such that

$$k \leq m_k \leq n_k,\ p(x_{m_k}, x_{n_k}) > \epsilon \geq p(x_{m_k}, x_{n_{k-1}}) \text{ for all } k \in \mathbb{N}. \tag{4}$$

Next, in the light of Lemma 2.18 we have

$$\lim_{k \to \infty} p(x_{m_k}, x_{n_k}) = \lim_{k \to \infty} p(x_{m_{k+1}}, x_{n_{k+1}}) = \epsilon. \tag{5}$$

Denote $r_k := p(x_{m_k}, x_{n_k})$. Since $\{x_n\}$ is $\mathcal{R}$-preserving and $\{x_n\} \subset T(X)$ (from (3.1)), in the light of locally T-transitivity of $\mathcal{R}$, we have $(x_{m_k}, x_{n_k}) \in \mathcal{R}$. Hence, applying contractive condition (e), we have

$$\begin{aligned} p(x_{m_{k+1}}, x_{n_{k+1}}) &= p(Tx_{m_k}, Tx_{n_k}) \\ &\leq \varphi(p(x_{m_k}, x_{n_k})) \\ &= \varphi(r_k) \end{aligned}$$

so that

$$p(x_{m_{k+1}}, x_{n_{k+1}}) \leq \varphi(r_k). \tag{6}$$

Using the facts that $r_k \to \epsilon$ in the real line as $k \to +\infty$ (from (3.5)), and $r_k > \epsilon$ for all $k \in \mathbb{N}$ (from (3.4)) and by the definition of Ω, we have

$$\limsup_{k \to \infty} \varphi(r_k) = \limsup_{r \to \epsilon^+} \varphi(r) < \epsilon. \tag{7}$$

Taking limit superior as $k \to +\infty$ in (3.6) and using (3.5) and (3.7), we have

$$\epsilon = \limsup_{k\to+\infty} p(x_{m_{k+1}}, x_{n_{k+1}}) \leq \limsup_{k\to+\infty} \varphi(r_k) < \epsilon$$

which is a contradiction. Therefore, by (L_3), of Lemma 2.18 we have $\{x_n\}$ is an $\mathcal{R}$-preserving Cauchy sequence in Y. As (Y, d) is $\mathcal{R}$-complete, we must have $x_n \to x$ as $n \to +\infty$ for some $x \in Y$.
Next, we claim that x is a fixed point of T. At first we consider that T is $\mathcal{R}$-continuous. As $\{x_n\}$ is $\mathcal{R}$-preserving with $x_n \xrightarrow{p} x$, $\mathcal{R}$-continuity of T implies that $x_{n+1} = Tx_n \xrightarrow{p} Tx$. Using the uniqueness of limit, we obtain $Tx = x$, i.e, x is a fixed point of T.
Alternately, let us assume that $\mathcal{R}_{|Y}$ is d-self-closed. So there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ with $[x_{n_k}, x] \in \mathcal{R}$ for all $k \in \mathbb{N}_0$. By using the fact that $[x_{n_k}, x] \in \mathcal{R}$ and contractive assumption (e), we have

$$p(x_{n_{k+1}}, Tx) = p(Tx_{n_k}, Tx) \leq \varphi(p(x_{n_k}, x)) \text{ for all } k \in \mathbb{N}_0.$$

We claim that

$$p(x_{n_{k+1}}, Tx) = p(x_{n_k}, x) \text{ for all } k \in \mathbb{N}. \tag{8}$$

On account of two different possibilities arising here, we consider a partition $\mathbb{N}^0, \mathbb{N}^+$ of $\mathbb{N}$, i.e., $\mathbb{N}^0 \cup \mathbb{N}^+ = \mathbb{N}$ and $\mathbb{N}^0 \cap \mathbb{N}^+ = \phi$ verifying that
(c_1) $p(x_{n_k}, x) = 0$ for all $n \in \mathbb{N}^0$,
(c_2) $p(x_{n_k}, x) > 0$ for all $n \in \mathbb{N}^+$,
In case (c_1) $p(Tx_{n_k}, Tx) = 0$ for all $k \in \mathbb{N}^0$ i.e. $p(x_{n_k}, x) = 0$ for all $k \in \mathbb{N}^0$, hence (3.8) holds for all $k \in \mathbb{N}^0$. In case (c_2), owing to definition of Ω we have $p(x_{n_k+1}, Tx) \leq \varphi(p(x_{n_k}, x)) < p(x_{n_k}, x)$ for all $n \in \mathbb{N}^+$. Finally (3.8) holds for all $n \in \mathbb{N}$.
Taking limit of (3.8) as $k \to +\infty$ and using $x_{n_k} \xrightarrow{p} x$, we have $x_{n_k+1} \xrightarrow{p} Tx$. By using the uniqueness of limit, we obtain $T(x) = x$ so that x is a fixed point of T. □

Remark 3.2. *Theorem 3.1 remains true if we replace the locally T-transitivity of $\mathcal{R}$ by any one of the following conditions retaining rest of the conditions:*
(i) $\mathcal{R}$ is transitive,
(ii) $\mathcal{R}$ is T-transitive,
(iii) $\mathcal{R}$ is locally transitive.

3.1. *Uniqueness result*

We state the uniqueness related result as follows :

Theorem 3.3. *In addition to the hypotheses of Theorem 3.1, suppose that the following condition holds:*
(u) $T(X)$ is $\mathcal{R}^s$-connected. Then T has a unique fixed point.

Proof. Let x and y be two fixed points of T, i.e., $F(T) \neq \phi$ and $x, y \in F(T)$, then for all $n \in \mathbb{N}_0$, we have

$$T^n x = x, \quad T^n y = y. \tag{9}$$

Clearly $x, y \in T(X)$. By assumption (u), there exists a path (say $z_0, z_1, z_2, ..., z_k$) of some finite length k in $\mathcal{R}^s$ from x to y so that

$$z_0 = x, \;\; z_k = y \text{ and } [z_i, z_{i+1}] \in \mathcal{R} \text{ for each } i (0 \leq i \leq k-1). \tag{10}$$

As $\mathcal{R}$ is T-closed, using Theorem 2.6 and Theorem 2.7, we have

$$[T^n z_i, T^n z_{i+1}] \in \mathcal{R} \text{ for each } i(0 \leq i \leq k-1) \text{ and for each } n \in \mathbb{N}_0. \tag{11}$$

Now we shall discuss the following two cases:
Case 1: In this case suppose that $(x, y) \in R^s$, then by Theorem 2.3 either $(T^n x, T^n y) \in \mathcal{R}$ or $(T^n y, T^n x,) \in \mathcal{R}$.

Using $\mathcal{R}$ is T-closed in the light of Theorem 2.7, we have
$(T^n x, T^n y) \in \mathcal{R}$ for $n = 0, 1, 2...$ and

$$\begin{aligned} p(x,y) = p(T^n x, T^n y) &\leq \varphi(d(T^{n-1}x, T^{n-1}y)) \\ &\leq \varphi(p(x,y)) < p(x,y). \end{aligned}$$

which leads to a contradiction. We obtain $x = y$, i.e., T has a unique fixed point.
Case 2: If $(x, y) \neq \mathcal{R}^s$ then there exists a path of length $k > 1$ in $\mathcal{R}^s$. In the light of (3.10) and (3.11), we define $t_n^i = p(T^n z_i, T^n z_{i+1})$. Also, $\mathcal{R}^s$ is T-closed implies that $t_n^i = p(T^n z_i, T^n z_{i+1})$. $\in \mathcal{R}^s$ for $i(0 \leq i \leq k-1)$ and $n = 0, 1, 2....$ Moreover, for any fix i we have

$$\begin{aligned} t_n^i &= p(T^n z_i, T^n z_{i+1}) \\ &\leq \varphi(p(T^{n-1}z_i, T^{n-1}z_{i+1})) \\ &= \varphi(t_{n-1}^i) \end{aligned}$$

so that

$$t_n^i \leq \varphi(t_{n-1}^i).$$

Consequently, $t_n^i = p(T^n z_i, T^n z_{i+1})$ is a non-negative non-increasing sequence and hence possess limit t. Taking limit $n \to \infty$ in last inequality, we have $t \leq \varphi(t) < t$, and hence $t = 0$ for each $i(0 \leq i \leq k-1)$.
Finally, making the use of triangular inequality in the light of the above conclusion, we obtain

$$p(x,y) = p(T^n z_0, T^n z_k) \leq t_n^0 + t_n^1 + ... + t_n^{k-1} \to o$$

as $n \to +\infty$. Hence T has a unique fixed point. □

Remark 3.4. *Theorem 3.3 remains true if we replace the condition (u) by one of the following conditions (besides retaining rest of the conditions):*
(u_1) *$\mathcal{R}|_{T(X)}$ is complete,*
(u_2) *$T(X)$ is $\mathcal{R}^s$-directed.*

Example 3.5. *Let $X = [0, +\infty)$ equipped with usual metric d. Then (X, d) is a complete metric space. Define a binary relation $\mathcal{R} = \{(x, y) \in X \times X : x \leq y\}$ and a mapping $T : X \to X$ such that*

$$T(x) = \frac{x}{1+x}, \ x \in X.$$

Then $\mathcal{R}$ is T-closed. Define $\varphi : [0, +\infty) \to [0, +\infty)$ by $\varphi(t) = \frac{t}{1+t}$, $t \in [0, \infty)$, and a w-distance $p : X \times X \to X$ by $p(x, y) = y$. Now for all $x, y \in X$ with $(x, y) \in \mathcal{R}$, we have

$$p(T(x), T(y)) = p\Big(\frac{x}{1+x}, \frac{y}{1+y}\Big) \leq \varphi(p(x,y)),$$
$$\frac{y}{1+y} \leq \varphi(y) = \frac{y}{1+y}.$$

So that T and φ satisfy assumption (e) of Theorem 3.1. Observe that all other assumptions of Theorem 3.1 are also satisfied. Therefore, T has a unique fixed point (namely $x = 0$).

Remark 3.6. *It is interesting to note that the mapping T in above example does not satisfy the contractive condition of Theorem 2.1 in Senapati and Dey [21].*
For example, if we consider $x = 0$ and $y = \epsilon$ where ϵ is arbitrary small but positive. Clearly, $(0, \epsilon) \in \mathcal{R}$ and if we take a constant λ such that $p(T(x), T(y)) \leq \lambda p(x, y)$, i.e., $\frac{\epsilon}{1+\epsilon} \leq \lambda\epsilon$ then $\lambda \geq \frac{1}{1+\epsilon}$ which amounts to say that $\lambda \geq 1$ so that $\lambda \notin [0, 1)$. Thus Example 3.5 vindicate the utility of Theorem 3.1 and Theorem 3.3 over the results of Sanapati and Dey [21] and many others.

Remark 3.7. *If we take $\varphi(t) = \lambda t$, in our main result Theorem 3.1, then we obtain the Theorem 2.1 of Senapati and Dey [21] and if we set $p(x, y) = d(x, y)$, and $\varphi(t) = \lambda t$, in our main result, we obtain the Theorem 3.1 of Alam and Imdad [2]. Hence our main result is an improved and generalized version of relation-theoretic metrical fixed-point theorems of Alam and Imdad [2], Senapati and Dey [21] and many others.*

4. An application

As an application, we present a unique solution for the first order periodic boundary value problem equipped with an arbitrary binary relation, wherein our main results are applicable. Considering the first order periodic boundary value problem as follows :

$$x'(t) = f(t, x(t)), \;\; t \in I = [0, T], \;\; x(0) = x(T), \tag{12}$$

where $T > 0$ and $f : I \times \mathbb{R} \to \mathbb{R}$ is a continuous function.
Let $C(I)$ denote the space of all continuous functions defined on I. We recall the following definitions.

Definition 4.1. [7]. *A function $\alpha \in C^1(I)$ is called a lower solution of (4.1), if*

$$\alpha'(t) \leq f(t, \alpha(t)), t \in I,$$
$$x(0) \leq x(T).$$

Definition 4.2. [7]. *A function $\alpha \in C^1(I)$ is called a upper solution of (4.1), if*

$$\alpha'(t) \geq f(t, \alpha(t)), t \in I,$$
$$x(0) \geq x(T).$$

Theorem 4.3. *In addition to the problem (4.1), suppose that there exist $\lambda > 0$ such that for all $x, y \in \mathbb{R}$ with $x \leq y$.*

$$0 \leq f(t, y) + \lambda y - [f(t, x) + \lambda x] \leq \lambda\varphi(y - x). \tag{13}$$

Then the existence of a lower solution or an upper solution of problem (4.1) ensures the existence and uniqueness of a solution of problem (4.1).

Proof. Problem (4.1) can be rewritten as

$$x'(t) + \lambda x(t) = f(t, x(t)) + \lambda x(t), \; t \in I = [0, T], \;\; x(0) = x(T).$$

This problem is equivalent to the integral equation

$$x(t) = \int_0^T G(t, s)[f(s, x(s)) + \lambda x(s)]ds$$

where

$$G(t, s) = \begin{cases} \frac{e^{\lambda(T+s-t)}}{e^{\lambda T}-1}, & 0 \leq s < t \leq T \\ \frac{e^{\lambda(s-t)}}{e^{\lambda T}-1}, & 0 \leq t < s \leq T. \end{cases}$$

Define a mapping $T : C(I) \to C(I)$ by

$$(Tx)(t) = \int_0^T G(t, s)[f(s, x(s)) + \lambda x(s)]ds,$$

and a binary relation

$$\mathcal{R} = \{(x, y) \in C(I) \times C(I) : x(t) \leq y(t), \text{ for all } t \in I\}.$$

(i) Note that $C(I)$ equipped with the sup-metric, i.e.,
$d(x, y) = sup \left|x(t) - y(t)\right|$ for $t \in I$ and $x, y \in C(I)$ is a complete metric space and hence $(C(I), d)$ is $\mathcal{R}$-complete.
(ii) Choose an $\mathcal{R}$-preserving sequence $\{x_n\}$ such that $x_n \xrightarrow{d} z$. Then for all $t \in I$, we get

$$x_0(t) \leq x_1(t) \leq x_2(t) \leq .... \leq x_n(t) \leq x_{n+1}(t) \leq ...$$

and convergence to $x(t)$ implies that $x_n(t) \leq z(t)$ for all $t \in I, n \in \mathbb{N}_0$, which amounts to saying that $[x_n, z] \in \mathcal{R}$ for all $n \in \mathbb{N}_0$. Hence, $\mathcal{R}$ is d-self-closed.
(iii) For any $(x, y) \in \mathcal{R}$, i.e. $x(t) \leq y(t)$ then by (4.2), we have

$$f(t, x(t)) + \lambda x(t) \leq f(t, y(t)) + \lambda y(t) \text{ for all } t \in I$$

and $G(t, s) > 0$ for $(t, s) \in I \times I$, we have

$$\begin{aligned}(Tx)(t) &= \int_0^T G(t,s)[f(s, x(s)) + \lambda x(s)]ds,\\ &\leq \int_0^T G(t,s)[f(s, y(s)) + \lambda y(s)]ds\\ &= (Ty)(t) \text{ for all } t \in I,\end{aligned}$$

which implies that $(Tx, Ty) \in \mathcal{R}$, i.e., $\mathcal{R}$ is T-closed.
(iv) Let $\alpha \in C^1(I)$ be a lower solution of (4.1), then we must have

$$\alpha'(t) + \lambda\alpha(t) \leq f(t, \alpha(t)) + \lambda\alpha(t) \text{ for all } t \in I.$$

Multiplying both sides by $e^{\lambda t}$, we have

$$(\alpha(t)e^{\lambda t})' \leq [f(t, \alpha(t)) + \lambda\alpha(t)]e^{\lambda t} \text{ for all } t \in I$$

which implies that

$$\alpha(t)e^{\lambda t} \leq \alpha(0) + \int_0^t [f(s, \alpha(s)) + \lambda\alpha(s)]e^{\lambda s}ds \text{ for all } t \in I. \tag{14}$$

As $\alpha(0) \leq \alpha(T)$, we have

$$\alpha(0)e^{\lambda T} \leq \alpha(T)e^{\lambda T} \leq \alpha(0) + \int_0^T [f(s, \alpha(s)) + \lambda\alpha(s)]e^{\lambda s}ds,$$

so that

$$\alpha(0) \leq \int_0^T \frac{e^{\lambda s}}{e^{\lambda T} - 1}[f(s, \alpha(s)) + \lambda\alpha(s)]ds. \tag{15}$$

Using (4.3) and (4.4), we have

$$\begin{aligned}\alpha(t)e^{\lambda t} &\leq \int_0^T \frac{e^{\lambda s}}{e^{\lambda T} - 1}[f(s, \alpha(s)) + \lambda\alpha(s)]ds + \int_0^t e^{\lambda s}[f(s, \alpha(s)) + \lambda\alpha(s)]ds\\ &= \int_0^t \frac{e^{\lambda(T+s)}}{e^{\lambda T} - 1}[f(s, \alpha(s)) + \lambda\alpha(s)]ds + \int_t^T \frac{e^{\lambda(s)}}{e^{\lambda T} - 1}[f(s, \alpha(s)) + \lambda\alpha(s)]ds,\end{aligned}$$

that is,

$$\begin{aligned}\alpha(t) &\leq \int_0^t \frac{e^{\lambda(T+s-t)}}{e^{\lambda T} - 1}[f(s, \alpha(s)) + \lambda\alpha(s)]ds + \int_t^T \frac{e^{\lambda(s-t)}}{e^{\lambda T} - 1}[f(s, \alpha(s)) + \lambda\alpha(s)]ds\\ &= \int_0^T G(t,s)[f(s, \alpha(s)) + \lambda\alpha(s)]ds\\ &= (T\alpha)(t)\end{aligned}$$

for all $t \in I$, i.e., $(\alpha(t), T\alpha(t)) \in \mathcal{R}$ for all $t \in I$ which implies that $X(T, \mathcal{R}) \neq \phi$.
(v) For all $(x, y) \in \mathcal{R}$,

$$\begin{aligned}
d(Tx, Ty) &= \sup_{t\in I} \left|(Tx)(t) - (Ty)(t)\right| = \sup_{t\in I}((Ty)(t) - (Tx)(t)) \\
&\leq \sup_{t\in I} \int_0^T G(t,s)[f(s,y(s)) + \lambda y(s) - f(s,x(s)) - \lambda x(s)]ds \\
&\leq \sup_{t\in I} \int_0^T G(t,s)\lambda\varphi(y(s) - x(s))ds \\
&= \lambda\varphi(d(x,y))\sup_{t\in I} \int_0^T G(t,s)ds \\
&= \lambda\varphi(d(x,y))\sup_{t\in I} \frac{1}{e^{\lambda T} - 1}\Big(\frac{1}{\lambda}e^{\lambda(T+s-t)}|_0^t + \frac{1}{\lambda}e^{\lambda(s-t)}|_t^T\Big) \\
&= \lambda\varphi(d(x,y))\sup_{t\in I} \frac{1}{\lambda(e^{\lambda T} - 1)}(e^{\lambda T} - 1) \\
&= \varphi(d(x,y)),
\end{aligned}$$

so that

$$d(Tx, Ty) \leq \varphi(d(x, y)).$$

Now, if we set $p(x, y) = d(x, y)$, then we have

$$p(Tx, Ty) \leq \varphi(p(x, y)) \text{ for all } x, y \in C(I) \text{ such that } (x, y) \in \mathcal{R},$$

where $\varphi \in \Omega$. Hence all the conditions of Theorem 3.1 are satisfied, consequently T has a fixed point. Finally following the proof of our earlier Theorem 3.3, T has a unique fixed point, which is in fact a unique solution of the problem (4.1). □

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Competing interests

The author declares that there is no conflict of interests.

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