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Convergence Theory of Iterative Methods based on Proper Splittings and Proper Multisplittings for Rectangular Linear Systems

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Abstract. Multisplitting methods are useful to solve differential-algebraic equations. In this connection, we discuss the theory of matrix splittings and multisplittings, which can be used for finding the iterative solution of a large class of rectangular (singular) linear system of equations of the form $Ax = b$. In this direction, many convergence results are proposed for different subclasses of proper splittings in the literature. But, in some practical cases, the convergence speed of the iterative scheme is very slow. To overcome this issue, several comparison results are obtained for different subclasses of proper splittings. This paper also presents a few such results. However, this idea fails to accelerate the speed of the iterative scheme in finding the iterative solution. In this regard, Climent and Perea [J. Comput. Appl. Math. 158 (2003), 43-48: MR2013603] introduced the notion of proper multisplittings to solve the system $Ax = b$ on parallel and vector machines, and established convergence theory for a subclass of proper multisplittings. With the aim to extend the convergence theory of proper multisplittings, this paper further adds a few results. Some of the results obtained in this paper are even new for the iterative theory of nonsingular linear systems.

1. Introduction

1.1. Background and motivation

Mathematical methods like simulations of the power systems, constrained mechanical systems, singular perturbations are based on combination of differential and algebraic equations (see [3] for more details). Geiser [14] very recently used waveform-relaxation methods and multisplitting methods to solve differential-algebraic equations. Multisplitting methods have their benefits in parallelizing their procedure. Motivated by the recent use of multisplitting methods, we discuss the theory of matrix splittings and multisplittings in a more general setting.

Historically, much of the progress in the iterative methods for finding the least squares solution of minimum norm of the rectangular system of linear equations of the form

$$Ax = b, \tag{1}$$

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where $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$ has been driven by the notion of proper splitting $A = U - V$ which is introduced by Berman and Plemmons [6] and states that a splitting $A = U - V$ of $A \in \mathbb{R}^{m\times n}$ is called a *proper splitting* if $R(U) = R(A)$ and $N(U) = N(A)$, where $R(A)$ and $N(A)$ denote the range space and the null space of A, respectively. Berman and Neumann [5] first gave a method construction of proper splittings. Different methods for constructing such splittings are also reported in [28] and [25]. Very recently, Mishra and Mishra [22] showed the uniqueness of a proper splitting under different sufficient conditions. Different extensions of proper splitings can be found in the articles [13] and [29]. The authors of [6] proved that the iterative scheme:

$$x^{k+1} = U^\dagger V x^k + U^\dagger b,\ k = 0, 1, 2, \ldots \tag{2}$$

converges to $A^\dagger b$, the least squares solution of minimum norm for a proper splitting $A = U - V$, for any initial vector x^0 if and only if $\rho(U^\dagger V) < 1$ (see Corollary 1, [6]). Here $\rho(A)$ denotes the *spectral radius* of a matrix $A \in \mathbb{R}^{n\times n}$. The above iterative scheme is said to be *convergent* if the spectral radius of the iteration matrix $U^\dagger V$ is strictly less than 1. The advantage of the iterative technique for computing the least squares solution of the rectangular system of linear equations is that it avoids the use of the normal system $A^TAx = A^Tb$, where A^TA is frequently ill-conditioned and influenced greatly by roundoff errors (see [19]). Such systems appear in deconvolution problems with a smooth kernel. Singular linear systems also appear in problems like the finite difference representation of Neumann problems.

1.2. Statement of Results

Improving the speed of the iterative scheme (2) is one of the challenging and interesting problems. In this context, many authors presented many comparison results. However, comparison results are not that much advantageous if a matrix has many matrix splittings as one can compare two matrix splittings at a time. To get rid of this problem, O'leary and White [26] introduced the theory of multisplittings. The triplet $(U_l, V_l, E_l)_{l=1}^p$ is called a *multisplitting* of $A \in \mathbb{R}^{n\times n}$ if
(i) $A = U_l - V_l$, for each $l = 1, 2, \ldots, p$, where each U_l is invertible,
(ii) $\sum_{l=1}^p E_l = I$, where the matrices E_l are diagonal and $E_l \geq 0$.
($B \geq 0$ ($B > 0$) means all entries of the matrix B are non-negative (positive).) A multisplitting is called as a *weak regular multisplitting of type I or type II* if each $A = U_l - V_l$ is a weak regular splitting of type I or type II, respectively. Then, they [26] showed that the iterative scheme:

$$x^{k+1} = Hx^k + Gb, \quad k = 0, 1, 2, \ldots, \tag{3}$$

where $H = \sum_{l=1}^p E_l U_l^{-1} V_l$ and $G = \sum_{l=1}^p E_l U_l^{-1}$, is convergent if $(U_l, V_l, E_l)_{l=1}^p$ is a weak regular multisplitting of type I and A is monotone. ($A \in \mathbb{R}^{n\times n}$ is called *monotone* if $Ax \geq 0 \Rightarrow x \geq 0$.) The book by Collatz [12] has details of how monotone matrices arise naturally in the study of finite difference approximation methods for certain elliptic partial differential equations. He also showed that A is monotone if and only if A^{-1} exists and $A^{-1} \geq 0$. Later on, Climent and Perea [10] extended the same theory for weak regular multisplitting of type II. In particular, they obtained the following result (which is a part of Theorem 3.2, [10]) among others.

Theorem 1.1 ([10]). *Let $(U_l, V_l, E_l)_{l=1}^p$ be a weak regular multisplitting of type II of $A \in \mathbb{R}^{n\times n}$ with $E_lA = AE_l$, for each $l = 1, 2, \ldots, p$. If A is monotone, then $\rho(H) < 1$.*

The notion of multisplittings for rectangular and singular matrices was first put forth in [11] by Climent and Perea. It was then carried forward by Baliarsingh and Jena [1] and Giri and Mishra [16]. A drawback of the multisplitting theory discussed in [1], [11] and [16] is that it works only for proper weak regular splittings of type I (see Section 2 for its definition). The authors did not address the problem of convergence when the matrix A does not have a matrix splitting of the above type.

1.3. Objective

The main objective of this paper is to short out the above-discussed issue. This can be done by showing that how the results involving weak regular multisplittings of type II in [10] can be extended to the case of the Moore-Penrose inverse. By doing so, we not only can expand convergence theory of multisplittings for rectangular/singular matrices but also can have a new characterization of an extension of a monotone matrix. Besides these, we prove a few new comparison results for matrix splittings even for nonsingular matrix case. In addition to these results, some new results on multisplittings theory are established.

1.4. Outline

The rest of the paper is broken down as follows. In Section 2, we define mathematical constructs including non-negative matrices, Moore-Penrose inverse and proper splittings which are required to state and prove the results in the subsequent sections. In Section 3, we prove several convergence and comparison results of proper weak regular splittings of different types. Section 4 discusses a few applications of these results to multisplitting theory of rectangular matrices. Finally, we end up with a conclusion section.

2. Preliminaries

This section contains basic constructs required to prove our main results. We begin with the notation $\mathbb{R}^{m\times n}$ which represents the set of all real matrices of order $m\times n$. We denote the transpose of $A \in \mathbb{R}^{m\times n}$ by A^T. Let $r(A)$ and $n(A)$ stand for the rank and nullity of a matrix A, respectively. Let L and M be complementary subspaces of $\mathbb{R}^n$, and $P_{L,M}$ be a projection onto L along M. Then $P_{L,M}A = A$ if and only if $R(A) \subseteq L$, and $AP_{L,M} = A$ if and only if $N(A) \supseteq M$. Two vectors x and y in $\mathbb{R}^n$ are orthogonal (perpendicular) if the angle between them is a right angle. It is denoted by $x \perp y$. By $L \perp M$, we mean every vector in L is orthogonal to every vector in M. In the case of $L \perp M$, $P_{L,M}$ will be denoted by P_L for notational simplicity. $\sigma(A)$ stands for the set of all eigenvalues of $A \in \mathbb{R}^{n\times n}$. It is well-known that $\rho(AB) = \rho(BA)$ for any two matrices A and B of appropriate order such that the products AB and BA are defined.

2.1. Non-negative matrices

$A \in \mathbb{R}^{m\times n}$ is called *non-negative* if $A \geq 0$. Let $B, C \in \mathbb{R}^{m\times n}$. We write $B \geq C$ if $B - C \geq 0$. The next results deal with non-negativity of a matrix and the spectral radius.

Theorem 2.1 (Theorem 2.20, [30]). *Let $B \in \mathbb{R}^{n\times n}$ and $B \geq 0$. Then*
(i) *B has a non-negative real eigenvalue equal to its spectral radius.*
(ii) *There exists a non-negative eigenvector for its spectral radius.*

Theorem 2.2 (Theorem 2.1.11, [8]). *Let $B \in \mathbb{R}^{n\times n}$, $B \geq 0$, $x \geq 0$ $(x \neq 0)$ and α be a positive scalar.*
(i) *If $\alpha x \leq Bx$, then $\alpha \leq \rho(B)$. Moreover, if $Bx > \alpha x$, then $\rho(B) > \alpha$.*
(ii) *If $Bx \leq \alpha x$, $x > 0$, then $\rho(B) \leq \alpha$.*

2.2. Moore-Penrose inverse

For $A \in \mathbb{R}^{m\times n}$, the unique matrix $Z \in \mathbb{R}^{n\times m}$ satisfying the following four equations known as Penrose equations:

$$AZA = A,\ ZAZ = Z,\ (AZ)^T = AZ \text{ and } (ZA)^T = ZA$$

is called the *Moore-Penrose inverse* of A. It always exists, and is denoted by $A^{\dagger}$. Certainly, when the matrix A is nonsingular, then $A^{\dagger} = A^{-1}$. A matrix $A \in \mathbb{R}^{m\times n}$ is said to be *semi-monotone* if $A^{\dagger} \geq 0$ (see [8]). Different characterizations of the notion of semi-monotonicity can be found in [7] and [25]. The following properties of $A^{\dagger}$ will be frequently used in the proofs of the next section: $R(A^T) = R(A^{\dagger})$; $N(A^T) = N(A^{\dagger})$; $AA^{\dagger} = P_{R(A)}$ and $A^{\dagger}A = P_{R(A^T)}$. For the historical note and for a detailed study of generalized inverses and its applications, one is referred to the excellent book [4]. Next result of this subsection characterizes the "reverse order law" for the Moore-Penrose inverse.

Theorem 2.3 (Theorem 1, [20]). *Let K and L be arbitrary matrices such that KL is defined. Then $(KL)^{\dagger} = L^{\dagger}K^{\dagger}$ if and only if $K^{\dagger}KLL^TK^T = LL^TK^T$ and $LL^{\dagger}K^TKL = K^TKL$.*

A few properties of proper splittings are summarized in the following theorem.

Theorem 2.4 (Theorem 1, [6]). *Let $A = U - V$ be a proper splitting of $A \in \mathbb{R}^{m\times n}$. Then*
(i) $A = U(I - U^\dagger V)$,
(ii) $I - U^\dagger V$ is invertible,
(iii) $A^\dagger = (I - U^\dagger V)^{-1}U^\dagger$.

Similarly, we have $A = (I - VU^\dagger)U, A^\dagger = U^\dagger(I - VU^\dagger)^{-1}$ (see Theorem 1, [9]), and $U^\dagger VA^\dagger = A^\dagger VU^\dagger$ (see Theorem 2.2 (f), [24]).

3. Main Results

In this section, we will discuss comparison results for proper weak regular splittings of different types and proper weak splittings of different types.

3.1. Comparison of proper weak regular splittings of different types

We begin with the definitions of proper weak regular splittings of type I and type II. Then convergence results are recalled before proving a comparison result for two different linear systems.

Definition 3.1. *A proper splitting $A = U - V$ of $A \in \mathbb{R}^{m\times n}$ is called a*
(i) proper weak regular splitting of type I if $U^\dagger \geq 0$ and $U^\dagger V \geq 0$ (Definition 1.2, [21])
(ii) proper weak regular splitting of type II if $U^\dagger \geq 0$ and $VU^\dagger \geq 0$ (Definition 3.5, [15]).

The convergence results for a proper weak regular splittings of above types are combined (see Corollary 4, [6] and Theorem 3.7, [15]), and is stated below.

Theorem 3.2. *Let $A = U - V$ be any of the above type of proper splittings. Then, $A^\dagger \geq 0$ if and only if $\rho(U^\dagger V) < 1$.*

Once the convergence criteria are fixed for a given class of splittings, we face another problem, i.e., which splitting one should pick if two or more splittings of the same class are known. This is again settled by many authors in the literature by the introduction of several comparison results. These results usually compare the spectral radius of each of the iteration matrix formed by two different splittings of the same class. The splitting which leads to the iteration matrix having the smaller spectral radius is preferred most. The major drawback of this theory is that it can compare only two splittings of the same class or even different class at a time. When a matrix has more splittings, it consumes more time for comparison. We discuss proper multisplitting theory in this paper which may avoid the above issue as well as give the iterative solution in a faster way. The next result deals with the rate of convergence of proper weak regular splittings of different types of two different matrices. The proof of the result is routine, therefore we omit it.

Theorem 3.3. *Let $A_1 = U_1 - V_1$ and $A_2 = U_2 - V_2$ be two proper weak regular splittings of different types of semi-monotone matrices $A_1 \in \mathbb{R}^{m\times n}$ and $A_2 \in \mathbb{R}^{m\times n}$ such that $A_2^\dagger - A_1^\dagger \geq 0$. If $U_1^\dagger \geq U_2^\dagger$, then $\rho(U_1^\dagger V_1) \leq \rho(U_2^\dagger V_2) < 1$.*

Corollary 3.4. *Let $A_1 = U_1 - V_1$ and $A_2 = U_2 - V_2$ be two weak regular splittings of different types of monotone matrices $A_1 \in \mathbb{R}^{n\times n}$ and $A_2 \in \mathbb{R}^{n\times n}$ such that $A_2^{-1} - A_1^{-1} \geq 0$. If $U_1^{-1} \geq U_2^{-1}$, then $\rho(U_1^{-1}V_1) \leq \rho(U_2^{-1}V_2) < 1$.*

3.2. Comparison of proper weak splittings of different types

Here we discuss comparison results of a more general class of matrices than the previous two mentioned in the last subsection. We next reproduce the same definitions along with their convergence criteria.

Definition 3.5. *A proper splitting $A = U - V$ of $A \in \mathbb{R}^{m\times n}$ is called a*
(i) proper weak splitting of type I if $U^\dagger V \geq 0$ (Definition 3.1, [23] & [18])
(ii) proper weak splitting of type II if $VU^\dagger \geq 0$ (Definition 3.14, [2] & [18]).

Lemma 3.6. *(i) Let $A = U - V$ be a proper weak splitting of type I of $A \in \mathbb{R}^{m\times n}$ and $A^\dagger U \geq 0$. Then $\rho(U^\dagger V) = \dfrac{\rho(A^\dagger U) - 1}{\rho(A^\dagger U)} < 1$. (Lemma 3.4, [23])*
(ii) Let $A = U - V$ be a proper weak splitting of type I (or type II) of $A \in \mathbb{R}^{m\times n}$. Then $A^\dagger V$ (or $VA^\dagger$) ≥ 0 if and only if $\rho(U^\dagger V) = \dfrac{\rho(A^\dagger V)}{1 + \rho(A^\dagger V)} < 1$. (Lemma 3.5, [23] & Remark 2, [9])

We begin with an example which points out a mistake in the proof of Theorem 3.15, [2] and is then rectified. Before that, we first recall the same result below.

Theorem 3.7 (Theorem 3.15, [2]). *Let $A_1, A_2 \in \mathbb{R}^{m\times n}$. Let $A_1 = U_1 - V_1$ be a proper weak splitting of type II and $0 \neq A_2 = U_2 - V_2$ be a proper weak splitting of type I . Suppose that A_1 and A_2 are semi-monotone matrices and $A_1^\dagger \geq A_2^\dagger$. If $A_2^\dagger V_2 \geq 0$ and $U_1^\dagger - U_2^\dagger \geq A_1^\dagger - A_2^\dagger$, then $\rho(U_1^\dagger V_1) \leq \rho(U_2^\dagger V_2) < 1$.*

We remark that in the proof of Theorem 3.15, [2] the irreducibility of matrices $\epsilon J A_2^\dagger + V_1 U_1^\dagger$ and $\epsilon A_1^\dagger J + U_2^\dagger V_2$ with $\epsilon > 0$ is not very obvious. By a simple example, we show that this assertion is incorrect.

Example 3.8. *Let $A = \begin{pmatrix} 6 & -2 & 0 \\ -3 & 4 & 0 \end{pmatrix}$. Then for $U = \begin{pmatrix} 7 & -1 & 0 \\ -3 & 4 & 0 \end{pmatrix}$, we have $A = U - V$ is a proper weak splitting of both type I and type II. Let $J = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$. Then for $\epsilon = 0.001$, we have $\epsilon A^\dagger J + U^\dagger V = \begin{pmatrix} 0.1603 & 0.1603 & 0.0003 \\ 0.1205 & 0.1205 & 0.0005 \\ 0 & 0 & 0 \end{pmatrix}$ which is reducible.*

A modified proof of Theorem 3.15, [2] is presented below.

Theorem 3.9. *Let $A_1, A_2 \in \mathbb{R}^{m\times n}$. Let $A_1 = U_1 - V_1$ be a proper weak splitting of type II and $0 \neq A_2 = U_2 - V_2$ be a proper weak splitting of type I . Suppose that A_1 and A_2 are semi-monotone matrices and $A_1^\dagger \geq A_2^\dagger$. If $A_2^\dagger V_2 \geq 0$ and $U_1^\dagger - U_2^\dagger \geq A_1^\dagger - A_2^\dagger$, then $\rho(U_1^\dagger V_1) \leq \rho(U_2^\dagger V_2) < 1$.*

Proof. By Lemma 3.6, we have $\rho(U_2^\dagger V_2) < 1$ as $A_2 = U_2 - V_2$ is a proper weak splitting of type I and $A_2^\dagger V_2 \geq 0$. Using the condition $U_1^\dagger - U_2^\dagger \geq A_1^\dagger - A_2^\dagger$, we obtain

$$\begin{aligned} U_2^\dagger V_2 A_2^\dagger &= U_2^\dagger (U_2 - A_2) A_2^\dagger \\ &= U_2^\dagger U_2 A_2^\dagger - U_2^\dagger A_2 A_2^\dagger \\ &= A_2^\dagger - U_2^\dagger \\ &\geq A_1^\dagger - U_1^\dagger \\ &= U_1^\dagger (U_1 - A_1) A_1^\dagger \\ &= U_1^\dagger V_1 A_1^\dagger. \end{aligned}$$

Since $A_1^\dagger \geq A_2^\dagger$, we get $U_2^\dagger V_2 A_1^\dagger \geq U_2^\dagger V_2 A_2^\dagger$. By using the above inequality we thus have

$$U_2^\dagger V_2 A_1^\dagger \geq U_1^\dagger V_1 A_1^\dagger = A_1^\dagger V_1 U_1^\dagger. \tag{4}$$

Also $V_1 U_1^\dagger \geq 0$. So, there exists an eigenvector $x \geq 0$ such that $V_1 U_1^\dagger x = \rho(V_1 U_1^\dagger) x$ by Theorem 2.1. Hence, $x \in R(V_1) \subseteq R(A_1)$. Now, post-multiplying (4) by x, we get

$$U_2^\dagger V_2 A_1^\dagger x \geq \rho(V_1 U_1^\dagger) A_1^\dagger x$$

which implies that $\rho(U_1^\dagger V_1) \leq \rho(U_2^\dagger V_2) < 1$ by Theorem 2.2. Note that $A_1^\dagger x \geq 0$ and $A_1^\dagger x \neq 0$. Otherwise $x \in R(A_1) \cap N(A_1^T)$, a contradiction. □

We again remark that Example 3.17, [2] is also presented in an incorrect form. In the same example, considered splittings are not proper weak splittings of both types. The example given below is a replacement of the same one.

Example 3.10. *Let* $A_1 = \begin{pmatrix} -3 & 6 & -3 \\ 6 & -3 & 6 \end{pmatrix} = \begin{pmatrix} -5 & 10 & -5 \\ 8 & -4 & 8 \end{pmatrix} - \begin{pmatrix} -2 & 4 & -2 \\ 2 & -1 & 2 \end{pmatrix} = U_1 - V_1$. *We have* $A_1^\dagger = \begin{pmatrix} 0.0556 & 0.1111 \\ 0.2222 & 0.1111 \\ 0.0556 & 0.1111 \end{pmatrix}$. *Let* $A_2 = \begin{pmatrix} -2 & 6 & -2 \\ 6 & -2 & 6 \end{pmatrix} = \begin{pmatrix} -3 & 12 & -3 \\ 12 & -3 & 12 \end{pmatrix} - \begin{pmatrix} -1 & 6 & -1 \\ 6 & -1 & 6 \end{pmatrix} = U_2 - V_2$. *Then* $A_2^\dagger = \begin{pmatrix} 0.0313 & 0.0938 \\ 0.1875 & 0.0625 \\ 0.0312 & 0.0938 \end{pmatrix}$, $A_1^\dagger - A_2^\dagger = \begin{pmatrix} 0.0243 & 0.0174 \\ 0.0347 & 0.0486 \\ 0.0243 & 0.0174 \end{pmatrix}$ *and* $U_1^\dagger - U_2^\dagger = \begin{pmatrix} 0.0222 & 0.0389 \\ 0.0444 & 0.0611 \\ 0.0222 & 0.0389 \end{pmatrix}$. *Here, we have* $A_1 = U_1 - V_1$ *is a proper weak splitting of type II and* $A_2 = U_2 - V_2$ *is a proper weak splittings of type I. We also have* $\rho(U_1^\dagger V_1) = 0.4000 \leq 0.5556 = \rho(U_2^\dagger V_2) < 1$, *but* $U_1^\dagger - U_2^\dagger \ngeq A_1^\dagger - A_2^\dagger$.

The next result drops one additional sufficient condition in Theorem 3.13, [15] that deals with the comparison of spectral radii of the iteration matrices of the above two types of splittings.

Theorem 3.11. *Let* $A = U_1 - V_1$ *be a proper weak splitting of type I and* $A = U_2 - V_2$ *be a proper weak regular splitting of type II of a semi-monotone matrix* $A \in \mathbb{R}^{m\times n}$. *If* $U_1^\dagger \geq U_2^\dagger$, *then* $\rho(U_1^\dagger V_1) \leq \rho(U_2^\dagger V_2) < 1$.

Proof. By Theorem 3.2, it follows that $\rho(U_1^\dagger V_1) < 1$ and $\rho(U_2^\dagger V_2) < 1$. It remains to show that $\rho(U_1^\dagger V_1) \leq \rho(U_2^\dagger V_2)$. Since $U_1^\dagger V_1 \geq 0$, there exists an eigenvector $x \geq 0$ such that $x^T U_1^\dagger V_1 = \rho(U_1^\dagger V_1) x^T$. Hence, $x \in R(V_1^T) \subseteq R(A^T)$. The condition $U_1^\dagger \geq U_2^\dagger$ yields

$$(I - U_1^\dagger V_1)A^\dagger \geq A^\dagger (I - V_2 U_2^\dagger),$$

i.e.,

$$U_1^\dagger V_1 A^\dagger \leq A^\dagger V_2 U_2^\dagger. \tag{5}$$

Pre-multiplying (5) by x^T gives

$$\rho(U_1^\dagger V_1) x^T A^\dagger \leq x^T A^\dagger V_2 U_2^\dagger.$$

Clearly, $x^T A^\dagger \geq 0$ and $x^T A^\dagger \neq 0$. Otherwise $x \in R(A^T) \cap N(A)$, a contradiction. Thus, by Theorem 2.2 (*i*), $\rho(U_1^\dagger V_1) \leq \rho(U_2^\dagger V_2)$. This completes the proof. □

The following is an immediate consequence of the above theorem when nonsingular matrices are considered.

Corollary 3.12. *(Theorem 3.7, [31]) Let* $A = U_1 - V_1$ *be a weak splitting of type I and* $A = U_2 - V_2$ *be a weak regular splitting of type II of a monotone matrix* $A \in \mathbb{R}^{n\times n}$. *If* $U_1^{-1} \geq U_2^{-1}$, *then* $\rho(U_1^{-1} V_1) \leq \rho(U_2^{-1} V_2) < 1$.

The next three results talk about the comparison of proper weak splittings of type I in different situations.

Theorem 3.13. *Let* $A = U_1 - V_1 = U_2 - V_2$ *be two proper weak splittings of type I of* $A \in \mathbb{R}^{m\times n}$. *Let* $P, Q \in \mathbb{R}^{m\times n}$ *such that* $R(A) = R(P) = R(Q)$ *and* $N(A) = N(P) = N(Q)$. *Then*

$$\rho(U_1^\dagger V_1) \leq \rho(U_2^\dagger V_2) < 1$$

provided the following conditions hold:
(*a*) $A^\dagger U_1 \geq 0,\ PA^\dagger U_1 P^\dagger \geq 0,\ A^\dagger U_2 \geq 0,\ QA^\dagger U_2 Q^\dagger \geq 0$.
(*b*) *There exist integers* $i \geq 0$ *and* $j \geq 1$ *such that*

$$P(A^\dagger U_1)^j P^\dagger Q(A^\dagger U_2)^i Q^\dagger \leq Q(A^\dagger U_2)^{i+j} Q^\dagger \tag{6}$$

or

$$Q(A^\dagger U_2)^i Q^\dagger P(A^\dagger U_1)^j P^\dagger \leq Q(A^\dagger U_2)^{i+j} Q^\dagger. \tag{7}$$

Proof. Assume that (6) is true. Since $PA^\dagger U_1 P^\dagger \geq 0$, there exists an eigenvector $x \geq 0$ such that

$$x^T PA^\dagger U_1 P^\dagger = \rho(PA^\dagger U_1 P^\dagger)x^T = \rho(A^\dagger U_1 P^\dagger P)x^T = \rho(A^\dagger U_1)x^T$$

by Theorem 2.1. From the above expression, we get $x \in R(A)$. Pre-multiplying (6) by x^T, we obtain

$$[\rho(A^\dagger U_1)]^j x^T Q(A^\dagger U_2)^i Q^\dagger \leq x^T Q(A^\dagger U_2)^{i+j} Q^\dagger,$$

i.e.,

$$[\rho(A^\dagger U_1)]^j y^T \leq y^T Q(A^\dagger U_2)^j Q^\dagger,$$

where $y^T = x^T Q(A^\dagger U_2)^i Q^\dagger$. Clearly, $y^T \geq 0$ and $y^T \neq 0$. Otherwise $0 = y^T = x^T Q(A^\dagger U_2)^i Q^\dagger$. Post-multiplying Q, we obtain $0 = x^T Q(A^\dagger U_2)^{i-1} A^\dagger U_2 Q^\dagger Q = x^T Q(A^\dagger U_2)^{i-1} A^\dagger U_2$. Post-multiplying $U_2^\dagger$, we have $0 = x^T Q(A^\dagger U_2)^{i-1} A^\dagger U_2 U_2^\dagger = x^T Q(A^\dagger U_2)^{i-1} A^\dagger$. Repeating the above process, we get $0 = x^T QA^\dagger$. Post-multiplying A, we get $0 = x^T Q$. Finally, post-multiplying $Q^\dagger$ and taking the transpose, we have $0 = x = QQ^\dagger x$ and is a contradiction. Hence, by Theorem 2.2, we obtain $[\rho(A^\dagger U_1)]^j \leq [\rho(A^\dagger U_2)]^j$, i.e., $\rho(A^\dagger U_1) \leq \rho(A^\dagger U_2)$. By Lemma 3.6 the required inequality holds. When (7) is true, one can similarly prove the required inequality. □

The following example illustrates the above theorem.

Example 3.14. *Let $A = U_1 - V_1 = U_2 - V_2$ be proper weak splittings of type I, with* $A = \begin{bmatrix} 5 & 2 \\ 3 & 8 \\ 1 & 2 \end{bmatrix}$, $U_1 = \begin{bmatrix} 25 & 16 \\ 15 & 30 \\ 5 & 8 \end{bmatrix}$, $V_1 = \begin{bmatrix} 20 & 14 \\ 12 & 22 \\ 4 & 6 \end{bmatrix}$, $U_2 = \begin{bmatrix} 56 & 10 \\ 54 & 40 \\ 16 & 10 \end{bmatrix}$ *and* $V_2 = \begin{bmatrix} 51 & 8 \\ 51 & 32 \\ 15 & 8 \end{bmatrix}$. *Let* $P = \begin{bmatrix} 80 & 64 \\ 65 & 86 \\ 20 & 24 \end{bmatrix}$ *and* $Q = \begin{bmatrix} 29 & 19 \\ 31 & -9 \\ 9 & -1 \end{bmatrix}$. *Then $R(A) = R(P) = R(Q)$ and $N(A) = N(P) = N(Q)$. We have* $A^\dagger U_1 = \begin{bmatrix} 5 & 2 \\ 0 & 3 \end{bmatrix} \geq 0$, $A^\dagger U_2 = \begin{bmatrix} 10 & 0 \\ 3 & 5 \end{bmatrix} \geq 0$, $PA^\dagger U_1 P^\dagger = \begin{bmatrix} 4.2092 & 0.8366 & 0.4444 \\ 0.9512 & 3.5547 & 0.8924 \\ 0.4714 & 0.8856 & 0.2361 \end{bmatrix} \geq 0$ *and* $QA^\dagger U_2 Q^\dagger = \begin{bmatrix} 7.0565 & 4.1859 & 1.4 \\ 1.2481 & 7.4324 & 1.8222 \\ 0.7088 & 1.9950 & 0.5111 \end{bmatrix} \geq 0$. *Also, for $i = 0$ and $j = 1$, we have* $P(A^\dagger U_1)^j P^\dagger Q(A^\dagger U_2)^i Q^\dagger = \begin{bmatrix} 4.2090 & 0.8361 & 0.4465 \\ 0.9511 & 3.5545 & 0.8934 \\ 0.4714 & 0.8855 & 0.2365 \end{bmatrix} \leq \begin{bmatrix} 7.0565 & 4.1859 & 1.4 \\ 1.2481 & 7.4324 & 1.8222 \\ 0.7088 & 1.9950 & 0.5111 \end{bmatrix} = Q(A^\dagger U_2)^{i+j} Q^\dagger$. *Therefore* $\rho(U_1^\dagger V_1) = 0.8 \leq 0.9 = \rho(U_2^\dagger V_2) < 1$.

Theorem 3.15. *Let $A = U_1 - V_1 = U_2 - V_2$ be two proper weak splittings of type I of $A \in \mathbb{R}^{m\times n}$. Let $P, Q \in \mathbb{R}^{m\times n}$ such that $R(A) = R(P) = R(Q)$ and $N(A) = N(P) = N(Q)$. Then*

$$\rho(U_1^\dagger V_1) \leq \rho(U_2^\dagger V_2) < 1$$

provided the following conditions hold:
(a) $A^\dagger V_1 \geq 0$, $PA^\dagger V_1 P^\dagger \geq 0$, $A^\dagger V_2 \geq 0$, $QA^\dagger V_2 Q^\dagger \geq 0$ and is irreducible.
(b) There exist integers $i \geq 0$ and $j \geq 1$ such that

$$P(A^\dagger V_1)^j P^\dagger Q(A^\dagger V_2)^i Q^\dagger \leq Q(A^\dagger V_2)^{i+j} Q^\dagger \tag{8}$$

or

$$Q(A^\dagger V_2)^i Q^\dagger P(A^\dagger V_1)^j P^\dagger \leq Q(A^\dagger V_2)^{i+j} Q^\dagger. \tag{9}$$

Proof. Assume that (8) is true. Since $QA^\dagger V_2 Q^\dagger \geq 0$ and is irreducible, there exists an eigenvector $z > 0$ such that $QA^\dagger V_2 Q^\dagger z = \rho(A^\dagger V_2)z$ by Theorem 2.7, [30]. Post-multiplying (8) by z, we obtain

$$[\rho(A^\dagger V_2)]^i P(A^\dagger V_1)^j P^\dagger z \leq [\rho(A^\dagger V_2)]^{i+j} z, \text{ i.e., } P(A^\dagger V_1)^j P^\dagger z \leq [\rho(A^\dagger V_2)]^j z.$$

Hence, by Theorem 2.2, we have $\rho(P(A^\dagger V_1)^j P^\dagger) = [\rho(A^\dagger V_1)]^j \leq [\rho(A^\dagger V_2)]^j$, i.e., $\rho(A^\dagger V_1) \leq \rho(A^\dagger V_2)$. By Lemma 3.6 (ii), the required inequality holds. The proof is similar in the case of the assumption (9). □

The example given below demonstrates the above theorem.

Example 3.16. *Let* $A = U_1 - V_1 = U_2 - V_2$ *be proper weak splittings of type I, with* $A = \begin{bmatrix} 4 & -3 \\ -5 & 6 \\ 4 & -3 \end{bmatrix}$, $U_1 = \begin{bmatrix} 8 & -6 \\ -10 & 12 \\ 8 & -6 \end{bmatrix}$, $V_1 = \begin{bmatrix} 4 & -3 \\ -5 & 6 \\ 4 & -3 \end{bmatrix}$, $U_2 = \begin{bmatrix} 8 & -12 \\ -10 & 24 \\ 8 & -12 \end{bmatrix}$ *and* $V_2 = \begin{bmatrix} 4 & -9 \\ -5 & 18 \\ 4 & -09 \end{bmatrix}$. *Let* $P = \begin{bmatrix} -2 & 5 \\ 7 & -4 \\ -2 & 5 \end{bmatrix}$ *and* $Q = \begin{bmatrix} -5 & 1.4 \\ 13 & 0.5 \\ -5 & 1.4 \end{bmatrix}$. *Then* $R(A) = R(P) = R(Q)$ *and* $N(A) = N(P) = N(Q)$. *We have* $A^{\dagger}V_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \geq 0$, $A^{\dagger}V_2 = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \geq 0$, $PA^{\dagger}V_1P^{\dagger} = \begin{bmatrix} 0.5 & 0 & 0.5 \\ 0 & 1 & 0 \\ 0.5 & 0 & 0.5 \end{bmatrix} \geq 0$ *and* $QA^{\dagger}V_2Q^{\dagger} = \begin{bmatrix} 1.3792 & 0.6763 & 1.3792 \\ 0.3140 & 1.2415 & 0.3140 \\ 1.3792 & 0.6763 & 1.3792 \end{bmatrix} \geq 0$. *Also, for* $i = 0$ *and* $j = 1$, *we have* $Q(A^{\dagger}V_2)^iQ^{\dagger}P(A^{\dagger}V_1)^jP^{\dagger} = \begin{bmatrix} 0.5 & 0 & 0.5 \\ 0 & 1 & 0 \\ 0.5 & 0 & 0.5 \end{bmatrix} \leq \begin{bmatrix} 1.3792 & 0.6763 & 1.3792 \\ 0.3140 & 1.2415 & 0.3140 \\ 1.3792 & 0.6763 & 1.3792 \end{bmatrix} = Q(A^{\dagger}V_2)^{i+j}Q^{\dagger}$. *Therefore* $\rho(U_1^{\dagger}V_1) = 0.5 \leq 0.75 = \rho(U_2^{\dagger}V_2) < 1$.

Theorem 3.17. *Let* $A = U_1 - V_1 = U_2 - V_2$ *be two proper weak splittings of type I of* $A \in \mathbb{R}^{m\times n}$. *Let* $P, Q \in \mathbb{R}^{m\times n}$ *such that* $R(A) = R(P) = R(Q)$ *and* $N(A) = N(P) = N(Q)$.
Then

$$\rho(U_1^{\dagger}V_1) \leq \rho(U_2^{\dagger}V_2) < 1$$

provided that the following conditions hold:
(*a*) $A^{\dagger}U_1 \geq 0,\ PA^{\dagger}U_1P^{\dagger} \geq 0,\ A^{\dagger}U_2 \geq 0,\ QA^{\dagger}U_2Q^{\dagger} \geq 0.$
(*b*) *There exist integers* $i \geq 0$ *and* $j \geq 1$ *such that*

$$P(A^{\dagger}U_1)^{i+j}P^{\dagger} \leq P(A^{\dagger}U_1)^iP^{\dagger}Q(A^{\dagger}U_2)^jQ^{\dagger} \tag{10}$$

or

$$P(A^{\dagger}U_1)^{i+j}P^{\dagger} \leq Q(A^{\dagger}U_2)^jQ^{\dagger}P(A^{\dagger}U_1)^iP^{\dagger}. \tag{11}$$

Proof. Assume that (*a*) and (10) hold. By Lemma 3.6, $\rho(U_i^{\dagger}V_i) < 1$ for $i = 1, 2$. Again, by Theorem 2.1, there exists an eigenvector $x \geq 0$ such that $x^TPA^{\dagger}U_1P^{\dagger} = \rho(PA^{\dagger}U_1P^{\dagger})x^T = \rho(A^{\dagger}U_1)x^T$. Pre-multiplying (10) by x^T, we obtain

$$[\rho(A^{\dagger}U_1)]^{i+j}x^T \leq [\rho(A^{\dagger}U_1)]^ix^TQ(A^{\dagger}U_2)^jQ^{\dagger}, \text{ i.e., } [\rho(A^{\dagger}U_1)]^jx^T \leq x^TQ(A^{\dagger}U_2)^jQ^{\dagger}.$$

Hence, by Theorem 2.2, we obtain $[\rho(A^{\dagger}U_1)]^j \leq [\rho(A^{\dagger}U_2)]^j$, i.e., $\rho(A^{\dagger}U_1) \leq \rho(A^{\dagger}U_2)$. By Lemma 3.6 (i), we have the required inequality. The proof with the hypothesis of the other condition goes parallel. □

The following example illustrates the above theorem.

Example 3.18. *Let* $A = U_1 - V_1 = U_2 - V_2$ *be proper weak splittings of type I, with* $A = \begin{bmatrix} 1 & 2 \\ -1 & 5 \\ 1 & 2 \end{bmatrix}$, $U_1 = \begin{bmatrix} 2.2 & 11 \\ -1.5 & 24 \\ 2.2 & 11 \end{bmatrix}$, $V_1 = \begin{bmatrix} 1.2 & 9 \\ -0.5 & 19 \\ 1.2 & 9 \end{bmatrix}$, $U_2 = \begin{bmatrix} 11 & 30 \\ -11 & 75 \\ 11 & 30 \end{bmatrix}$ *and* $V_2 = \begin{bmatrix} 10 & 28 \\ -10 & 70 \\ 10 & 28 \end{bmatrix}$. *Let* $P = \begin{bmatrix} 1 & 4 \\ -1 & 10 \\ 1 & 4 \end{bmatrix}$ *and* $Q = \begin{bmatrix} 2 & 6 \\ -2 & 15 \\ 2 & 6 \end{bmatrix}$. *Then* $R(A) = R(P) = R(Q)$ *and* $N(A) = N(P) = N(Q)$. *We have* $A^{\dagger}U_1 = \begin{bmatrix} 2 & 1 \\ 0.1 & 5 \end{bmatrix} \geq 0$, $A^{\dagger}U_2 = \begin{bmatrix} 11 & 0 \\ 0 & 15 \end{bmatrix} \geq 0$, $PA^{\dagger}U_1P^{\dagger} = \begin{bmatrix} 1.6071 & 0.8143 & 1.6071 \\ 1.3929 & 3.7857 & 1.3929 \\ 1.6071 & 0.8143 & 1.6071 \end{bmatrix} \geq 0$ *and* $QA^{\dagger}U_2Q^{\dagger} = \begin{bmatrix} 6.0714 & 1.1429 & 6.0714 \\ 1.4286 & 13.8571 & 1.4286 \\ 6.0714 & 1.1429 & 6.0714 \end{bmatrix} \geq 0$. *Also, for* $i = 0$ *and* $j = 1$, *we have* $P(A^\dagger U_1)^{i+j}P^\dagger = \begin{bmatrix} 1.6071 & 0.8143 & 1.6071 \\ 1.3929 & 3.7857 & 1.3929 \\ 1.6071 & 0.8143 & 1.6071 \end{bmatrix} \leq \begin{bmatrix} 6.0714 & 1.1429 & 6.0714 \\ 1.4286 & 13.8571 & 1.4286 \\ 6.0714 & 1.1429 & 6.0714 \end{bmatrix} = P(A^\dagger U_1)^i P^\dagger Q(A^\dagger U_2)^j Q^\dagger$. *Therefore* $\rho(U_1^\dagger V_1) = 0.8013 \leq 0.9333 = \rho(U_2^\dagger V_2)$.

The next result can be proved analogously.

Theorem 3.19. *Let* $A = U_1 - V_1 = U_2 - V_2$ *be two proper weak splittings of type I of* $A \in \mathbb{R}^{m\times n}$. *Let* $P, Q \in \mathbb{R}^{m\times n}$ *such that* $R(A) = R(P) = R(Q)$ *and* $N(A) = N(P) = N(Q)$. *Then*

$$\rho(U_1^\dagger V_1) \leq \rho(U_2^\dagger V_2) < 1$$

provided the following conditions hold:
(*a*) $A^\dagger V_1 \geq 0,\ PA^\dagger V_1 P^\dagger \geq 0,\ A^\dagger V_2 \geq 0,\ QA^\dagger V_2 Q^\dagger \geq 0.$
(*b*) *There exist integers* $i \geq 0$ *and* $j \geq 1$ *such that*

$$P(A^\dagger V_1)^{i+j}P^\dagger \leq P(A^\dagger V_1)^i P^\dagger Q(A^\dagger V_2)^j Q^\dagger$$

or

$$P(A^\dagger V_1)^{i+j}P^\dagger \leq Q(A^\dagger V_2)^j Q^\dagger P(A^\dagger V_1)^i P^\dagger.$$

The following example illustrates the above theorem.

Example 3.20. *Let* $A = U_1 - V_1 = U_2 - V_2$ *be proper weak splittings of type I, with* $A = \begin{bmatrix} 3 & -1 \\ 2 & 3 \\ 4 & 5 \end{bmatrix}$, $U_1 = \begin{bmatrix} 9 & 2 \\ 6 & 5 \\ 12 & 9 \end{bmatrix}$, $V_1 = \begin{bmatrix} 6 & 3 \\ 4 & 2 \\ 8 & 4 \end{bmatrix}$, $U_2 = \begin{bmatrix} 14 & 4 \\ 13 & 10 \\ 25 & 18 \end{bmatrix}$ *and* $V_2 = \begin{bmatrix} 11 & 5 \\ 11 & 7 \\ 21 & 13 \end{bmatrix}$. *Let* $P = \begin{bmatrix} -3 & 1 \\ -2 & -3 \\ -4 & -5 \end{bmatrix}$ *and* $Q = \begin{bmatrix} -6 & 2 \\ -4 & -6 \\ -8 & -10 \end{bmatrix}$. *Then* $R(A) = R(P) = R(Q)$ *and* $N(A) = N(P) = N(Q)$. *We have* $A^\dagger V_1 = \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix} \geq 0$, $A^\dagger V_2 = \begin{bmatrix} 4 & 2 \\ 1 & 1 \end{bmatrix} \geq 0$, $PA^\dagger V_1 P^\dagger = \begin{bmatrix} 0.9753 & 0.2654 & 0.6358 \\ 0.6502 & 0.1770 & 0.4239 \\ 1.3004 & 0.3539 & 0.8477 \end{bmatrix} \geq 0$ *and* $QA^\dagger V_2 Q^\dagger = \begin{bmatrix} 1.8889 & 0.4444 & 1.1111 \\ 1.4856 & 0.6132 & 1.3292 \\ 2.9095 & 1.1399 & 2.4979 \end{bmatrix} \geq 0$. *Also, for* $i = 0$ *and* $j = 1$, *we have* $P(A^\dagger V_1)^{i+j}P^\dagger = \begin{bmatrix} 0.9753 & 0.2654 & 0.6358 \\ 0.6502 & 0.1770 & 0.4239 \\ 1.3004 & 0.3539 & 0.8477 \end{bmatrix} \leq \begin{bmatrix} 1.8889 & 0.4444 & 1.1111 \\ 1.4856 & 0.6132 & 1.3292 \\ 2.9095 & 1.1399 & 2.4979 \end{bmatrix} = Q(A^\dagger V_2)^j Q^\dagger P(A^\dagger V_1)^i P^\dagger$. *Therefore* $\rho(U_1^\dagger V_1) = 0.6667 \leq 0.8202 = \rho(U_2^\dagger V_2)$.

4. Applications to Proper Multisplitting Theory

In this section, we first recall the theory of proper multisplittings [11], and then using convergence and comparison results introduced in Section 3 for a single splitting as a tool, we introduce convergence and comparison results for proper multisplittings. The definition of a proper multisplitting of a rectangular matrix introduced by Climent and Perea [11] is as follows:

Definition 4.1 (Definition 2, [11]). *The triplet* $(U_l, V_l, E_l)_{l=1}^p$ *is a proper multisplitting of* $A \in \mathbb{R}^{m\times n}$ *if*
(*i*) $A = U_l - V_l$ *is a proper splitting, for each* $l = 1, 2, \ldots, p$,
(*ii*) $E_l \geq 0$, *for each* $l = 1, 2, \ldots, p$ *is a diagonal* $n \times n$ *matrix, and* $\sum_{l=1}^{p} E_l = I$, *where* I *is the* $n \times n$ *identity matrix.*

We say a proper multisplitting, a *proper weak regular multisplitting of type I or a proper weak regular multisplitting of type II*, if each one of the proper splitting $A = U_l - V_l$ is a proper weak regular splitting of type I or type II, respectively. Climent and Perea [11] considered the following parallel iterative scheme:

$$x^{k+1} = Hx^k + Gb, \quad k = 1, 2, \ldots, \tag{12}$$

where $(U_l, V_l, E_l)_{l=1}^p$ is a proper multisplitting of $A \in \mathbb{R}^{m\times n}$, $H = \sum_{l=1}^{p} E_l U_l^\dagger V_l$ and $G = \sum_{l=1}^{p} E_l U_l^\dagger$. The same authors [11] obtained the following result for a proper weak regular multisplitting of type I.

Lemma 4.2 (Lemma 1, [11]). *Let $(U_l, V_l, E_l)_{l=1}^p$ be a proper weak regular multisplitting of type I of $A \in \mathbb{R}^{m\times n}$ with $H = \sum_{l=1}^{p} E_l U_l^\dagger V_l$. Then*
(i) $H \geq 0$ and therefore $H^j \geq 0$, for $j = 0, 1, \ldots$,
(ii) $\sum_{l=1}^{p} E_l U_l^\dagger A = GA = (I - H)A^\dagger A$,
(iii) $(I + H + H^2 + \cdots + H^n)(I - H) = I - H^{n+1}$.

We next prove a lemma for a proper weak regular multisplitting of type II of a real square singular matrix A with the assumption of an additional condition $AE_l = E_l A$ for each $l = 1, 2, ..., p$.

Lemma 4.3. *Let $(U_l, V_l, E_l)_{l=1}^p$ be a proper weak regular multisplitting of type II of $A \in \mathbb{R}^{n\times n}$ with $S = \sum_{l=1}^{p} E_l V_l U_l^\dagger$ and $AE_l = E_l A$, for each $l = 1, 2, \ldots, p$. Then*
(i) $S \geq 0$ and therefore $S^j \geq 0$, for $j = 0, 1, \ldots$,
(ii) $A \sum_{l=1}^{p} E_l U_l^\dagger = AG = (I - S)AA^\dagger$,
(iii) $(I + S + S^2 + \cdots + S^n)(I - S) = I - S^{n+1}$.

Proof. (i) Clearly, as $V_l U_l^\dagger \geq 0$. Hence $S^j \geq 0$, for $j = 0, 1, \ldots$.
(ii) Using $AE_l = E_l A$, we have

$$\begin{aligned} AG &= A \sum_{l=1}^{p} E_l U_l^\dagger \\ &= \sum_{l=1}^{p} AE_l U_l^\dagger \\ &= \sum_{l=1}^{p} E_l A U_l^\dagger \\ &= \sum_{l=1}^{p} E_l (I - V_l U_l^\dagger) U_l U_l^\dagger \\ &= (I - \sum_{l=1}^{p} E_l V_l U_l^\dagger) U_l U_l^\dagger \\ &= (I - \sum_{l=1}^{p} E_l V_l U_l^\dagger) AA^\dagger \\ &= (I - S)AA^\dagger. \end{aligned}$$

(iii) This follows from (i). □

Another lemma which will be used to prove Theorem 4.9 is presented below.

Lemma 4.4. *Let $A, B \in \mathbb{R}^{n\times n}$ such that $AB = BA$. If $R(A) \subseteq R(B^T)$ and $R(A^T) \subseteq R(B)$, then $AB^\dagger = B^\dagger A$.*

Proof. The condition $R(A) \subseteq R(B^T)$ implies $B^\dagger BA = P_{R(B^T)}A = A$. Similarly, the other condition $R(A^T) \subseteq R(B)$ yields $ABB^\dagger = AP_{R(B)} = A$. Therefore, $AB^\dagger = B^\dagger BAB^\dagger = B^\dagger ABB^\dagger = B^\dagger A$. □

Putting $A = E$, a diagonal matrix and $B = A$ in the above lemma, we have the following corollary.

Corollary 4.5. *Let $A \in \mathbb{R}^{n\times n}$ and $E \in \mathbb{R}^{n\times n}$ be a diagonal matrix such that $EA = AE$. If $R(E) \subseteq R(A) \cap R(A^T)$, then $EA^\dagger = A^\dagger E$.*

A real square matrix A is called *EP (or range symmetric)* if $R(A) = R(A^T)$. This class of matrices contains many subclasses like the symmetric matrices, skew-symmetric matrices, normal matrices and all nonsingular matrices. Several extensions of this class can be found out in the literature. A characterization of an EP matrix is the following. A matrix A is EP if and only if it commutes with its Moore-Penrose inverse $A^\dagger$. This is one of the main reasons for studying the class of EP matrices. Another interesting property of this class, is that the Moore-Penrose inverse of an EP matrix coincides with two other types of generalized inverses, the Drazin inverse and the Group inverse (see [4]). Therefore, the computation of the Moore-Penrose inverse is very useful for many types of applications such as the system of linear equations, the system of linear differential equations and linear difference equations. All our next results are for the above class of matrices. In the case of an EP matrix A, the condition $R(E_l) \subseteq R(A) \cap R(A^T)$ will simply reduce to $R(E_l) \subseteq R(A)$. By adding another hypothesis $EA = AE$ to Corollary 4.5, we have the following result and can be proved easily.

Lemma 4.6. *Let A be an EP matrix and E be a diagonal matrix. If $EA = AE$ and $R(E) \subseteq R(A)$, then*
(i) $EA^\dagger A = A^\dagger AE$.
(ii) $EAA^\dagger = AA^\dagger E$.

We are now ready to prove the following result which contains a few properties of $H = \sum_{l=1}^{p} E_l U_l^\dagger V_l$ and $S = \sum_{l=1}^{p} E_l V_l U_l^\dagger$.

Lemma 4.7. *Let $(U_l, V_l, E_l)_{l=1}^p$ be a proper multisplitting of an EP matrix $A \in \mathbb{R}^{n\times n}$ such that $R(E_l) \subseteq R(A)$ and $E_l A = AE_l$, for each $l = 1, 2, \ldots, p$. Then*
(i) $AA^\dagger S = S = SAA^\dagger$ and $A^\dagger AH = H = HA^\dagger A$.
(ii) $S = AHA^\dagger$, $H = A^\dagger SA$, and $\rho(S) = \rho(H)$.

Proof. (*i*) Lemma 4.6 yields $A^\dagger AE_l = E_l A^\dagger A$ and $AA^\dagger E_l = E_l AA^\dagger$, for each $l = 1, 2, \ldots, p$. So

$$\begin{aligned} AA^\dagger S &= AA^\dagger \sum_{l=1}^{p} E_l V_l U_l^\dagger \\ &= \sum_{l=1}^{p} E_l AA^\dagger V_l U_l^\dagger \\ &= \sum_{l=1}^{p} E_l V_l U_l^\dagger (= S) \end{aligned}$$

$$= \sum_{l=1}^{p} E_l V_l U_l^{\dagger} U_l U_l^{\dagger}$$
$$= \sum_{l=1}^{p} E_l V_l U_l^{\dagger} A A^{\dagger}$$
$$= SAA^{\dagger}.$$

Again,

$$\begin{aligned} A^{\dagger}AH &= A^{\dagger}A \sum_{l=1}^{p} E_l U_l^{\dagger} V_l \\ &= \sum_{l=1}^{p} E_l A^{\dagger} A U_l^{\dagger} V_l \\ &= \sum_{l=1}^{p} E_l U_l^{\dagger} V_l (= H) \\ &= \sum_{l=1}^{p} E_l U_l^{\dagger} V_l A^{\dagger} A \\ &= HA^{\dagger}A. \end{aligned}$$

(ii) Using the conditions $U_l^{\dagger} V_l A^{\dagger} = A^{\dagger} V_l U_l^{\dagger}$ and $AE_l = E_l A$, we have

$$\begin{aligned} AHA^{\dagger} &= A \sum_{l=1}^{p} E_l U_l^{\dagger} V_l A^{\dagger} \\ &= \sum_{l=1}^{p} E_l A A^{\dagger} V_l U_l^{\dagger} \\ &= \sum_{l=1}^{p} E_l V_l U_l^{\dagger} \\ &= S. \end{aligned}$$

Also, $H = A^{\dagger}AHA^{\dagger}A = A^{\dagger}SA$, by using (i).

To prove $\rho(S) = \rho(H)$, let λ be an eigenvalue of S corresponding to the eigenvector x. Then $\lambda x = Sx = AHA^{\dagger}x$, so $x \in R(A)$. Now, pre-multiplying $A^{\dagger}$ to $\lambda x = AHA^{\dagger}x$, we get $\lambda y = A^{\dagger}AHy = Hy$, where $y = A^{\dagger}x$. Suppose that $0 = y = A^{\dagger}x$. Then $x \in N(A^{\dagger}) = N(A^T)$. But $x \in R(A)$. Thus $x = 0$, a contradiction. Hence $y \neq 0$. Therefore y is an eigenvector of H corresponding to the eigenvalue λ and thus $\sigma(S) \setminus \{0\} \subseteq \sigma(H) \setminus \{0\}$. The other way can be proved similarly by considering μ, an eigenvalue of H corresponding to the eigenvector z. Therefore $\rho(S) = \rho(H)$. □

For nonsingular matrices, the above result reduces to the following corollary.

Corollary 4.8. *Let $(U_l, V_l, E_l)_{l=1}^{p}$ be a multisplitting of a nonsingular matrix $A \in \mathbb{R}^{n\times n}$ such that $E_l A = AE_l$, for each $l = 1, 2, \ldots, p$. Then $S = AHA^{-1}$, $H = A^{-1}SA$ and $\rho(S) = \rho(H)$.*

We are now in a position to address the main problem stated in the introduction (i.e., Theorem 1.1). This is answered in the next result, and this result provides a characterization of semi-monotone matrices in terms of proper multisplittings. Further, the same result yields convergence criteria of a proper weak regular multisplitting of type II of a real square singular matrix.

Theorem 4.9. *Let $(U_l, V_l, E_l)_{l=1}^{p}$ be a proper weak regular multisplitting of type II of an EP matrix $A \in \mathbb{R}^{n\times n}$ with $R(E_l) \subseteq R(A)$ and $E_lA = AE_l$, for each $l = 1, 2, \dots, p$. Then, A is semi-monotone if and only if $\rho(S) < 1$, where $S = \sum_{l=1}^{p} E_l V_l U_l^{\dagger}$.*

Proof. By Lemma 4.6, we get $A^{\dagger}AE_l = E_lA^{\dagger}A$. Hence $A^{\dagger}AG = A^{\dagger}A\sum_{l=1}^{p} E_lU_l^{\dagger} = \sum_{l=1}^{p} E_lA^{\dagger}AU_l^{\dagger} = \sum_{l=1}^{p} E_lU_l^{\dagger} = G$. From Lemma 4.3 (*ii*), we obtain $AG = (I - S)AA^{\dagger}$. So, $G = A^{\dagger}AG = A^{\dagger}(I - S)AA^{\dagger} = A^{\dagger} - A^{\dagger}S = A^{\dagger}(I - S)$, as $SAA^{\dagger} = S$ by Lemma 4.7 (*i*). Since $G \geq 0$ and $S \geq 0$, we have

$$\begin{aligned}
0 &\leq G(I + S + S^2 + \cdots + S^n)\\
&= A^{\dagger}(I - S)(I + S + S^2 + \cdots + S^n)\\
&= A^{\dagger}(I - S^{n+1}) \leq A^{\dagger}.
\end{aligned}$$

So, the partial sum of the series $\sum_{m=0}^{\infty} S^m$ is uniformly bounded and hence $\rho(S) < 1$. Conversely, $S \geq 0$ and $\rho(S) < 1$ imply $(I - S)^{-1} \geq 0$. Hence $A^{\dagger} = G(I - S)^{-1} \geq 0$. □

We now have Theorem 3.2, [10] as a corollary.

Corollary 4.10 (Theorem 3.2, [10]). *Let $(U_l, V_l, E_l)_{l=1}^{p}$ be a weak regular multisplitting of type II of $A \in \mathbb{R}^{n\times n}$ with $E_lA = AE_l$, for each $l = 1, 2, \dots, p$. Then, A is monotone if and only if $\rho(S) < 1$, where $S = \sum_{l=1}^{p} E_lV_lU_l^{-1}$.*

Recently, Giri and Mishra [16] have shown that the iteration matrix H in (12) induces a unique proper weak regular splitting of type I under some sufficient conditions, and the same result is recalled next.

Theorem 4.11 (Theorem 3.17, [16]). *Let $(U_l, V_l, E_l)_{l=1}^{p}$ be a proper weak regular multisplitting of type I of a semi-monotone matrix $A \in \mathbb{R}^{m\times n}$. Then the unique splitting $A = B - C$ induced by H with $B = A(I - H)^{-1}$ is a convergent proper weak regular splitting of type I if $R(E_l) \subseteq R(A^T)$, for each $l = 1, 2, \dots, p$.*

An obvious question arises at this stage is that what can we say about the type of the induced splitting $A = X - Y$ induced by S. This is settled next.

Theorem 4.12. *Let $(U_l, V_l, E_l)_{l=1}^{p}$ be a proper weak regular multisplitting of type II of a semi-monotone and EP matrix $A \in \mathbb{R}^{n\times n}$. Then the unique splitting $A = X - Y$ induced by S with $X = (I - S)^{-1}A$ is a convergent proper weak regular splitting of type II if $R(E_l) \subseteq R(A)$ and $E_lA = AE_l$, for each $l = 1, 2, \dots, p$.*

Proof. We have $E_lA^{\dagger}A = A^{\dagger}AE_l$ and $E_lAA^{\dagger} = AA^{\dagger}E_l$, for each $l = 1, 2, \dots, p$ by Lemma 4.6. By Lemma 4.7 (*i*), we obtain $A^{\dagger}AS = S = SAA^{\dagger}$. Therefore, pre-multiplying Lemma 4.3 (*ii*) by $A^{\dagger}$, we get $G = A^{\dagger}(I - S)AA^{\dagger}$. From equation (12), we obtain $X^{\dagger} = G = A^{\dagger}(I - S)$. Also, Theorem 4.9 shows that $\rho(S) < 1$ and hence $(I - S)$ is invertible. Let $K = A^{\dagger}$ and $L = I - S$. We then have

$$\begin{aligned}
K^{\dagger}KLL^TK^T &= AA^{\dagger}(I - S)(I - S)^T{A^{\dagger}}^T\\
&= (AA^{\dagger} - AA^{\dagger}S)(I - S)^T{A^{\dagger}}^T\\
&= (I - S)(AA^{\dagger})^T(I - S)^T{A^{\dagger}}^T\\
&= (I - S)(A^{\dagger}(I - S)AA^{\dagger})^T\\
&= (I - S)(A^{\dagger}AA^{\dagger}(I - S))^T\\
&= (I - S)(I - S)^T{A^{\dagger}}^T\\
&= LL^TK^T.
\end{aligned}$$

The other condition of the reverse order law (Theorem 2.3) follows clearly. Hence $X = (I - S)^{-1}A$.

Next, we have to prove $R(X) = R(A)$ and $N(X) = N(A)$. To show $R(X) = R(A)$, it is sufficient to obtain $N(X^T) = N(A^T)$. Let $x \in N(X^T) = N(X^\dagger)$. Then $X^\dagger x = A^\dagger(I - S)x = 0$. Pre-multiplying $A^\dagger(I - S)x = 0$ by $(I - S)^{-1}A$, we get $(I - S)^{-1}AA^\dagger(I - S)x = (I - S)^{-1}(I - S)AA^\dagger x = AA^\dagger x = 0$. Then $0 = A^\dagger AA^\dagger x = A^\dagger x = 0$. Hence $N(X^T) \subseteq N(A^T)$. Also,

$$r(A^T) \geq r(X^T) = r(X^T(I - S)^T) = r(A^T).$$

Which implies $n(A^T) = n(X^T)$ and thus $N(X^T) = N(A^T)$. As $X = (I - S)^{-1}A$, so $N(X) = N(A)$.

To prove uniqueness, suppose that there exists another induced proper splitting $A = \tilde{X} - \tilde{Y}$ such that $\tilde{X}^\dagger = G$. Then $\tilde{Y}\tilde{X}^\dagger = (\tilde{X} - A)\tilde{X}^\dagger = \tilde{X}\tilde{X}^\dagger - A\tilde{X}^\dagger = AA^\dagger - AA^\dagger(I - S) = AA^\dagger S = S$ and $S\tilde{X} = \tilde{Y}\tilde{X}^\dagger\tilde{X} = \tilde{Y} = \tilde{X} - A$. So $\tilde{X} = A + S\tilde{X}$, i.e., $(I - S)\tilde{X} = A$. Hence, $\tilde{X} = (I - S)^{-1}A = X$ and thus, S induces the unique proper splitting $A = X - Y$.

Therefore, $X^\dagger = G \geq 0$ and $YX^\dagger = (X - A)X^\dagger = XX^\dagger - AX^\dagger = AA^\dagger - AA^\dagger(I - S) = AA^\dagger S = S \geq 0$. Henceforth, the splitting $A = X - Y$ induced by S is a proper weak regular of type II. By Theorem 4.9, we thus have $\rho(YX^\dagger) = \rho(S) < 1$. □

The above theorem admits the following corollary in the case of nonsingular matrices.

Corollary 4.13. *Let $(U_l, V_l, E_l)_{l=1}^p$ be a weak regular multisplitting of type II of a monotone matrix $A \in \mathbb{R}^{n\times n}$ with $E_lA = AE_l$. Then the unique splitting $A = X - Y$ induced by S with $X = (I - S)A^{-1}$ is a convergent weak regular splitting of type II.*

Remark 4.14. *One can observe that, the matrices H and S in Theorem 4.12 induce the same proper splitting. To prove this, let $B = A(I - H)^{-1}$. As $S = SAA^\dagger = AA^\dagger S$, $H = A^\dagger AH = HA^\dagger A$ and $S = AHA^\dagger$, so $S^k = AH^kA^\dagger$, for any positive integer k. Since $\rho(S) < 1$, then $X = (I - S)^{-1}A = \sum_{k=0}^{\infty} S^kA = \sum_{k=0}^{\infty} AH^kA^\dagger A = \sum_{k=0}^{\infty} AH^k = A(I - H)^{-1} = B$, as $\rho(S) = \rho(H)$ by Lemma 4.7 (ii). Hence $R(X) = R(B) = R(A)$ and $N(X) = N(B) = N(A)$, and thus $A = X - Y$ is a proper splitting.*

The spectral radii of the iteration matrices of two different proper weak regular multisplittings of the same coefficient matrix A is compared below.

Theorem 4.15. *Let $(U_l^{(1)}, V_l^{(1)}, E_l)_{l=1}^p$ be a proper weak regular multisplitting of type I and let $(U_l^{(2)}, V_l^{(2)}, E_l)_{l=1}^p$ be a proper weak regular multisplitting of type II of a semi-monotone and EP matrix $A \in \mathbb{R}^{n\times n}$ such that $R(E_l) \subseteq R(A)$ and $E_lA = AE_l$, for each $l = 1, 2, \ldots, p$. If*

$$(U_l^{(2)})^\dagger \leq (U_l^{(1)})^\dagger, \quad \textit{for each} \quad l = 1, 2, \ldots, p,$$

then

$$\rho(H_1) \leq \rho(H_2) < 1.$$

Proof. By Theorem 4.11 and Theorem 4.12, the splittings $A = B_1 - C_1 = B_2 - C_2$ induced by H_1 and S_2 are convergent proper weak regular splittings of type I and type II, respectively. We also have $\rho(S_2) = \rho(H_2)$ by Lemma 4.7. So, we need to prove that $\rho(H_1) \leq \rho(H_2)$. From the given condition $(U_l^{(2)})^\dagger \leq (U_l^{(1)})^\dagger$, we obtain

$$\sum_{l=1}^{p} E_l(U_l^{(2)})^\dagger \leq \sum_{l=1}^{p} E_l(U_l^{(1)})^\dagger.$$

This implies $G_2 \leq G_1$, i.e., $B_2^\dagger \leq B_1^\dagger$. By Theorem 3.11, we thus have

$$\rho(H_1) \leq \rho(H_2) < 1.$$

□

The first part of Theorem 2.21, [27] is obtained as a corollary to the above result.

Corollary 4.16 (Theorem 2.21, [27]). *Let $(U_l^{(1)}, V_l^{(1)}, E_l)_{l=1}^p$ be a weak regular multisplitting of type I and let $(U_l^{(2)}, V_l^{(2)}, E_l)_{l=1}^p$ be a weak regular multisplitting of type II of a monotone matrix $A \in \mathbb{R}^{n\times n}$ such that $E_l A = AE_l$, for each $l = 1, 2, \ldots, p$. If*

$$(U_l^{(2)})^{-1} \leq (U_l^{(1)})^{-1}, \quad \text{for each } l = 1, 2, \ldots, p,$$

then

$$\rho(H_1) \leq \rho(H_2) < 1.$$

Theorem 4.17. *Let $(U_l, V_l, E_l)_{l=1}^p$ be a proper weak regular multisplitting of type II of a semi-monotone and EP matrix $A \in \mathbb{R}^{n\times n}$ such that $R(E_l) \subseteq R(A)$ and $E_l A = AE_l$, for each $l = 1, 2, \ldots, p$. If there exists a non-negative matrix $M \in \mathbb{R}^{n\times n}$ such that $V_l U_l^{\dagger}(A + M) \geq M$, for each $l = 1, 2, \ldots, p$, then the unique splitting $A = X - Y$ induced by S with $X = (I - S)^{-1}A$ is proper regular.*

Proof. By Theorem 4.12, we have $\rho(S) < 1$ and the unique splitting $A = X - Y$ induced by S is proper weak regular of type II. Then

$$\begin{aligned} Y &= X - A \\ &= (I - S)^{-1}A - A \\ &= (I - S)^{-1}SA \\ &\geq M \geq 0 \end{aligned}$$

and thus $A = X - Y$ is proper regular. □

Theorem 4.17 admits the following corollary in the case of nonsingular matrices.

Corollary 4.18. *Let $(U_l, V_l, E_l)_{l=1}^p$ be a weak regular multisplitting of type II of a monotone matrix $A \in \mathbb{R}^{n\times n}$. If there exists a non-negative matrix $M \in \mathbb{R}^{n\times n}$ such that $V_l U_l^{-1}(A + M) \geq M$, for each $l = 1, 2, \ldots, p$, then the unique splitting $A = X - Y$ induced by S with $X = (I - S)^{-1}A$ is a regular splitting.*

By substituting $M = 0$ in Theorem 4.17, we obtain the next result.

Corollary 4.19. *Let $(U_l, V_l, E_l)_{l=1}^p$ be a proper weak regular multisplitting of type II of a semi-monotone EP matrix $A \in \mathbb{R}^{n\times n}$. If $V_l U_l^{\dagger} V_l \leq V_l$, for each $l = 1, 2, \ldots, p$, then the unique splitting $A = X - Y$ induced by S with $X = (I - S)^{-1}S$ is a proper regular splitting.*

We further obtain the same conclusion as in Theorem 4.17 under different set of assumptions.

Theorem 4.20. *Let $(U_l, V_l, E_l)_{l=1}^p$ be a proper weak regular multisplitting of type II of a semi-monotone EP matrix $A \in \mathbb{R}^{n\times n}$. Assume that, for each l, $E_l = \alpha_l I$ with $\alpha_l > 0$ and $\sum_{l=1}^p \alpha_l = 1$. Let $V_a = \sum_{l=1}^p \alpha_l V_l$, and $V_l \leq V_b$, for each $l = 1, 2, \ldots, p$. If there exists a non-negative matrix $U \in \mathbb{R}^{n\times n}$ such that $V_l U_l^{\dagger}(V_b - U) \leq V_a - U$, for each $l = 1, 2, \ldots, p$, then the unique splitting $A = X - Y$ induced by S with $X = (I - S)^{-1}A$ is a proper regular splitting.*

Proof. As $Y = X - A = (I-S)^{-1}SA$, we have

$$\begin{aligned}
Y &= (I-S)^{-1}\sum_{l=1}^{p}\alpha_l V_l U_l^{\dagger} A \\
&= (I-S)^{-1}\sum_{l=1}^{p}\alpha_l V_l U_l^{\dagger}(I - V_l U_l^{\dagger})U_l \\
&= (I-S)^{-1}\sum_{l=1}^{p}\alpha_l V_l (I - U_l^{\dagger} V_l) \\
&= (I-S)^{-1}\sum_{l=1}^{p}\alpha_l V_l - (I-S)^{-1}\sum_{l=1}^{p}\alpha_l V_l U_l^{\dagger} V_l \\
&\geq (I-S)^{-1}V_a - (I-S)^{-1}\sum_{l=1}^{p}\alpha_l V_l U_l^{\dagger} V_b \\
&= (I-S)^{-1}V_a - (I-S)^{-1}SV_b \\
&= (I-S)^{-1}(V_a - SV_b) \geq U \geq 0.
\end{aligned}$$

□

As a corollary, we have the following for nonsingular matrices.

Corollary 4.21. *Let $(U_l, V_l, E_l)_{l=1}^{p}$ be a weak regular multisplitting of type II of a monotone matrix $A \in \mathbb{R}^{n\times n}$. Assume that, for each l, $E_l = \alpha_l I$ with $\alpha_l > 0$ and $\sum_{l=1}^{p}\alpha_l = 1$. Let $V_a = \sum_{l=1}^{p}\alpha_l V_l$, and $V_l \leq V_b$, for each $l = 1, 2, \ldots, p$. If there exists a non-negative matrix $U \in \mathbb{R}^{n\times n}$ such that $V_l U_l^{-1}(V_b - U) \leq V_a - U$, for each $l = 1, 2, \ldots, p$, then the unique splitting $A = X - Y$ induced by S with $X = (I-S)^{-1}A$ is a regular splitting.*

5. Conclusion

This paper adds many new results to the convergence theory of proper multisplittings. The important findings are summarized as follows:

- Comparison results for proper weak regular splittings of same and different types are presented.
- Some of these results are then used to prove a few comparison results for proper multisplittings.
- Characterizations of semi-monotone matrices are obtained using proper weak regular multisplittings of different types.
- The induced splitting is also shown to be a convergent proper weak regular splitting of type II under some sufficient conditions.

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