



Existence of Positive Solutions to the Fractional Laplacian With Positive Dirichlet Data

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Abstract. We consider the fractional Laplacian with positive Dirichlet data

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} u = \lambda u^p & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = \varphi & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases}$$

where $p > 1, 0 < \alpha < \min\{2, n\}$, $\Omega \subset \mathbb{R}^n$ is a smooth bounded domain, φ is a nonnegative function, positive somewhere and satisfying some other conditions. We prove that there exists $\lambda^* > 0$ such that for any $0 < \lambda < \lambda^*$, the problem admits at least one positive classical solution; for $\lambda > \lambda^*$, the problem admits no classical solution. Moreover, for $1 < p \leq \frac{n+\alpha}{n-\alpha}$, there exists $0 < \bar{\lambda} \leq \lambda^*$ such that for any $0 < \lambda < \bar{\lambda}$, the problem admits a second positive classical solution. From the results obtained, we can see that the existence results of the fractional Laplacian with positive Dirichlet data are quite different from the fractional Laplacian with zero Dirichlet data.

1. Introduction

In this paper we consider the problem

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} u = \lambda u^p & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = \varphi & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \quad (1)$$

where $0 < \alpha < \min\{2, n\}, p > 1, \Omega \subset \mathbb{R}^n$ is a smooth bounded domain and λ is a positive parameter.

For $\alpha = 2$, it is well known that if $\varphi \equiv 0$, for $2 < p + 1 < 2^* := \frac{2n}{n-2}$ ($n \geq 3$), the problem (1) always admits a positive solution; for $p + 1 \geq 2^* := \frac{2n}{n-2}$, the problem (1) admits no any positive solution provided Ω is star-shaped by using the Pohozaev identity [7]. But if $\varphi \not\equiv 0$, the situation is quite different. One important result obtained by Caffarelli and Spruck [3] is that: if $\varphi \in C^{1+\beta}(\partial\Omega) \geq 0$ ($0 < \beta < 1$) is positive somewhere, then the problem (1) with $p + 1 = 2^*$ admits one positive solution for suitable chosen small positive number λ and any smooth bounded domain Ω .

For $0 < \alpha < 2$, if $\varphi \equiv 0$ and $2 < p + 1 < 2^*(\alpha) := \frac{2n}{n-\alpha}$, then the problem (1) also admits a positive solution (see for example [12, 14] and the references therein); for $p + 1 \geq 2^*(\alpha)$, the problem (1) admits no

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any bounded positive solution provided Ω is star-shaped due to a Pohozaev identity (see [10]). Thus it is again interesting to consider the case $\varphi \neq 0$, which is the concern of this paper.

We first introduce the fractional Sobolev space:

$$H^{\frac{\alpha}{2}}(\mathbb{R}^n) := \left\{ u : \mathbb{R}^n \rightarrow \mathbb{R} : \int_{\mathbb{R}^n} |u(x)|^2 dx < \infty, \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x - y|^{n+\alpha}} dy dx < \infty \right\}$$

endowed with the natural norm

$$\|u\|_{H^{\frac{\alpha}{2}}(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |u(x)|^2 dx + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x - y|^{n+\alpha}} dy dx \right)^{\frac{1}{2}}. \tag{2}$$

The fractional Laplacian operator is defined as

$$(-\Delta)^{\frac{\alpha}{2}} u(x) = c_{n,\alpha} P.V. \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+\alpha}} dy = c_{n,\alpha} \lim_{\varepsilon \rightarrow 0^+} \int_{\mathbb{R}^n \setminus B_\varepsilon(x)} \frac{u(x) - u(y)}{|x - y|^{n+\alpha}} dy,$$

where *P.V.* stands for the Cauchy principal value, $c_{n,\alpha}$ is a dimensional constant that depends on n and α .

We list some spaces introduced in [4]:

$$V(\Omega) = \{v : \mathbb{R}^n \rightarrow \mathbb{R} : v|_\Omega \in L^2(\Omega), \frac{v(x) - v(y)}{|x - y|^{\frac{n+\alpha}{2}}} \in L^2(\Omega \times \mathbb{R}^n)\},$$

$$H(\Omega) = \{u \in V(\mathbb{R}^n) : u = 0 \text{ for a.e. } x \in \mathbb{R}^n \setminus \Omega\}.$$

Define the norm in $V(\mathbb{R}^n)$ as (2). Then $V(\mathbb{R}^n) = H(\mathbb{R}^n) = H^{\frac{\alpha}{2}}(\mathbb{R}^n)$. If Ω is a Lipschitz domain, we also have $H(\Omega) = H_0^{\frac{\alpha}{2}}(\Omega)$ [4].

Define a bilinear form by

$$\varepsilon(u, v) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u(x) - u(y))v(x)}{|x - y|^{n+\alpha}} dy dx. \tag{3}$$

For convenience, we assume Ω is a smooth bounded domain here and hereafter. Assume $f \in H^*(\Omega)$, the dual space of $H(\Omega)$. The following two definitions are again from [4].

Definition 1.1. $u \in H(\Omega)$ is called a solution of

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} u = f & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{4}$$

if

$$\varepsilon(u, \varphi) = \langle f, \varphi \rangle \quad \text{for all } \varphi \in H(\Omega). \tag{5}$$

Definition 1.2. Let $g \in V(\Omega)$. A function $u \in V(\Omega)$ is called a solution of

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} u = f & \text{in } \Omega, \\ u = g & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{6}$$

if $u - g \in H(\Omega)$ and (5) holds.

The solution defined by Definition 1.1 is a weak solution [4].

We assume through this paper $0 < \alpha < \min\{2, n\}$ and that

(φ_1). $\varphi(x) \in L^\infty(\mathbb{R}^n) \cap V(\Omega)$;

(φ_2). $\varphi(x) \in C^{\alpha+\gamma}(\mathbb{R}^n)$ for some $\gamma > 0$ small, $\varphi(x) \geq 0$ and positive somewhere.

Theorem 1.1. Assume that (φ_1) and (φ_2) hold. Then there exists $\lambda^* > 0$ such that for any $0 < \lambda < \lambda^*$, the problem (1) admits a minimal positive classical solution; for $\lambda > \lambda^*$, the problem (1) admits no positive classical solution.

Here we say u is a classical solution to (1) if u is regular in the interior of Ω , continuous up to the boundary, and (1) holds pointwise.

Theorem 1.2. Assume that (φ_1) and (φ_2) hold. Then for $\lambda = \lambda^*$ the problem (1) admits at least one positive weak solution.

Theorem 1.3. Assume that (φ_1) and (φ_2) hold. If $1 < p < 2^*(\alpha) - 1$, then there exists $0 < \bar{\lambda} \leq \lambda^*$ such that for any $0 < \lambda < \bar{\lambda}$, the problem (1) admits a second positive classical solution. If $p = 2^*(\alpha) - 1$, then for any $0 < \lambda < \lambda^*$, the problem (1) admits a second positive classical solution.

Remark 1.1. Ros-Oton, Serra[9] considered the problem (1) as $\varphi = 0$ and $f : [0, \infty) \rightarrow \mathbb{R}$ satisfying

$$f \in C^1, \text{ nondecreasing}, f(0) > 0, \lim_{u \rightarrow \infty} \frac{f(u)}{u} = +\infty,$$

and obtained similar results as Theorem 1.1 and Theorem 1.2. But the problem considered in [9] is different from that considered here.

2. Proof of Theorems 1.1 and 1.2

We first give a Poincaré-Sobolev inequality (see [6]).

Lemma 2.1. Let $0 < \alpha < \min\{2, n\}$. Then there exists a positive constant $C(n, \alpha)$ such that, for any measurable and compactly supported function $u(x) : \mathbb{R}^n \rightarrow \mathbb{R}$, there holds

$$\|u\|_{L^q(\mathbb{R}^n)}^2 \leq C \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u(x) - u(y))^2}{|x - y|^{n+\alpha}} dy dx, \tag{7}$$

where $q \in [1, 2^*(\alpha)]$.

Lemma 2.2. Assume $0 < \alpha < \min\{2, n\}$, $\Omega \subset \mathbb{R}^n$ is an open smooth bounded domain, $\varphi \in V(\Omega)$ satisfies (φ_1) and (φ_2) . Then the problem

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} u = 0 & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = \varphi & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{8}$$

admits one positive classical solution $u \in C^{\alpha+\beta}(\mathbb{R}^n)$ for some $\beta > 0$.

Proof. Firstly, from [4] we know that (8) admits a weak solution $u \in V(\Omega)$. Secondly, we have $u \in L^\infty(\mathbb{R}^n)$ by $\varphi \in L^\infty(\mathbb{R}^n)$, and thus $u \in C^{\alpha+\beta}(\mathbb{R}^n)$ for some $\beta > 0$ by standard argument, see for example [8]. \square

Proof of Theorem 1.1.

(1) We first show that (1) admits no classical solution if $\lambda > 0$ large enough. In fact, we can choose the eigenfunction $\tilde{\phi}(x)$ corresponding to the first eigenvalue μ_1 to

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} \phi = \mu \phi & \text{in } \Omega, \\ \phi > 0 & \text{in } \Omega, \\ \phi = 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases} \tag{9}$$

It is easy to know that $\mu_1 > 0$ and $\tilde{\phi}(x) > 0$ in Ω from [13]. Multiplying (1) by $\tilde{\phi}$ and integrating it we have

$$\int_{\Omega} (-\Delta)^{\frac{\alpha}{2}} u(x) \tilde{\phi}(x) dx = \int_{\Omega} \lambda u^p(x) \tilde{\phi}(x) dx. \tag{10}$$

Notice that

$$\int_{\Omega} (-\Delta)^{\frac{\alpha}{2}} u(x) \tilde{\phi}(x) dx = \int_{\mathbb{R}^n} u(x) (-\Delta)^{\frac{\alpha}{2}} \tilde{\phi}(x) dx = \mu_1 \int_{\Omega} u(x) \tilde{\phi}(x) dx. \tag{11}$$

Combining (10) with (11) we know

$$\int_{\Omega} (\lambda u^p(x) - \mu_1 u(x)) \tilde{\phi}(x) dx = 0. \tag{12}$$

Choose a smooth bounded domain $\tilde{\Omega} \subset\subset \Omega$. By maximum principle there exists $c_0 > 0$ such that

$$u \geq c_0 > 0 \quad x \in \tilde{\Omega}. \tag{13}$$

Then

$$\begin{aligned} & \int_{\Omega} (\lambda u^p(x) - \mu_1 u(x)) \tilde{\phi}(x) dx \\ &= \int_{\tilde{\Omega}} (\lambda u^p(x) - \mu_1 u(x)) \tilde{\phi}(x) dx + \int_{\Omega \setminus \tilde{\Omega}} (\lambda u^p(x) - \mu_1 u(x)) \tilde{\phi}(x) dx \\ &\geq \int_{\tilde{\Omega}} (\lambda u^p(x) - \mu_1 u(x)) \tilde{\phi}(x) dx - \int_{\Omega \setminus \tilde{\Omega}} \mu_1 u(x) \tilde{\phi}(x) dx \\ &> 0 \end{aligned}$$

as $\lambda > 0$ large enough, which is a contradiction to (12).

(2) Choose a bounded smooth domain $\Omega_1 \supset\supset \Omega$. Take the eigenfunction $\bar{\phi}(x)$ corresponding to the first eigenvalue μ'_1 to

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} \phi = \mu \phi & \text{in } \Omega_1, \\ \phi > 0 & \text{in } \Omega_1, \\ \phi = 0 & \text{in } \mathbb{R}^n \setminus \Omega_1. \end{cases} \tag{14}$$

By Lemma 2.2 we know that the problem

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} h = 0 & \text{in } \Omega, \\ h > 0 & \text{in } \Omega, \\ h = \varphi & \text{in } \mathbb{R}^n \setminus \Omega \end{cases} \tag{15}$$

admits a classical solution $h(x)$.

Take k large enough such that $\bar{u}(x) := k\bar{\phi}(x) + h(x) \geq \varphi(x)$, $x \in \mathbb{R}^n \setminus \Omega$. Now take $\lambda > 0$ small enough such that

$$(-\Delta)^{\frac{\alpha}{2}} \bar{u}(x) = k(-\Delta)^{\frac{\alpha}{2}} \bar{\phi}(x) = k\mu'_1 \bar{\phi}(x) \geq \lambda(k\bar{\phi}(x) + h(x))^p, \quad x \in \Omega, \tag{16}$$

which implies that $\bar{u}(x)$ is a supersolution to (1). On the other hand, taking $u_0(x) = h(x)$ as a subsolution to (1) we consider the problem

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} u_m = \lambda u_{m-1}^p & \text{in } \Omega, \\ u_m > 0 & \text{in } \Omega, \\ u_m = \varphi & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases} \tag{17}$$

By using the maximum principle we conclude

$$h = u_0 \leq u_1 \leq \dots \leq u_m \leq \dots \leq \bar{u} \leq C.$$

To prove the existence of weak solution we first rewrite the problem (1) as

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}}(u_m - h) = \lambda u_{m-1}^p & \text{in } \Omega, \\ u_m - h > 0 & \text{in } \Omega, \\ u_m - h = 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases} \tag{18}$$

Multiplying (18) by $u_m(x) - h(x)$ and integrating it we have

$$\int_{\mathbb{R}^n} (-\Delta)^{\frac{\alpha}{2}}(u_m(x) - h(x))(u_m(x) - h(x))dx = \lambda \int_{\Omega} u_{m-1}^p (u_m(x) - h(x))dx \leq C,$$

where $C > 0$ is a constant, which then implies that

$$\begin{aligned} & \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{((u_m(x) - h(x)) - (u_m(y) - h(y)))^2}{|x - y|^{n+\alpha}} dy dx \\ &= 2 \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u_m(x) - h(x))^2 - (u_m(x) - h(x))(u_m(y) - h(y))}{|x - y|^{n+\alpha}} dy dx \\ &= C(n, \alpha) \int_{\mathbb{R}^n} (-\Delta)^{\frac{\alpha}{2}}(u_m(x) - h(x))(u_m(x) - h(x))dx \leq C(n, \alpha). \end{aligned} \tag{19}$$

Thus we have $u_m(x) - h(x)$ is uniformly bounded in $H_0^{\frac{\alpha}{2}}(\Omega)$. Assume that $u_m(x) - h(x)$ converges weakly to $u_\lambda(x) - h(x)$ in $H_0^{\frac{\alpha}{2}}(\Omega)$. Then it is easy to see that $u_\lambda(x) \in H^{\frac{\alpha}{2}}(\mathbb{R}^n)$ is a weak solution to (1). This weak solution is obviously classical by standard argument by the assumptions (φ_1) and (φ_2) .

It is also easy to see that $u_\lambda(x)$ is the minimal solution to (1). Otherwise, if $\tilde{u}(x)$ is another solution to (1), we can take $\tilde{u}(x)$ and $u_0 = h$ as supersolution and subsolution to (1), respectively. And then it is easy to prove that $u_\lambda(x) \leq \tilde{u}(x)$.

Now define

$$\Lambda^* = \{ \lambda \mid \lambda > 0, (1) \text{ has a classical solution} \}.$$

We show that Λ^* is a nontrivial bounded interval. In fact, if $\lambda_0 \in \Lambda^*$ and u_{λ_0} is the corresponding classical solution to (1) at $\lambda = \lambda_0$, then for any $0 < \lambda < \lambda_0$, u_{λ_0} is a supersolution to (1) at λ , which implies the existence of solution to (1) at λ by the same procedure as above. And then $\lambda \in \Lambda^*$. Define

$$\lambda^* = \sup\{\lambda \in \Lambda^*\} > 0.$$

Then the above arguments show that for any $0 < \lambda < \lambda^*$, the problem (1) admits at least one positive classical solution; for $\lambda > \lambda^*$, the problem (1) admits no positive classical solution. This completes the proof of Theorem 1.1.

Proof of Theorem 1.2.

For any $\lambda \in \Lambda^*$, we have

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}}(u_\lambda - h) = \lambda u_\lambda^p & \text{in } \Omega, \\ u_\lambda - h > 0 & \text{in } \Omega, \\ u_\lambda - h = 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases} \tag{20}$$

As in the proof of Theorem 1.1 we know that $\|u_\lambda - h\|_{H_0^{\frac{\alpha}{2}}(\Omega)} \leq C$ uniformly, where C is independent of λ .

Now assume that $u_\lambda - h$ converges weakly to $u_{\lambda^*}(x) - h(x)$ in $H_0^{\frac{\alpha}{2}}(\Omega)$.

Then for any $\psi(x) \in H_0^{\frac{\alpha}{2}}(\Omega)$,

$$\begin{aligned} & \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u_\lambda(x) - h(x) - (u_\lambda(y) - h(y)))\psi(x)}{|x - y|^{n+\alpha}} dy dx \\ \rightarrow & \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u_{\lambda^*}(x) - h(x) - (u_{\lambda^*}(y) - h(y)))\psi(x)}{|x - y|^{n+\alpha}} dy dx \end{aligned}$$

and

$$\int_{\Omega} \lambda u_\lambda^p \psi(x) dx \rightarrow \int_{\Omega} \lambda^* u_{\lambda^*}^p \psi(x) dx$$

as $\lambda \rightarrow \lambda^*$, which then implies that u_{λ^*} is a weak solution to (1) as $\lambda = \lambda^*$.

3. Proof of Theorems 1.3

We first prove Theorem 1.3 when $1 < p < \frac{n+\alpha}{n-\alpha} = 2^*(\alpha) - 1$.

Proof of Theorem 1.3 when $1 < p < 2^*(\alpha) - 1$. We look for a second solution to (1) of the form $u = u^* + v$, where u^* is the first weak solution found in the above section, $v > 0$, and v satisfies

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} v = \lambda(u^* + v)^p - \lambda(u^*)^p & \text{in } \Omega, \\ v = 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases} \tag{21}$$

The energy functional corresponding to (21) is

$$\begin{aligned} J(v) = & \frac{1}{4} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(v(x) - v(y))^2}{|x - y|^{n+\alpha}} dy dx - \frac{\lambda}{p + 1} \int_{\mathbb{R}^n} (u^* + v^+)^{p+1} dx \\ & + \lambda \int_{\mathbb{R}^n} (u^*)^p v^+ dx + \frac{\lambda}{p + 1} \int_{\mathbb{R}^n} (u^*)^{p+1} dx. \end{aligned} \tag{22}$$

Notice that if v is a nontrivial critical point of J , then v is a weak solution to (21). It is obvious that

$$\begin{aligned} J(0) &= 0, \\ J(tv) &= \frac{1}{4} t^2 \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(v(x) - v(y))^2}{|x - y|^{n+\alpha}} dy dx - \frac{\lambda}{p + 1} \int_{\mathbb{R}^n} (u^* + tv^+)^{p+1} dx \\ & \quad + \lambda t \int_{\mathbb{R}^n} (u^*)^p v^+ dx + \frac{\lambda}{p + 1} \int_{\mathbb{R}^n} (u^*)^{p+1} dx \\ & \leq \frac{1}{4} t^2 \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(v(x) - v(y))^2}{|x - y|^{n+\alpha}} dy dx - \frac{\lambda t^{p+1}}{p + 1} \int_{\mathbb{R}^n} (v^+)^{p+1} dx \\ & \rightarrow -\infty \quad (t \rightarrow \infty). \end{aligned} \tag{23}$$

We rewrite (22) as

$$\begin{aligned} & J(v) \tag{24} \\ = & \frac{1}{4} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(v(x) - v(y))^2}{|x - y|^{n+\alpha}} dy dx - \frac{\lambda}{p + 1} \int_{\mathbb{R}^n} (v^+)^{p+1} dx \\ & - \frac{\lambda}{p + 1} \left(\int_{\mathbb{R}^n} (u^* + v^+)^{p+1} dx - \int_{\mathbb{R}^n} (v^+)^{p+1} dx - (p + 1) \int_{\mathbb{R}^n} (u^*)^p v^+ dx - \int_{\mathbb{R}^n} (u^*)^{p+1} dx \right) \\ := & \frac{1}{4} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(v(x) - v(y))^2}{|x - y|^{n+\alpha}} dy dx - \frac{\lambda}{p + 1} \int_{\mathbb{R}^n} (v^+)^{p+1} dx - I. \end{aligned}$$

Now we divide into two cases.

(i). $p + 1 \geq 3$. We have

$$\begin{aligned} I &\leq \frac{\lambda}{p+1} \left(\int_{\Omega} (p+1)u^*(v^+)^p dx + C \int_{\Omega} (u^*)^{p-1}(v^+)^2 dx + C \int_{\Omega} (u^*)^2(v^+)^{p-1} dx \right) \\ &\leq \lambda \max_{x \in \Omega} u^* \int_{\Omega} (v^+)^p dx + \frac{\lambda C}{p+1} \left(\int_{\Omega} (u^*)^{p+1} dx \right)^{\frac{p-1}{p+1}} \left(\int_{\Omega} (v^+)^{p+1} dx \right)^{\frac{2}{p+1}} \\ &\quad + \frac{\lambda C}{p+1} \max_{x \in \Omega} (u^*)^2 \int_{\Omega} (v^+)^{p-1} dx \\ &\leq \lambda C_1 \|(v^+)\|_{H_0^{\frac{\alpha}{2}}(\Omega)}^p + \lambda C_2 \|(v^+)\|_{H_0^{\frac{\alpha}{2}}(\Omega)}^2 + \lambda C_3 \|(v^+)\|_{H_0^{\frac{\alpha}{2}}(\Omega)}^{p-1}, \end{aligned}$$

where we use the following elementary inequality for $p + 1 \geq 3$, (see for example [5])

$$(a + b)^{p+1} - a^{p+1} - b^{p+1} - (p + 1)a^p b - (p + 1)ab^p \leq C(a^2 b^{p-1} + a^{p-1} b^2), \text{ for } a, b \geq 0.$$

Thus

$$\begin{aligned} &J(v) \\ &\geq \frac{1}{4} \|v\|_{H_0^{\frac{\alpha}{2}}(\Omega)}^2 - \lambda C_4 \|(v^+)\|_{H_0^{\frac{\alpha}{2}}(\Omega)}^{p+1} - \lambda C_1 \|(v^+)\|_{H_0^{\frac{\alpha}{2}}(\Omega)}^p - \lambda C_2 \|(v^+)\|_{H_0^{\frac{\alpha}{2}}(\Omega)}^2 \\ &\quad - \lambda C_3 \|(v^+)\|_{H_0^{\frac{\alpha}{2}}(\Omega)}^{p-1}. \end{aligned}$$

(ii). $2 < p + 1 < 3$.

$$I \leq \frac{C\lambda}{p+1} \int_{\Omega} (u^*)^{p-1}(v^+)^2 dx \leq \lambda C_2 \|(v^+)\|_{H_0^{\frac{\alpha}{2}}(\Omega)}^2,$$

where another elementary inequality is used for $2 < p + 1 < 3$,

$$(a + b)^{p+1} - a^{p+1} - b^{p+1} - (p + 1)a^p b \leq Ca^{p-1} b^2, \text{ for } a, b > 0,$$

where $C > 0$ some large positive constant. Hence

$$J(v) \geq \frac{1}{4} \|v\|_{H_0^{\frac{\alpha}{2}}(\Omega)}^2 - \lambda C_4 \|(v^+)\|_{H_0^{\frac{\alpha}{2}}(\Omega)}^{p+1} - \lambda C_2 \|(v^+)\|_{H_0^{\frac{\alpha}{2}}(\Omega)}^2.$$

So for both of the above two cases, if $\lambda > 0$ small enough, then there exists $\rho > 0$ small such that

$$J(v) \geq \frac{1}{4} \|v\|_{H_0^{\frac{\alpha}{2}}(\Omega)}^2 - \frac{1}{8} \|(v^+)\|_{H_0^{\frac{\alpha}{2}}(\Omega)}^2 \geq \frac{1}{8} \rho^2, \text{ for } \|v\|_{H_0^{\frac{\alpha}{2}}(\Omega)} = \rho.$$

Thus by using the Mountain Pass Lemma, $J(v)$ admits a nontrivial critical point, that is, the problem (21) admits a nontrivial weak solution. Standard arguments give that this weak solution is also classical. The maximum principle implies that $v > 0$ in Ω . So we know that the problem (1) admits the second positive classical solution. \square

We now consider the case $p = \frac{n+\alpha}{n-\alpha} = 2^*(\alpha) - 1$. We first list some lemmas.

Lemma 3.1. *If $\{v_n\}$ is a Palais-Smale sequence of $J(v)$ in $H_0^{\frac{\alpha}{2}}(\Omega)$, that is, $J(v_n) \rightarrow c$, $DJ(v_n) \rightarrow 0$, then $\{v_n\}$ are bounded in $H_0^{\frac{\alpha}{2}}(\Omega)$.*

Proof. Since $J(v_n) \rightarrow c, DJ(v_n) \rightarrow 0$, we have

$$\begin{aligned}
 J(v_n) &= \frac{1}{4} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(v_n(x) - v_n(y))^2}{|x - y|^{n+\alpha}} dy dx - \frac{\lambda}{2^*(\alpha)} \int_{\mathbb{R}^n} (u^* + (v_n^+))^{2^*(\alpha)} dx \\
 &\quad + \lambda \int_{\mathbb{R}^n} (u^*)^{2^*(\alpha)-1} (v_n^+) dx + \frac{\lambda}{2^*(\alpha)} \int_{\mathbb{R}^n} (u^*)^{2^*(\alpha)} dx \\
 &\rightarrow c,
 \end{aligned} \tag{25}$$

$$\begin{aligned}
 DJ(v_n)v_n^+ &= \frac{1}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(v_n(x) - v_n(y))^2}{|x - y|^{n+\alpha}} dy dx - \lambda \int_{\mathbb{R}^n} (u^* + v_n^+)^{2^*(\alpha)-1} v_n^+ dx \\
 &\quad + \lambda \int_{\mathbb{R}^n} (u^*)^{2^*(\alpha)-1} v_n^+ dx \\
 &= o(\|v_n^+\|_{H_0^{\frac{\alpha}{2}}(\Omega)}).
 \end{aligned} \tag{26}$$

By (25) and (26) we have

$$\begin{aligned}
 &c + o(\|v_n^+\|_{H_0^{\frac{\alpha}{2}}(\Omega)}) \\
 &= \frac{\lambda}{2} \int_{\mathbb{R}^n} (u^* + v_n^+)^{2^*(\alpha)-1} v_n^+ dx + \frac{\lambda}{2} \int_{\mathbb{R}^n} (u^*)^{2^*(\alpha)-1} v_n^+ dx \\
 &\quad - \frac{\lambda}{2^*(\alpha)} \int_{\mathbb{R}^n} (u^* + v_n^+)^{2^*(\alpha)} dx + \frac{\lambda}{2^*(\alpha)} \int_{\mathbb{R}^n} (u^*)^{2^*(\alpha)} dx \\
 &= \left(\frac{1}{2} - \frac{1}{2^*(\alpha)}\right) \lambda \int_{\mathbb{R}^n} (u^* + v_n^+)^{2^*(\alpha)-1} v_n^+ dx - \frac{\lambda}{2^*(\alpha)} \int_{\mathbb{R}^n} (u^* + v_n^+)^{2^*(\alpha)-1} u^* dx \\
 &\quad + \frac{\lambda}{2} \int_{\mathbb{R}^n} (u^*)^{2^*(\alpha)-1} v_n^+ dx + \frac{\lambda}{2^*(\alpha)} \int_{\mathbb{R}^n} (u^*)^{2^*(\alpha)} dx \\
 &\geq \frac{\alpha\lambda}{2n} \int_{\mathbb{R}^n} (v_n^+)^{2^*(\alpha)} dx - \frac{\lambda}{2^*(\alpha)} \int_{\mathbb{R}^n} (u^* + v_n^+)^{2^*(\alpha)-1} u^* dx.
 \end{aligned} \tag{27}$$

Notice that

$$\begin{aligned}
 &\int_{\mathbb{R}^n} (u^* + v_n^+)^{2^*(\alpha)-1} u^* dx \\
 &\leq \int_{\mathbb{R}^n} (2^{2^*(\alpha)-1} (u^*)^{2^*(\alpha)-1} + 2^{2^*(\alpha)-1} (v_n^+)^{2^*(\alpha)-1}) u^* dx \\
 &= 2^{2^*(\alpha)-1} \int_{\mathbb{R}^n} (u^*)^{2^*(\alpha)} dx + 2^{2^*(\alpha)-1} \int_{\mathbb{R}^n} (v_n^+)^{2^*(\alpha)-1} u^* dx \\
 &\leq 2^{2^*(\alpha)-1} \int_{\mathbb{R}^n} (u^*)^{2^*(\alpha)} dx + 2^{2^*(\alpha)-1} \left(\int_{\mathbb{R}^n} (u^*)^{2^*(\alpha)} dx \right)^{\frac{1}{2^*(\alpha)}} \left(\int_{\mathbb{R}^n} (v_n^+)^{2^*(\alpha)} dx \right)^{\frac{2^*(\alpha)-1}{2^*(\alpha)}},
 \end{aligned}$$

which combining with (27) gives

$$\int_{\mathbb{R}^n} (v_n^+)^{2^*(\alpha)} dx \leq C.$$

Since

$$\begin{aligned} & \left| \lambda \int_{\mathbb{R}^n} (u^* + v_n^+)^{2^*(\alpha)-1} v_n^+ dx - \lambda \int_{\mathbb{R}^n} (u^*)^{2^*(\alpha)-1} v_n^+ dx \right| \\ & \leq \lambda \int_{\mathbb{R}^n} (u^* + v_n^+)^{2^*(\alpha)-1} v_n^+ dx + \lambda \int_{\mathbb{R}^n} (u^*)^{2^*(\alpha)-1} v_n^+ dx \\ & \leq \lambda 2^{2^*(\alpha)-1} \int_{\mathbb{R}^n} ((u^*)^{2^*(\alpha)-1} v_n^+ + (v_n^+)^{2^*(\alpha)}) dx + \lambda \int_{\Omega} (u^*)^{2^*(\alpha)-1} v_n^+ dx \\ & \leq \lambda (2^{2^*(\alpha)-1} + 1) \left(\int_{\mathbb{R}^n} (u^*)^{2^*(\alpha)} dx \right)^{\frac{1}{2^*(\alpha)}} \left(\int_{\mathbb{R}^n} (v_n^+)^{2^*(\alpha)} dx \right)^{\frac{2^*(\alpha)-1}{2^*(\alpha)}} + \lambda 2^{2^*(\alpha)-1} \int_{\Omega} (v_n^+)^{2^*(\alpha)} dx \\ & \leq C, \end{aligned}$$

by (26) we then know

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(v_n(x) - v_n(y))^2}{|x - y|^{n+\alpha}} dy dx \leq C,$$

that is, $\{v_n\}$ are bounded in $H_0^{\frac{\alpha}{2}}(\Omega)$. \square

Lemma 3.2. *If $\{v_n\}$ is a Palais-Smale sequence of $J(v)$ in $H_0^{\frac{\alpha}{2}}(\Omega)$, then there exists $v \in H_0^{\frac{\alpha}{2}}(\Omega)$ such that $v_n \rightharpoonup v$ in $H_0^{\frac{\alpha}{2}}(\Omega)$ as $n \rightarrow \infty$, $DJ(v) = 0$; and if*

$$J(v_n) \rightarrow c \in \left(0, \frac{\alpha}{4n} \left(\frac{1}{2\lambda}\right)^{\frac{n-\alpha}{\alpha}} S_{\alpha}^n\right)$$

as $n \rightarrow \infty$, where $S = \inf_{v \in H^{\frac{\alpha}{2}}(\mathbb{R}^n), \|v\|_{L^{2^*(\alpha)}(\mathbb{R}^n)}=1} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|v(x)-v(y)|^2}{|x-y|^{n+\alpha}} dy dx$ is the best constant of the fractional Sobolev inequality, then $v \not\equiv 0$ and v is a nontrivial weak solution to (21).

Proof. By Lemma 3.1 we know

$$v_n \rightharpoonup v \text{ in } H_0^{\frac{\alpha}{2}}(\Omega), \text{ as } n \rightarrow \infty. \tag{28}$$

So for any $\varphi \in C_0^\infty(\Omega)$,

$$\begin{aligned} & DJ(v_n)\varphi \\ & = \frac{1}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(v_n(x) - v_n(y))(\varphi(x) - \varphi(y))}{|x - y|^{n+\alpha}} dy dx \\ & \quad - \lambda \int_{\mathbb{R}^n} (u^* + v_n^+)^{2^*(\alpha)-1} \varphi dx + \lambda \int_{\mathbb{R}^n} (u^*)^{2^*(\alpha)-1} \varphi dx \\ & \rightarrow \frac{1}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(v(x) - v(y))(\varphi(x) - \varphi(y))}{|x - y|^{n+\alpha}} dy dx \\ & \quad - \lambda \int_{\mathbb{R}^n} (u^* + v^+)^{2^*(\alpha)-1} \varphi dx + \lambda \int_{\mathbb{R}^n} (u^*)^{2^*(\alpha)-1} \varphi dx \\ & = DJ(v)\varphi = 0, \end{aligned}$$

which then gives that $DJ(v) = 0$.

Now we show $v \not\equiv 0$ if $c \in \left(0, \frac{\alpha}{4n} \left(\frac{1}{2\lambda}\right)^{\frac{n-\alpha}{\alpha}} S_{\alpha}^n\right)$. Assume by contradiction that $v \equiv 0$. Since $DJ(v_n)(v_n^+ + u^*) = o(1)$, we have

$$\begin{aligned} & \frac{1}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(v_n(x) - v_n(y))^2}{|x - y|^{n+\alpha}} dy dx + \frac{1}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(v_n(x) - v_n(y))(u^*(x) - u^*(y))}{|x - y|^{n+\alpha}} dy dx \\ & - \lambda \int_{\mathbb{R}^n} (u^* + v_n^+)^{2^*(\alpha)} dx + \lambda \int_{\mathbb{R}^n} (u^*)^{2^*(\alpha)-1} (u^* + v_n^+) dx \\ & = o(1). \end{aligned}$$

By (28), as $n \rightarrow \infty$,

$$\begin{aligned} & \frac{1}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(v_n(x) - v_n(y))(u^*(x) - u^*(y))}{|x - y|^{n+\alpha}} dy dx \\ \rightarrow & \frac{1}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(v(x) - v(y))(u^*(x) - u^*(y))}{|x - y|^{n+\alpha}} dy dx = 0. \end{aligned} \tag{29}$$

Notice that for $1 \leq q < \frac{2n}{n-\alpha}$,

$$v_n^+ \rightarrow 0, \text{ in } L^q(\Omega), \text{ as } n \rightarrow \infty.$$

Hence

$$\lambda \int_{\mathbb{R}^n} (u^*)^{2^*(\alpha)-1} (u^* + v_n^+) dx \rightarrow \lambda \int_{\mathbb{R}^n} (u^*)^{2^*(\alpha)} dx, \text{ as } n \rightarrow \infty. \tag{30}$$

Since $v_n(x)$ is bounded, by Brézis-Lieb Lemma [1] we have

$$\int_{\mathbb{R}^n} (u^* + v_n^+)^{2^*(\alpha)} dx - \int_{\mathbb{R}^n} (u^*)^{2^*(\alpha)} dx - \int_{\mathbb{R}^n} (v_n^+)^{2^*(\alpha)} dx = o(1). \tag{31}$$

Thus by (29), (30), (31), we have

$$\frac{1}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(v_n(x) - v_n(y))^2}{|x - y|^{n+\alpha}} dy dx - \lambda \int_{\mathbb{R}^n} (v_n^+)^{2^*(\alpha)} dx = o(1). \tag{32}$$

Therefore

$$o(1) \geq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(v_n(x) - v_n(y))^2}{|x - y|^{n+\alpha}} dy dx \left(\frac{1}{2} - \lambda S^{-\frac{2^*(\alpha)}{2}} \left(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(v_n(x) - v_n(y))^2}{|x - y|^{n+\alpha}} dy dx \right)^{\frac{n-1}{2}} \right),$$

where S is the best constant of the fractional Sobolev inequality, that is,

$$S = \inf_{v \in H^{\frac{\alpha}{2}}(\mathbb{R}^n), \|v\|_{L^{2^*(\alpha)}(\mathbb{R}^n)} = 1} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|v(x) - v(y)|^2}{|x - y|^{n+\alpha}} dy dx.$$

Hence as $n \rightarrow \infty$,

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(v_n(x) - v_n(y))^2}{|x - y|^{n+\alpha}} dy dx \rightarrow 0,$$

or

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(v_n(x) - v_n(y))^2}{|x - y|^{n+\alpha}} \geq \left(\frac{1}{2\lambda} \right)^{\frac{n-\alpha}{\alpha}} S^{\frac{n}{\alpha}} + o(1).$$

Now to continue we consider two cases.

i). $\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(v_n(x) - v_n(y))^2}{|x - y|^{n+\alpha}} dy dx \rightarrow 0.$

$$J(v_n) \rightarrow 0, \text{ as } n \rightarrow \infty,$$

which contradicts to $J(v_n) \rightarrow c > 0.$

ii). $\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(v_n(x) - v_n(y))^2}{|x - y|^{n+\alpha}} \geq \left(\frac{1}{2\lambda} \right)^{\frac{n-\alpha}{\alpha}} S^{\frac{n}{\alpha}} + o(1).$

By (32), as $n \rightarrow \infty$ we have

$$\begin{aligned} J(v_n) &\geq \frac{1}{4} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(v_n(x) - v_n(y))^2}{|x - y|^{n+\alpha}} dy dx - \frac{\lambda}{2^*(\alpha)} \int_{\mathbb{R}^n} (v_n)^{2^*(\alpha)} dx + o(1) \\ &= \frac{\alpha}{4n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(v_n(x) - v_n(y))^2}{|x - y|^{n+\alpha}} dy dx + o(1) \\ &\geq \frac{\alpha}{4n} \left(\frac{1}{2\lambda}\right)^{\frac{n-\alpha}{\alpha}} S_\alpha^\frac{n}{\alpha} + o(1), \end{aligned}$$

which contradicts to

$$c < \frac{\alpha}{4n} \left(\frac{1}{2\lambda}\right)^{\frac{n-\alpha}{\alpha}} S_\alpha^\frac{n}{\alpha}.$$

Hence we conclude that $v \not\equiv 0$, that is, v is a nontrivial solution to the problem (21). \square

Lemma 3.3. *The functional $J(v)$ admits a Palais-Smale sequence in $H_0^{\frac{\alpha}{2}}(\Omega)$ at level $0 < c < \frac{\alpha}{4n} \left(\frac{1}{2\lambda}\right)^{\frac{n-\alpha}{\alpha}} S_\alpha^\frac{n}{\alpha}$.*

Proof. By the fractional Sobolev inequality,

$$\begin{aligned} J(v) &\geq \frac{1}{4} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(v(x) - v(y))^2}{|x - y|^{n+\alpha}} dy dx - \frac{\lambda}{2^*(\alpha)} \int_{\mathbb{R}^n} (u^* + v^+)^{2^*(\alpha)} dx \\ &\geq \frac{1}{4} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(v(x) - v(y))^2}{|x - y|^{n+\alpha}} dy dx - \frac{C\lambda}{2^*(\alpha)} \left(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(v(x) - v(y))^2}{|x - y|^{n+\alpha}} dy dx \right)^{\frac{2^*(\alpha)}{2}} \\ &= \frac{1}{4} \rho^2 - \frac{C\lambda}{2^*(\alpha)} \rho^{2^*(\alpha)} \quad (\forall v \in \partial B_\rho(0)) \\ &> 0, \end{aligned}$$

for ρ small enough. Thus by using Theorem 2.2 in [2], we know that the functional $J(v)$ admits a Palais-Smale sequence at level $c > 0$. Again by using Theorem 2.2 in [2], it is left to prove for some $v \neq 0, v \in H_0^{\frac{\alpha}{2}}(\Omega)$ that

$$\max_{t \geq 0} J(tv) < \frac{\alpha}{4n} \left(\frac{1}{2\lambda}\right)^{\frac{n-\alpha}{\alpha}} S_\alpha^\frac{n}{\alpha}.$$

Assume $B(x_0, 4\delta) \subset \Omega$. As in [11], we take $V_\epsilon(x) = \eta(x)U_\epsilon(x)$, where $\eta(x) \in C^\infty(\mathbb{R}^n), 0 \leq \eta(x) \leq 1, x \in \mathbb{R}^n; \eta(x) = 1$ when $x \in B(x_0, \delta), \eta(x) = 0$ when $x \in \mathbb{R}^n \setminus B(x_0, 2\delta)$, and

$$U_\epsilon(x) = \kappa \left(\frac{\epsilon}{\epsilon^2 + |x - x_0|^2}\right)^{\frac{n-\alpha}{2}}, \kappa > 0,$$

is a positive solution to the problem

$$(-\Delta)^{\frac{\alpha}{2}} u = |u|^{2^*(\alpha)-2} u, \text{ in } \mathbb{R}^n$$

and satisfying

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|U_\epsilon(x) - U_\epsilon(y)|^2}{|x - y|^{n+\alpha}} dy dx = \int_{\mathbb{R}^n} |U_\epsilon(x)|^{2^*(\alpha)} dx = S_\alpha^\frac{n}{\alpha}.$$

By [11], we know as $\epsilon \rightarrow 0$,

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|V_\epsilon(x) - V_\epsilon(y)|^2}{|x - y|^{n+\alpha}} dy dx \leq S_\alpha^\frac{n}{\alpha} - O(\epsilon^{n-\alpha}), \tag{33}$$

$$\int_{\mathbb{R}^n} |V_\epsilon(x)|^{2^*(\alpha)} dx = S_\alpha^\frac{n}{\alpha} - O(\epsilon^n). \tag{34}$$

Now we only need to prove for some $\epsilon > 0$ that

$$\max_{t \geq 0} J(tV_\epsilon) < \frac{\alpha}{4n} \left(\frac{1}{2\lambda}\right)^{\frac{n-\alpha}{\alpha}} S_\alpha^n.$$

Suppose for any $\epsilon > 0$, there exists $t_\epsilon > 0$ such that

$$J(t_\epsilon V_\epsilon) \geq \frac{\alpha}{4n} \left(\frac{1}{2\lambda}\right)^{\frac{n-\alpha}{\alpha}} S_\alpha^n.$$

Claim: $t_\epsilon \rightarrow t_0 > 0$.

In fact, if up to a subsequence, $t_\epsilon \rightarrow +\infty$ or $t_\epsilon \rightarrow 0$, we know by using (23), (33) and (34) that

$$J(t_\epsilon V_\epsilon) \rightarrow -\infty, \text{ or } J(t_\epsilon V_\epsilon) \rightarrow 0,$$

respectively, which contradicts to $J(t_\epsilon V_\epsilon) \geq \frac{\alpha}{4n} \left(\frac{1}{2\lambda}\right)^{\frac{n-\alpha}{\alpha}} S_\alpha^n$.

Notice that

$$\begin{aligned} J(t_\epsilon V_\epsilon) &= \frac{1}{4} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(t_\epsilon V_\epsilon(x) - t_\epsilon V_\epsilon(y))^2}{|x - y|^{n+\alpha}} dy dx - \frac{\lambda}{2^*(\alpha)} \int_{\mathbb{R}^n} (u^* + t_\epsilon V_\epsilon)^{2^*(\alpha)} dx \\ &\quad + \lambda \int_{\mathbb{R}^n} (u^*)^{2^*(\alpha)-1} t_\epsilon V_\epsilon + \frac{\lambda}{2^*(\alpha)} \int_{\mathbb{R}^n} (u^*)^{2^*(\alpha)} dx. \end{aligned}$$

Then by using an elementary inequality

$$(a + b)^{2^*(\alpha)} - a^{2^*(\alpha)} - b^{2^*(\alpha)} - 2^*(\alpha)a^{2^*(\alpha)-1}b \geq 0, \quad a, b \geq 0,$$

we have

$$J(t_\epsilon V_\epsilon) \leq \frac{1}{4} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{t_\epsilon^2 (V_\epsilon(x) - V_\epsilon(y))^2}{|x - y|^{n+\alpha}} dy dx - \frac{\lambda}{2^*(\alpha)} \int_{\mathbb{R}^n} (t_\epsilon V_\epsilon)^{2^*(\alpha)} dx.$$

Then by using (33) and (34),

$$\begin{aligned} J(t_\epsilon V_\epsilon) &\leq \left(\frac{1}{4}t_\epsilon^2 - \frac{\lambda}{2^*(\alpha)}t_\epsilon^{2^*(\alpha)}\right)S_\alpha^n + O(\epsilon^n) - O(\epsilon^{n-\alpha}) \\ &< \frac{\alpha}{4n} \left(\frac{1}{2\lambda}\right)^{\frac{n-\alpha}{\alpha}} S_\alpha^n, \end{aligned}$$

as $\epsilon > 0$ small enough, where the last inequality holds since the function

$$f(t_\epsilon) = \frac{1}{4}t_\epsilon^2 - \frac{\lambda}{2^*(\alpha)}t_\epsilon^{2^*(\alpha)}$$

attains its maximum at $t_\epsilon = \left(\frac{1}{2\lambda}\right)^{\frac{1}{2^*(\alpha)-2}}$. Thus we obtain a contradiction. \square

Proof of Theorem 1.3 when $p = 2^*(\alpha) - 1$. By Lemma 3.1, Lemma 3.2 and Lemma 3.3 we conclude that the problem (21) has a nontrivial weak solution. Thus as the case $p = 2^*(\alpha) - 1$, we conclude that the problem (1) admits the second positive classical solution. \square

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