



Lightlike Submanifolds of Metallic Semi-Riemannian Manifolds

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Abstract. Our aim in this paper is to investigate some special types of lightlike submanifolds in metallic semi-Riemannian manifolds. We study invariant lightlike submanifolds and screen semi-invariant lightlike hypersurfaces of metallic semi-Riemannian manifolds and give examples. We obtain some conditions for the induced connection to be a metric connection and present integrability conditions for the distributions involved in the definitions of such types.

1. Introduction

The existence of a wide range of applications in mathematics and physics has made the Riemannian and semi-Riemannian geometries an important research area for differential geometry. But by the 1970s, Einstein's general relativity theory shifted the interest on Riemannian and semi-Riemannian geometries to the Lorentzian geometry. Later, Duggal and Bejancu published a book on lightlike geometry in 1996 [10] and filled an important gap in the theory of submanifolds. In this book, the geometric objects for obtaining the Gauss-Codazzi equations of a lightlike submanifold are defined by means of a non-degenerate screen distribution.

The main difference between the theory of lightlike submanifolds and semi-Riemannian submanifolds arises due to the fact that in the first case, a part of the normal vector bundle TN^\perp lies in the tangent bundle TN of the submanifold N of a semi-Riemannian manifold \tilde{N} , whereas in the second case $TN \cap TN^\perp = \{0\}$. Thus, the basic problem of lightlike submanifolds is to replace the intersecting part by a vector subbundle whose sections are nowhere tangent to N . To construct a nonintersecting lightlike transversal vector bundle of the tangent bundle, Duggal and Bejancu used an extrinsic approach while Küpeli used an intrinsic approach [21]. Since then, many authors have studied the geometry of lightlike hypersurfaces and lightlike submanifolds. Recent studies have been updated in [11]. Many studies on lightlike submanifolds have been reported by many geometers (see [1, 3, 13, 14, 23] and the references therein). In this paper, we follow the approach given by Duggal and Bejancu in [10]. We note that lightlike hypersurfaces are examples of physical models of Killing horizons in general relativity [16] and the relationship between Killing horizons and black holes is based on Hawking's area theorem [18].

It is known that the number $\phi = (1 + \sqrt{5})/2 = 1,618033\dots$ is a solution of the equation $x^2 - x - 1 = 0$ and it is called as golden ratio. The golden ratio is very interesting because of its use in art works and frequent

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occurrence in the nature. Thus, Crasmareanu and Hretcanu defined a golden manifold N by a tensor field Φ on N satisfies $\Phi^2 = \Phi + I$ in [7]. In the same paper, the authors showed that ϕ and $1 - \phi$ are eigenvalues of Φ . Then, in [29] Şahin and Akyol introduced golden maps between golden Riemannian manifolds and showed that such maps are harmonic maps. Finally, lightlike hypersurfaces of a golden semi-Riemannian manifold were studied by Poyraz and Yaşar in [22].

In 1997 Spinadel introduced metallic means family or metallic proportions as a generalization of the golden mean in [24]-[28]. Let p and q be positive integers. Then, members of the metallic means family are the positive solutions of the equation $x^2 - px - q = 0$ and these numbers, which are called (p, q) metallic numbers, are denoted by

$$\sigma_{p,q} = \frac{p + \sqrt{p^2 + 4q}}{2}. \quad (1.1)$$

The members of the metallic means family take the name of a metal, like the golden mean, the silver mean, the bronze mean, the copper mean and many others. A metallic manifold \check{N} has a tensor field \check{J} such that the equality $\check{J}^2 = p\check{J} + qI$ is satisfied, where p and q positive integers and the eigenvalues of automorphism \check{J} of the tangent bundle $T\check{N}$ are $\sigma_{p,q}$ and $p - \sigma_{p,q}$ [9]. Moreover, if (\check{N}, \check{g}) is a Riemannian manifold endowed with a metallic structure \check{J} such that the Riemannian metric \check{g} is \check{J} -compatible, i.e., $\check{g}(\check{J}X, Y) = \check{g}(X, \check{J}Y)$, for any $X, Y \in \chi(\check{N})$, then (\check{g}, \check{J}) is called metallic Riemannian structure and $(\check{N}, \check{g}, \check{J})$ is a metallic Riemannian manifold. Metallic structure on the ambient Riemannian manifold provides important geometrical results on the submanifolds, since it is an important tool while investigating the geometry of submanifolds. Invariant, anti-invariant, semi-invariant, slant and semi-slant submanifolds of a metallic Riemannian manifold are studied in [6, 19, 20]. Some types of lightlike submanifolds of a metallic semi-Riemannian manifold are introduced in [2, 12] and the authors obtained important characterizations on such submanifolds of metallic semi-Riemannian manifolds with examples.

In this paper, we study some special types of lightlike submanifolds in metallic semi-Riemannian manifolds. In section 2, we give basic information needed for the rest of the paper. In section 3 and section 4, we research invariant lightlike submanifolds and screen semi-invariant lightlike hypersurfaces of metallic semi-Riemannian manifolds, respectively. In these sections, we give some characterizations and investigate the geometry of leaves of distributions which arise from definitions. In general, the induced connection of a lightlike submanifold is not a metric connection. Therefore it is an important problem to find conditions for the induced connection to be a metric connection. So, we find necessary and sufficient conditions for the induced connection to be a metric connection. Finally, we note that the paper contains examples.

2. Preliminaries

Let \check{N} be a differentiable manifold and \check{J} be a $(1, 1)$ type tensor field on \check{N} . If the following equation is satisfied, then \check{J} is called a metallic structure on \check{N} :

$$\check{J}^2 = p\check{J} + qI, \quad (2.1)$$

where p, q are positive integers and I is the identity operator on the Lie algebra $\chi(\check{N})$ of the vector fields on \check{N} . If \check{J} is a self-adjoint operator with respect to semi-Riemann metric \check{g} of a semi-Riemann manifold \check{N} , that is,

$$\check{g}(\check{J}U, V) = \check{g}(U, \check{J}V), \quad (2.2)$$

is satisfied, then \check{g} is said to be \check{J} -compatible and $(\check{N}, \check{J}, \check{g})$ is called a metallic semi-Riemannian manifold. Using (2.2), we can write

$$\check{g}(\check{J}U, \check{J}V) = p\check{g}(U, \check{J}V) + q\check{g}(U, V), \quad (2.3)$$

for any $U, V \in \Gamma(T\check{N})$.

It is well known that the non-degenerate metric \check{g} of a $(m + n)$ -dimensional semi-Riemann manifold \check{N} is not always induced as a non-degenerate metric on an m -dimensional submanifold N of \check{N} . If the induced metric g is degenerate on N and $\text{rank}(\text{Rad}(TN)) = r, 1 \leq r \leq m$, then (N, g) is called a lightlike submanifold of (\check{N}, \check{g}) , where the radical distribution $\text{Rad}(TN)$ and the normal bundle TN^\perp of the tangent bundle TN are defined by

$$\text{Rad}(TN) = TN \cap TN^\perp$$

and

$$TN^\perp = \cup_{x \in N} \{u \in T_x \check{N} \mid \check{g}(u, v) = 0, \forall v \in T_x N\}.$$

Since TN and TN^\perp are degenerate vector subbundles, there exist complementary non-degenerate distributions $S(TN)$ and $S(TN^\perp)$ of $\text{Rad}(TN)$ in TN and TN^\perp , respectively, which are called the screen distribution and screen transversal bundle (or co-screen distribution) of N such that

$$TN = S(TN) \perp \text{Rad}(TN), \quad TN^\perp = S(TN^\perp) \perp \text{Rad}(TN).$$

On the other hand, consider an orthogonal complementary bundle $S(TN)^\perp$ to $S(TN)$ in $T\check{N}$ such that

$$S(TN)^\perp = S(TN^\perp) \perp S(TN^\perp)^\perp, \tag{2.4}$$

where $S(TN^\perp)^\perp$ is the orthogonal complementary to $S(TN^\perp)$ in $S(TN^\perp)^\perp$.

We now recall the following important result.

Theorem 2.1. *Let $(N, g, S(TN), S(TN^\perp))$ be a r -lightlike submanifold of a semi-Riemannian manifold (\check{N}, \check{g}) . Then, there exists a complementary vector bundle $\text{ltr}(TN)$ called a lightlike transversal bundle of $\text{Rad}(TN)$ in $S(TN^\perp)^\perp$ and a basis of $\Gamma(\text{ltr}(TN)|_U)$ consists of smooth sections $\{N_1, \dots, N_r\}$ of $S(TN^\perp)^\perp|_U$ such that*

$$\check{g}(\xi_i, N_j) = \delta_{ij}, \quad \check{g}(N_i, N_j) = 0, \quad i, j = 1, \dots, r,$$

where $\{\xi_1, \dots, \xi_r\}$ is a basis of $\Gamma(\text{Rad}(TN))$ [10, page 144].

This result implies that there exists a complementary (but not orthogonal) vector bundle $\text{tr}(TN)$ to TN in $T\check{N}|_N$, which is called transversal vector bundle, such that the following decompositions are held:

$$\text{tr}(TN) = \text{ltr}(TN) \perp S(TN^\perp) \tag{2.5}$$

and

$$S(TN^\perp)^\perp = \text{Rad}(TN) \oplus \text{ltr}(TN). \tag{2.6}$$

Thus, using (2.4), (2.5) and (2.6), we get

$$\begin{aligned} T\check{N}|_N &= S(TN) \perp S(TN^\perp)^\perp \\ &= S(TN) \perp \{\text{Rad}(TN) \oplus \text{ltr}(TN)\} \perp S(TN^\perp) \\ &= TN \oplus \text{tr}(TN). \end{aligned} \tag{2.7}$$

A submanifold $(N, g, S(TN), S(TN^\perp))$ is called

- (1): r -lightlike if $r < \min\{m, n\}$,
- (2): Co-isotropic if $r = n < m$, i.e., $S(TN^\perp) = \{0\}$,
- (3): Isotropic if $r = m < n$, i.e., $S(TN) = \{0\}$ and
- (4): Totally lightlike if $r = m = n$, i.e., $S(TN) = \{0\} = S(TN^\perp)$.

The Gauss and Weingarten equations of N are given by

$$\bar{\nabla}_U V = \nabla_U V + h(U, V), \quad \forall U, V \in \Gamma(TN), \tag{2.8}$$

and

$$\bar{\nabla}_U N = -A_N U + \nabla_U^t N, \quad \forall U \in \Gamma(TN), \quad N \in \Gamma(tr(TN)), \tag{2.9}$$

where $\{\nabla_U V, A_N U\}$ and $\{h(U, V), \nabla_U^t N\}$ are belong to $\Gamma(TN)$ and $\Gamma(tr(TN))$, respectively. ∇ and ∇^t are linear connections on N and on the vector bundle $tr(TN)$, respectively. The second fundamental form h is a symmetric $\mathcal{F}(N)$ -bilinear form on $\Gamma(TN)$ with values in $\Gamma(tr(TN))$ and the shape operator A_V is a linear endomorphism of $\Gamma(TN)$.

If we consider (2.7) and using the projection morphisms denoted by

$$L : tr(TN) \rightarrow ltr(TN), \quad S : tr(TN) \rightarrow S(TN^\perp),$$

then, for any $U, V \in \Gamma(TN), N \in \Gamma(ltr(TN))$ and $W \in \Gamma(S(TN^\perp))$, we can write

$$\bar{\nabla}_U V = \nabla_U V + h^l(U, V) + h^s(U, V), \tag{2.10}$$

$$\bar{\nabla}_U N = -A_N U + \nabla_U^l(N) + D^s(U, N), \tag{2.11}$$

$$\bar{\nabla}_U W = -A_W U + D^l(U, W) + \nabla_U^s(W), \tag{2.12}$$

where $\{\nabla_U^l N, D^l(U, W)\}$ and $\{D^s(U, N), \nabla_U^s W\}$ are parts of $ltr(TN)$ and $S(TN^\perp)$, respectively and $h^l(U, V) = Lh(U, V) \in \Gamma(ltr(TN)), h^s(U, V) = Sh(U, V) \in \Gamma(S(TN^\perp))$. Denote the projection of TN on $S(TN)$ by \bar{P} . Then, by using (2.8), (2.10)-(2.12) and taking account that $\bar{\nabla}$ is a metric connection we obtain

$$\check{g}(h^s(U, V), W) + \check{g}(Y, D^l(U, W)) = g(A_W U, Y), \tag{2.13}$$

$$\check{g}(D^s(U, N), W) = \check{g}(N, A_W U), \tag{2.14}$$

$$\nabla_U \bar{P}V = \nabla_U^* \bar{P}V + h^*(U, \bar{P}V), \tag{2.15}$$

and

$$\nabla_U \xi = -A_\xi^* U + \nabla_U^t \xi, \tag{2.16}$$

for $U, V \in \Gamma(TN)$ and $\xi \in \Gamma(Rad(TN))$, where ∇^* and ∇^{*t} are induced connections on $S(TN)$ and $Rad(TN)$, respectively. On the other hand, h^* and A^* are $\Gamma(Rad(TN))$ -valued and $\Gamma(S(TN))$ -valued $\mathcal{F}(N)$ -bilinear forms on $\Gamma(TN) \times \Gamma(S(TN))$ and $\Gamma(Rad(TN)) \times \Gamma(TN)$, respectively. h^* is called local second fundamental form on $S(TN)$ and A^* is second fundamental form of $Rad(TN)$.

By using above equations we obtain

$$\check{g}(h^l(U, \bar{P}V), \xi) = g(A_\xi^* U, \bar{P}V), \tag{2.17}$$

$$\check{g}(h^*(U, \bar{P}V), N) = g(A_N U, \bar{P}V), \tag{2.18}$$

$$\check{g}(h^l(U, \xi), \xi) = 0, \quad A_\xi^* \xi = 0. \tag{2.19}$$

In general, the induced connection ∇ on N is not a metric connection. Since $\bar{\nabla}$ is a metric connection, by using (2.10), we get

$$(\nabla_U g)(V, Z) = \check{g}(h^l(U, V), Z) + \check{g}(h^l(U, Z), V). \tag{2.20}$$

However, it is important to note that ∇^* is a metric connection on $S(TN)$.

In case of N is being a lightlike hypersurface, we recall that $h^l(U, V) = B(U, V)N$, for all $U, V \in \Gamma(TN)$, and $h^s = 0$, where B is called the second fundamental form of N .

3. Invariant Lightlike Submanifolds of Metallic Semi-Riemannian Manifolds

We begin by defining the invariant lightlike submanifold of a metallic semi-Riemannian manifold.

Definition 3.1. Let $(\check{N}, \check{J}, \check{g})$ be a metallic semi-Riemannian manifold and (N, g) be a lightlike submanifold of \check{N} . Then, we say that N is an invariant lightlike submanifold of \check{N} , if the following conditions are satisfied:

$$\check{J}(S(TN)) = S(TN) \quad (3.1)$$

and

$$\check{J}(\text{Rad}(TN)) = \text{Rad}(TN). \quad (3.2)$$

Corollary 3.2. Let $(\check{N}, \check{J}, \check{g})$ be a metallic semi-Riemannian manifold and (N, g) be an invariant lightlike submanifold of \check{N} . Then, the lightlike transversal distribution $\text{ltr}(TN)$ is invariant with respect to \check{J} .

Proof. We assume that N is an invariant lightlike submanifold of \check{N} . Then, for any $U \in \Gamma(S(TN))$, $\xi \in \Gamma(\text{Rad}(TN))$ and $N \in \Gamma(\text{ltr}(TN))$, we have

$$\check{g}(\check{J}N, \xi) = \check{g}(N, \check{J}\xi) \neq 0 \quad (3.3)$$

and

$$\check{g}(\check{J}N, U) = \check{g}(N, \check{J}U) = 0. \quad (3.4)$$

From (3.3) and (3.4), it is clear that $\check{J}N$ has no component in $\Gamma(\text{Rad}(TN))$ and $\Gamma(S(TN))$, respectively. On the other hand, since $\text{Rad}(TN) \perp S(TN^\perp)$, then $S(TN^\perp)$ does not contain $\check{J}N$. Thus, the proof is completed. \square

Example 3.3. Let $\check{N} = R_1^5$ be a metallic semi-Riemannian manifold of signature $(-, +, +, +, +)$ with respect to the canonical basis $\{\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial x_5\}$. Consider a metallic structure \check{J} defined by

$$\check{J}(x_1, x_2, x_3, x_4, x_5) = (\sigma x_1, \sigma x_2, \sigma x_3, (p - \sigma)x_4, (p - \sigma)x_5).$$

Let N be a submanifold of $(R_1^5, \check{J}, \check{g})$ given by

$$x_1 = u_3, \quad x_2 = -\sin \alpha u_1 + \cos \alpha u_3,$$

$$x_3 = \cos \alpha u_1 + \sin \alpha u_3, \quad x_4 = u_2, \quad x_5 = 0.$$

Then TN is spanned by $\{Z_1, Z_2, Z_3\}$, where

$$Z_1 = -\sin \alpha \partial x_2 + \cos \alpha \partial x_3,$$

$$Z_2 = \partial x_4,$$

$$Z_3 = \partial x_1 + \cos \alpha \partial x_2 + \sin \alpha \partial x_3.$$

Hence N is a 1-lightlike submanifold of R_1^5 with

$$\text{Rad}(TN) = \text{Span}\{Z_3\}$$

and

$$S(TN) = \text{Span}\{Z_1, Z_2\}.$$

It is easy to see that

$$\check{J}Z_3 = \sigma Z_3 \in \Gamma(\text{Rad}(TN)),$$

$$\check{J}Z_1 = \sigma Z_1 \in \Gamma(S(TN)),$$

$$\check{J}Z_2 = (p - \sigma)Z_2 \in \Gamma(S(TN)),$$

which mean that $S(TN)$ and $\text{Rad}(TN)$ is invariant with respect to \check{J} . On the other hand, by direct calculations, we get the lightlike transversal bundle and screen-transversal distribution are spanned by

$$N = \frac{1}{2}\{-\partial x_1 + \cos \alpha \partial x_2 + \sin \alpha \partial x_3\}, \quad W = \partial x_5,$$

respectively. It is clear that $\text{ltr}(TN)$ and $S(TN^\perp)$ are invariant distributions. Thus, N is an invariant lightlike submanifold of \check{N} .

Let $(\check{N}, \check{J}, \check{g})$ be a metallic semi-Riemannian manifold and N be an invariant lightlike submanifold of \check{N} . In this paper, we assume that

$$\bar{\nabla} \check{J} = 0, \tag{3.5}$$

which implies that $\bar{\nabla}_U \check{J}V = \check{J}\bar{\nabla}_U V$ and similarly $\bar{\nabla}_U \check{J}N = \check{J}\bar{\nabla}_U N$, for all $U, V \in \Gamma(TN), N \in \Gamma(\text{ltr}(TN))$.

Now we denote the projection morphisms on $S(TN)$ and $\text{Rad}(TN)$ by T and Q , respectively. Then, for any $U \in \Gamma(TN)$ we write

$$U = TU + QU, \tag{3.6}$$

where $TU \in \Gamma(S(TN))$ and $QU \in \Gamma(\text{Rad}(TN))$.

Applying \check{J} to (3.6) we get

$$\check{J}U = \check{J}TU + \check{J}QU. \tag{3.7}$$

If we denote $\check{J}TU$ and $\check{J}QU$ by SU and LU , respectively, then we can rewrite (3.7) as

$$\check{J}U = SU + LU, \tag{3.8}$$

where $SU \in \Gamma(S(TN))$ and $LU \in \Gamma(\text{Rad}(TN))$.

If we differentiate (3.8) and using (3.5), (2.8), (2.10)-(2.12), for any $U, V \in \Gamma(TN)$, we have

$$\begin{aligned} S\nabla_U V + L\nabla_U V + \check{J}h^l(U, V) + \check{J}h^s(U, V) &= \nabla_U^* SV + h^*(U, SV) + h^l(U, SV) + h^s(U, SV) \\ &- A_{LV}^* U + \nabla_U^t LV + h^l(U, LV) + h^s(U, LV). \end{aligned} \tag{3.9}$$

Considering the tangential, lightlike transversal and screen transversal parts of this equation we obtain the following.

Lemma 3.4. Let $(\check{N}, \check{J}, \check{g})$ be a metallic semi-Riemannian manifold and (N, g) be an invariant lightlike submanifold of \check{N} . Then, we have

$$S\nabla_U V = \nabla_U^* SV - A_{LV}^* U, \tag{3.10}$$

$$L\nabla_U V = h^*(U, SV) + \nabla_U^t LV, \tag{3.11}$$

$$\check{J}h^l(U, V) = h^l(U, \check{J}V), \tag{3.12}$$

$$\check{J}h^s(U, V) = h^s(U, \check{J}V), \tag{3.13}$$

$$\check{J}\nabla_U V = \nabla_U^* SV - A_{LV}^* U + h^*(U, SV) + \nabla_U^t LV, \tag{3.14}$$

where $U, V \in \Gamma(TN)$.

Theorem 3.5. Let N be an invariant lightlike submanifold of a metallic semi-Riemannian manifold \check{N} . Then, the radical distribution $Rad(TN)$ is integrable if and only if either

$$A_{jX}^* Y = A_{jY}^* X \text{ and } A_X^* Y = A_Y^* X$$

or

$$A_{jX}^* Y - A_{jY}^* X = p(A_X^* Y - A_Y^* X),$$

for any $X, Y \in \Gamma(Rad(TN))$ and $Z \in \Gamma(S(TN))$.

Proof. We know that the distribution $Rad(TN)$ is integrable if and only if $[X, Y] \in \Gamma(Rad(TN))$, for all $X, Y \in \Gamma(Rad(TN))$, that is,

$$\check{g}([X, Y], Z) = 0.$$

Thus, for any $Z \in \Gamma(S(TN))$, using (2.3) and (3.5), we have

$$\begin{aligned} \check{g}([X, Y], Z) &= \check{g}(\bar{\nabla}_X Y - \bar{\nabla}_Y X, Z) \\ &= \frac{1}{q} \check{g}(\bar{\nabla}_X jY - \bar{\nabla}_Y jX, jZ) + \frac{p}{q} \check{g}(\bar{\nabla}_Y X - \bar{\nabla}_X Y, jZ) \\ &= 0. \end{aligned}$$

If we use (2.10), we get

$$0 = \check{g}(\nabla_X jY, jZ) - p\check{g}(\nabla_X Y, jZ) - \check{g}(\nabla_Y jX, jZ) + p\check{g}(\nabla_Y X, jZ).$$

Finally, using (2.16) in the last equation, we obtain

$$\check{g}(-A_{jY}^* X + A_{jX}^* Y, jZ) + p\check{g}(A_Y^* X - A_X^* Y, jZ) = 0,$$

which completes the proof. \square

Theorem 3.6. Let N be an invariant lightlike submanifold of a metallic semi-Riemannian manifold \check{N} . Then, the screen distribution $S(TN)$ is integrable if and only if either h^* is symmetric and self-adjoint or

$$h^*(U, jV) - h^*(V, jU) = p(h^*(U, V) - h^*(V, U)),$$

for all $U, V \in \Gamma(S(TN))$ and $N \in \Gamma(ltr(TN))$.

Proof. $S(TN)$ is integrable if and only if $\check{g}([U, V], N) = 0$, for all $U, V \in \Gamma(S(TN))$ and $N \in \Gamma(ltr(TN))$. Using (2.3) and (3.5), we get

$$\begin{aligned} \check{g}([U, V], N) &= \check{g}(\bar{\nabla}_U V - \bar{\nabla}_V U, N) \\ &= \frac{1}{q} \check{g}(\bar{\nabla}_U jV - \bar{\nabla}_V jU, jN) + \frac{p}{q} \check{g}(\bar{\nabla}_V U - \bar{\nabla}_U V, jN) \\ &= 0. \end{aligned}$$

If we use (2.15), we have

$$\check{g}(h^*(U, jV) - h^*(V, jU), jN) - p\check{g}(h^*(U, V) - h^*(V, U), jN) = 0,$$

which completes the proof. \square

Theorem 3.7. Let N be an invariant lightlike submanifold of a metallic semi-Riemannian manifold \check{N} . Then, the induced connection ∇ on N is a metric connection if and only if

$$A_{\check{\xi}}^* U = pA_{\xi}^* U,$$

for all $U \in \Gamma(TN)$ and $\xi \in \Gamma(Rad(TN))$.

Proof. Assume that ∇ is a metric connection. Then, $\forall \nabla_U \xi \in \Gamma(Rad(TN))$, for $U \in \Gamma(TN)$ and $\xi \in \Gamma(Rad(TN))$. Thus, using (2.10) and (2.3), we derive

$$\frac{1}{q} \check{g}(\check{\nabla}_U \check{J}\xi, \check{J}Z) - \frac{p}{q} \check{g}(\check{\nabla}_U \xi, \check{J}Z) = 0,$$

for $Z \in \Gamma(S(TN))$.

Then if we use (2.10) again and (2.16) in the equation above, we get

$$\frac{1}{q} \check{g}(A_{\check{\xi}}^* U, \check{J}Z) - \frac{p}{q} \check{g}(A_{\xi}^* U, \check{J}Z) = 0,$$

which completes the proof. The converse of the assertion is obvious. \square

Theorem 3.8. Let N be an invariant lightlike submanifold of a metallic semi-Riemannian manifold \check{N} . Then, the radical distribution $Rad(TN)$ defines a totally geodesic foliation on N if and only if

$$h^l(\xi, \check{J}U) = ph^l(\xi, U),$$

for all $U \in \Gamma(S(TN))$ and $\xi \in \Gamma(Rad(TN))$.

Proof. We assume that $Rad(TN)$ defines a totally geodesic foliation on N . That is, for $\xi, \xi_1 \in \Gamma(Rad(TN))$, $\nabla_{\xi} \xi_1 \in \Gamma(Rad(TN))$. Since $\check{\nabla}$ is a metric connection, one can easily see that

$$\check{g}(\nabla_{\xi} \xi_1, U) = \check{g}(\check{\nabla}_{\xi} \xi_1, U) = \check{g}(\xi_1, \nabla_{\xi} U) = 0.$$

Using (2.3), (2.10) and (2.15), we have

$$\check{g}(\check{J}\xi, h^l(\xi_1, \check{J}U)) - p\check{g}(\check{J}\xi, h^l(\xi_1, U)) = 0,$$

and assertion is proved. The proof of the converse part can be made similarly. \square

Theorem 3.9. Let N be an invariant lightlike submanifold of a metallic semi-Riemannian manifold \check{N} . Then, the screen distribution $S(TN)$ defines a totally geodesic foliation on N if and only if

$$h^*(U, \check{J}V) = ph^*(U, V)$$

for $U, V \in \Gamma(S(TN))$.

Proof. $S(TN)$ defines a totally geodesic foliation on N if and only if $\nabla_U V \in \Gamma(S(TN))$, for $U, V \in \Gamma(S(TN))$. If we consider $\check{\nabla}$ is a metric connection, we get

$$\check{g}(\nabla_U V, N) = \check{g}(\check{\nabla}_U V, N) = 0,$$

for $N \in \Gamma(ltr(TN))$.

Using (2.3), (2.10) and (2.15), we obtain

$$\check{g}(\nabla_U \check{J}V + h^*(U, \check{J}V), \check{J}N) - p\check{g}(\nabla_U V + h^*(U, V), \check{J}N) = 0,$$

which completes the proof. The converse proof is obvious. \square

4. Screen Semi-Invariant Lightlike Hypersurfaces of Metallic Semi-Riemannian Manifolds

We recall the definition of a screen semi-invariant hypersurface in a metallic semi-Riemannian manifold.

Definition 3.10. [2] Let $(\check{N}, \check{J}, \check{g})$ be a metallic semi-Riemannian manifold and (N, g) be a lightlike hypersurface of \check{N} . Then, we say that N is a screen semi-invariant lightlike hypersurface of \check{N} , if the following conditions are satisfied:

$$\check{J}(\text{Rad}(TN)) \subseteq S(TN) \quad (4.1)$$

and

$$\check{J}(\text{ltr}(TN)) \subseteq S(TN). \quad (4.2)$$

From definition above for a screen semi-invariant lightlike hypersurface of a metallic semi-Riemannian manifold, we can define a non-degenerate distribution L_0 such that $S(TN)$ is decomposed as:

$$S(TN) = L_0 \perp L_1 \oplus L_2, \quad (4.3)$$

where $L_1 = \check{J}(\text{Rad}(TN))$ and $L_2 = \check{J}(\text{ltr}(TN))$.

Proposition 3.11. [2] Let (N, g) be a screen semi-invariant lightlike hypersurface of a metallic semi-Riemannian manifold $(\check{N}, \check{J}, \check{g})$. Then, $\check{J}L_0$ is invariant with respect to \check{J} .

Proof. Let consider any vector field U of $\Gamma(L_0)$. From (4.3) and (2.2), we derive

$$\check{g}(\check{J}U, \xi) = 0 \quad \text{and} \quad \check{g}(\check{J}U, N) = 0, \quad (4.4)$$

for $\xi \in \Gamma(\text{Rad}(TN))$ and $N \in \Gamma(\text{ltr}(TN))$. That is, $\check{J}U \notin \Gamma(\text{ltr}(TN) \cup \text{Rad}(TN))$.

Similarly, using (4.3) and (2.3), we have

$$\check{g}(\check{J}U, \check{J}\xi) = 0 \quad \text{and} \quad \check{g}(\check{J}U, \check{J}N) = 0. \quad (4.5)$$

That is, $\check{J}U \notin \Gamma(\check{J}(\text{ltr}(TN)) \cup \check{J}(\text{Rad}(TN)))$ and proof is completed. \square

Thus, TN can be written as:

$$TN = L_1 \oplus L_2 \perp L_0 \perp \text{Rad}(TN). \quad (4.6)$$

If we denote the invariant distribution of TN by L such as

$$L = L_0 \perp \text{Rad}(TN) \perp \check{J}(\text{Rad}(TN)), \quad (4.7)$$

then, (4.6) is reduced to

$$TN = L \oplus L_2. \quad (4.8)$$

Thus, from (2.7) and (4.8), we have the following decomposition:

$$T\check{N} = L_1 \oplus L_2 \perp L_0 \perp \{\text{Rad}(TN) \oplus \text{ltr}(TN)\}. \quad (4.9)$$

Example 3.12. Let $\check{N} = R_2^5$ be a metallic semi-Riemannian manifold of signature $(-, +, -, +, +)$ and metallic structure \check{J} is defined as

$$\check{J}(x_1, x_2, x_3, x_4, x_5) = (\sigma x_1, \sigma x_2, (p - \sigma)x_3, \sigma x_4, (p - \sigma)x_5).$$

Let N be a hypersurface of $(R_2^5, \check{J}, \check{g})$ given by

$$x_5 = (p - \sigma)x_1 + (p - \sigma)x_2 + x_3.$$

Then we get

$$\begin{aligned} U_1 &= \partial x_1 + (p - \sigma)\partial x_5, & U_2 &= \partial x_2 + (p - \sigma)\partial x_5, \\ U_3 &= \partial x_3 + \partial x_5, & U_4 &= \partial x_4. \end{aligned}$$

If we choose

$$\xi = (p - \sigma)U_1 - (p - \sigma)U_2 + U_3 = (p - \sigma)\partial x_1 - (p - \sigma)\partial x_2 + \partial x_3 + \partial x_5$$

and

$$\begin{aligned} W_1 &= U_4, \\ W_2 &= -\partial x_1 + \partial x_2 + (p - \sigma)\partial x_3 + (p - \sigma)\partial x_5, \\ W_3 &= -\frac{1}{2(1 + (p - \sigma)^2)}(-\partial x_1 - \partial x_2 + (p - \sigma)\partial x_3 - (p - \sigma)\partial x_5) \end{aligned}$$

then $S(TN) = Sp\{W_1, W_2, W_3\}$ and $Rad(TN) = Sp\{\xi\}$. By direct calculations we get

$$N = -\frac{1}{2(1 + (p - \sigma)^2)}\{(p - \sigma)\partial x_1 + (p - \sigma)\partial x_2 + \partial x_3 - \partial x_5\}.$$

By choosing $q = 1$, we obtain $\check{J}\xi = W_2$ and $\check{J}N = W_3$, which imply that N is a screen semi-invariant lightlike hypersurface of \check{N} .

Let us denote the projection morphisms on L and L_2 by B and R , respectively. Then $U \in \Gamma(TN)$ can be written as:

$$U = BU + RU, \tag{4.10}$$

where $BU \in \Gamma(L)$ and $RU \in \Gamma(L_2)$.

Applying \check{J} to (4.10) we derive

$$\check{J}U = \check{J}BU + \check{J}RU. \tag{4.11}$$

If we denote $\check{J}BU$ and $\check{J}RU$ by S_1U and R_1U , respectively, then we can rewrite (4.11) as

$$\check{J}U = S_1U + R_1U, \tag{4.12}$$

where $S_1U \in \Gamma(L)$ and $R_1U \in \Gamma(ltr(TN))$.

Let N be a screen semi-invariant lightlike hypersurface of a metallic semi-Riemannian manifold \check{N} . Using (3.5), (2.10) and (2.15), $\forall U, V \in \Gamma(TN)$, we obtain

$$\check{J}\nabla_U V + \check{J}B(U, V) = \nabla_U^* \check{J}V + h^*(U, \check{J}V) + B(U, \check{J}V). \tag{4.13}$$

Considering the tangential and lightlike transversal parts of (4.13), we give following.

Proposition 3.13. Let N be a screen semi-invariant lightlike hypersurface of a metallic semi-Riemannian manifold \check{N} . Then we have

$$\check{J}\nabla_U V = \nabla_U^* \check{J}V + h^*(U, \check{J}V) - \check{J}B(U, V), \tag{4.14}$$

$$B(U, \check{J}V) = 0, \tag{4.15}$$

for all $U, V \in \Gamma(TN)$.

Theorem 3.14. [2] Let N be a screen semi-invariant lightlike hypersurface of a metallic semi-Riemannian manifold \check{N} . Then, the invariant distribution L is integrable if and only if

$$B(\check{J}W, \check{J}U) = pB(U, \check{J}W) + qB(U, W)$$

for all $U, W \in \Gamma(L)$.

Proof. L is integrable iff $[U, W] \in \Gamma(L)$, for all $U, W \in \Gamma(L)$. That is, $\check{g}([\check{J}W, U], \check{J}\xi) = 0$, for $\xi \in \Gamma(Rad(TN))$. Then, using (2.2), (3.5) and (2.3), we get

$$\check{g}(\check{\nabla}_{\check{J}W} \check{J}U, \xi) - p\check{g}(\check{\nabla}_U W, \check{J}\xi) - q\check{g}(\check{\nabla}_U W, \xi) = 0.$$

Finally, if we use (2.10) in the last equation, the proof is completed. \square

Theorem 3.15. Let N be a screen semi-invariant lightlike hypersurface of a metallic semi-Riemannian manifold \check{N} . Then, the radical distribution $Rad(TN)$ is integrable if and only if

$$\nabla_\xi^* \check{J}\xi_1 - \nabla_{\xi_1}^* \check{J}\xi = p(A_\xi^* \xi_1 - A_{\xi_1}^* \xi) \tag{4.16}$$

or

$$\check{J}\nabla_\xi^* \check{J}\xi_1 - \check{J}\nabla_{\xi_1}^* \check{J}\xi = p(\nabla_\xi^* \check{J}\xi_1 - \nabla_{\xi_1}^* \check{J}\xi) \tag{4.17}$$

for any $\xi, \xi_1 \in \Gamma(Rad(TN))$.

Proof. We assume that $Rad(TN)$ is integrable. Then, $\check{g}([\xi, \xi_1], Z) = 0$ for for all $\xi, \xi_1 \in \Gamma(Rad(TN))$, $Z \in \Gamma(S(TN))$. If we use (2.3), we have

$$0 = \check{g}(\check{\nabla}_\xi \check{J}\xi_1, \check{J}Z) - p\check{g}(\check{\nabla}_\xi \xi_1, \check{J}Z) - \check{g}(\check{\nabla}_{\xi_1} \check{J}\xi, \check{J}Z) + p\check{g}(\check{\nabla}_{\xi_1} \xi, \check{J}Z) \tag{4.18}$$

Then, using (2.10) and (2.15), we obtain

$$\check{g}(\nabla_\xi^* \check{J}\xi_1 + pA_{\xi_1}^* \xi - \nabla_{\xi_1}^* \check{J}\xi - pA_\xi^* \xi_1, \check{J}Z) = 0,$$

which satisfies (4.16).

On the other hand, if we use (2.2), (2.10) and (2.15) in (4.18), we get

$$\begin{aligned} 0 &= \check{g}(\nabla_\xi^* \check{J}\xi_1 - \nabla_{\xi_1}^* \check{J}\xi, \check{J}Z) \\ &+ p\check{g}(\nabla_{\xi_1}^* \check{J}\xi + h^*(\xi_1, \check{J}\xi) - \check{J}B(\xi_1, \xi) - \nabla_\xi^* \check{J}\xi_1 + h^*(\xi, \check{J}\xi_1) - \check{J}B(\xi, \xi_1), Z). \end{aligned}$$

Since B is symmetric and from (4.14), we have

$$\check{g}(\check{J}\nabla_\xi^* \check{J}\xi_1 - \nabla_{\xi_1}^* \check{J}\xi + p\nabla_{\xi_1}^* \check{J}\xi - p\nabla_\xi^* \check{J}\xi_1, Z) = 0,$$

which satisfies (4.17) and the proof is completed. \square

Theorem 3.16. Let N be a screen semi-invariant lightlike hypersurface of a metallic semi-Riemannian manifold \check{N} . Then, the screen distribution $S(TN)$ is integrable if and only if

$$\nabla_U^* \check{J}W - \nabla_W^* \check{J}U = p(\nabla_U^* W - \nabla_W^* U) \quad (4.19)$$

or

$$\nabla_U^* \check{J}W = \nabla_W^* \check{J}U \quad (4.20)$$

for any $U, W \in \Gamma(S(TN))$.

Proof. We know that $S(TN)$ is integrable iff $\check{g}([U, W], N) = 0$, for for all $U, W \in \Gamma(S(TN))$, $N \in \Gamma(\text{ltr}(TN))$. Using (2.3) and (3.5), we get

$$\check{g}(\bar{\nabla}_U \check{J}W - \bar{\nabla}_W \check{J}U, \check{J}N) - p\check{g}(\bar{\nabla}_U W - \bar{\nabla}_W U, \check{J}N) = 0. \quad (4.21)$$

If we use (2.10) and (2.15) in (4.21),

$$\check{g}(\nabla_U^* \check{J}W - \nabla_W^* \check{J}U - p\nabla_U^* W + p\nabla_W^* U, \check{J}N) = 0$$

is obtained and (4.19) is satisfies.

On the other hand, using (2.2), (2.10) and (2.15) in (4.21), we get

$$\check{g}(\nabla_U^* \check{J}W - \nabla_W^* \check{J}U, \check{J}N) = 0.$$

Thus, (4.20) is satisfied and the proof is completed. \square

Theorem 3.17. Let N be a screen semi-invariant lightlike hypersurface of a metallic semi-Riemannian manifold \check{N} . Then, the induced connection ∇ on N is a metric connection if and only if one of the followings is satisfied:

$$\nabla_U^* \check{J}\xi = -pA_\xi^* U \quad (4.22)$$

or

$$qA_\xi^* U = 0, \quad (4.23)$$

for all $U \in \Gamma(TN)$ and $\xi \in \Gamma(\text{Rad}(TN))$.

Proof. ∇ is a metric connection iff $\nabla_U \xi \in \Gamma(\text{Rad}(TN))$ that is, $g(\nabla_U \xi, Z) = 0$, for $U \in \Gamma(TN)$, $\xi \in \Gamma(\text{Rad}(TN))$ and $Z \in \Gamma(S(TN))$. If we consider (2.10), we have

$$\check{g}(\bar{\nabla}_U \xi, Z) = 0. \quad (4.24)$$

Then, using (2.3), (3.5), (2.10) and (2.15), we obtain

$$\check{g}(\nabla_U^* \check{J}\xi + pA_\xi^* U, \check{J}Z) = 0,$$

which satisfies (4.22).

Now, using (2.3) in (4.24), we get

$$\check{g}(\check{J}\bar{\nabla}_U \xi - p\check{J}\bar{\nabla}_U \xi, Z) = 0.$$

Using (2.10), (2.15) and (2.16), we derive

$$0 = \check{g}(\check{J}\nabla_U^* \check{J}\xi + \check{J}h^*(U, \check{J}\xi) + \check{J}B(U, \check{J}\xi) + p\check{J}A_\xi^* U - p\check{J}\nabla_U^* \xi - p\check{J}B(U, \xi), Z).$$

Finally, if we use (2.3) and (4.15) in the last equation,

$$\check{g}(-qA_\xi^* U + q\nabla_U^* \xi, Z) = 0$$

is obtained and (4.23) is satisfied. \square

Theorem 3.18. Let N be a screen semi-invariant lightlike hypersurface of a metallic semi-Riemannian manifold \check{N} . Then, the radical distribution $Rad(TN)$ defines a totally geodesic foliation on N if and only if

$$\nabla_{\xi}^* \check{J}U = p\nabla_{\xi}^* U,$$

for all $U \in \Gamma(S(TN))$ and $\xi \in \Gamma(Rad(TN))$.

Proof. We assume that $Rad(TN)$ defines a totally geodesic foliation on N . That is, for $\xi, \xi_1 \in \Gamma(Rad(TN))$, $\nabla_{\xi} \xi_1 \in \Gamma(Rad(TN))$. Since $\bar{\nabla}$ is a metric connection, we can write

$$g(\nabla_{\xi} \xi_1, U) = \check{g}(\bar{\nabla}_{\xi} \xi_1, U) = \check{g}(\xi_1, \bar{\nabla}_{\xi} U) = 0.$$

Using (2.3), (2.10) and (2.15), we have

$$\check{g}(\check{J}\xi, \nabla_{\xi}^* \check{J}U - p\nabla_{\xi}^* U) = 0$$

and assertion is proved. \square

Theorem 3.19. Let N be a screen semi-invariant lightlike hypersurface of a metallic semi-Riemannian manifold \check{N} . Then, the screen distribution $S(TN)$ defines a totally geodesic foliation on N if and only if

$$\nabla_U^* \check{J}V = p\nabla_U^* V,$$

for all $U, V \in \Gamma(S(TN))$.

Proof. $S(TN)$ defines a totally geodesic foliation on N iff for $U, V \in \Gamma(S(TN))$, $\nabla_U V \in \Gamma(S(TN))$. If we consider $\bar{\nabla}$ is a metric connection, we get

$$\check{g}(\nabla_U V, N) = \check{g}(\bar{\nabla}_U V, N) = 0,$$

for $N \in \Gamma(ltr(TN))$. Using (2.3), (2.10) and (2.15),

$$\check{g}(\nabla_U^* \check{J}V, \check{J}N) - p\check{g}(\nabla_U^* V, \check{J}N) = 0,$$

which completes the proof. \square

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