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On the Convergence of Series of Moments for Row Sums of Random Variables

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Abstract. Given a triangular array $\{X_{n,k},\ 1 \leqslant k \leqslant n, n \geqslant 1\}$ of random variables satisfying $\mathbb{E}|X_{n,k}|^p < \infty$ for some $p \geqslant 1$ and sequences $\{b_n\}$, $\{c_n\}$ of positive real numbers, we shall prove that $\sum_{n=1}^{\infty} c_n \mathbb{E}\left[\left|\sum_{k=1}^{n}(X_{n,k} - \mathbb{E}\,X_{n,k})\right| / b_n - \varepsilon\right]_+^p < \infty$, where $x_+ = \max(x, 0)$. Our results are announced in a general setting, allowing us to obtain the convergence of the series in question under various types of dependence.

1. Introduction

In [5], Li and Spătaru proved the following statement: if $\{X_n,\ n \geqslant 1\}$ is a sequence of independent and identically distributed (i.i.d.) random variables with $\mathbb{E}\,X_1 = 0$ and $p > 0$, $0 < q < 2$, $r \geqslant 1$ are such that $qr \geqslant 1$, then

$$\begin{cases} \mathbb{E}|X_1|^p < \infty \ \text{ if } p > qr \\ \mathbb{E}|X_1|^{qr} \log(1 + |X_1|) < \infty \ \text{ if } p = qr \\ \mathbb{E}|X_1|^{qr} < \infty \ \text{ if } p < qr \end{cases} \tag{1.1}$$

is equivalent to

$$\int_{\varepsilon}^{\infty} \sum_{n=1}^{\infty} n^{r-2} \mathbb{P}\left\{\left|\sum_{k=1}^{n} X_k\right| > x^{1/p} n^{1/q}\right\} \mathrm{d}x < \infty \quad \text{for all } \varepsilon > 0.$$

A few years later, Chen and Wang [2] showed that, letting $p > 0$, $\{X_n,\ n \geqslant 1\}$ be a random sequence and $\{b_n\}$, $\{c_n\}$ be sequences of positive real numbers,

$$\int_{\varepsilon}^{\infty} \sum_{n=1}^{\infty} c_n \mathbb{P}\left\{|X_n| > x^{1/p} b_n\right\} \mathrm{d}x < \infty \quad \text{for all } \varepsilon > 0$$

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and

$$\sum_{n=1}^{\infty} c_n \mathbb{E}\left[\max\left(\frac{|X_n|}{b_n} - \varepsilon, 0\right)\right]^p < \infty \text{ for all } \varepsilon > 0$$

are equivalent. Hence, putting $x_+ = \max(x, 0)$, Li and Spătaru's result can be restated as: if $\{X_n,\ n \geqslant 1\}$ is a sequence of i.i.d. random variables with $\mathbb{E}\, X_1 = 0$ and $p > 0$, $0 < q < 2$, $r \geqslant 1$ are such that $qr \geqslant 1$, then (1.1) is equivalent to

$$\sum_{n=1}^{\infty} n^{r-2}\mathbb{E}\left(n^{-1/q}\left|\sum_{k=1}^{n} X_k\right| - \varepsilon\right)_+^p < \infty \text{ for all } \varepsilon > 0. \tag{1.2}$$

The extension of (1.2) to arrays of (dependent) random variables has been emerged in literature over the last years (see [9], [11], [12], [13] or more recently [14]). Our purpose in this paper is to give general sufficient conditions to obtain

$$\sum_{n=1}^{\infty} c_n\mathbb{E}\left[\frac{\left|\sum_{k=1}^{n}(X_{n,k} - \mathbb{E}\, X_{n,k})\right|}{b_n} - \varepsilon\right]_+^p < \infty \text{ for all } \varepsilon > 0 \tag{1.3}$$

when $\{X_{n,k},\ 1 \leqslant k \leqslant n, n \geqslant 1\}$ is a triangular array of random variables and $\{b_n\}$, $\{c_n\}$ are sequences of positive constants. Namely, we shall assume that row sums of a suitable truncated triangular array of random variables satisfies classical moment inequalities, scilicet, a von Bahr-Esseen type inequality [10] or a Rosenthal type inequality (see, for instance, [8] page 59). These are general assumptions which cover well-known dependent structures, particularly extended negatively dependence or pairwise negatively quadrant dependence (see Lemma 3.6 in last section).

In the sequel, we shall denote the indicator random variable of an event A by I_A and, for each $t > 0$, we shall define also the function $g_t(x) = \max(\min(x, t), -t)$ which describes the truncation at level t.

2. Main results

In our first two statements, we shall establish the convergence of series (1.3) by assuming that, for any $t > 0$, the (truncated) array of random variables $\{g_t(X_{n,k}),\ 1 \leqslant k \leqslant n, n \geqslant 1\}$ satisfies a von Bahr-Esseen type inequality, i.e. there is a sequence of positive numbers $\{\alpha_n\}$ such that for some $q > 1$,

$$\mathbb{E}\left|\sum_{k=1}^{n} [g_t(X_{n,k}) - \mathbb{E}\, g_t(X_{n,k})]\right|^q \leqslant \alpha_n \sum_{k=1}^{n} \mathbb{E}\, |g_t(X_{n,k})|^q \tag{2.1}$$

for all $n \geqslant 1$ and $t > 0$.

Theorem 2.1. *Let $p > 1$, $\{X_{n,k},\ 1 \leqslant k \leqslant n, n \geqslant 1\}$ be an array of random variables satisfying $\mathbb{E}|X_{n,k}|^p < \infty$ for each $1 \leqslant k \leqslant n$ and all $n \geqslant 1$, and verifying (2.1) for a $q > p$ and some sequence $\{\alpha_n\}$ of positive numbers. If $\{b_n\}$, $\{c_n\}$ are real sequences of positive numbers such that*

(a) $\sum_{n=1}^{\infty}\sum_{k=1}^{n} \alpha_n c_n b_n^{-q} \int_0^{b_n^q} \mathbb{P}\{|X_{n,k}|^q > t\}\, \mathrm{d}t < \infty$,

(b) $\sum_{n=1}^{\infty}\sum_{k=1}^{n} c_n \mathbb{E}\, |X_{n,k}| I_{\{|X_{n,k}| > b_n\}} / b_n < \infty$,

(c) $\sum_{n=1}^{\infty}\sum_{k=1}^{n} (1 + \alpha_n) c_n b_n^{-p} \int_{b_n^p}^{\infty} \mathbb{P}\{|X_{n,k}|^p > t\}\, \mathrm{d}t < \infty$,

then

$$\sum_{n=1}^{\infty} c_n\mathbb{E}\left[\frac{\left|\sum_{k=1}^{n}(X_{n,k} - \mathbb{E}\, X_{n,k})\right|}{b_n} - \varepsilon\right]_+^p < \infty$$

for all $\varepsilon > 0$.

Proof. Fixing $\varepsilon > 0$, we have

$$\begin{aligned}\mathbb{E}\left[\left|\sum_{k=1}^{n}(X_{n,k} - \mathbb{E}\,X_{n,k})\right| - \varepsilon b_n\right]_+^p &= \int_0^\infty \mathbb{P}\left\{\left|\sum_{k=1}^{n}(X_{n,k} - \mathbb{E}\,X_{n,k})\right| > \varepsilon b_n + t^{1/p}\right\}\mathrm{d}t \\ &\leqslant b_n^p \mathbb{P}\left\{\left|\sum_{k=1}^{n}(X_{n,k} - \mathbb{E}\,X_{n,k})\right| > \varepsilon b_n\right\} + \int_{b_n^p}^\infty \mathbb{P}\left\{\left|\sum_{k=1}^{n}(X_{n,k} - \mathbb{E}\,X_{n,k})\right| > t^{1/p}\right\}\mathrm{d}t.\end{aligned} \tag{2.2}$$

Defining $X'_{n,k} := g_{b_n}(X_{n,k})$ and $X''_{n,k} = X_{n,k} - X'_{n,k}$, Chebyshev inequality and assumption (2.1) entail

$$\begin{aligned}&\mathbb{P}\left\{\left|\sum_{k=1}^{n}(X_{n,k} - \mathbb{E}\,X_{n,k})\right| > \varepsilon b_n\right\} \\ &\leqslant \mathbb{P}\left\{\left|\sum_{k=1}^{n}(X'_{n,k} - \mathbb{E}\,X'_{n,k})\right| > \frac{\varepsilon b_n}{2}\right\} + \mathbb{P}\left\{\left|\sum_{k=1}^{n}(X''_{n,k} - \mathbb{E}\,X''_{n,k})\right| > \frac{\varepsilon b_n}{2}\right\} \\ &\leqslant \frac{2^q}{\varepsilon^q b_n^q}\mathbb{E}\left|\sum_{k=1}^{n}(X'_{n,k} - \mathbb{E}\,X'_{n,k})\right|^q + \frac{2}{\varepsilon b_n}\mathbb{E}\left|\sum_{k=1}^{n}(X''_{n,k} - \mathbb{E}\,X''_{n,k})\right| \\ &\leqslant \frac{2^q \alpha_n}{\varepsilon^q b_n^q}\sum_{k=1}^{n}\mathbb{E}\,|X'_{n,k}|^q + \frac{4}{\varepsilon b_n}\sum_{k=1}^{n}\mathbb{E}\,|X''_{n,k}| \\ &\leqslant \frac{2^{2q-1}\alpha_n}{\varepsilon^q b_n^q}\sum_{k=1}^{n}\left[\mathbb{E}\,|X_{n,k}|^q I_{\{|X_{n,k}|\leqslant b_n\}} + b_n^q \mathbb{P}\{|X_{n,k}| > b_n\}\right] + \frac{4}{\varepsilon b_n}\sum_{k=1}^{n}\mathbb{E}\,|X_{n,k}| I_{\{|X_{n,k}|>b_n\}} \\ &= \frac{2^{2q-1}\alpha_n}{\varepsilon^q b_n^q}\sum_{k=1}^{n}\int_0^{b_n^q}\mathbb{P}\{|X_{n,k}|^q > t\}\,\mathrm{d}t + \frac{4}{\varepsilon b_n}\sum_{k=1}^{n}\mathbb{E}\,|X_{n,k}| I_{\{|X_{n,k}|>b_n\}}.\end{aligned} \tag{2.3}$$

Setting $Y'_{n,k} := g_{t^{1/p}}(X_{n,k})$ and $Y''_{n,k} = X_{n,k} - Y'_{n,k}$, it follows

$$\begin{aligned}&\int_{b_n^p}^\infty \mathbb{P}\left\{\left|\sum_{k=1}^{n}(X_{n,k} - \mathbb{E}\,X_{n,k})\right| > t^{1/p}\right\}\mathrm{d}t \\ &\leqslant \int_{b_n^p}^\infty \mathbb{P}\left\{\left|\sum_{k=1}^{n}(Y'_{n,k} - \mathbb{E}\,Y'_{n,k})\right| > \frac{t^{1/p}}{2}\right\}\mathrm{d}t + \int_{b_n^p}^\infty \mathbb{P}\left\{\left|\sum_{k=1}^{n}(Y''_{n,k} - \mathbb{E}\,Y''_{n,k})\right| > \frac{t^{1/p}}{2}\right\}\mathrm{d}t.\end{aligned}$$

Hence,

$$\begin{aligned}&\int_{b_n^p}^\infty \mathbb{P}\left\{\left|\sum_{k=1}^{n}\left(Y'_{n,k} - \mathbb{E}\,Y'_{n,k}\right)\right| > \frac{t^{1/p}}{2}\right\}\mathrm{d}t \\ &\leqslant 2^q \int_{b_n^p}^\infty t^{-q/p}\mathbb{E}\left|\sum_{k=1}^{n}\left(Y'_{n,k} - \mathbb{E}\,Y'_{n,k}\right)\right|^q \mathrm{d}t \\ &\leqslant 2^q \alpha_n \int_{b_n^p}^\infty t^{-q/p}\sum_{k=1}^{n}\mathbb{E}|Y'_{n,k}|^q \mathrm{d}t \\ &\leqslant 2^{2q-1}\alpha_n \int_{b_n^p}^\infty t^{-q/p}\sum_{k=1}^{n}\left[\mathbb{E}|X_{n,k}|^q I_{\{|X_{n,k}|\leqslant t^{1/p}\}} + t^{q/p}\mathbb{P}\left\{\left|X_{n,k}\right| > t^{1/p}\right\}\right]\mathrm{d}t\end{aligned} \tag{2.4}$$

$$
\begin{aligned}
&= 2^{2q-1}q\alpha_n \sum_{k=1}^{n} \int_{b_n^p}^{\infty} t^{-q/p} \int_0^{t^{1/p}} s^{q-1}\,\mathbb{P}\left\{\left|X_{n,k}\right| > s\right\} \mathrm{d}s\,\mathrm{d}t \\
&= 2^{2q-1}q\alpha_n \sum_{k=1}^{n} \int_0^{\infty} s^{q-1}\,\mathbb{P}\left\{\left|X_{n,k}\right| > s\right\} \int_{\max\left(b_n^p, s^p\right)}^{\infty} t^{-q/p}\mathrm{d}t\,\mathrm{d}s \\
&= 2^{2q-1}q\alpha_n \sum_{k=1}^{n} \left[\frac{p b_n^{p-q}}{q-p} \int_0^{b_n} s^{q-1}\,\mathbb{P}\left\{\left|X_{n,k}\right| > s\right\} \mathrm{d}s + \frac{p}{q-p}\int_{b_n}^{\infty} s^{p-1}\mathbb{P}\left\{\left|X_{n,k}\right| > s\right\}\,\mathrm{d}s\right] \\
&= \frac{p\,2^{2q-1}\alpha_n b_n^{p-q}}{q-p} \sum_{k=1}^{n} \int_0^{b_n^q} \mathbb{P}\left\{\left|X_{n,k}\right| > t^{1/q}\right\} \mathrm{d}t + \frac{q\,2^{2q-1}\alpha_n}{q-p} \sum_{k=1}^{n} \int_{b_n^p}^{\infty} \mathbb{P}\left\{\left|X_{n,k}\right| > t^{1/p}\right\}\,\mathrm{d}t.
\end{aligned}
$$

On the other hand, $\left|Y''_{n,k}\right| \leqslant \left|X_{n,k}\right| I_{\{|X_{n,k}|>t^{1/p}\}}$, we obtain for every $p > 1$

$$
\begin{aligned}
&\int_{b_n^p}^{\infty} \mathbb{P}\left\{\left|\sum_{k=1}^{n}\left(Y''_{n,k} - \mathbb{E}\,Y''_{n,k}\right)\right| > \frac{t^{1/p}}{2}\right\}\mathrm{d}t \\
&\qquad \leqslant 4\int_{b_n^p}^{\infty} t^{-1/p} \sum_{k=1}^{n} \mathbb{E}\left|Y''_{n,k}\right|\mathrm{d}t \\
&\qquad \leqslant 4\sum_{k=1}^{n} \int_{b_n^p}^{\infty} t^{-1/p}\mathbb{E}\left|X_{n,k}\right| I_{\{|X_{n,k}|>t^{1/p}\}}\mathrm{d}t \\
&\qquad = 4\sum_{k=1}^{n}\left(\int_{b_n^p}^{\infty} t^{-1/p}\int_{t^{1/p}}^{\infty} \mathbb{P}\left\{\left|X_{n,k}\right| > s\right\}\mathrm{d}s\,\mathrm{d}t + \int_{b_n^p}^{\infty}\mathbb{P}\left\{\left|X_{n,k}\right| > t^{1/p}\right\}\mathrm{d}t\right) \qquad (2.5)\\
&\qquad = 4\sum_{k=1}^{n}\left(\int_{b_n}^{\infty} \mathbb{P}\left\{\left|X_{n,k}\right| > s\right\}\int_{b_n^p}^{s^p} t^{-1/p}\mathrm{d}t\,\mathrm{d}s + \int_{b_n^p}^{\infty}\mathbb{P}\left\{\left|X_{n,k}\right| > t^{1/p}\right\}\mathrm{d}t\right) \\
&\qquad \leqslant 4\sum_{k=1}^{n}\left(\frac{p}{p-1}\int_{b_n}^{\infty} s^{p-1}\mathbb{P}\left\{\left|X_{n,k}\right| > s\right\}\mathrm{d}s + \int_{b_n^p}^{\infty}\mathbb{P}\left\{\left|X_{n,k}\right| > t^{1/p}\right\}\mathrm{d}t\right) \\
&\qquad = \frac{4p}{p-1}\sum_{k=1}^{n}\int_{b_n^p}^{\infty}\mathbb{P}\left\{\left|X_{n,k}\right| > t^{1/p}\right\}\mathrm{d}t.
\end{aligned}
$$

Thus, by gathering (2.2), (2.3), (2.4) and (2.5) we get

$$
\begin{aligned}
&\sum_{n=1}^{\infty} c_n \mathbb{E}\left[\frac{\left|\sum_{k=1}^{n}(X_{n,k} - \mathbb{E}\,X_{n,k})\right|}{b_n} - \varepsilon\right]_+^p \\
&= \sum_{n=1}^{\infty} \frac{c_n}{b_n^p}\mathbb{E}\left[\left|\sum_{k=1}^{n}(X_{n,k} - \mathbb{E}\,X_{n,k})\right| - \varepsilon b_n\right]_+^p \\
&\leqslant \sum_{n=1}^{\infty}\left(c_n\mathbb{P}\left\{\left|\sum_{k=1}^{n}(X_{n,k} - \mathbb{E}\,X_{n,k})\right| > \varepsilon b_n\right\} + \frac{c_n}{b_n^p}\int_{b_n^p}^{\infty}\mathbb{P}\left\{\left|\sum_{k=1}^{n}(X_{n,k} - \mathbb{E}\,X_{n,k})\right| > t^{1/p}\right\}\mathrm{d}t\right) \qquad (2.6)\\
&\leqslant \left(\frac{2^{2q-1}}{\varepsilon^q} + \frac{p2^{2q-1}}{q-p}\right)\sum_{n=1}^{\infty}\sum_{k=1}^{n}\frac{\alpha_n c_n}{b_n^q}\int_0^{b_n^q}\mathbb{P}\left\{\left|X_{n,k}\right|^q > t\right\}\mathrm{d}t + \frac{4}{\varepsilon}\sum_{n=1}^{\infty}\sum_{k=1}^{n}\frac{c_n}{b_n}\mathbb{E}\,|X_{n,k}|I_{\{|X_{n,k}|>b_n\}} \\
&\quad + \left(\frac{q\,2^{2q-1}}{q-p} + \frac{4p}{p-1}\right)\sum_{n=1}^{\infty}\sum_{k=1}^{n}\frac{(1+\alpha_n)c_n}{b_n^p}\int_{b_n^p}^{\infty}\mathbb{P}\left\{\left|X_{n,k}\right|^p > t\right\}\,\mathrm{d}t \\
&< \infty,
\end{aligned}
$$

according to assumptions (a), (b) and (c). The proof is complete. □

Theorem 2.2. *Let* $\{X_{n,k},\ 1 \leqslant k \leqslant n, n \geqslant 1\}$ *be an array of random variables satisfying* $\mathbb{E}|X_{n,k}| < \infty$ *for each* $1 \leqslant k \leqslant n$ *and all* $n \geqslant 1$*, and verifying* (2.1) *for a* $q > 1$ *and some sequence* $\{\alpha_n\}$ *of positive numbers. If* $\{b_n\}$*,* $\{c_n\}$ *are real sequences of positive numbers such that*

(a) $\sum_{n=1}^{\infty} \sum_{k=1}^{n} \alpha_n c_n b_n^{-q} \int_0^{b_n^q} \mathbb{P}\{|X_{n,k}|^q > t\}\, \mathrm{d}t < \infty$,

(b) $\sum_{n=1}^{\infty} \sum_{k=1}^{n} c_n \mathbb{E}\,|X_{n,k}| I_{\{|X_{n,k}|>b_n\}}/b_n < \infty$,

(c) $\sum_{n=1}^{\infty} \sum_{k=1}^{n} (\alpha_n c_n/b_n) \int_{b_n}^{\infty} \mathbb{P}\{|X_{n,k}| > t\}\, \mathrm{d}t < \infty$,

then

$$\sum_{n=1}^{\infty} c_n \mathbb{E}\left[\frac{\left|\sum_{k=1}^{n}(X_{n,k} - \mathbb{E}\,X_{n,k})\right|}{b_n} - \varepsilon\right]_+ < \infty$$

for all $\varepsilon > 0$.

Proof. All steps in the proof of Theorem 2.1 remains true for $p = 1$ except the upper bound (2.5). Supposing $Y'_{n,k} := g_t(X_{n,k})$ and $Y''_{n,k} = X_{n,k} - Y'_{n,k}$ we have, for any $t \geqslant b_n$,

$$\begin{aligned}\left|\sum_{k=1}^{n}\left(Y''_{n,k} - \mathbb{E}\,Y''_{n,k}\right)\right| &\leqslant \sum_{k=1}^{n}\left(|Y''_{n,k}| + \mathbb{E}|Y''_{n,k}|\right)\\ &\leqslant \sum_{k=1}^{n}\left(\left|X_{n,k}\right| I_{\{|X_{n,k}|>t\}} + \mathbb{E}\left|X_{n,k}\right| I_{\{|X_{n,k}|>t\}}\right)\\ &\leqslant \sum_{k=1}^{n}\left(\left|X_{n,k}\right| I_{\{|X_{n,k}|>b_n\}} + \mathbb{E}\left|X_{n,k}\right| I_{\{|X_{n,k}|>b_n\}}\right).\end{aligned}$$

Hence,

$$\begin{aligned}&\int_{b_n}^{\infty} \mathbb{P}\left\{\left|\sum_{k=1}^{n}\left(Y''_{n,k} - \mathbb{E}\,Y''_{n,k}\right)\right| > \frac{t}{2}\right\}\mathrm{d}t\\ &\quad\leqslant \int_{b_n}^{\infty} \mathbb{P}\left\{\sum_{k=1}^{n}\left(\left|X_{n,k}\right| I_{\{|X_{n,k}|>b_n\}} + \mathbb{E}\left|X_{n,k}\right| I_{\{|X_{n,k}|>b_n\}}\right) > \frac{t}{2}\right\}\mathrm{d}t\\ &\quad= 2\int_{\frac{b_n}{2}}^{\infty} \mathbb{P}\left\{\sum_{k=1}^{n}\left(\left|X_{n,k}\right| I_{\{|X_{n,k}|>b_n\}} + \mathbb{E}\left|X_{n,k}\right| I_{\{|X_{n,k}|>b_n\}}\right) > s\right\}\mathrm{d}s \qquad (2.7)\\ &\quad\leqslant 2\,\mathbb{E}\left[\sum_{k=1}^{n}\left(\left|X_{n,k}\right| I_{\{|X_{n,k}|>b_n\}} + \mathbb{E}\left|X_{n,k}\right| I_{\{|X_{n,k}|>b_n\}}\right)\right]\\ &\quad= 4\sum_{k=1}^{n} \mathbb{E}\left|X_{n,k}\right| I_{\{|X_{n,k}|>b_n\}}\end{aligned}$$

and

$$c_n \mathbb{E}\left[\frac{\left|\sum_{k=1}^{n}(X_{n,k} - \mathbb{E}\,X_{n,k})\right|}{b_n} - \varepsilon\right]_+$$

$$
\begin{aligned}
&= \frac{c_n}{b_n}\mathbb{E}\left[\left|\sum_{k=1}^{n}(X_{n,k} - \mathbb{E}\,X_{n,k})\right| - \varepsilon b_n\right]_+ \\
&\leqslant \left(\frac{2^{2q-1}}{\varepsilon^q} + \frac{2^{2q-1}}{q-1}\right)\sum_{k=1}^{n}\frac{\alpha_n c_n}{b_n^q}\int_0^{b_n^q}\mathbb{P}\left\{\left|X_{n,k}\right|^q > t\right\}\mathrm{d}t + \left(\frac{4}{\varepsilon}+4\right)\sum_{k=1}^{n}\frac{c_n}{b_n}\mathbb{E}\,|X_{n,k}|I_{\{|X_{n,k}|>b_n\}} \\
&\quad + \frac{q\,2^{2q-1}}{q-1}\sum_{k=1}^{n}\frac{\alpha_n c_n}{b_n}\int_{b_n}^{\infty}\mathbb{P}\left\{\left|X_{n,k}\right| > t\right\}\mathrm{d}t
\end{aligned}
$$

by employing (2.2), (2.4) with $p = 1$ and (2.3), (2.7). The thesis is established. □

The next two results, give us conditions for the convergence of (1.3) under the assumption that, for every $t > 0$, the array of random variables $\{g_t(X_{n,k}),\ 1 \leqslant k \leqslant n, n \geqslant 1\}$ satisfies a Rosenthal type inequality. Specifically, we shall admit that there are sequences of positive numbers $\{\beta_n\}$ and $\{\xi_n\}$ such that for some $q > 2$,

$$
\mathbb{E}\left|\sum_{k=1}^{n}\left[g_t(X_{n,k}) - \mathbb{E}\,g_t(X_{n,k})\right]\right|^q \leqslant \beta_n\sum_{k=1}^{n}\mathbb{E}\,|g_t(X_{n,k})|^q + \xi_n\left[\sum_{k=1}^{n}\mathbb{E}\,|g_t(X_{n,k})|^2\right]^{q/2} \tag{2.8}
$$

for all $n \geqslant 1$ and $t > 0$.

Theorem 2.3. *Let $p > 1$, $\{X_{n,k},\ 1 \leqslant k \leqslant n, n \geqslant 1\}$ be an array of random variables satisfying $\mathbb{E}|X_{n,k}|^p < \infty$ for each $1 \leqslant k \leqslant n$ and all $n \geqslant 1$, and verifying* (2.8) *for a $q > \max\{p, 2\}$ and some sequences $\{\beta_n\}$, $\{\xi_n\}$ of positive numbers. If $\{b_n\}$, $\{c_n\}$ are real sequences of positive numbers such that*

(a) $\sum_{n=1}^{\infty}\sum_{k=1}^{n}\beta_n c_n b_n^{-q}\int_0^{b_n^q}\mathbb{P}\{|X_{n,k}|^q > t\}\,\mathrm{d}t < \infty$,

(b) $\sum_{n=1}^{\infty}\xi_n c_n b_n^{-p}\int_0^{b_n^{p-q}}\left(\sum_{k=1}^{n}\int_0^{t^{2/(p-q)}}\mathbb{P}\{X_{n,k}^2 > s\}\mathrm{d}s\right)^{q/2}\mathrm{d}t < \infty$,

(c) $\sum_{n=1}^{\infty}\xi_n c_n b_n^{-q}\left(\sum_{k=1}^{n}\int_0^{b_n^2}\mathbb{P}\{X_{n,k}^2 > t\}\,\mathrm{d}t\right)^{q/2} < \infty$,

(d) $\sum_{n=1}^{\infty}\sum_{k=1}^{n}c_n\mathbb{E}\,|X_{n,k}|I_{\{|X_{n,k}|>b_n\}}/b_n < \infty$,

(e) $\sum_{n=1}^{\infty}\sum_{k=1}^{n}(1+\beta_n)c_n b_n^{-p}\int_{b_n^p}^{\infty}\mathbb{P}\{|X_{n,k}|^p > t\}\,\mathrm{d}t < \infty$,

then

$$
\sum_{n=1}^{\infty}c_n\mathbb{E}\left[\frac{\left|\sum_{k=1}^{n}(X_{n,k} - \mathbb{E}\,X_{n,k})\right|}{b_n} - \varepsilon\right]_+^p < \infty
$$

for all $\varepsilon > 0$.

Proof. The proof follows in exactly the same manner as the proof of Theorem 2.1 except for upper bounds (2.3) and (2.4) which must be replaced. Letting $X'_{n,k} := g_{b_n}(X_{n,k})$ and $X''_{n,k} = X_{n,k} - X'_{n,k}$, assumption (2.8) ensures

$$
\begin{aligned}
&\mathbb{P}\left\{\left|\sum_{k=1}^{n}(X_{n,k} - \mathbb{E}\,X_{n,k})\right| > \varepsilon b_n\right\} \\
&\quad\leqslant \mathbb{P}\left\{\left|\sum_{k=1}^{n}(X'_{n,k} - \mathbb{E}\,X'_{n,k})\right| > \frac{\varepsilon b_n}{2}\right\} + \mathbb{P}\left\{\left|\sum_{k=1}^{n}(X''_{n,k} - \mathbb{E}\,X''_{n,k})\right| > \frac{\varepsilon b_n}{2}\right\}
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant \frac{2^q}{\varepsilon^q b_n^q}\mathbb{E}\left|\sum_{k=1}^{n}(X'_{n,k}-\mathbb{E}\,X'_{n,k})\right|^q+\frac{2}{\varepsilon b_n}\mathbb{E}\left|\sum_{k=1}^{n}(X''_{n,k}-\mathbb{E}\,X''_{n,k})\right| \\
&\leqslant \frac{2^q\beta_n}{\varepsilon^q b_n^q}\sum_{k=1}^{n}\mathbb{E}\,|X'_{n,k}|^q+\frac{2^q\xi_n}{\varepsilon^q b_n^q}\left(\sum_{k=1}^{n}\mathbb{E}\,|X'_{n,k}|^2\right)^{q/2}+\frac{4}{\varepsilon b_n}\sum_{k=1}^{n}\mathbb{E}\,|X''_{n,k}| \qquad (2.9)\\
&\leqslant \frac{2^{2q-1}\beta_n}{\varepsilon^q b_n^q}\sum_{k=1}^{n}\left[\mathbb{E}\,|X_{n,k}|^q I_{\{|X_{n,k}|\leqslant b_n\}}+b_n^q\mathbb{P}\{|X_{n,k}|>b_n\}\right]+\frac{4}{\varepsilon b_n}\sum_{k=1}^{n}\mathbb{E}\,|X_{n,k}|I_{\{|X_{n,k}|>b_n\}}\\
&\quad+\frac{2^{3q/2}\xi_n}{\varepsilon^q b_n^q}\left(\sum_{k=1}^{n}\mathbb{E}\,X_{n,k}^2 I_{\{|X_{n,k}|\leqslant b_n\}}+b_n^2\sum_{k=1}^{n}\mathbb{P}\{|X_{n,k}|>b_n\}\right)^{q/2}\\
&=\frac{2^{2q-1}\beta_n}{\varepsilon^q b_n^q}\sum_{k=1}^{n}\int_0^{b_n^q}\mathbb{P}\{|X_{n,k}|^q>t\}\,\mathrm{d}t+\frac{4}{\varepsilon b_n}\sum_{k=1}^{n}\mathbb{E}\,|X_{n,k}|I_{\{|X_{n,k}|>b_n\}}\\
&\quad+\frac{2^{3q/2}\xi_n}{\varepsilon^q b_n^q}\left(\sum_{k=1}^{n}\int_0^{b_n^2}\mathbb{P}\left\{X_{n,k}^2>t\right\}\mathrm{d}t\right)^{q/2}.
\end{aligned}
$$

On the other hand, considering $Y'_{n,k} := g_{t^{1/p}}(X_{n,k})$ we have

$$
\begin{aligned}
&\int_{b_n^p}^{\infty}\mathbb{P}\left\{\left|\sum_{k=1}^{n}\left(Y'_{n,k}-\mathbb{E}\,Y'_{n,k}\right)\right|>\frac{t^{1/p}}{2}\right\}\mathrm{d}t\\
&\leqslant 2^q\int_{b_n^p}^{\infty}t^{-q/p}\mathbb{E}\left|\sum_{k=1}^{n}\left(Y'_{n,k}-\mathbb{E}\,Y'_{n,k}\right)\right|^q\mathrm{d}t\\
&\leqslant 2^q\beta_n\int_{b_n^p}^{\infty}t^{-q/p}\sum_{k=1}^{n}\mathbb{E}|Y'_{n,k}|^q\mathrm{d}t+2^q\xi_n\int_{b_n^p}^{\infty}t^{-q/p}\left(\sum_{k=1}^{n}\mathbb{E}|Y'_{n,k}|^2\right)^{q/2}\mathrm{d}t\\
&\leqslant \frac{p\,2^{2q-1}\beta_n b_n^{p-q}}{q-p}\sum_{k=1}^{n}\int_0^{b_n^q}\mathbb{P}\left\{\left|X_{n,k}\right|>t^{1/q}\right\}\mathrm{d}t+\frac{q\,2^{2q-1}\beta_n}{q-p}\sum_{k=1}^{n}\int_{b_n^p}^{\infty}\mathbb{P}\left\{\left|X_{n,k}\right|>t^{1/p}\right\}\mathrm{d}t\\
&\quad+2^{3q/2}\xi_n\int_{b_n^p}^{\infty}t^{-q/p}\left[\sum_{k=1}^{n}\left(\mathbb{E}\,X_{n,k}^2 I_{\{|X_{n,k}|\leqslant t^{1/p}\}}+t^{2/p}\mathbb{P}\left\{\left|X_{n,k}\right|>t^{1/p}\right\}\right)\right]^{q/2}\mathrm{d}t \qquad (2.10)\\
&=\frac{p\,2^{2q-1}\beta_n b_n^{p-q}}{q-p}\sum_{k=1}^{n}\int_0^{b_n^q}\mathbb{P}\left\{\left|X_{n,k}\right|>t^{1/q}\right\}\mathrm{d}t+\frac{q\,2^{2q-1}\beta_n}{q-p}\sum_{k=1}^{n}\int_{b_n^p}^{\infty}\mathbb{P}\left\{\left|X_{n,k}\right|>t^{1/p}\right\}\mathrm{d}t\\
&\quad+2^{3q/2}\xi_n\int_{b_n^p}^{\infty}t^{-q/p}\left(\int_0^{t^{2/p}}\sum_{k=1}^{n}\mathbb{P}\left\{X_{n,k}^2>s\right\}\mathrm{d}s\right)^{q/2}\mathrm{d}t\\
&=\frac{p\,2^{2q-1}\beta_n b_n^{p-q}}{q-p}\sum_{k=1}^{n}\int_0^{b_n^q}\mathbb{P}\left\{\left|X_{n,k}\right|>t^{1/q}\right\}\mathrm{d}t+\frac{q\,2^{2q-1}\beta_n}{q-p}\sum_{k=1}^{n}\int_{b_n^p}^{\infty}\mathbb{P}\left\{\left|X_{n,k}\right|>t^{1/p}\right\}\mathrm{d}t\\
&\quad+\frac{p2^{3q/2}\xi_n}{q-p}\int_0^{b_n^{p-q}}\left(\sum_{k=1}^{n}\int_0^{v^{2/(p-q)}}\mathbb{P}\left\{X_{n,k}^2>s\right\}\mathrm{d}s\right)^{q/2}\mathrm{d}v.
\end{aligned}
$$

Employing (2.2), (2.5), (2.9) and (2.10) as in (2.6) the conclusion follows. The proof is complete. □

Theorem 2.4. *Let $\{X_{n,k},\ 1\leqslant k\leqslant n, n\geqslant 1\}$ be an array of random variables satisfying $\mathbb{E}|X_{n,k}|<\infty$ for each $1\leqslant k\leqslant n$ and all $n\geqslant 1$, and verifying* (2.8) *for a $q>2$ and some sequences $\{\beta_n\}$, $\{\xi_n\}$ of positive numbers. If $\{b_n\}$, $\{c_n\}$ are real* *sequences of positive numbers such that*

(a) $\sum_{n=1}^{\infty} \sum_{k=1}^{n} \beta_n c_n b_n^{-q} \int_0^{b_n^q} \mathbb{P}\{|X_{n,k}|^q > t\} \, \mathrm{d}t < \infty$,

(b) $\sum_{n=1}^{\infty} (\xi_n c_n / b_n) \int_0^{b_n^{1-q}} \left(\sum_{k=1}^{n} \int_0^{t^{2/(1-q)}} \mathbb{P}\{X_{n,k}^2 > s\} \mathrm{d}s \right)^{q/2} \mathrm{d}t < \infty$

(c) $\sum_{n=1}^{\infty} \xi_n c_n b_n^{-q} \left(\sum_{k=1}^{n} \int_0^{b_n^2} \mathbb{P}\{X_{n,k}^2 > t\} \, \mathrm{d}t \right)^{q/2} < \infty$,

(d) $\sum_{n=1}^{\infty} \sum_{k=1}^{n} c_n \mathbb{E}\, |X_{n,k}| I_{\{|X_{n,k}| > b_n\}} / b_n < \infty$,

(e) $\sum_{n=1}^{\infty} \sum_{k=1}^{n} (\beta_n c_n / b_n) \int_{b_n}^{\infty} \mathbb{P}\{|X_{n,k}| > t\} \, \mathrm{d}t < \infty$,

then

$$\sum_{n=1}^{\infty} c_n \mathbb{E} \left[\frac{\left| \sum_{k=1}^{n} (X_{n,k} - \mathbb{E}\, X_{n,k}) \right|}{b_n} - \varepsilon \right]_+ < \infty$$

for all $\varepsilon > 0$.

Proof. The thesis is a consequence of (2.2), (2.10) with $p = 1$ and (2.7), (2.9). □

Remark 2.5. *Notice that if* $\{X_{n,k}, \ 1 \leqslant k \leqslant n, n \geqslant 1\}$ *is an array of row-wise extended negatively dependent random variables with dominating sequence* $\{M_n, n \geqslant 1\}$ *(see [6]), then* (2.8) *holds with* $q \geqslant 2$ *and* $\beta_n = \xi_n = C(q)(1 + M_n)$ *with* $C(q)$ *a positive constant depending only on* q *(see Lemma 2 of [6]); further,* (2.1) *still holds for these dependent structures with* $1 \leqslant q \leqslant 2$ *and* $\alpha_n = C(q)(1 + M_n)$*, where* $C(q) > 0$ *depends only on* q.

Supposing $0 < p \leqslant 1$, $\varepsilon > 0$ and b_n a real sequence of positive numbers, Lemma 2.1 of [9] and elementary inequality $(x + y)^p \leqslant x^p + y^p$, $x, y \geqslant 0$ lead us to

$$\mathbb{E} \left(\left| \sum_{k=1}^{n} X_{n,k} \right| - \varepsilon b_n \right)_+^p \leqslant \mathbb{E} \left(\left| \sum_{k=1}^{n} X_{n,k} I_{\{|X_{n,k}| \leqslant b_n\}} \right| - \varepsilon b_n \right)_+^p + \mathbb{E} \left| \sum_{k=1}^{n} X_{n,k} I_{\{|X_{n,k}| > b_n\}} \right|^p .$$

By taking $q > p$, we obtain

$$\begin{aligned}
&\mathbb{E} \left(\left| \sum_{k=1}^{n} X_{n,k} I_{\{|X_{n,k}| \leqslant b_n\}} \right| - \varepsilon b_n \right)_+^p \\
&\quad \leqslant b_n^p \mathbb{P} \left\{ \left| \sum_{k=1}^{n} X_{n,k} I_{\{|X_{n,k}| \leqslant b_n\}} \right| > \varepsilon b_n \right\} + \int_{b_n^p}^{\infty} \mathbb{P} \left\{ \left| \sum_{k=1}^{n} X_{n,k} I_{\{|X_{n,k}| \leqslant b_n\}} \right| > t^{1/p} \right\} \mathrm{d}t \\
&\quad \leqslant \varepsilon^{-q} b_n^{p-q} \mathbb{E} \left| \sum_{k=1}^{n} X_{n,k} I_{\{|X_{n,k}| \leqslant b_n\}} \right|^q + \mathbb{E} \left| \sum_{k=1}^{n} X_{n,k} I_{\{|X_{n,k}| \leqslant b_n\}} \right|^q \int_{b_n^p}^{\infty} t^{-q/p} \mathrm{d}t \\
&\quad = \varepsilon^{-q} b_n^{p-q} \mathbb{E} \left| \sum_{k=1}^{n} X_{n,k} I_{\{|X_{n,k}| \leqslant b_n\}} \right|^q + \frac{p b_n^{p-q}}{q - p} \mathbb{E} \left| \sum_{k=1}^{n} X_{n,k} I_{\{|X_{n,k}| \leqslant b_n\}} \right|^q \\
&\quad = \left(\varepsilon^{-q} + \frac{p}{q - p} \right) b_n^{p-q} \mathbb{E} \left| \sum_{k=1}^{n} X_{n,k} I_{\{|X_{n,k}| \leqslant b_n\}} \right|^q
\end{aligned}$$

which yields

$$\mathbb{E} \left(\left| \sum_{k=1}^{n} X_{n,k} \right| - \varepsilon b_n \right)_+^p \leqslant \left(\varepsilon^{-q} + \frac{p}{q - p} \right) b_n^{p-q} \mathbb{E} \left| \sum_{k=1}^{n} X_{n,k} I_{\{|X_{n,k}| \leqslant b_n\}} \right|^q + \mathbb{E} \left| \sum_{k=1}^{n} X_{n,k} I_{\{|X_{n,k}| > b_n\}} \right|^p . \tag{2.11}$$

Hence, we can still announce the result hereinafter whose proof follows from inequality (2.11); we omit the details.

Theorem 2.6. *Let* $0 < p < 1$, $\{X_{n,k},\ 1 \leqslant k \leqslant n, n \geqslant 1\}$ *be an array of random variables satisfying* $\mathbb{E}|X_{n,k}|^p < \infty$ *for each* $1 \leqslant k \leqslant n$ *and all* $n \geqslant 1$. *If* $\{b_n\}$, $\{c_n\}$ *are real sequences of positive numbers such that*

(a) $\sum_{n=1}^{\infty}\sum_{k=1}^{n} c_n b_n^{-q}\mathbb{E}|X_{n,k}|^q I_{\{|X_{n,k}|\leqslant b_n\}} < \infty$ *for some* $p < q \leqslant 1$,

(b) $\sum_{n=1}^{\infty}\sum_{k=1}^{n} c_n b_n^{-p}\mathbb{E}\,|X_{n,k}|^p I_{\{|X_{n,k}|>b_n\}} < \infty$,

then

$$\sum_{n=1}^{\infty} c_n\mathbb{E}\left[\frac{\left|\sum_{k=1}^{n} X_{n,k}\right|}{b_n} - \varepsilon\right]_+^p < \infty$$

for all $\varepsilon > 0$.

Remark 2.7. *Under the assumptions of Theorem 2.1 (or Theorem 2.3) we obviously have, for any* $0 < r \leqslant p$,

$$\sum_{n=1}^{\infty} c_n\mathbb{E}\left[\frac{\left|\sum_{k=1}^{n}(X_{n,k} - \mathbb{E}\,X_{n,k})\right|}{b_n} - \varepsilon\right]_+^r < \infty \tag{2.12}$$

for all $\varepsilon > 0$, *because*

$$\begin{aligned}
&\mathbb{E}\left(\left|\sum_{k=1}^{n}(X_{n,k} - \mathbb{E}\,X_{n,k})\right| - \varepsilon b_n\right)_+^r \\
&\quad\leqslant b_n^r\mathbb{P}\left\{\left|\sum_{k=1}^{n}(X_{n,k} - \mathbb{E}\,X_{n,k})\right| > \varepsilon b_n\right\} + \int_{b_n^r}^{\infty}\mathbb{P}\left\{\left|\sum_{k=1}^{n}(X_{n,k} - \mathbb{E}\,X_{n,k})\right| > t^{1/r}\right\}\mathrm{d}t \\
&\quad= b_n^r\mathbb{P}\left\{\left|\sum_{k=1}^{n}(X_{n,k} - \mathbb{E}\,X_{n,k})\right| > \varepsilon b_n\right\} + \frac{r}{p}\int_{b_n^p}^{\infty} s^{r/p-1}\mathbb{P}\left\{\left|\sum_{k=1}^{n}(X_{n,k} - \mathbb{E}\,X_{n,k})\right| > s^{1/p}\right\}\mathrm{d}s \quad \left(t = s^{r/p}\right) \\
&\quad\leqslant b_n^r\mathbb{P}\left\{\left|\sum_{k=1}^{n}(X_{n,k} - \mathbb{E}\,X_{n,k})\right| > \varepsilon b_n\right\} + b_n^{r-p}\int_{b_n^p}^{\infty}\mathbb{P}\left\{\left|\sum_{k=1}^{n}(X_{n,k} - \mathbb{E}\,X_{n,k})\right| > s^{1/p}\right\}\mathrm{d}s
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{n=1}^{\infty} c_n\mathbb{E}\left[\frac{\left|\sum_{k=1}^{n}(X_{n,k} - \mathbb{E}\,X_{n,k})\right|}{b_n} - \varepsilon\right]_+^r \\
&\quad\leqslant \sum_{n=1}^{\infty}\left(c_n\mathbb{P}\left\{\left|\sum_{k=1}^{n}(X_{n,k} - \mathbb{E}\,X_{n,k})\right| > \varepsilon b_n\right\} + \frac{c_n}{b_n^p}\int_{b_n^p}^{\infty}\mathbb{P}\left\{\left|\sum_{k=1}^{n}(X_{n,k} - \mathbb{E}\,X_{n,k})\right| > s^{1/p}\right\}\mathrm{d}s\right).
\end{aligned}$$

In the same way, (2.12) *holds for every* $0 < r \leqslant 1$ *whenever the assumptions of Theorem 2.2 (or Theorem 2.4) are met.*

3. Applications

It is straightforward to see that

$$\int_{u^p}^{\infty}\mathbb{P}\{|X_{n,k}|^p > t\}\,\mathrm{d}t \leqslant \mathbb{E}\,|X_{n,k}|^p I_{\{|X_{n,k}|>u\}}, \tag{3.1}$$

$$\mathbb{P}\{|X_{n,k}| > u\} \leqslant \frac{\mathbb{E}\,|X_{n,k}|^p I_{\{|X_{n,k}|>u\}}}{u^p} \tag{3.2}$$

and

$$\int_0^{u^q} \mathbb{P}\{|X_{n,k}|^q > t\}\,\mathrm{d}t \leqslant u^{q-p}\mathbb{E}\,|X_{n,k}|^p I_{\{|X_{n,k}|>u\}} + \mathbb{E}\,|X_{n,k}|^q I_{\{|X_{n,k}|\leqslant u\}} \tag{3.3}$$

for any $p, q, u > 0$. Thus, if $\{\alpha_n\}$ is a constant sequence then Theorems 2.1 and 2.2 can be gathered in the following result.

Corollary 3.1. *Let $p \geqslant 1$, $\{X_{n,k},\ 1 \leqslant k \leqslant n, n \geqslant 1\}$ be an array of random variables satisfying $\mathbb{E}|X_{n,k}|^p < \infty$ for each $1 \leqslant k \leqslant n$ and all $n \geqslant 1$, and verifying* (2.1) *for a $q > p$ and some constant sequence $\{\alpha_n\}$. If $\{b_n\}$, $\{c_n\}$ are real sequences of positive numbers such that*

(i) $\sum_{n=1}^{\infty}\sum_{k=1}^{n} c_n b_n^{-q}\mathbb{E}\,|X_{n,k}|^q I_{\{|X_{n,k}|\leqslant b_n\}} < \infty$,

(ii) $\sum_{n=1}^{\infty}\sum_{k=1}^{n} c_n b_n^{-p}\mathbb{E}\,|X_{n,k}|^p I_{\{|X_{n,k}|> b_n\}} < \infty$,

then

$$\sum_{n=1}^{\infty} c_n \mathbb{E}\left[\frac{\left|\sum_{k=1}^{n}(X_{n,k} - \mathbb{E}\,X_{n,k})\right|}{b_n} - \varepsilon\right]_+^p < \infty$$

for all $\varepsilon > 0$.

Proof. Since $\{\alpha_n\}$ is a constant sequence, (ii) ensures assumption (c) of Theorems 2.1 and 2.2 via (3.1). According to (3.3), (i) and (ii) together guarantee assumption (a) of Theorems 2.1 and 2.2. Finally, assumption (b) of Theorems 2.1 and 2.2 follows from (ii) by noting that

$$\frac{|X_{n,k}| I_{\{|X_{n,k}|>b_n\}}}{b_n} \leqslant \frac{|X_{n,k}|^p I_{\{|X_{n,k}|>b_n\}}}{b_n^p}.$$

Hence, Corollary 3.1 is proved. □

Similarly, we can also join Theorems 2.3 and 2.4 when sequences $\{\beta_n\}$ and $\{\xi_n\}$ are both constant.

Corollary 3.2. *Let $p \geqslant 1$, $\{X_{n,k},\ 1 \leqslant k \leqslant n, n \geqslant 1\}$ be an array of random variables satisfying $\mathbb{E}|X_{n,k}|^p < \infty$ for each $1 \leqslant k \leqslant n$ and all $n \geqslant 1$, and verifying* (2.8) *for a $q > \max\{p, 2\}$ and some constant sequences $\{\beta_n\}$, $\{\xi_n\}$. If $\{b_n\}$, $\{c_n\}$ are real sequences of positive numbers such that*

(i) $\sum_{n=1}^{\infty}\sum_{k=1}^{n} c_n b_n^{-q}\mathbb{E}\,|X_{n,k}|^q I_{\{|X_{n,k}|\leqslant b_n\}} < \infty$,

(ii) $\sum_{n=1}^{\infty}\sum_{k=1}^{n} c_n b_n^{-p}\mathbb{E}\,|X_{n,k}|^p I_{\{|X_{n,k}|> b_n\}} < \infty$,

(iii) $\sum_{n=1}^{\infty} c_n b_n^{-pq/2}\left(\sum_{k=1}^{n}\mathbb{E}\,|X_{n,k}|^p I_{\{|X_{n,k}|> b_n\}}\right)^{q/2} < \infty$,

(iv) $\sum_{n=1}^{\infty} c_n b_n^{-q}\left(\sum_{k=1}^{n}\mathbb{E}\,X_{n,k}^2 I_{\{|X_{n,k}|\leqslant b_n\}}\right)^{q/2} < \infty$,

then

$$\sum_{n=1}^{\infty} c_n \mathbb{E}\left[\frac{\left|\sum_{k=1}^{n}(X_{n,k} - \mathbb{E}\,X_{n,k})\right|}{b_n} - \varepsilon\right]_+^p < \infty$$

for all $\varepsilon > 0$.

Proof. From (iii) and (iv), we have that conditions (b), (c) of Theorems 2.3 and 2.4 are verified since

$$
\begin{aligned}
& b_n^{-p} \int_0^{b_n^{p-q}} \left(\sum_{k=1}^{n} \int_0^{t^{2/(p-q)}} \mathbb{P}\left\{X_{n,k}^2 > s\right\} \mathrm{d}s \right)^{q/2} \mathrm{d}t \\
& \leqslant 2^{(q-2)/2} b_n^{-p} \int_0^{b_n^{p-q}} \left(\sum_{k=1}^{n} t^{2/(p-q)} \mathbb{P}\left\{X_{n,k}^2 > t^{2/(p-q)}\right\} \right)^{q/2} \mathrm{d}t \\
& \quad + 2^{(q-2)/2} b_n^{-p} \int_0^{b_n^{p-q}} \left(\sum_{k=1}^{n} \mathbb{E}\, X_{n,k}^2 I_{\{|X_{n,k}| \leqslant t^{1/(p-q)}\}} \right)^{q/2} \mathrm{d}t \\
& \leqslant 2^{(q-2)/2} b_n^{-p} \int_0^{b_n^{p-q}} \left(\sum_{k=1}^{n} t^{(2-p)/(p-q)} \mathbb{E}\, |X_{n,k}|^p I_{\{|X_{n,k}| > t^{1/(p-q)}\}} \right)^{q/2} \mathrm{d}t \\
& \quad + 2^{(q-2)/2} b_n^{-p} \int_0^{b_n^{p-q}} \left(\sum_{k=1}^{n} \mathbb{E}\, X_{n,k}^2 I_{\{|X_{n,k}| \leqslant b_n\}} + \sum_{k=1}^{n} \mathbb{E}\, X_{n,k}^2 I_{\{b_n < |X_{n,k}| \leqslant t^{1/(p-q)}\}} \right)^{q/2} \mathrm{d}t \\
& \leqslant 2^{(q-2)/2} b_n^{-p} \left(\sum_{k=1}^{n} \mathbb{E}\, |X_{n,k}|^p I_{\{|X_{n,k}| > b_n\}} \right)^{q/2} \int_0^{b_n^{p-q}} t^{q(2-p)/(2p-2q)} \mathrm{d}t \\
& \quad + 2^{q-2} \left(\frac{\sum_{k=1}^{n} \mathbb{E}\, X_{n,k}^2 I_{\{|X_{n,k}| \leqslant b_n\}}}{b_n^2} \right)^{q/2} + 2^{q-2} b_n^{-p} \int_0^{b_n^{p-q}} \left(\sum_{k=1}^{n} \mathbb{E}\, X_{n,k}^2 I_{\{b_n < |X_{n,k}| \leqslant t^{1/(p-q)}\}} \right)^{q/2} \mathrm{d}t \\
& \leqslant 2^{(q-2)/2} b_n^{-p} \left(\sum_{k=1}^{n} \mathbb{E}\, |X_{n,k}|^p I_{\{|X_{n,k}| > b_n\}} \right)^{q/2} \frac{2(q-p) b_n^{p-pq/2}}{p(q-2)} + 2^{q-2} \left(\frac{\sum_{k=1}^{n} \mathbb{E}\, X_{n,k}^2 I_{\{|X_{n,k}| \leqslant b_n\}}}{b_n^2} \right)^{q/2} \\
& \quad + 2^{q-2} b_n^{-p} \left(\sum_{k=1}^{n} \mathbb{E}\, |X_{n,k}|^p I_{\{|X_{n,k}| > b_n\}} \right)^{q/2} \left[b_n^{p-pq/2} + \int_0^{b_n^{p-q}} t^{q(2-p)/(2p-2q)} \, \mathrm{d}t \right] \\
& = \frac{2^{q/2}(q-p)}{p(q-2)} \left(\frac{\sum_{k=1}^{n} \mathbb{E}\, |X_{n,k}|^p I_{\{|X_{n,k}| > b_n\}}}{b_n^p} \right)^{q/2} + 2^{q-2} \left(\frac{\sum_{k=1}^{n} \mathbb{E}\, X_{n,k}^2 I_{\{|X_{n,k}| \leqslant b_n\}}}{b_n^2} \right)^{q/2} \\
& \quad + 2^{q-2} \left(\frac{\sum_{k=1}^{n} \mathbb{E}\, |X_{n,k}|^p I_{\{|X_{n,k}| > b_n\}}}{b_n^p} \right)^{q/2} + \frac{2^{q-1}(q-p)}{p(q-2)} \left(\frac{\sum_{k=1}^{n} \mathbb{E}\, |X_{n,k}|^p I_{\{|X_{n,k}| > b_n\}}}{b_n^p} \right)^{q/2} \\
& = \left[\frac{(2^{q/2} + 2^{q-1})(q-p)}{p(q-2)} + 2^{q-2} \right] \left(\frac{\sum_{k=1}^{n} \mathbb{E}\, |X_{n,k}|^p I_{\{|X_{n,k}| > b_n\}}}{b_n^p} \right)^{q/2} \\
& \quad + 2^{q-2} \left(\frac{\sum_{k=1}^{n} \mathbb{E}\, X_{n,k}^2 I_{\{|X_{n,k}| \leqslant b_n\}}}{b_n^2} \right)^{q/2}
\end{aligned}
$$

and

$$
\begin{aligned}
& b_n^{-q} \left(\sum_{k=1}^{n} \int_0^{b_n^2} \mathbb{P}\left\{X_{n,k}^2 > t\right\} \mathrm{d}t \right)^{q/2} \\
& \leqslant 2^{(q-2)/2} b_n^{-q} \left(\sum_{k=1}^{n} b_n^2 \mathbb{P}\left\{|X_{n,k}| > b_n\right\} \right)^{q/2} + 2^{(q-2)/2} b_n^{-q} \left(\sum_{k=1}^{n} \mathbb{E}\, X_{n,k}^2 I_{\{|X_{n,k}| \leqslant b_n\}} \right)^{q/2}
\end{aligned}
$$

$$\leqslant 2^{(q-2)/2} b_n^{-q} \left(b_n^{2-p} \sum_{k=1}^{n} \mathbb{E}\,|X_{n,k}|^p I_{\{|X_{n,k}|>b_n\}} \right)^{q/2} + 2^{(q-2)/2} \left(\frac{\sum_{k=1}^{n} \mathbb{E}\, X_{n,k}^2 I_{\{|X_{n,k}|\leqslant b_n\}}}{b_n^2} \right)^{q/2}$$

$$\leqslant 2^{(q-2)/2} \left(\frac{\sum_{k=1}^{n} \mathbb{E}\,|X_{n,k}|^p I_{\{|X_{n,k}|>b_n\}}}{b_n^p} \right)^{q/2} + 2^{(q-2)/2} \left(\frac{\sum_{k=1}^{n} \mathbb{E}\, X_{n,k}^2 I_{\{|X_{n,k}|\leqslant b_n\}}}{b_n^2} \right)^{q/2}.$$

The remaining assumptions of Theorems 2.3 and 2.4 follow from (i), (ii) as in the proof of Corollary 3.1. The proof is complete. □

Let $\{\Psi_{n,k}(x),\ 1 \leqslant k \leqslant n, n \geqslant 1\}$ be an array of functions defined on $[0,\infty)$ satisfying for all $n \geqslant 1$ and every $1 \leqslant k \leqslant n$,

$$\Psi_{n,k}(0) = 0, \quad 0 < \frac{\Psi_{n,k}(t)}{t^p} \uparrow \ \text{ and } \ \frac{\Psi_{n,k}(t)}{t^q} \downarrow \ \text{ as } \ 0 < t \uparrow \tag{3.4}$$

for some $1 \leqslant p < q$.

Corollary 3.3. *Let $\{\Psi_{n,k}(x),\ 1 \leqslant k \leqslant n, n \geqslant 1\}$ be an array of functions defined on $[0,\infty)$ verifying* (3.4) *for some $1 \leqslant p < q \leqslant 2$, and $\{X_{n,k},\ 1 \leqslant k \leqslant n, n \geqslant 1\}$ be an array of zero-mean random variables satisfying* (2.1) *for such q and some constant sequence α_n. If $\{b_n\}$, $\{c_n\}$ are real sequences of positive numbers such that*

(1) $\sum_{n=1}^{\infty} c_n \sum_{k=1}^{n} \mathbb{E}\Psi_{n,k}(|X_{n,k}|)/\Psi_{n,k}(b_n) < \infty$,

then $\sum_{n=1}^{\infty} c_n \mathbb{E}\left(\left|\sum_{k=1}^{n} X_{n,k}\right| / b_n - \varepsilon\right)_+^p < \infty$ for all $\varepsilon > 0$.

Proof. From $\Psi_{n,k}(t)/t^q \downarrow$ as $0 < t \uparrow$, it follows

$$\frac{\mathbb{E}|X_{n,k}|^q I_{\{|X_{n,k}|\leqslant b_n\}}}{b_n^q} \leqslant \frac{\mathbb{E}\Psi_{n,k}(|X_{n,k}| I_{\{|X_{n,k}|\leqslant b_n\}})}{\Psi_{n,k}(b_n)} \tag{3.5}$$

for all $1 \leqslant k \leqslant n$ and $n \geqslant 1$. On the other hand, $\Psi_{n,k}(t)/t^p \uparrow$ as $0 < t \uparrow$ entails

$$\frac{\mathbb{E}|X_{n,k}|^p I_{\{|X_{n,k}|> b_n\}}}{b_n^p} \leqslant \frac{\mathbb{E}\Psi_{n,k}(|X_{n,k}| I_{\{|X_{n,k}|> b_n\}})}{\Psi_{n,k}(b_n)}, \tag{3.6}$$

$$\mathbb{E}\Psi_{n,k}(|X_{n,k}| I_{\{|X_{n,k}|\leqslant b_n\}}) \leqslant \mathbb{E}\Psi_{n,k}(|X_{n,k}|), \tag{3.7}$$

$$\mathbb{E}\Psi_{n,k}(|X_{n,k}| I_{\{|X_{n,k}|> b_n\}}) \leqslant \mathbb{E}\Psi_{n,k}(|X_{n,k}|), \tag{3.8}$$

for each $1 \leqslant k \leqslant n$ and $n \geqslant 1$. Hence, (3.5) and (3.7) yield

$$\frac{\mathbb{E}|X_{n,k}|^q I_{\{|X_{n,k}|\leqslant b_n\}}}{b_n^q} \leqslant \frac{\mathbb{E}\Psi_{n,k}(|X_{n,k}|)}{\Psi_{n,k}(b_n)}$$

for any $1 \leqslant k \leqslant n$ and $n \geqslant 1$, which assures assumption (i) of Corollary 3.1. Moreover, (3.6) and (3.8) imply

$$\frac{\mathbb{E}|X_{n,k}|^p I_{\{|X_{n,k}|> b_n\}}}{b_n^p} \leqslant \frac{\mathbb{E}\Psi_{n,k}(|X_{n,k}| I_{\{|X_{n,k}|> b_n\}})}{\Psi_{n,k}(b_n)} \leqslant \frac{\mathbb{E}\Psi_{n,k}(|X_{n,k}|)}{\Psi_{n,k}(b_n)} \tag{3.9}$$

for every $1 \leqslant k \leqslant n$ and $n \geqslant 1$. Thus, assumption (ii) of Corollary 3.1 holds via (3.9) and (1). The proof is complete. □

Corollary 3.4. *Let $\{\Psi_{n,k}(x),\ 1 \leqslant k \leqslant n, n \geqslant 1\}$ be an array of functions defined on $[0,\infty)$ verifying (3.4) for some $p \geqslant 1$ and $q > \max\{2,p\}$, and $\{X_{n,k},\ 1 \leqslant k \leqslant n, n \geqslant 1\}$ be an array of zero-mean random variables satisfying (2.8) for such q and some constant sequences β_n, ξ_n. If $\{b_n\}$, $\{c_n\}$ are real sequences of positive numbers such that*

(1) $\sum_{n=1}^{\infty} c_n \sum_{k=1}^{n} \mathbb{E}\Psi_{n,k}(|X_{n,k}|)/\Psi_{n,k}(b_n) < \infty$,

(2) $\sum_{n=1}^{\infty} c_n \left(\sum_{k=1}^{n} \mathbb{E}\, X_{n,k}^2 I_{\{|X_{n,k}|\leqslant b_n\}}/b_n^2\right)^{q/2} < \infty$,

(3) $\sum_{n=1}^{\infty} c_n \left[\sum_{k=1}^{n} \mathbb{E}\Psi_{n,k}(|X_{n,k}|)/\Psi_{n,k}(b_n)\right]^{q/2} < \infty$,

then $\sum_{n=1}^{\infty} c_n \mathbb{E}\left(\left|\sum_{k=1}^{n} X_{n,k}\right| /b_n - \varepsilon\right)_+^p < \infty$.

Proof. The thesis is a consequence of Corollary 3.2 by arguing as in the proof of Corollary 3.3. □

Remark 3.5. *We observe that Theorem 3 of [12] can be obtained via Corollaries 3.3 and 3.4 by taking $c_n = 1$ for all $n \geqslant 1$ and $\Psi_{n,k}(x)$ not depending on n,k satisfying $\Psi_{n,k}(0) = 0$; indeed, for such sequence c_n, the assumption $\sum_{n=1}^{\infty} c_n \left[\sum_{k=1}^{n} \mathbb{E}\Psi_{n,k}(|X_{n,k}|)/\Psi_{n,k}(b_n)\right]^{q/2} < \infty$ can be dropped in Corollary 3.4.*

The lemma below gives us a von Bahr-Esseen type inequality for row-wise pairwise negative quadrant dependent (NQD) triangular arrays (see, for instance, [7]). The proof can be performed as in Theorem 2.1 of [1] by employing the truncation $X'_{n,k} = g_{x^{1/r}}(X_{n,k})$, $1 < r < 2$ and $X''_{n,k} = X_{n,k} - X'_{n,k}$, being thus omitted.

Lemma 3.6. *Let $1 \leqslant r \leqslant 2$ and $\{X_{n,k},\ 1 \leqslant k \leqslant n,\ n \geqslant 1\}$ be a triangular array of zero-mean row-wise pairwise NQD random variables such that $\mathbb{E}\,|X_{n,k}|^r < \infty$ for all $n \geqslant 1$ and any $1 \leqslant k \leqslant n$. Then*

$$\mathbb{E}\left|\sum_{k=1}^{n} X_{n,k}\right|^r \leqslant C(r) \sum_{k=1}^{n} \mathbb{E}\,|X_{n,k}|^r, \quad n \geqslant 1$$

where $C(r) > 0$ depends only on r.

Remark 3.7. *It is worthy to note that using Lemma 3.6 in Theorems 2.1 and 2.2, we can extend Theorem 3.7 of [1] to sequences $\{X_n,\ n \geqslant 1\}$ of pairwise NQD and identically distributed random variables, by admitting $X_{n,k} = X_k$, $p = r$, $c_n = n^{t-2}$ and $b_n = n^{1/\rho}$ $(0 < \rho < 2)$ with $1 \leqslant r \leqslant 2$, $t \geqslant 1$, and $t\rho < 2$.*

Remark 3.8. *Let us point out that all statements presented throughout can be properly extended without effort to general arrays $\{X_{n,j},\ 1 \leqslant j \leqslant k_n,\ n \geqslant 1\}$, where $\{k_n\}$ is a sequence of positive integers such that $k_n \to \infty$ as $n \to \infty$. Considering also an array $\{\Psi_{n,j}(x),\ 1 \leqslant j \leqslant k_n, n \geqslant 1\}$ of functions defined on $[0,\infty)$ satisfying (3.4) for every $1 \leqslant j \leqslant k_n$ and all $n \geqslant 1$, we conclude from Lemma 3.6 that our Corollary 3.3 extends Theorem 1.1 of [14].*

Corollary 3.9. *Let $1 < r < 2$ and $\{X_{n,k},\ 1 \leqslant k \leqslant n,\ n \geqslant 1\}$ be a triangular array of row-wise pairwise NQD random variables such that $\mathbb{E}\,|X_{n,k}|^r < \infty$ for all $n \geqslant 1$ and any $1 \leqslant k \leqslant n$. If $1 \leqslant p < r$ and $\{b_n\}$ is a real sequence of positive constants satisfying,*

(1) $\sum_{n=1}^{\infty} \sum_{k=1}^{n} b_n^{-r} \int_0^{b_n^r} \mathbb{P}\{|X_{n,k}|^r > t\}\, \mathrm{d}t < \infty$,

(2) $\sum_{n=1}^{\infty} \sum_{k=1}^{n} \mathbb{E}\,|X_{n,k}| I_{\{|X_{n,k}|>b_n\}}/b_n < \infty$,

(3) $\sum_{n=1}^{\infty} \sum_{k=1}^{n} b_n^{-p} \int_{b_n^p}^{\infty} \mathbb{P}\{|X_{n,k}|^p > t\}\, \mathrm{d}t < \infty$,

then $\sum_{n=1}^{\infty} \mathbb{E}\left[\left|\sum_{k=1}^{n}(X_{n,k} - \mathbb{E}\,X_{n,k})\right| /b_n - \varepsilon\right]_+^p < \infty$ *for all $\varepsilon > 0$.*

Proof. From previous Lemma 3.6 we obtain (2.1) with $q = r$ and $\alpha_n = C(r)$. The thesis follows from Theorems 2.1 and 2.2 by taking $c_n = 1$ for all $n \geqslant 1$. □

4. Final comments

In 1947, Hsu and Robbins [4] introduced the concept of complete convergence (see also [3] for a survey). By taking $c_n = 1$ for all $n \geqslant 1$ in (1.3), we obtain that $\left|\sum_{k=1}^{n}(X_{n,k} - \mathbb{E}\,X_{n,k})\right|/b_n$ converges completely to zero: indeed, setting $A_n(\varepsilon) := \left\{\omega\colon \left|\sum_{k=1}^{n}(X_{n,k} - \mathbb{E}\,X_{n,k})\right|/b_n - \varepsilon \geqslant 0\right\}$, we have, for each $\delta > 0$,

$$\begin{aligned}\mathbb{E}\left[\frac{\left|\sum_{k=1}^{n}(X_{n,k} - \mathbb{E}\,X_{n,k})\right|}{b_n} - \varepsilon\right]_+^p &\geqslant \delta\,\mathbb{P}\left\{\left[\frac{\left|\sum_{k=1}^{n}(X_{n,k} - \mathbb{E}\,X_{n,k})\right|}{b_n} - \varepsilon\right]_+^p > \delta\right\} \\ &= \delta\,\mathbb{P}\left[\left\{\left[\frac{\left|\sum_{k=1}^{n}(X_{n,k} - \mathbb{E}\,X_{n,k})\right|}{b_n} - \varepsilon\right]_+^p > \delta\right\} \cap A_n(\varepsilon)\right] \\ &\quad + \delta\,\mathbb{P}\left[\left\{\left[\frac{\left|\sum_{k=1}^{n}(X_{n,k} - \mathbb{E}\,X_{n,k})\right|}{b_n} - \varepsilon\right]_+^p > \delta\right\} \cap A_n(\varepsilon)^{\complement}\right] \\ &= \delta\,\mathbb{P}\left\{\frac{\left|\sum_{k=1}^{n}(X_{n,k} - \mathbb{E}\,X_{n,k})\right|}{b_n} > \varepsilon + \delta^{1/p}\right\}\end{aligned} \qquad (p \geqslant 1).$$

Furthermore, by using the elementary inequality $|x|^p \leqslant \max(1, 2^{p-1})\left[(|x| - \varepsilon)_+^p + \varepsilon^p\right]$ for every real number x and any $p, \varepsilon > 0$, it follows

$$\mathbb{E}\left|\frac{\sum_{k=1}^{n}(X_{n,k} - \mathbb{E}\,X_{n,k})}{b_n}\right|^p \leqslant 2^{p-1}\left\{\mathbb{E}\left[\frac{\left|\sum_{k=1}^{n}(X_{n,k} - \mathbb{E}\,X_{n,k})\right|}{b_n} - \varepsilon\right]_+^p + \varepsilon^p\right\} \qquad (p \geqslant 1)$$

so that, our statements also guarantee the convergence in mean of order $p \geqslant 1$ (to zero) for triangular arrays of random variables under the considered assumptions.

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