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Bifurcation of a Class of Stochastic Delay Differential Equations

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Abstract. In this paper, we study the bifurcation of a class of two-dimensional stochastic delay differential equations. Firstly, we translate the original system into an Itô limiting diffusion system by applying stochastic Taylor expansion, small time delay expansion, polar coordinate transformation, and stochastic averaging procedure. Then we discuss the dynamical bifurcation by analyzing the qualitative changes of invariant measures, and investigate the phenomenological bifurcation by utilizing Fokker–Planck equation. The obtained conclusions are completely new, which generalize and improve some existing results.

1. Introduction

In this paper, we investigate the interplay of noise and delay on stochastic bifurcation, and consider the following stochastic delay differential equations (SDDEs)

$$\begin{cases} \mathrm{d}x(t) = \left[f_1(x(t), y(t)) + g_1(x(t-\tau_1))\right]\mathrm{d}t + h_1(x(t), y(t))\mathrm{d}W_1(t), \\ \mathrm{d}y(t) = \left[f_2(x(t), y(t)) + g_2(y(t-\tau_2))\right]\mathrm{d}t + h_2(x(t), y(t))\mathrm{d}W_2(t), \end{cases} \tag{1}$$

where $f_i \in C^3(\mathbb{R}\times\mathbb{R},\mathbb{R})$, $g_i \in C^1(\mathbb{R},\mathbb{R})$, $h_i \in C^1(\mathbb{R}\times\mathbb{R},\mathbb{R})$, $i = 1, 2$, τ_1 and τ_2 are two small time delays, $W_1(t)$ and $W_2(t)$ are mutually independent standard real-valued Wiener processes on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with the filtration $\{\mathcal{F}_t\}_{t\geq 0}$, i.e., $\mathbb{E}[W_1(t)] = \mathbb{E}[W_2(t)] = 0$, $\mathbb{E}[W_1(t)W_1(t+s)] = \delta(s)$, $\mathbb{E}[W_2(t)W_2(t+s)] = \delta(s)$ and $\mathbb{E}[W_1(t)W_2(t+s)] = 0$. We further suppose that f_i, g_i and h_i $(i = 1, 2)$ satisfy the global Lipschitz condition and the linear growth condition which always ensure the existence and uniqueness of the solutions to system (1), and also assume that $f_i(0,0) = h_i(0,0) = g_i(0) = 0$, which means that the point $O(0,0)$ is the fixed point of system (1).

It is known that stochastic bifurcation plays an important role in describing the qualitative changes of parameterized families of stochastic dynamical system. In the last decade, stochastic bifurcation has attracted extensive investigations, and some interesting results have been obtained for various stochastic differential equations, see, e.g., [2–4, 10, 12, 14, 16–18, 20–26]. For example, Huang et al. [12] employed Lyapunov exponent and singular boundary theory to investigate the stability and bifurcation of a stochastic differential equation modelling a Hopfield neural network with two neurons. Xu [21] explored the

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phenomenological bifurcation of a stochastic logistic model with correlated colored noises by performing detailed structural analysis of the stationary probability distribution. Using stochastic averaging method, Li et al. [14] studied the bifurcation of a business cycle model with stochastically parametric excitation. The main methods of these references are to transform the original system into a weakly perturbed Hamiltonian system or polar coordinate form, and then apply Lyapunov exponent, singular boundary theory, invariant measure or Fokker–Planck equation to analyze the dynamical behavior of the corresponding amplitude equation.

Note that the articles mentioned above deal with the bifurcation of stochastic differential equations without delay. As far as we know, the results on stochastic bifurcation of SDDEs are few. In fact, stochastic process without time delay contribution can be modelled as Markov process, which will simplify analytical calculation because the transition probability between two states is delta-correlated. However, it is no longer true for stochastic differential system with delay because the rate of change of state variable depends on not only the current but also the historical states. By introducing a multiple scale expansion method, Gaudreault et al. [7] investigated the bifurcation for a modified stochastic van der Pol oscillator with delayed feedback. Unfortunately, the analytical method is not applicable to system (1) because the system does not necessarily present or include different time scales. Feng et al. [5] adopted a technique by which the delayed feedback bang-bang control force was expressed approximately in terms of the state variable without delay. Meanwhile, Gaudreault [6] and Guillouzic et al. [9] proposed an approximate approach which employed the small delay expansion to expand stochastic delay process in one-dimensional SDDEs. Therefore, in this paper it is a crucial problem for us how to deal with the time delay term of system (1).

In contrast to the case of one-dimensional SDDEs, there exists greater difficulty in analyzing stochastic bifurcation of two-dimensional ones. In this paper, we shall develop the method of small time delay expansion to investigate the bifurcation of two-dimensional SDDEs. More precisely, we will show that system (7) undergoes a dynamical bifurcation as the feasible parameter passes through a critical value. Particularly, if $\tau_1 = \tau_2 = 0$ the dynamical bifurcation is identical to the case of without delay in paper [16], and if $h_i = 0$ $(i = 1, 2)$ the stochastic dynamical bifurcation will degenerate into a deterministic pitchfork bifurcation illustrated in [11]. We also will prove system (1) and system (7) undergo the consistent phenomenological bifurcations, and the results are identical to those of paper [16] when $\tau_1 = \tau_2 = 0$.

The structure of this paper is as follows. In the subsequent section, we obtain stochastic averaging Itô diffusion equations by applying stochastic Taylor expansion, small time delay expansion, standard rescalings, polar coordinate transformation, and stochastic averaging method. Section 3 is devoted to the dynamical bifurcation of the averaging modulus equation associated with system (7) by means of the properties of invariant measures, and the phenomenological bifurcations of systems (1) and (7) by virtue of their Fokker–Planck equations.

2. Preliminaries

In this section, our main task is to translate system (1) into an averaging Itô diffusion system. Taking the Taylor expansions of f_i, g_i and h_i at the point $O(0, 0)$, we rewrite system (1) as

$$\begin{cases} \mathrm{d}x = \Big\{\Big[a_{110}x + a_{101}y + a_{120}x^2 + a_{111}xy + a_{102}y^2 + a_{130}x^3 + a_{121}x^2y + a_{112}xy^2 \\ \qquad +a_{103}y^3 + o(3)\Big] + \big[b_{11}x_{\tau_1} + o(1)\big]\Big\}\,\mathrm{d}t + \big[c_{110}x + c_{101}y + o(1)\big]\,\mathrm{d}W_1(t), \\ \mathrm{d}y = \Big\{\Big[a_{210}x + a_{201}y + a_{220}x^2 + a_{211}xy + a_{202}y^2 + a_{230}x^3 + a_{221}x^2y + a_{212}xy^2 \\ \qquad +a_{203}y^3 + o(3)\Big] + \big[b_{21}y_{\tau_2} + o(1)\big]\Big\}\,\mathrm{d}t + \big[c_{210}x + c_{201}y + o(1)\big]\,\mathrm{d}W_2(t), \end{cases} \tag{1}$$

where $x_{\tau_1} = x(t - \tau_1)$, $y_{\tau_2} = y(t - \tau_2)$, $o(3)$ and $o(1)$ represent the higher order terms, and

$$a_{jks} = \frac{1}{k!s!} \cdot \frac{\partial^{k+s}}{\partial x^k \partial y^s}\bigg|_{(x,y)=(0,0)} f_j(x, y), \ \ b_{11} = \frac{\mathrm{d}}{\mathrm{d}x}\bigg|_{x=0} g_1(x);$$

$$c_{jmn} = \frac{1}{m!n!} \cdot \frac{\partial^{m+n}}{\partial x^m \partial y^n}\bigg|_{(x,y)=(0,0)} h_j(x, y), \ \ b_{21} = \frac{\mathrm{d}}{\mathrm{d}y}\bigg|_{y=0} g_2(y)$$

for $j \in \{1,2\}$, $k,s \in \{0,1,2,3\}$ with $(k,s) \neq (0,0)$, and $m,n \in \{0,1\}$ with $(m,n) \neq (0,0)$. The truncated equations of system (1) are as follows:

$$\begin{cases} \mathrm{d}x =(a_{110}x + a_{101}y + a_{120}x^2 + a_{111}xy + a_{102}y^2 + a_{130}x^3 + a_{121}x^2y + a_{112}xy^2 + a_{103}y^3 + b_{11}x_{\tau_1})\mathrm{d}t \\ \qquad + (c_{110}x + c_{101}y)\mathrm{d}W_1(t), \\ \mathrm{d}y =(a_{210}x + a_{201}y + a_{220}x^2 + a_{211}xy + a_{202}y^2 + a_{230}x^3 + a_{221}x^2y + a_{212}xy^2 + a_{203}y^3 + b_{21}y_{\tau_2})\mathrm{d}t \\ \qquad + (c_{210}x + c_{201}y)\mathrm{d}W_2(t). \end{cases} \tag{2}$$

By utilizing the idea of small delay expansion [6] , system (2) can be translated into

$$\begin{cases} \mathrm{d}x =(K_{110}x + K_{101}y + K_{120}x^2 + K_{111}xy + K_{102}y^2 + K_{130}x^3 + K_{121}x^2y + K_{112}xy^2 + K_{103}y^3)\mathrm{d}t \\ \qquad + (\kappa_{110}x + \kappa_{101}y)\mathrm{d}W_1(t), \\ \mathrm{d}y =(K_{210}x + K_{201}y + K_{220}x^2 + K_{211}xy + K_{202}y^2 + K_{230}x^3 + K_{221}x^2y + K_{212}xy^2 + K_{203}y^3)\mathrm{d}t \\ \qquad + (\kappa_{210}x + \kappa_{201}y)\mathrm{d}W_2(t), \end{cases} \tag{3}$$

where $K_{jks} = (1 - \tau_j b_{j1}) \cdot a_{jks}$ and $\kappa_{jmn} = (1 - \tau_j b_{j1}) \cdot c_{jmn}$ for $j \in \{1,2\}$, $k,s \in \{0,1,2,3\}$ with $(k,s) \neq (0,0)$, and $m,n \in \{0,1\}$ with $(m,n) \neq (0,0)$, excluding $K_{110} = (1 - \tau_1 b_{11})(a_{110} + b_{11})$ and $K_{201} = (1 - \tau_2 b_{21})(a_{201} + b_{21})$.

To detect the stochastic dynamical behaviors, we need to translate system (3) into a weakly perturbed system. Similar to the argument of Guckenheimer and Liang in[8, 15], we introduce the following standard rescalings $x = \bar{x}$, $y = \bar{y}$, $t = \bar{t}$, $K_{jks} = \varepsilon \bar{K}_{jks}$, $\kappa_{jmn} = \varepsilon^{\frac{1}{2}} \bar{\kappa}_{jmn}$, where $j \in \{1,2\}$, $k,s \in \{0,1,2,3\}$, $m,n \in \{0,1\}$, ε is a sufficiently small positive number. For convenience, we omit the bars from the rescaled variables and obtain

$$\begin{cases} \mathrm{d}x =\varepsilon(K_{110}x + K_{101}y + K_{120}x^2 + K_{111}xy + K_{102}y^2 + K_{130}x^3 + K_{121}x^2y + K_{112}xy^2 + K_{103}y^3)\mathrm{d}t \\ \qquad + \varepsilon^{\frac{1}{2}}(\kappa_{110}x + \kappa_{101}y)\mathrm{d}W_1(t), \\ \mathrm{d}y =\varepsilon(K_{210}x + K_{201}y + K_{220}x^2 + K_{211}xy + K_{202}y^2 + K_{230}x^3 + K_{221}x^2y + K_{212}xy^2 + K_{203}y^3)\mathrm{d}t \\ \qquad + \varepsilon^{\frac{1}{2}}(\kappa_{210}x + \kappa_{201}y)\mathrm{d}W_2(t). \end{cases} \tag{4}$$

By combining the polar coordinate transformation $x = r\cos\theta$ and $y = r\sin\theta$ with Itô formula, we obtain

$$\begin{cases} \mathrm{d}r = & \varepsilon \Big\{ \Big[K_{110}\cos^2\theta + (K_{101} + K_{210})\cos\theta\sin\theta + K_{201}\sin^2\theta \\ & +\frac{1}{2}\kappa_{210}^2\cos^4\theta + \kappa_{210}\kappa_{201}\cos^3\theta\sin\theta + \frac{1}{2}\kappa_{101}^2\sin^4\theta \\ & +\frac{1}{2}(\kappa_{110}^2 + \kappa_{201}^2)\cos^2\theta\sin^2\theta + \kappa_{110}\kappa_{101}\cos\theta\sin^3\theta \Big] r \\ & + \Big[K_{120}\cos^3\theta + (K_{111} + K_{220})\cos^2\theta\sin\theta \\ & +(K_{102} + K_{211})\cos\theta\sin^2\theta + K_{202}\sin^3\theta \Big] r^2 + \Big[K_{130}\cos^4\theta \\ & +(K_{121} + K_{230})\cos^3\theta\sin\theta + (K_{112} + K_{221})\cos^2\theta\sin^2\theta \\ & +(K_{103} + K_{212})\cos\theta\sin^3\theta + K_{203}\sin^4\theta \Big] r^3 \Big\} \mathrm{d}t \\ & +\varepsilon^{\frac{1}{2}}(\kappa_{110}\cos^2\theta + \kappa_{101}\cos\theta\sin\theta) r\mathrm{d}W_1(t) \\ & +\varepsilon^{\frac{1}{2}}(\kappa_{210}\cos\theta\sin\theta + \kappa_{201}\sin^2\theta) r\mathrm{d}W_2(t), \\ \mathrm{d}\theta = & \varepsilon \Big\{ \Big[K_{210}\cos^2\theta + (-K_{110} + K_{201})\cos\theta\sin\theta - K_{101}\sin^2\theta \\ & +(\kappa_{110}^2 - \kappa_{210}^2)\cos^3\theta\sin\theta + (\kappa_{101}^2 - \kappa_{201}^2)\cos\theta\sin^3\theta \\ & +2(\kappa_{110}\kappa_{101} - \kappa_{210}\kappa_{201})\cos^2\theta\sin^2\theta \Big] \\ & + \Big[K_{220}\cos^3\theta + (K_{211} - K_{120})\cos^2\theta\sin\theta \\ & +(K_{202} - K_{111})\cos\theta\sin^2\theta - K_{102}\sin^3\theta \Big] r + \Big[K_{230}\cos^4\theta \\ & +(K_{221} - K_{130})\cos^3\theta\sin\theta + (K_{212} - K_{121})\cos^2\theta\sin^2\theta \\ & +(K_{203} - K_{112})\cos\theta\sin^3\theta - K_{103}\sin^4\theta \Big] r^2 \Big\} \mathrm{d}t \\ & -\varepsilon^{\frac{1}{2}}(\kappa_{110}\cos\theta\sin\theta + \kappa_{101}\sin^2\theta)\mathrm{d}W_1(t) \\ & +\varepsilon^{\frac{1}{2}}(\kappa_{210}\cos^2\theta + \kappa_{201}\cos\theta\sin\theta)\mathrm{d}W_2(t). \end{cases} \tag{5}$$

Follows from the Khasminskii limiting theorem [13] that the stochastic response process $\{r(t), \theta(t)\}$ of system (5) converges weakly to a two-dimensional Markov diffusion process when ε is sufficiently small. Therefore, we obtain

$$\begin{cases} \mathrm{d}r = \left[\left(\mu_1 + \dfrac{\mu_2}{16}\right)r + \dfrac{\mu_3}{8}r^3\right]\mathrm{d}t + \left(\dfrac{\mu_4}{8}r^2\right)^{\frac{1}{2}} \mathrm{d}W_r(t), \\ \mathrm{d}\theta = \left(\dfrac{1}{4}\mu_5 + \dfrac{\mu_6}{8}r^2\right)\mathrm{d}t + \left(\dfrac{\mu_2}{8}\right)^{\frac{1}{2}} \mathrm{d}W_\theta(t), \end{cases} \tag{6}$$

where $W_r(t)$ and $W_\theta(t)$ are mutually independent standard Wiener processes, and

$$\begin{aligned} \mu_1 =& \frac{1}{2}(K_{110} + K_{201}), \ \ \mu_2 = \kappa_{110}^2 + \kappa_{201}^2 + \kappa_{101}^2 + 3\kappa_{210}^2, \\ \mu_3 =& 3K_{130} + K_{112} + K_{221} + K_{203}, \ \ \mu_4 = 3\kappa_{110}^2 + \kappa_{101}^2 + \kappa_{210}^2 + \kappa_{201}^2, \\ \mu_5 =& -2K_{101} + 2K_{210} + \kappa_{110}\kappa_{101} - \kappa_{210}\kappa_{201}, \ \ \mu_6 = -K_{103} + K_{212} - K_{121} + 3K_{230}. \end{aligned}$$

In view of the existence of random factors, we always assume that μ_2 and μ_4 are not equal to zero in the subsequent analysis, which implies that both μ_2 and μ_4 are positive. Note that in system (6) the averaging amplitude equation is uncoupled with the phase equation. Therefore, we can analyze the stochastic dynamics of system (1) by discussing the averaging amplitude equation of system (6), i.e.,

$$\mathrm{d}r = \left[\left(\mu_1 + \frac{\mu_2}{16}\right)r + \frac{\mu_3}{8}r^3\right]\mathrm{d}t + \left(\frac{\mu_4}{8}r^2\right)^{\frac{1}{2}} \mathrm{d}W_r(t). \tag{7}$$

3. Stochastic bifurcation

In this section, we mainly investigate the bifurcation behaviors of some related systems from the dynamical viewpoint and phenomenological approach. Henceforth, we always assume $\mu_3 < 0$ and $\mu_4 > 0$.

3.1. D-bifurcation

Let $v_t = \left(-\frac{\mu_3}{8}\right)^{\frac{1}{2}} r$, then system (7) can be translated into the following Stratonovich stochastic differential equation

$$\mathrm{d}v_t = \left[\left(\mu_1 + \frac{\mu_2}{16} - \frac{\mu_4}{16}\right)v_t - v_t^3\right]\mathrm{d}t + \left(\frac{\mu_4}{8}\right)^{\frac{1}{2}} v_t \circ \mathrm{d}W_r(t). \tag{1}$$

Let $\alpha = \mu_1 + \frac{\mu_2}{16} - \frac{\mu_4}{16}$ and $\sigma = \left(\frac{\mu_4}{8}\right)^{\frac{1}{2}}$, then the system can be rewritten as

$$\mathrm{d}v_t = \left(\alpha v_t - v_t^3\right)\mathrm{d}t + \sigma v_t \circ \mathrm{d}W_r(t), \tag{2}$$

which generates the following local random dynamical system (RDS)

$$\varphi_\alpha(t,\omega)v = \frac{v\exp(\alpha t + \sigma W_r(t))}{\left(1 + 2v^2\int_0^t \exp(2\alpha s + 2\sigma W_r(s))\mathrm{d}s\right)^{\frac{1}{2}}}, \tag{3}$$

where v is the initial value of v_t. The domain $D_\alpha(t,\omega)$ and the range $R_\alpha(t,\omega)$ of RDS $\varphi_\alpha(t,\omega) : D_\alpha(t,\omega) \to R_\alpha(t,\omega)$ will be determined as follows:

$$D_\alpha(t,\omega) = \begin{cases} \mathbb{R}, & t \geq 0, \\ (-d_\alpha(t,\omega), d_\alpha(t,\omega)), & t < 0, \end{cases} \tag{4}$$

and

$$R_\alpha(t,\omega) = D_\alpha(-t,\vartheta(t)\omega) = \begin{cases} (-r_\alpha(t,\omega), r_\alpha(t,\omega)), & t > 0, \\ \mathbb{R}, & t \leq 0, \end{cases} \tag{5}$$

where

$$d_\alpha(t,\omega) = \frac{1}{\left(2\left|\int_0^t \exp(2\alpha s + 2\sigma W_r(s))\mathrm{d}s\right|\right)^{\frac{1}{2}}} > 0,$$

and

$$r_\alpha(t,\omega) = d_\alpha(-t,\vartheta(t)\omega) = \frac{\exp(\alpha s + \sigma W_r(s))}{\left(2\left|\int_0^t \exp(2\alpha s + 2\sigma W_r(s))\mathrm{d}s\right|\right)^{\frac{1}{2}}} > 0.$$

Set $E_\alpha(\omega) = \cap_{t\in\mathbb{R}} D_\alpha(t,\omega)$, which is the set of initial values v ensuring the non-explosion of RDS $\varphi_\alpha(t,\omega)v$, i.e.,

$$E_\alpha(\omega) = \begin{cases} (-d_\alpha^-(t,\omega), d_\alpha^-(t,\omega)), & \alpha > 0, \\ \{0\}, & \alpha \leq 0, \end{cases} \tag{6}$$

where

$$0 < d_\alpha^-(t,\omega) = \frac{1}{\left(2\left|\int_0^\infty \exp(2\alpha s + 2\sigma W_r(s))\mathrm{d}s\right|\right)^{\frac{1}{2}}} < \infty.$$

By using Theorem 1.8.4 of [1], we have

(i) For $\alpha > 0$, there exist three random Dirac measures $\mu_{1,\omega}^\alpha = \delta_0$, $\mu_{2,\omega}^\alpha = \delta_{-d_\alpha^-(\omega)}$ and $\mu_{3,\omega}^\alpha = \delta_{d_\alpha^-(\omega)}$, which are all Markov measures;

(ii) For $\alpha \leq 0$, there exists exactly one invariant measure $\mu_{1,\omega}^\alpha = \delta_0$.

We note that the linearized RDS $\chi_t = D\varphi_\alpha(t,\omega,v)\chi$ satisfies

$$\chi_t = \left(\alpha - 3(\varphi_\alpha(t,\omega,v))^2\right)\chi_t\mathrm{d}t + \sigma\chi_t \circ \mathrm{d}W_r(t), \tag{7}$$

which implies

$$\chi_t = D\varphi_\alpha(t,\omega,v)\chi = \chi \exp\left(\alpha t + \sigma W_r(\omega) - 3\int_0^t (\varphi_\alpha(s,\omega,v))^2\mathrm{d}s\right).$$

Therefore, we deduce that the Lyapunov exponents of $\mu_{i,\omega}^\alpha$ $(i = 1,2,3)$ satisfy

$$\lambda(\mu_{i,\omega}^\alpha) = \lim_{t\to\infty}\frac{1}{t}\ln\|D\varphi_\alpha(t,\omega,v)\chi\| = \alpha - 3\lim_{t\to\infty}\frac{1}{t}\int_0^t (\varphi_\alpha(s,\omega,v))^2\mathrm{d}s = \alpha - 3\mathbb{E}v^2 \tag{8}$$

if the implicit condition (IC) $v^2 \in L^1(\mathbb{P})$ holds. Thus, we obtain

(i) For $\alpha \in \mathbb{R}$, the Dirac measure $\mu_{1,\omega}^\alpha = \delta_0$ is satisfied, then $\lambda(\mu_{1,\omega}^\alpha) = \alpha$. We easily obtain that $\mu_{1,\omega}^\alpha$ is stable for $\alpha < 0$ and unstable for $\alpha > 0$;

(ii) For $\alpha > 0$, the invariant measure $\mu_{2,\omega}^{\alpha} = \delta_{-d_{\alpha}^{-}(\omega)}$ is $\mathcal{F}_{-\infty}^{0}$ measurable. Hence, the density function $p_2(v)$ of $\mu_{2,\omega}^{\alpha}$ satisfies the Fokker–Planck equation

$$\frac{\partial p_2(v)}{\partial t} = -\frac{\partial}{\partial v}\left(\left(\alpha v + \frac{\sigma^2}{2}v - v^3\right)p_2(v)\right) + \frac{1}{2}\frac{\partial^2}{\partial v^2}\left(\sigma^2 v^2 p_2(v)\right),$$

which possesses the unique solution

$$p_2(v) = \begin{cases} C_{\alpha} v^{\frac{2\alpha}{\sigma^2}-1}\exp\left(-\frac{v^2}{\sigma^2}\right), & v \geq 0, \\ 0, & v < 0 \end{cases}$$

with the normalizing parameter $C_{\alpha} = \left(\sigma^{\frac{2\alpha}{\sigma^2}}\Gamma\left(\frac{\alpha}{\sigma^2}\right)\right)^{-1}$. And because $\mathbb{E}_{\mu_{2,\omega}^{\alpha}} v^2 = \mathbb{E}(-d_{\alpha}^{-}(\omega))^2 = \int_0^{\infty} v^2 p_2(v)\mathrm{d}v < \infty$ always implicitly holds. Furthermore,

$$(d_{\alpha}^{-}(\vartheta(t)\omega))^2 = \frac{\exp(2\alpha s + 2\sigma W_r(s))}{2\int_{-\infty}^{t}\exp(2\alpha s + 2\sigma W_r(s))\mathrm{d}s} = \frac{\Psi'(t)}{2\Psi(t)},$$

where $\Psi(t) = \int_{-\infty}^{t}\exp(2\alpha s + 2\sigma W_r(s))\mathrm{d}s$. From the ergodic theory, it follows that $\mathbb{E}(d_{\alpha}^{-}(\omega))^2 = \frac{1}{2}\lim_{t\to\infty}\frac{1}{t}\ln\Psi(t) = \alpha$. By (8), we immediately obtain $\lambda(\mu_{2,\omega}^{\alpha}) = -2\alpha < 0$, which implies that $\mu_{2,\omega}^{\alpha}$ is stable for $\alpha > 0$;

(iii) For $\alpha > 0$, the invariant measure $\mu_{3,\omega}^{\alpha} = \delta_{d_{\alpha}^{-}(\omega)}$ is also $\mathcal{F}_{-\infty}^{0}$ measurable, and its density $p_3(v)$ is equal to $p_2(-v)$. Moreover, we have $\mathbb{E}(-d_{\alpha}^{-}(\omega))^2 = \mathbb{E}(d_{\alpha}^{-}(\omega))^2 = \alpha$, and hence $\lambda(\mu_{3,\omega}^{\alpha}) = -2\alpha < 0$. Similarly, we deduce that the invariant measure $\mu_{3,\omega}^{\alpha}$ is stable for $\alpha > 0$.

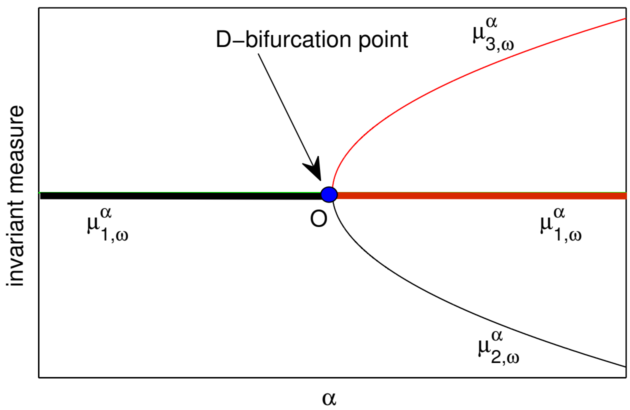

Fig. 1: The graph of D-bifurcation of stochastic system (7).

Therefore, we can draw the following result.

Theorem 3.1. *If $\alpha < 0$, i.e., $\mu_4 > 16\mu_1 + \mu_2$, the RDS $\varphi_{\alpha}(t,\omega)$ possesses exactly one invariant measure $\mu_{1,\omega}^{\alpha}$, which is stable. If $\alpha > 0$, i.e., $\mu_4 < 16\mu_1 + \mu_2$, the RDS $\varphi_{\alpha}(t,\omega)$ possesses three random Dirac measures $\mu_{1,\omega}^{\alpha}$, $\mu_{2,\omega}^{\alpha}$ and $\mu_{3,\omega}^{\alpha}$, where $\mu_{1,\omega}^{\alpha}$ is unstable, $\mu_{2,\omega}^{\alpha}$ and $\mu_{3,\omega}^{\alpha}$ are stable. Therefore, the RDS $\varphi_{\alpha}(t,\omega)$ undergoes a D-bifurcation at the point $\alpha_D = 0$, i.e., stochastic system (7) undergoes a stochastic pitchfork bifurcation (the details to see [1]) as the parameter μ_4 passes through the value of $16\mu_1 + \mu_2$.*

Remark 3.2. *From the above analysis, we discover that the location of D-bifurcation depends on the parameters μ_1, μ_2 and μ_4, which implies that the location of bifurcation point is associated with time delays τ_1 and τ_2. In particular, if $\tau_1 = \tau_2 = 0$, then the dynamical bifurcation of system (7) is identical to the one of without delay in paper [16].*

Remark 3.3. *If the random perturbation of system (1) disappears, i.e., $h_i(x, y) = 0$ (i=1,2), then $\mu_2 = \mu_4 = 0$ and stochastic system (7) will be translated into the following deterministic differential equation*

$$\mathrm{d}r = \left(\mu_1 r + \frac{\mu_3}{8} r^3\right)\mathrm{d}t.$$

It is easy to draw a conclusion that the above system undergoes a deterministic pitchfork bifurcation as the parameter μ_1 passes through the bifurcation value $\mu_1 = 0$. Therefore, in a sense the stochastic pitchfork bifurcation can be seen as the generalization of deterministic pitchfork bifurcation.

3.2. P-bifurcation

Firstly, we investigate the phenomenological bifurcation of stochastic system (7). The Fokker–Planck equation of density function $p(r)$ can be denoted by

$$\frac{\partial p(r)}{\partial t} = -\frac{\partial}{\partial r}\left[\left(\left(\mu_1 + \frac{\mu_2}{16}\right) r + \frac{\mu_3}{8} r^3\right) p(r)\right] + \frac{1}{2}\frac{\partial^2}{\partial r^2}\left(\frac{\mu_4}{8} r^2 p(r)\right),$$

and the density function $p(r)$ satisfies the degenerate equation

$$-\frac{\partial}{\partial r}\left[\left(\left(\mu_1 + \frac{\mu_2}{16}\right) r + \frac{\mu_3}{8} r^3\right) p(r)\right] + \frac{1}{2}\frac{\partial^2}{\partial r^2}\left(\frac{\mu_4}{8} r^2 p(r)\right) = 0.$$

By utilizing the existence condition of invariant measures in D-bifurcation analysis, the density function $p(r)$ can be written as

$$p(r) = \begin{cases} \delta(r), & \mu_4 \geq 16\mu_1 + \mu_2, \\ \dfrac{r^{\frac{16\mu_1+\mu_2-2\mu_4}{\mu_4}} \cdot \exp\left(\frac{\mu_3}{\mu_4} r^2\right)}{\Gamma\left(\frac{16\mu_1+\mu_2-\mu_4}{2\mu_4}\right)\left(-\frac{\mu_4}{\mu_3}\right)^{\frac{16\mu_1+\mu_2-\mu_4}{2\mu_4}}}, & \mu_4 < 16\mu_1 + \mu_2. \end{cases} \tag{9}$$

To obtain the extreme value point of density function $p(r)$, we need to solve $\frac{\mathrm{d}p(r)}{\mathrm{d}r} = 0$, i.e.,

$$\frac{\left[\frac{16\mu_1+\mu_2-2\mu_4}{\mu_4} + \frac{2\mu_3}{\mu_4} r^2\right] r^{\frac{16\mu_1+\mu_2-3\mu_4}{\mu_4}} \cdot \exp\left(\frac{\mu_3}{\mu_4} r^2\right)}{\Gamma\left(\frac{16\mu_1+\mu_2-\mu_4}{2\mu_4}\right)\left(-\frac{\mu_4}{\mu_3}\right)^{\frac{16\mu_1+\mu_2-\mu_4}{2\mu_4}}} = 0. \tag{10}$$

Therefore, we get that $r = 0$ or

$$r = r^* \triangleq \sqrt{\frac{2\mu_4 - 16\mu_1 - \mu_2}{2\mu_3}} \tag{11}$$

for $\mu_4 < 8\mu_1 + \frac{1}{2}\mu_2$, moreover the condition $\frac{\mathrm{d}^2 p(r)}{\mathrm{d}r^2}|_{r=r^*} < 0$ holds. Therefore, it is very easy to obtain the following conclusions:

(i) If $8\mu_1 + \frac{1}{2}\mu_2 \leq \mu_4 < 16\mu_1 + \mu_2$, the density function $p(r)$ tends to infinite as $r \to 0^+$. In this case, the random trajectories of system (7) are concentrated at the neighborhood of the point o. The result is shown in Fig.2.

(ii) If $\frac{16\mu_1+\mu_2}{3} \leq \mu_4 < 8\mu_1 + \frac{1}{2}\mu_2$, the density function $p(r)$ possesses a maximum value at the point $r^* = \sqrt{\frac{2\mu_4-16\mu_1-\mu_2}{2\mu_3}}$ and a minimum value at the point $r_* = 0$, but the derivative of $p(r)$ does not exist at the point $r_* = 0$. In this case, the random trajectories of system (7) are concentrated at the neighborhood of the point r^*. These results are shown in Fig.3.

(iii) If $\mu_4 < \frac{16\mu_1+\mu_2}{3}$, the density function $p(r)$ possesses a maximum value at the point $r^* = \sqrt{\frac{2\mu_4-16\mu_1-\mu_2}{2\mu_3}}$ and a minimum value at the point $r_* = 0$. Here, we easily find that the density function $p(r)$ becomes a smooth function at the point $r_* = 0$. These results are shown in Fig.4.

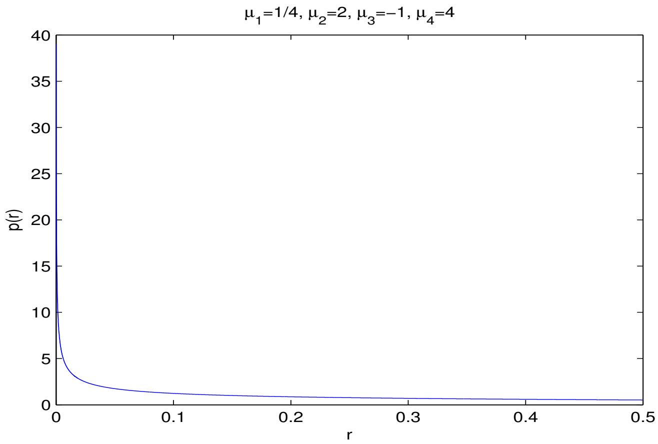

Fig. 2: The graph of stationary probability distribution $p(r)$ of system (7) for $8\mu_1 + \frac{1}{2}\mu_2 \leq \mu_4 < 16\mu_1 + \mu_2$.

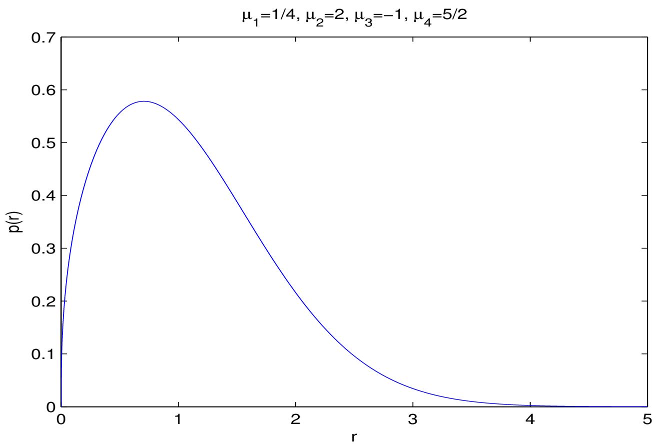

Fig. 3: The graph of stationary probability distribution $p(r)$ of system (7) for $\frac{16\mu_1+\mu_2}{3} \leq \mu_4 < 8\mu_1 + \frac{1}{2}\mu_2$.

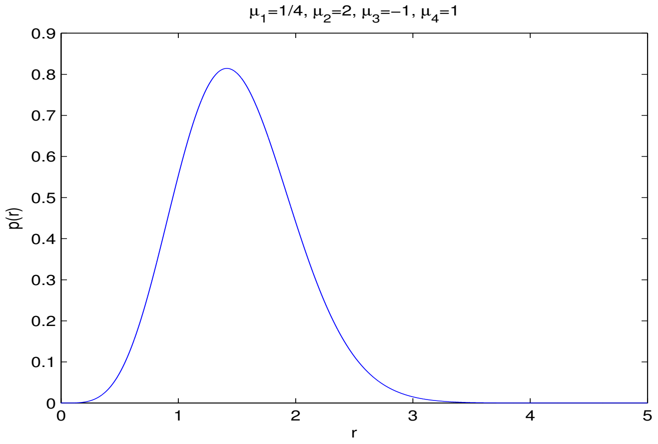

Fig. 4: The graph of stationary probability distribution $p(r)$ of system (7) for $\mu_4 < \frac{16\mu_1+\mu_2}{3}$.

In summary, we can obtain the following result.

Theorem 3.4. *If $\mu_3 < 0$ and $\mu_4 > 0$, system (7) undergoes stochastic phenomenological bifurcations as the parameter μ_4 passes through the values of $8\mu_1 + \frac{1}{2}\mu_2$ and $\frac{16\mu_1+\mu_2}{3}$. In particular, there appears stochastic Hopf bifurcation when $\mu_4 = 8\mu_1 + \frac{1}{2}\mu_2$.*

Remark 3.5. *We observe that as the parameter μ_4 passes through the value of $16\mu_1 + \mu_2$, the density function $p(r)$ changes from a delta function $\delta(r)$ to a power law with an exponential cut off at large r, which implies that system (7) also undergoes a stochastic phenomenological bifurcation in a generalized sense.*

Remark 3.6. *It is not difficult to find that the phenomenological bifurcation of system (7) is identical to the one of without delay in paper [16] when $\tau_1 = \tau_2 = 0$.*

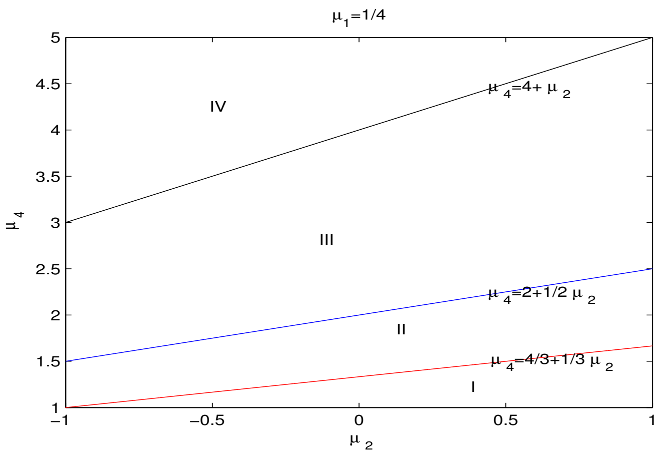

Fig. 5: The graph of P-bifurcation of stochastic system (7).

Remark 3.7. *In Fig.5, the P-bifurcation diagram of system (7) with $\mu_1 = 1/4$ was depicted, and the plane of control parameters μ_2 and μ_4 was divided into four parts: I,II, III, IV. In the regions I and II, the density function $p(r)$ possesses a maximum at the point r^* and a minimum at the point $r_* = 0$. In the regions III and IV, there does not exist extreme value point for the density $p(r)$.*

In what follows, we analyze the features of the joint density $\rho(x,y)$ to Cartesian coordinates x and y. By using the relation $\rho(x,y) = |J|\varrho(r,\theta)$ and $p(r) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \varrho(r,\theta)\mathrm{d}\theta$, where $|J| = \frac{1}{r}$ and $\varrho(r,\theta)$ is the joint density to r and θ (see [19]), we obtain

$$\rho(x,y) = \frac{(x^2+y^2)^{\frac{16\mu_1+\mu_2-3\mu_4}{2\mu_4}} \cdot \exp\left(\frac{\mu_3}{\mu_4}(x^2+y^2)\right)}{\pi\Gamma\left(\frac{16\mu_1+\mu_2-\mu_4}{2\mu_4}\right)\left(-\frac{\mu_4}{\mu_3}\right)^{\frac{16\mu_1+\mu_2-\mu_4}{2\mu_4}}}.$$

Thus, we obtain the following conclusions:

(i) If $\mu_4 \geq \frac{16\mu_1+\mu_2}{3}$, the joint density $\rho(x,y)$ close to infinite as $x \to 0$ and $y \to 0$. The result is shown in Fig.6 and Fig.9.

(ii) If $\frac{16\mu_1+\mu_2}{4} \leq \mu_4 < \frac{16\mu_1+\mu_2}{3}$, a maximum value appears at the points of stable limit cycle $x^2+y^2 = \frac{3\mu_4-16\mu_1-\mu_2}{2\mu_3}$, and a minimum value arises at the point $O(0,0)$. Meantime, we notice that the partial derivatives of $\rho(x,y)$ are discontinuous at the point $O(0,0)$. These results are shown in Fig.7 and Fig.10.

(ii) If $\mu_4 < \frac{16\mu_1+\mu_2}{4}$, the joint density $\rho(x,y)$ possesses a maximum value, which appears at the points of stable limit cycle $x^2+y^2 = \frac{3\mu_4-16\mu_1-\mu_2}{2\mu_3}$, and a minimum value at the original point $O(0,0)$ where the partial derivatives of $\rho(x,y)$ are continuous. These results are shown in Fig.8 and Fig.11.

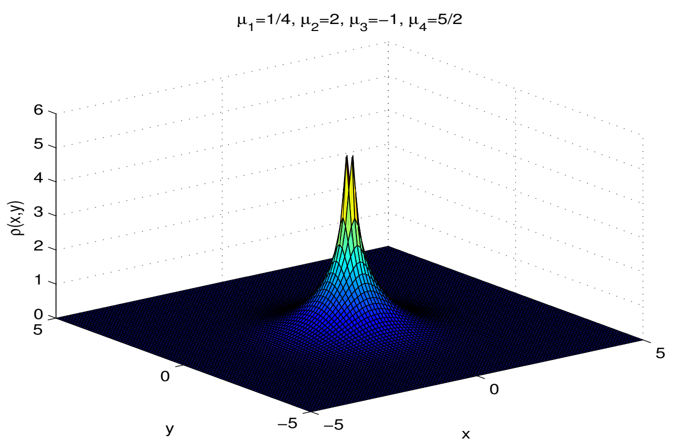

Fig. 6: The graph of joint density $\rho(x,y)$ of system (1) for $\mu_4 \geq \frac{16\mu_1+\mu_2}{3}$.

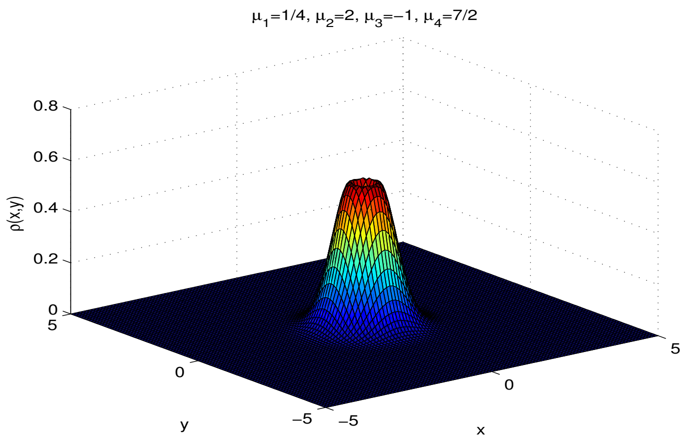

Fig. 7: The graph of joint density $\rho(x, y)$ of system (1) for $\frac{16\mu_1+\mu_2}{4} \leq \mu_4 < \frac{16\mu_1+\mu_2}{3}$.

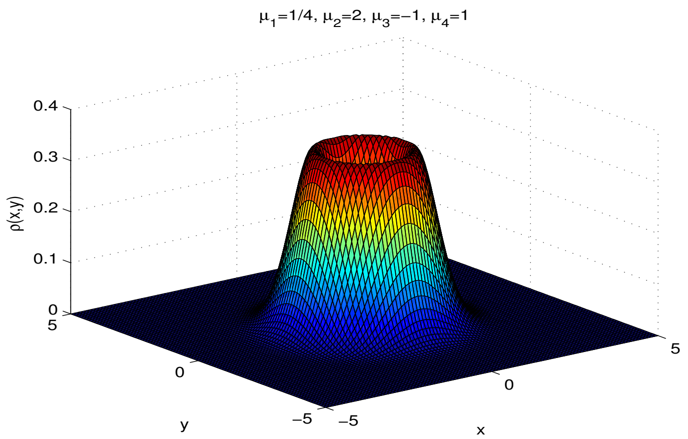

Fig. 8: The graph of joint density $\rho(x, y)$ of system (1) for $\mu_4 < \frac{16\mu_1+\mu_2}{4}$.

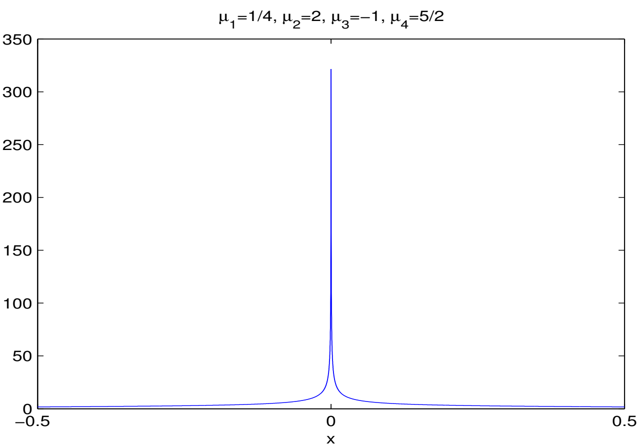

Fig. 9: The graph of joint density $\rho(x, 0)$ of system (1) for $\mu_4 \geq \frac{16\mu_1+\mu_2}{3}$.

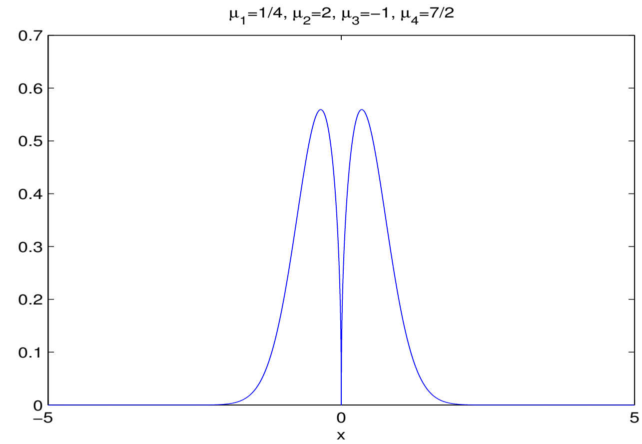

Fig. 10: The graph of joint density $\rho(x, 0)$ of system (1) for $\frac{16\mu_1+\mu_2}{4} \leq \mu_4 < \frac{16\mu_1+\mu_2}{3}$.

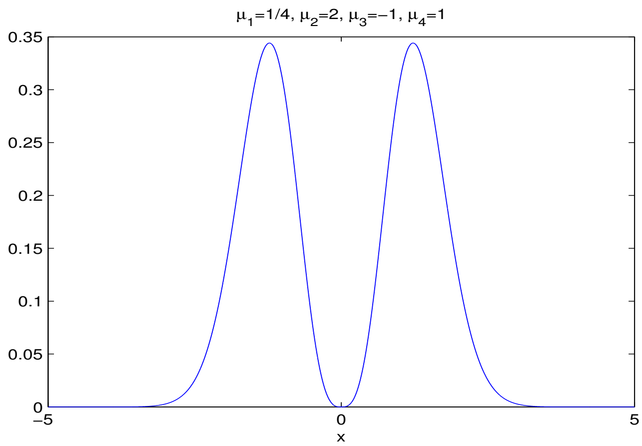

Fig. 11: The graph of joint density $\rho(x, 0)$ of system (1) for $\mu_4 < \frac{16\mu_1+\mu_2}{4}$.

We have the following result.

Theorem 3.8. *If $\mu_3 < 0$ and $\mu_4 > 0$, stochastic system (1) undergoes phenomenological bifurcations as the parameter μ_4 passes through the values of $\frac{16\mu_1+\mu_2}{3}$ and $\frac{16\mu_1+\mu_2}{4}$. In particular, there appears stochastic Hopf bifurcation when $\mu_4 = \frac{16\mu_1+\mu_2}{3}$.*

Remark 3.9. *If $\tau_1 = \tau_2 = 0$, then the phenomenological bifurcation of system (1) is identical to the one of without delay in paper [16].*

Remark 3.10. *It is worth pointing out that systems (1) and (7) undergo the consistent phenomenological bifurcations, but the appearance of the maximum at points of the limit cycle in the joint density $\rho(x, y)$ does not match the appearance of the maximum for the density $p(r)$.*

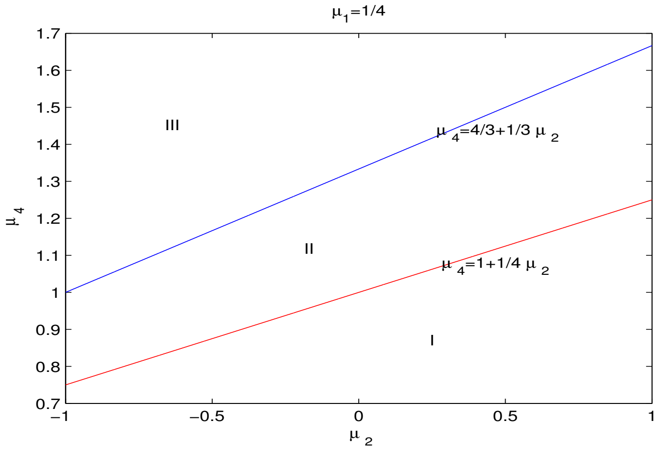

Fig. 12: The graph of P-bifurcation of stochastic system (1).

Remark 3.11. *In Fig.12, the P-bifurcation diagram of system (1) with $\mu_1 = 1/4$ was depicted, the plane of control parameters μ_2 and μ_4 was divided into three parts: I,II, III. In the regions I and II, the joint density $\rho(x, y)$ possesses a maximum value at the points of stable limit cycle $x^2 + y^2 = \frac{3\mu_4 - 16\mu_1 - \mu_2}{2\mu_3}$ and a minimum value at the origin $O(0, 0)$. In the region III, there does not exist extreme value point.*

4. Conclusions

In this paper, we mainly investigate the dynamical and phenomenological bifurcations of a class of two-dimensional stochastic delay differential equations. It is shown that our results are new and extend some previously known results in the literature. However, we only consider the case that time delay appear in the drift term, which still cannot describe accurately the real dynamical behavior for some practical system. Therefore, our future work is to introduce two or more time delays into the drift and diffusion terms. We believe that the treatment will bring more abundant dynamical behaviors for the related system.

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