Filomat 34:6 (2020), 1755–1769
https://doi.org/10.2298/FIL2006755T

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Crossed Modules, Double Group-Groupoids and Crossed Squares

Sedat Temel[a], Tunçar Şahan[b], Osman Mucuk[c]

[a]*Department of Mathematics, Recep Tayyip Erdogan University, Rize, Turkey*
[b]*Department of Mathematics, Aksaray University, Aksaray, Turkey*
[c]*Department of Mathematics, Erciyes University, Kayseri, Turkey*

Abstract. The purpose of this paper is to obtain the notion of crossed module over group-groupoids considering split extensions and prove a categorical equivalence between these types of crossed modules and double group-groupoids. This equivalence enables us to produce various examples of double groupoids. We also prove that crossed modules over group-groupoids are equivalent to crossed squares.

1. Introduction

In this paper we are interested in crossed modules of group-groupoids associated with the split extensions and producing new examples of double groupoids in which the sets of squares, edges and points are group-groupoids.

The idea of crossed module over groups was initially introduced by Whitehead in [29, 30] during the investigation of the properties of second relative homotopy groups for topological spaces. The categorical equivalence between crossed modules over groups and group-groupoids which are widely called in literature *2-groups* [3], *$\mathcal{G}$-groupoids* or *group objects* in the category of groupoids [9], was proved by Brown and Spencer in [9, Theorem 1]. Following this equivalence normal and quotient objects in these two categories have been recently compared and associated objects in the category of group-groupoids have been characterized in [22]. This categorical equivalence has also been extended by Porter in [26, Section 3] to a more general algebraic category $\mathbf{C}$ called category of groups with operations whose idea goes back to Higgins [15] and Orzech [24, 25]. This result is used for example in [1] as a tool to extend some results about topological groups to the topological groups with operations. Recently, group-groupoid aspect of the monodromy groupoid was developed in [19]. Also monodromy groupoids of internal groupoids within topological groups with operations were investigated in [18].

Double groupoids which can be thought as a groupoid objects in the category of groupoids were introduced by Ehresmann in [12, 13]. Double categories have been used in mathematical physics as an application of categorical methods to deeper understanding of genuine features of some problems. As an example, one can see the reference [16] for an extension of topological quantum field theories via double categories. Also according to [6, Chapter 6], the structure of crossed module is inadequate to give a proof of 2-dimensional Seifert-van-Kampen Theorem and hence one needs the idea of double groupoid. For the

2010 *Mathematics Subject Classification*. Primary 20L05; Secondary 22A22, 18D35
Keywords. Group-groupoid, crossed module, double group-groupoid, crossed square
Received: 10 April 2020; Revised: 06 November 2020; Accepted: 29 November 2020
Communicated by Ljubiša D.R. Kočinac
Email addresses: sedat.temel@erdogan.edu.tr (Sedat Temel), tuncarsahan@gmail.com (Tunçar Şahan), mucuk@erciyes.edu.tr (Osman Mucuk)

purpose of obtaining some examples of double groupoids, Brown and Spencer in [9] proved the categorical equivalence between crossed modules over groups and special double groupoids in the sense that the horizontal and vertical groupoids agree and the set of points is singleton. Then the categorical equivalence of crossed modules over groupoids and double groupoids with thin structures was proved in [6, Chapter 6].

From Loday [17] we know that cat^1-groups are equivalent to crossed modules over groups and cat^2-groups to crossed squares. More generally Ellis and Steiner in [14] proved that cat^n-groups are equivalent to crossed n-cubes. The readers are also refereed to [4] for algebraic structures related to groupoids and algebraic descriptions of homotopy n-types. Groupoid versions of Lemma 2.2 and Proposition 5.2 in [17] were given by Temel in [28]. In addition, a similar result in the category of groupoids was obtained by Aytekin [2] in terms of simplicial objects. He proved that the category of crossed modules over groupoids is naturally equivalent to that of simplicial groupoids with Moore complex of length 1.

In this paper following Porter's methods in [26] we introduce the notion of crossed module over group-groupoids via split extensions of short exact sequences and obtain the double groupoids associated with these crossed modules. Moreover we have a categorical equivalence between these types of crossed modules and double group-groupoids, which enables us to have some varieties of examples for double groupoids. We finally prove that crossed modules over group-groupoids is also equivalent to the crossed squares and therefore to cat^2-groups. With the help of the above-mentioned equivalence, we believe that this study will shed light on the (co)homology theory of crossed squares.

2. Preliminaries

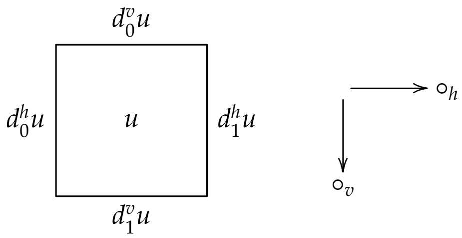
Let C be a finitely complete category. By an *internal category* $\mathcal{D} = (D_0, D_1, d_0, d_1, \varepsilon, m)$ in C we mean a class of objects D_0 called object of objects and a class of morphisms D_1 called object of arrows, together with initial and final point maps; $d_0, d_1 \colon D_1 \to D_0$, object inclusion map $\varepsilon \colon D_0 \to D_1$ in C (for $x \in D_0$ the element x denoted by 1_x) as morphisms of C,

$$\begin{tikzcd} D_1 \arrow[r, shift left=1, "d_0"] \arrow[r, shift right=1, "d_1"'] & D_0 \arrow[l, bend right, "\varepsilon"'] \end{tikzcd}$$

such that $d_0\varepsilon = d_1\varepsilon = 1_{D_0}$ and a morphism $m \colon D_1 \times_{D_0} D_1 \to D_1$ of C called the composition map (usually expressed as $m(a, b) = b \circ a$) where $D_1 \times_{D_0} D_1$ is the pullback of d_0, d_1 such that $\varepsilon d_0(a) \circ a = a = a \circ \varepsilon d_0(a)$. An *internal groupoid* in C is an internal category with a morphism $\eta \colon D_1 \to D_1$, $\eta(a) = a^{-1}$ of C called inverse such that $a^{-1} \circ a = 1_{s(a)}$, $a \circ a^{-1} = 1_{t(a)}$.

A *group-groupoid* G is an internal groupoid in the category of groups, i.e., G and G_0 are groups with the property that the initial and final point maps, object inclusion map, the inversion and partial composition ($G_{d_0} \times_{d_1} G \to G$) are group morphisms. An alternative name used in literature for a group-groupoid is 2-group [3]. In a group-groupoid the group operation is written additively while the composition in the groupoid by " $\circ$ " as above. Group-groupoid morphisms are usual functors which are also group homomorphisms. We recall that a crossed module over groups originally defined by Whitehead [29, 30], consists of two groups A and B, an action of B on A denoted by $b \cdot a$ for $a \in A$ and $b \in B$; and a morphism $\partial \colon A \to B$ of groups such that $\partial(b \cdot a) = b + \partial(a) - b$ and $\partial(a) \cdot a_1 = a + a_1 - a$ for all $a, a_1 \in A$ and $b \in B$. We denote such a crossed module by (A, B, ∂). A morphism (f_1, f_2) from (A, B, ∂) to (A', B', ∂') is a pair of morphisms of groups $f_1 \colon A \to A'$ and $f_2 \colon B \to B'$ such that $f_2\partial = \partial' f_1$ and $f_1(b \cdot a) = f_2(b) \cdot f_1(a)$ for $a \in A$ and $b \in B$.

A *double groupoid* denoted by $\mathcal{G} = (S, H, V, P)$ has the sets S, H, V and P of squares, horizontal edges, vertical edges and points, respectively. The set S of the squares has groupoid structures on H and on V. Also H and V are groupoids on P and these four groupoid structures are compatible with each other. In a double groupoid a square u has bounding edges as follows



and the horizontal and vertical compositions of the squares are denoted by $v \circ_h u$ and $v \circ_v u$ [5].

In particular, if the horizontal and vertical groupoids coincide, then it is said to be a *special double groupoid*. According to Brown and Spencer [9] crossed modules over groups and special double groupoids where the set of points is singleton are categorically equivalent. By the detail of the proof for given a crossed module $\partial\colon A \to B$ there is a special double groupoid $\mathcal{G}$ in which the set S of squares consists of the elements

$$\left(\alpha\,;\,\begin{pmatrix} a & \begin{smallmatrix} c \\ b \end{smallmatrix} & d \end{pmatrix}\right)$$

for $a, b, c, d \in B$ with $\partial(\alpha) = b^{-1}a^{-1}cd$.

The horizontal and vertical compositions of squares are respectively defined to be

$$\left(\beta\,;\,c \;\begin{smallmatrix} f \\ h \end{smallmatrix}\; g\right) \circ_h \left(\alpha\,;\,a \;\begin{smallmatrix} b \\ d \end{smallmatrix}\; c\right) = \left(\alpha\beta^{d^{-1}}\,;\,a \;\begin{smallmatrix} bf \\ dh \end{smallmatrix}\; g\right)$$

$$\left(\tau\,;\,j \;\begin{smallmatrix} d \\ i \end{smallmatrix}\; h\right) \circ_v \left(\alpha\,;\,a \;\begin{smallmatrix} b \\ d \end{smallmatrix}\; c\right) = \left(\alpha^{j}\tau\,;\,aj \;\begin{smallmatrix} b \\ i \end{smallmatrix}\; ch\right).$$

See [6, Chapter 6] for more discussions on double groupoids in which horizontal and vertical edges are same.

3. Extensions and Crossed Modules of Group-Groupoids

The idea of groups with operations goes back to [15] and [24] (see also [25]) and it is adapted in [26] and [11, p.21] as follows:

A category **C** of groups with a set of operations Ω and with a set E of identities such that E includes the group laws, and the following conditions hold for the set Ω_i of i-ary operations in Ω is said to be a *category of groups with operations*:

(a) $\Omega = \Omega_0 \cup \Omega_1 \cup \Omega_2$;

(b) The group operations written additively $0, -$ and $+$ are the elements of Ω_0, Ω_1 and Ω_2, respectively. Let $\Omega_2' = \Omega_2 \backslash \{+\}$, $\Omega_1' = \Omega_1 \backslash \{-\}$ and assume that if $\star \in \Omega_2'$, then $\star^\circ$ defined by $a \star^\circ b = b \star a$ is also in Ω_2'. Also assume that $\Omega_0 = \{0\}$;

(c) For each $\star \in \Omega_2'$, E includes the identity $a \star (b + c) = a \star b + a \star c$;

(d) For each $\omega \in \Omega_1'$ and $\star \in \Omega_2'$, E includes the identities $\omega(a + b) = \omega(a) + \omega(b)$ and $\omega(a) \star b = \omega(a \star b)$.

Topological version of this definition was given in [20].

According to [26] for groups with operations A and B an *extension* of A by B is an exact sequence

$$\mathbf{0} \longrightarrow A \xrightarrow{\imath} E \xrightarrow{p} B \longrightarrow \mathbf{0}$$

in which p is surjective and $\imath$ is the kernel of p. It is *split* if there exists a morphism $s\colon B \to E$ such that $ps = 1_B$. For given such a split extension an action of B on A called *derived action* which is due to Orzech [24, p.293] is defined by

$$\begin{aligned} b \cdot a &= s(b) + a - s(b) \\ b \star a &= s(b) \star a. \end{aligned} \tag{3.1}$$

for $b \in B$, $a \in A$ and $\star \in \Omega_2'$.

In the rest of this section applying the methods of [26] to the group-groupoids we obtain the notion of crossed modules for them: Let G and H be two group-groupoids. We define an *extension* of H by G to be a short exact sequence of group-groupoids

$$\mathcal{E}: \qquad \mathbf{1} \longrightarrow G \xrightarrow{\iota} K \xrightarrow{p} H \longrightarrow \mathbf{1}$$

where $\mathbf{1}$ represents a singleton group-groupoid. Hence $G = \operatorname{Ker} p$ and p is an epimorphism; and therefore G can be considered as a normal subgroup-groupoid of K. For given such an extension we have the group extensions

$$\mathcal{E}_1: \qquad 0 \longrightarrow G \xrightarrow{\iota_1} K \xrightarrow{p_1} H \longrightarrow 0$$

$$\mathcal{E}_0: \qquad 0 \longrightarrow G_0 \xrightarrow{\iota_0} K_0 \xrightarrow{p_0} H \longrightarrow 0$$

along with the morphisms of them

$$\begin{tikzcd}
\mathcal{E}_1: & 0 \arrow[r] & G \arrow[r, "\iota_1"] \arrow[d, shift right, "d_1"'] \arrow[d, shift left, "d_0"] & K \arrow[r, "p_1"] \arrow[d, shift right, "d_1"'] \arrow[d, shift left, "d_0"] & H \arrow[r] \arrow[d, shift right, "d_1"'] \arrow[d, shift left, "d_0"] & 0 \\
\mathcal{E}_0: & 0 \arrow[r] & G_0 \arrow[r, "\iota_0"'] \arrow[u, bend left, "\varepsilon"] & K_0 \arrow[r, "p_0"'] \arrow[u, bend left, "\varepsilon"] & H \arrow[r] \arrow[u, bend left, "\varepsilon"] & 0
\end{tikzcd}$$

Hence a group-groupoid extension $\mathcal{E}$ can be thought as an internal groupoid in the category of group extensions.

Replacing groups with operations in [26] with group-groupoids we define an extension to be *split* if there exists a morphism $s\colon H \to K$ of group-groupoids such that $ps = 1_H$

$$\begin{tikzcd}
\mathcal{E}: & \mathbf{1} \arrow[r] & G \arrow[r, "\iota"] & K \arrow[r, "p"'] & H \arrow[l, bend right, "s"'] \arrow[r] & \mathbf{1}
\end{tikzcd}$$

In this case both extensions of groups $\mathcal{E}_1$ and $\mathcal{E}_0$ above become split.

We now obtain semidirect product of group-groupoids as follows: Let $\mathcal{E}$ be a split extension of H by G. Then the functor $\theta\colon G \times H \to K$ defined by $\theta(a, b) = a + s(b)$ on arrows and by $\theta(x, y) = x + s(y)$ on objects has an inverse $\theta^{-1}(c) = (c - sp(c), p(c))$ for $c \in K$ where $G \times H$ is the product category; and $G \times H$ is a group-groupoid with the group addition defined by

$$\begin{aligned}
(a, b) + (a_1, b_1) &= \theta^{-1}\left(\theta\left((a, b) + (a_1, b_1)\right)\right) \\
&= \theta^{-1}\left(\theta(a, b) + \theta(a_1, b_1)\right) \\
&= \theta^{-1}\left(a + s(b) + a_1 + s(b_1)\right) \\
&= (a + (s(b) + a_1 - s(b)), b + b_1)
\end{aligned}$$

for all $a, a_1 \in G$ and $b, b_1 \in H$. Moreover θ is an isomorphism of group-groupoids. Hence $G \times H$ becomes a group-groupoid inherited by H. We call the group-groupoid $G \times H$ obtained *semidirect product* of group-groupoids G and H and denote by $G \rtimes H$. So we have a split extension

$$\begin{tikzcd}
\mathcal{E}_{G \rtimes H}: & \mathbf{1} \arrow[r] & G \arrow[r, "\iota"] & G \rtimes H \arrow[r, "p"'] & H \arrow[l, bend right, "s"'] \arrow[r] & \mathbf{1}
\end{tikzcd}$$

associated with $G \rtimes H$.

Lemma 3.1. *For a split extension $\mathcal{E}$ of H by G, the group H acts on the group G.*

Proof. The action of H on G is defined by

$$b \cdot a = s(b) + a - s(b)$$

for $a \in G, b \in H$. □

We call such an action *derived action* of group-groupoids.

Lemma 3.2. *For group-groupoids G and H, if the group H acts on G, then H_0 also acts on G_0 by an action*

$$y \cdot x = d_0(\varepsilon(y \cdot x)) = d_0(\varepsilon(y) \cdot \varepsilon(x))$$

for all $y \in H_0$ and $x \in G_0$.

Proof. It is a consequence of the fact that $1_y \cdot 1_x = 1_{y \cdot x}$ for $x \in G_0$ and $y \in H_0$. □

Hence we can define group-groupoid action as follows:

Definition 3.3. Let G and H be two group-groupoids. If there is a group action of H on G, then we say that H acts on G.

Lemma 3.4. *An action of H on G is a derived action if and only if $G \rtimes H$ is a group-groupoid with the group addition given by*

$$(a, b) + (a_1, b_1) = (a + b \cdot a_1, b + b_1)$$

for all $a, a_1 \in G$, $b, b_1 \in H$.

Example 3.5. For any group-groupoid G, the conjugation action of G on itself given by $a \cdot a_1 = a + a_1 - a$ for all $a, a_1 \in G$ is a derived action. Hence there is a split extension of G by G

$$\mathcal{E}_G : \qquad \mathbf{1} \longrightarrow G \xrightarrow{\iota} G \rtimes G \underset{p}{\overset{s}{\rightleftarrows}} G \longrightarrow \mathbf{1}$$

associated with the conjugation action, where $\iota(a) = (a, 0)$, $p(a, a_1) = a_1$, $s(a) = (0, a)$ for all $a, a_1 \in G$.

Definition 3.6. Let

$$\mathcal{E} : \qquad \mathbf{1} \longrightarrow G \xrightarrow{\iota} K \underset{p}{\overset{s}{\rightleftarrows}} H \longrightarrow \mathbf{1}$$

and

$$\mathcal{E}' : \qquad \mathbf{1} \longrightarrow G' \xrightarrow{\iota} K' \underset{p}{\overset{s}{\rightleftarrows}} H' \longrightarrow \mathbf{1}$$

be two split extensions of group-groupoids. We define a morphism $(\alpha, \beta, \gamma) \colon \mathcal{E} \to \mathcal{E}'$ of split extensions to be consisting of morphisms $\alpha \colon G \to G'$, $\beta \colon K \to K'$ and $\gamma \colon H \to H'$ such that the following diagram commutes.

$$\begin{tikzcd}
\mathcal{E} : \arrow[d, "(\alpha,\beta,\gamma)"'] & \mathbf{1} \arrow[r] & G \arrow[r, "\iota"] \arrow[d, "\alpha"'] & K \arrow[r, shift right, "p"'] \arrow[d, "\beta"'] & H \arrow[l, shift right, "s"'] \arrow[r] \arrow[d, "\gamma"] & \mathbf{1} \\
\mathcal{E}' : & \mathbf{1} \arrow[r] & G' \arrow[r, "\iota'"] & K' \arrow[r, shift right, "p'"'] & H' \arrow[l, shift right, "s'"'] \arrow[r] & \mathbf{1}
\end{tikzcd}$$

Remark 3.7. If there is a derived action of H on G, then the morphism $(1_G,\theta,1_H)\colon \mathcal{E}_{G\rtimes H}\to\mathcal{E}$ of split extensions denoted below is an isomorphism of split extensions.

$$\begin{tikzcd}
\mathcal{E}_{G\rtimes H}: \arrow[d, "(1_G,\theta,1_H)"'] & \mathbf{1} \arrow[r] & G \arrow[r, "\iota"] \arrow[d, "1_G"'] & G\rtimes H \arrow[r, shift right, "p"'] \arrow[d, "\theta"'] & H \arrow[l, shift right, "s"'] \arrow[r] \arrow[d, "1_H"] & \mathbf{1} \\
\mathcal{E}: & \mathbf{1} \arrow[r] & G \arrow[r, "\iota"] & K \arrow[r, shift right, "p"'] & H \arrow[l, shift right, "s"'] \arrow[r] & \mathbf{1}
\end{tikzcd}$$

Following the idea in [26] we now define crossed module of group-groupoids as follows:

Definition 3.8. Let G and H be two group-groupoids with an action of H on G. We call a morphism $\partial\colon G\to H$ of group-groupoids *crossed module* of them whenever

$$(1_G, 1_G\times\partial,\partial)\colon \mathcal{E}_G\to\mathcal{E}_{G\rtimes H} \quad\text{and}\quad (\partial,\partial\times 1_H, 1_H)\colon \mathcal{E}_{G\rtimes H}\to\mathcal{E}_H$$

denoted below are morphisms of split extensions.

$$\begin{tikzcd}
\mathcal{E}_G: \arrow[d, "(1_G,1_G\times\partial,\partial)"'] & \mathbf{1} \arrow[r] & G \arrow[r, "\iota"] \arrow[d, "1_G"'] & G\rtimes G \arrow[r, shift right, "p"'] \arrow[d, "1_G\times\partial"'] & G \arrow[l, shift right, "s"'] \arrow[r] \arrow[d, "\partial"] & \mathbf{1} \\
\mathcal{E}_{G\rtimes H}: \arrow[d, "(\partial,\partial\times 1_H,1_H)"'] & \mathbf{1} \arrow[r] & G \arrow[r, "\iota"] \arrow[d, "\partial"'] & G\rtimes H \arrow[r, shift right, "p"'] \arrow[d, "\partial\times 1_H"'] & H \arrow[l, shift right, "s"'] \arrow[r] \arrow[d, "1_H"] & \mathbf{1} \\
\mathcal{E}_H: & \mathbf{1} \arrow[r] & H \arrow[r, "\iota"] & H\rtimes H \arrow[r, shift right, "p"'] & H \arrow[l, shift right, "s"'] \arrow[r] & \mathbf{1}
\end{tikzcd}$$

We write (G,H,∂) for such a crossed module. By the assessment of the above morphisms of split extensions we can state the crossed module over group-groupoids as follows:

Proposition 3.9. *Let $\partial\colon G\to H$ be a morphism of group-groupoids such that H acts on G. Then (G,H,∂) is a crossed module over group-groupoids if and only if (G,H,∂_1) is a crossed module over groups.*

We should point out that in a crossed module (G,H,∂) over group-groupoids, $\partial_0\colon G_0\to H_0$ is also a crossed module of groups.

Lemma 3.10. *Let (G,H,∂) be a crossed module over group-groupoids. Then H_0 acts on G by*

$$y\cdot a=\varepsilon^H(y)\cdot a=1_y\cdot a$$

for $a\in G$, $y\in H_0$ and H acts on G_0 by

$$b\cdot x=d_1^H(b)\cdot x$$

for all $b\in H$, $x\in G_0$.

Proof. These can be seen by easy calculations. So proof is omitted. □

We now give the following examples of crossed modules for group-groupoids:

Example 3.11. Let G be a group-groupoid and N a normal subgroup-groupoid of G in the sense of [22], i.e., H is a normal subgroup of N and hence N_0 is a subgroup of G_0. Then (N,G,inc) is a crossed module over group-groupoids where $\mathrm{inc}\colon N\hookrightarrow G$ is the inclusion functor and the action of G on N is conjugation. In particularly $(G,G,1_G)$ and $(\mathbf{1},G,0)$ are crossed modules over group-groupoids.

Example 3.12. Let (A, B, ∂) be a crossed module over groups. Then $A \rtimes A$ and $B \rtimes B$ are group-groupoids on A and on B, respectively; and then $(A \rtimes A, B \rtimes B, \alpha \times \alpha)$ becomes a crossed module over group-groupoids.

To give a topological example of crossed modules over group-groupoids we first recall that by a *topological crossed module* we a mean a crossed module (A, B, ∂) in which A and B are topological groups, the action of B on A is continuous and ∂ is a continuous morphism of topological groups.

Example 3.13. It is known from [10] that if X is a topological group, then the fundamental groupoid πX is a group-groupoid . Therefore if (A, B, ∂) is a topological crossed module, then $(\pi A, \pi B, \pi(\partial))$ becomes a crossed module of group-groupoids.

Example 3.14. A group can be thought as a discrete group-groupoid in which the arrows are only identities. Hence every crossed module over groups (A, B, ∂) can be considered as a crossed module over group-groupoids.

Example 3.15. For a group A, the direct product $G = A \times A$ becomes a group-groupoid on A with $d_0(a, b) = a$, $d_1(a, b) = b$, $\varepsilon(a) = (a, a)$, $n(a, b) = (b, a)$ and $(b, c) \circ (a, b) = (a, c)$. Hence a crossed module over groups (A, B, ∂) gives rise to a crossed module over group-groupoids replacing A and B with the associated group-groupoids.

A morphism $(f, g)\colon (G, H, \partial) \to (G', H', \partial')$ between crossed modules over group-groupoids is defined to be a pair of group-groupoid morphisms $f\colon G \to G'$ and $g\colon H \to H'$ with the property that $(f, g)\colon (G, H, \partial) \to (G', H', \partial')$ is a morphisms of crossed modules over groups. We write **XMod(GpGd)** for the category of crossed modules over group-groupoids and morphisms between them to be the morphisms defined above.

4. Crossed Modules and Double Group-Groupoids

In this section we define a double group-groupoid to be an internal groupoid in the category of group-groupoids; and then prove that these are categorically equivalent to associated crossed modules.

If G is an internal groupoid in the category of group-groupoids, then the following structural groupoid maps are morphisms of group-groupoids provided $G_0 = H$

$$\begin{tikzcd} G_{d_0}\times_{d_1} G \arrow[r, "m"] & G \arrow[loop below, "n"] \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] & H \arrow[l, bend right=60, "\varepsilon"'] \end{tikzcd}$$

Here we have four different but compatible group-groupoid structures (G, G_0), (H, H_0), (G, H) and (G_0, H_0).

$$\mathcal{G}\colon\quad \begin{tikzcd} G \arrow[r, shift left=2, "d_0"] \arrow[r, shift right=2, "d_1"'] \arrow[d, shift right=2, "d_0"'] \arrow[d, shift left=2, "d_1"] & H \arrow[l, "\varepsilon_1"' description] \arrow[d, shift right=2, "d_0"'] \arrow[d, shift left=2, "d_1"] \\ G_0 \arrow[u, "\varepsilon" description] \arrow[r, shift left=2, "d_0"] \arrow[r, shift right=2, "d_1"'] & H_0 \arrow[l, "\varepsilon_0" description] \arrow[u, "\varepsilon" description] \end{tikzcd}$$

Hence we define a *double group-groupoid* to be consisting of four different, but compatible, group-groupoids (S, H), (S, V), (H, P) and (V, P) such that the following diagram of group-groupoids commutes

$$\mathcal{G}\colon\quad \begin{tikzcd} S \arrow[r, shift left=2, "d_0^h"] \arrow[r, shift right=2, "d_1^h"'] \arrow[d, shift right=2, "d_0^v"'] \arrow[d, shift left=2, "d_1^v"] & H \arrow[l, "\varepsilon^h" description] \arrow[d, shift right=2, "d_0^H"'] \arrow[d, shift left=2, "d_1^H"] \\ V \arrow[u, "\varepsilon^v" description] \arrow[r, shift left=2, "d_0^V"] \arrow[r, shift right=2, "d_1^V"'] & P \arrow[l, "\varepsilon^V" description] \arrow[u, "\varepsilon^H" description] \end{tikzcd}$$

Horizontal and vertical compositions together with group operations have the following interchange laws:

$$\begin{aligned} (\beta \circ_v \alpha) \circ_h (\beta_1 \circ_v \alpha_1) &= (\beta \circ_h \beta_1) \circ_v (\alpha \circ_h \alpha_1) \\ (\beta \circ_v \alpha) + (\beta_1 \circ_v \alpha_1) &= (\beta + \beta_1) \circ_v (\alpha + \alpha_1) \\ (\beta \circ_h \alpha) + (\beta_1 \circ_h \alpha_1) &= (\beta + \beta_1) \circ_h (\alpha + \alpha_1) \end{aligned} \tag{4.1}$$

whenever one side of the equations make sense.

We can now give the following examples of double group-groupoids:

Example 4.1. If G is a group-groupoid, then $\mathcal{G} = (G, G, G_0, G_0)$ is a double group-groupoid with the trivial structural maps

$$\mathcal{G}: \begin{tikzcd}[row sep=huge, column sep=huge] G \arrow[r, shift left=2, "1"] \arrow[r, shift right=2, "1"'] \arrow[d, shift right=2, "d_0^G"'] \arrow[d, shift left=2, "d_1^G"] & G \arrow[l, "1" description] \arrow[d, shift right=2, "d_0^G"'] \arrow[d, shift left=2, "d_1^G"] \\ G_0 \arrow[u, "\varepsilon^G" description] \arrow[r, shift left=2, "1"] \arrow[r, shift right=2, "1"'] & G_0 \arrow[l, "1" description] \arrow[u, "\varepsilon^G" description] \end{tikzcd}$$

Example 4.2. Let (A, B, ∂) be a topological crossed module. Then $\pi(A, B, \partial) = (\pi(A \rtimes B), \pi(B), A \rtimes B, B)$ becomes a double group-groupoid

$$\mathcal{G}: \begin{tikzcd}[row sep=huge, column sep=huge] \pi(A \rtimes B) \arrow[r, shift left=2, "d_0^h"] \arrow[r, shift right=2, "d_1^h"'] \arrow[d, shift right=2, "d_0^v"'] \arrow[d, shift left=2, "d_1^v"] & \pi(B) \arrow[l, "\varepsilon^h" description] \arrow[d, shift right=2, "d_0^H"'] \arrow[d, shift left=2, "d_1^H"] \\ A \rtimes B \arrow[u, "\varepsilon^v" description] \arrow[r, shift left=2, "d_0^V"] \arrow[r, shift right=2, "d_1^V"'] & B \arrow[l, "\varepsilon^V" description] \arrow[u, "\varepsilon^H" description] \end{tikzcd}$$

Example 4.3. Let (G, H, ∂) be a crossed module over group-groupoids. Then we have a double group-groupoid as follows

$$\begin{tikzcd}[row sep=huge, column sep=huge] G \rtimes H \arrow[r, shift left=2, "d_0^h"] \arrow[r, shift right=2, "d_1^h"'] \arrow[d, shift right=3, "d_0^G \times d_0^H"'] \arrow[d, shift left=3, "d_1^G \times d_1^H"] & H \arrow[l, "\varepsilon^h" description] \arrow[d, shift right=2, "d_0^H"'] \arrow[d, shift left=2, "d_1^H"] \\ G_0 \rtimes H_0 \arrow[u, "\varepsilon^v \times \varepsilon^H" description] \arrow[r, shift left=2, "d_0^V"] \arrow[r, shift right=2, "d_1^V"'] & H_0 \arrow[l, "\varepsilon^V" description] \arrow[u, "\varepsilon^H" description] \end{tikzcd}$$

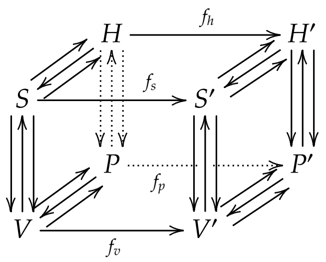
where $d_0^h(a, b) = b$, $d_1^h(a, b) = \partial_1(a) + b$, $\varepsilon^h(b) = (0, b)$, $m^h((a_1, b_1), (a, b)) = (a_1 + a, b)$ for $b_1 = \partial_1(a) + b$ and $d_0^V(x, y) = y$, $d_1^V(x, y) = \partial_0(x) + y$, $\varepsilon^V(y) = (0, y)$, $m^V((x_1, y_1), (x, y)) = (x_1 + x, y)$ for $y_1 = \partial_0(x) + y$. A square (a, b) in $G \rtimes H$ is

$$\begin{tikzcd}[row sep=huge, column sep=huge] y \arrow[r, "b"] \arrow[d, "(x,y)"'] \arrow[dr, phantom, "(a,b)"] & y_1 \arrow[d, "(x_1,y_1)"] \\ \partial_0(x) + y \arrow[r, "\partial_1(a)+b"'] & \partial_0(x_1) + y_1 \end{tikzcd}$$

for $a \in G(x, x_1)$ and $b \in H(y, y_1)$. If (a, b), (a_1, b_1) and (a', b') are squares in $G \rtimes H$ with $a \in G(x, x_1)$, $a_1 \in G(x_1, x_2)$, $a' \in G(x', x_1')$, $b \in H(y, y_1)$, $b_1 \in H(y_1, y_2)$ and $b' = \partial_1(a) + b$ then

$$(a_1, b_1) \circ_h (a, b) = (a_1 \circ a, b_1 \circ b),$$
$$(a', b') \circ_v (a, b) = (a' + a, b).$$

Let $\mathcal{G}$ and $\mathcal{G}'$ be two double group-groupoids. A morphism form $\mathcal{G}$ to $\mathcal{G}'$ is a double groupoid morphism $\mathcal{F} = (f_s, f_h, f_v, f_p) \colon \mathcal{G} \to \mathcal{G}'$ such that $f_s \colon S \to S'$, $f_h \colon H \to H'$, $f_v \colon V \to V'$ and $f_p \colon P \to P'$ are group homomorphisms. Such a morphism of double group-groupoids may be denoted by a diagram as follows:



We write **DbGpGd** for the category with objects double groupoids and morphisms as arrows.

Lemma 4.4. *In a double group-groupoid, as a consequence of interchange laws (Eq. 4.1), the vertical and horizontal compositions of squares can be written in terms of the group operations as*

$$\beta_1 \circ_h \beta = \beta_1 - \varepsilon^h d_1^h(\beta) + \beta = \beta - \varepsilon^h d_1^h(\beta) + \beta_1,$$
$$\alpha_1 \circ_v \alpha = \alpha_1 - \varepsilon^v d_1^v(\alpha) + \alpha = \alpha - \varepsilon^v d_1^v(\alpha) + \alpha_1$$

for all $\alpha, \alpha_1, \beta, \beta_1 \in G$ such that $d_1^v(\alpha) = d_0^v(\alpha_1)$ and $d_1^h(\beta) = d_0^h(\beta_1)$. □

Thus the horizontal inverse of $\beta \in S$ is

$$\beta^{-h} = \varepsilon^h d_0^h(\beta) - \beta + \varepsilon^h d_1^h(\beta) = \varepsilon^h d_1^h(\beta) - \beta + \varepsilon^h d_0^h(\beta)$$

and the vertical inverse of $\alpha \in S$ is

$$\alpha^{-v} = \varepsilon^v d_0^v(\alpha) - \alpha + \varepsilon^v d_1^v(\alpha) = \varepsilon^v d_1^v(\alpha) - \alpha + \varepsilon^v d_0^v(\alpha).$$

In particular, if $\alpha \in \operatorname{Ker} d_0^v$ and $\beta \in \operatorname{Ker} d_0^h$ then

$$\beta^{-h} = -\beta + \varepsilon^h d_1^h(\beta) = \varepsilon^h d_1^h(\beta) - \beta \tag{4.2}$$

and

$$\alpha^{-v} = -\alpha + \varepsilon^v d_1^v(\alpha) = \varepsilon^v d_1^v(\alpha) - \alpha. \tag{4.3}$$

In Lemma 4.4 if we take β with $d_1^h(\beta) = 0$ and α with $d_1^v(\alpha) = 0$ then we obtain the following result.

Corollary 4.5. *Let $\mathcal{G}$ be a double group-groupoid and $\alpha, \alpha_1, \beta, \beta_1 \in S$ with $d_1^v(\alpha) = 0 = d_0^v(\alpha_1)$ and $d_1^h(\beta) = 0 = d_0^h(\beta_1)$. Then*

$$\beta_1 \circ_h \beta = \beta_1 + \beta = \beta + \beta_1,$$
$$\alpha_1 \circ_v \alpha = \alpha_1 + \alpha = \alpha + \alpha_1$$

i.e. squares in $\operatorname{Ker} d_0^v$ (resp. $\operatorname{Ker} d_0^h$) and in $\operatorname{Ker} d_1^v$ (resp. $\operatorname{Ker} d_1^h$) are commutative.

Following corollary is a consequence of Equations (4.2),(4.3) and Corollary 4.5.

Corollary 4.6. *Let $\mathcal{G}$ be a double group-groupoid. Then*

$$\alpha + \alpha_1 - \alpha = \varepsilon^v d_1^v(\alpha) + \alpha_1 - \varepsilon^v d_1^v(\alpha)$$

and

$$\beta + \beta_1 - \beta = \varepsilon^h d_1^h(\beta) + \beta_1 - \varepsilon^h d_1^h(\beta)$$

for all $\alpha, \alpha_1 \in \operatorname{Ker} d_0^v$ and $\beta, \beta_1 \in \operatorname{Ker} d_0^h$.

Theorem 4.7. *The category* **XMod(GpGd)** *of crossed modules over group-groupoids is equivalent to the category* **DbGpGd** *of double group-groupoids.*

Proof. Example 4.3 gives rise to a functor $\theta\colon$ **XMod(GpGd)** $\to$ **DbGpGd** which associates a crossed module over group-groupoids with a double groupoid.

Conversely, for a double group-groupoid $\mathcal{G} = (S, H, V, P)$, we define a crossed module (K, H, ∂) associated with $\mathcal{G}$ where

$$K = (\operatorname{Ker} d_0^h, \operatorname{Ker} d_0^V, d_0^v, d_1^v, \varepsilon^v, m^v, n^v),$$
$$H = (H, P, d_0^H, d_1^H, \varepsilon^H, m^H, n^H)$$

and $\partial = (\partial_1 = d_1^h, \partial_0 = d_1^V)$. Such a crossed module can be visualized as in the following diagram

$$\begin{tikzcd}
\operatorname{Ker} d_0^h \arrow[r, "\partial_1"] \arrow[d, shift right=4, "d_0^v"'] \arrow[d, shift left=4, "d_1^v"] & H \arrow[d, shift right=4, "d_0^H"'] \arrow[d, shift left=4, "d_1^H"] \\
\operatorname{Ker} d_0^V \arrow[r, "\partial_0"] \arrow[u, "\varepsilon^v"] & P \arrow[u, "\varepsilon^H"]
\end{tikzcd}$$

Here the action of H on $\operatorname{Ker} d_0^h$ is given by

$$b \cdot a = \varepsilon^h(b) + a - \varepsilon^h(b)$$

for $b \in H$ and $a \in \operatorname{Ker} d_0^h$. Hence we have a functor $\gamma\colon$ **DbGpGd** $\to$ **XMod(GpGd)**

We now show that these functors are equivalences of categories. In order to define a natural equivalence $S\colon \theta\gamma \Rightarrow 1_{\mathbf{DbGpGd}}$, for an object $\mathcal{G}$ in **DbGpGd** a morphism $S_{\mathcal{G}} = (f_s, 1, f_v, 1)\colon \theta\gamma(\mathcal{G}) \to \mathcal{G}$ is defined by $f_v(a, x) = a + \varepsilon^V(x)$ and $f_s(\alpha, b) = \alpha - \varepsilon^h(b)$ for $(a, x) \in V \rtimes P$ and $(\alpha, b) \in S \rtimes H$.

Conversely, a natural equivalence $T\colon 1_{\mathbf{XMod(GpGd)}} \Rightarrow \gamma\theta$ can be defined such a way that for a crossed module $C = (G, H, \partial)$, the morphism $T_C = (f, g)\colon C \to \gamma\theta(C)$ is given by $f_1(a) = (0, a)$ for $a \in G$ and g is the identity.

Other details are straightforward and so are omitted. □

5. Crossed Modules and Crossed Squares

In this section we prove that the category **XMod(GpGd)** of crossed modules over group-groupoids and the category **X²Mod(Gp)** of crossed squares over groups are equivalent.

We now recall the definition of crossed squares from [7]. Further we will prove that the category **XMod(GpGd)** of crossed modules over group-groupoids and the category **X²Mod(Gp)** of crossed squares over groups are equivalent.

Definition 5.1. A crossed square over groups

$$\mathcal{X}: \quad \begin{tikzcd} L \arrow[r, "\lambda"] \arrow[d, "\lambda'"'] & M \arrow[d, "\mu"] \\ N \arrow[r, "\nu"'] & P \end{tikzcd}$$

consists of four morphisms of groups $\lambda\colon L \to M$, $\lambda'\colon L \to N$, $\mu\colon M \to P$ and $\nu\colon N \to P$, such that $\nu\lambda' = \mu\lambda$ together with actions of the group P on L, M, N on the left, conventionally, (and hence actions of M on L and N via μ and of N on L and M via ν) and a function $h\colon M \times N \to L$. These are subject to the following axioms:

[CS 1] λ, λ' are P-equivariant and μ, ν and $\kappa = \mu\lambda$ are crossed modules,

[CS 2] $\lambda h(m,n) = m + n \cdot (-m)$, $\lambda' h(m,n) = m \cdot n - n$,

[CS 3] $h(\lambda(l), n) = l + n \cdot (-l)$, $h(m, \lambda'(l)) = m \cdot l - l$,

[CS 4] $h(m + m', n) = m \cdot h(m', n) + h(m,n)$, $h(m, n + n') = h(m,n) + n \cdot h(m, n')$,

[CS 5] $h(p \cdot m, p \cdot n) = p \cdot h(m,n)$ and

for all $l \in L$, $m, m' \in M$, $n, n' \in N$ and $p \in P$.

A normal subcrossed module denotes a crossed square whilst a normal subgroup denotes a crossed module. With this idea a normal subcrossed square will be in form a crossed 3-cube, and so on.

Example 5.2. Let (A, B, ∂) be crossed module and (S, T, σ) a normal subcrossed module of (A, B, ∂). Then

$$\begin{tikzcd} S \arrow[r, "\sigma"] \arrow[d, hook, "\mathrm{inc}"'] & T \arrow[d, hook, "\mathrm{inc}"] \\ A \arrow[r, "\partial"'] & B \end{tikzcd}$$

forms a crossed square of groups where the action of B on S is induced action from the action of B on A and the action of B on T is conjugation. The h map is defined by

$$\begin{array}{rccl} h: & T \times A & \to & S \\ & (t, a) & \mapsto & h(t,a) = t \cdot a - a \end{array}$$

for all $t \in T$ and $a \in A$.

A topological example of crossed squares is the fundamental crossed square which is defined in [17] as follows: Suppose given a commutative square of spaces

$$\boldsymbol{X}: \quad \begin{tikzcd} C \arrow[r, "f"] \arrow[d, "g"'] & A \arrow[d, "a"] \\ B \arrow[r, "b"'] & X \end{tikzcd}$$

Let $F(f)$ be the homotopy fibre of f and $F(X)$ the homotopy fibre of $F(g) \to F(a)$. Then the commutative square of groups

$$\Pi X: \quad \begin{tikzcd} \pi_1 F(X) \arrow[r] \arrow[d] & \pi_1 F(g) \arrow[d] \\ \pi_1 F(f) \arrow[r] & \pi_1(C) \end{tikzcd}$$

associated to X is naturally equipped with a structure of crossed square. ΠX is called the fundamental crossed square of X [7].

Example 5.3. Let (A, B, α) be a topological crossed module. Then we know that $(\pi A, A, d_0^A, d_1^A, \varepsilon^A, m^A)$ and $(\pi B, B, d_0^B, d_1^B, \varepsilon^B, m^B)$ are fundamental group-groupoids. Then

$$\begin{tikzcd} \operatorname{Ker} d_0^A \arrow[r, "\pi\alpha"] \arrow[d, "d_1^A"'] & \operatorname{Ker} d_0^B \arrow[d, "d_1^B"] \\ A \arrow[r, "\alpha"'] & B \end{tikzcd}$$

has a crossed square structure where $h([\beta], a) = [\beta \cdot a - a]$ for all $[\beta] \in \operatorname{Ker} d_0^B$ and $a \in A$. Here the path $(\beta \cdot a - a)\colon [0, 1] \to A$ is given by $(\beta \cdot a - a)(r) = \beta(r) \cdot a - a$ for all $r \in [0, 1]$.

A morphism of crossed squares from $\mathcal{X}_1$ to $\mathcal{X}_2$ consists of four group homomorphisms $f_L\colon L_1 \to L_2$, $f_M\colon M_1 \to M_2$, $f_N\colon N_1 \to N_2$ and $f_P\colon P_1 \to P_2$ which are compatible with the actions and the functions h_1 and h_2.

$$\begin{tikzcd} & M_1 \arrow[rr, "f_M"] \arrow[dd, dotted, "\mu_1"] & & M_2 \arrow[dd, "\mu_2"] \\ L_1 \arrow[ur, "\lambda_1"] \arrow[rr, "f_L"] \arrow[dd, "\lambda'_1"'] & & L_2 \arrow[ur, "\lambda_2"'] \arrow[dd, "\lambda'_2"'] & \\ & P_1 \arrow[rr, dotted, "f_P"] & & P_2 \\ N_1 \arrow[ur, "\nu_1"] \arrow[rr, "f_N"'] & & N_2 \arrow[ur, "\nu_2"'] & \end{tikzcd}$$

Category of crossed squares over groups with morphisms between crossed squares defined above is denoted by $\mathbf{X^2Mod(Gp)}$. Crossed squares are equivalent to the crossed modules over crossed modules.

Theorem 5.4. *The category* $\mathbf{XMod(GpGd)}$ *of crossed modules over group-groupoids is equivalent to the category* $\mathbf{X^2Mod(Gp)}$ *of crossed squares over groups.*

Proof. Define a functor $\delta\colon \mathbf{XMod(GpGd)} \to \mathbf{X^2Mod(Gp)}$ as in the following way: Let (G, H, ∂) be a crossed module over group-groupoids. If we set $L = \operatorname{Ker} d_0^G$, $M = \operatorname{Ker} d_0^H$, $N = G_0$, $P = H_0$, $\lambda = \partial_1$, $\lambda' = d_1^G$, $\mu = d_1^H$ and $\nu = \partial_0$ then

$$\delta((G, H, \partial)) = \mathcal{X}: \quad \begin{tikzcd} L \arrow[r, "\lambda"] \arrow[d, "\lambda'"'] & M \arrow[d, "\mu"] \\ N \arrow[r, "\nu"'] & P \end{tikzcd}$$

is a crossed square. Here the action of P on N is already given and the action of M on L is induced action. Actions of P on M and on L are given by

$$p \cdot m = \varepsilon^H(p) + m - \varepsilon^H(p) \quad \text{and} \quad p \cdot l = \varepsilon^H(p) \cdot l$$

respectively, for $p \in P$, $m \in M$ and $l \in L$. Moreover,

$$\begin{array}{rccl} h & : & M \times N & \rightarrow & L \\ & & (m,n) & \mapsto & h(m,n) = m \cdot \varepsilon^G(n) - \varepsilon^G(n). \end{array}$$

Now we will verify that $\mathcal{X}$ satisfies the conditions **[CS1]-[CS6]** of Definition 5.1. Let $l \in L$, $m, m' \in M$, $n, n' \in N$ and $p \in P$. Then

[CS 1] It is easy to see that μ, ν and $\kappa = \mu\lambda$ are crossed modules. Now we will show that λ and λ' are P-equivariant. Let $p \in P$ and $l \in L$. Then

$$\lambda(p \cdot l) = \partial_1(\varepsilon^H(p) \cdot l) = \varepsilon^H(p) + \partial_1(l) - \varepsilon^H(p) = p \cdot \lambda(l)$$

and

$$\lambda'(p \cdot l) = d_1^G(\varepsilon^H(p) \cdot l) = d_1^H \varepsilon^H(p) \cdot d_1^G(l) = p \cdot \lambda'(l)$$

[CS 2] Let $m \in M$ and $n \in N$. Then

$$\begin{aligned} \lambda h(m,n) &= \partial_1(m \cdot \varepsilon^G(n) - \varepsilon^G(n)) \\ &= \partial_1(m \cdot \varepsilon^G(n)) - \partial_1(\varepsilon^G(n)) \\ &= m + \partial_1(\varepsilon^G(n)) - m - \partial_1(\varepsilon^G(n)) \\ &= m + n \cdot (-m) \end{aligned}$$

and

$$\begin{aligned} \lambda' h(m,n) &= d_1^G(m \cdot \varepsilon^G(n) - \varepsilon^G(n)) \\ &= d_1^G(m \cdot \varepsilon^G(n)) - d_1^G(\varepsilon^G(n)) \\ &= d_1^H(m) \cdot d_1^G(\varepsilon^G(n)) - d_1^G(\varepsilon^G(n)) \\ &= d_1^H(m) \cdot n - n \\ &= m \cdot n - n \end{aligned}$$

[CS 3] Let $l \in L$ and $n \in N$. Then

$$\begin{aligned} h(\lambda(l), n) &= \partial_1(l) \cdot \varepsilon^G(n) - \varepsilon^G(n) \\ &= l + \varepsilon^G(n) - l - \varepsilon^G(n) \\ &= l + n \cdot (-l) \end{aligned}$$

and

$$\begin{aligned} h(m, \lambda'(l)) &= m \cdot \varepsilon^G(d_1^G(l)) - \varepsilon^G(d_1^G(l)) \\ &= (m \cdot \varepsilon^G(d_1^G(l)) - \varepsilon^G(d_1^G(l)) + l) - l \\ &= (m \cdot \varepsilon^G(d_1^G(l)) \circ l) - l \\ &= ((m \cdot \varepsilon^G(d_1^G(l))) \circ (1_0 \cdot l)) - l \\ &= ((m \circ 1_0) \cdot (\varepsilon^G(d_1^G(l)) \circ l)) - l \\ &= m \cdot l - l \end{aligned}$$

[CS 4] Let $m, m' \in M$ and $n \in N$. Then

$$\begin{aligned} h(m+m',n) &= (m+m')\cdot\varepsilon^G(n) - \varepsilon^G(n) \\ &= m\cdot(m'\cdot\varepsilon^G(n)) - \varepsilon^G(n) \\ &= m\cdot(m'\cdot\varepsilon^G(n)) + m\cdot(-\varepsilon^G(n)+\varepsilon^G(n)) - \varepsilon^G(n) \\ &= m\cdot(m'\cdot\varepsilon^G(n) - \varepsilon^G(n)) + m\cdot\varepsilon^G(n) - \varepsilon^G(n) \\ &= m\cdot h(m',n) + h(m,n) \end{aligned}$$

and

$$\begin{aligned} h(m,n+n') &= m\cdot\varepsilon^G(n+n') - \varepsilon^G(n+n') \\ &= m\cdot(\varepsilon^G(n)+\varepsilon^G(n')) - \varepsilon^G(n') - \varepsilon^G(n) \\ &= (m\cdot\varepsilon^G(n) - \varepsilon^G(n)) + \varepsilon^G(n) + (m\cdot\varepsilon^G(n') - \varepsilon^G(n')) - \varepsilon^G(n) \\ &= h(m,n) + n\cdot h(m,n') \end{aligned}$$

[CS 5] Let $m \in M$, $n \in N$ and $p \in P$. Then

$$\begin{aligned} h(p\cdot m, p\cdot n) &= h(\varepsilon^H(p) + m - \varepsilon^H(p), p\cdot n) \\ &= (\varepsilon^H(p) + m - \varepsilon^H(p))\cdot\varepsilon^G(p\cdot n) - \varepsilon^G(p\cdot n) \\ &= (\varepsilon^H(p) + m - \varepsilon^H(p))\cdot(\varepsilon^H(p)\cdot\varepsilon^G(n)) - (\varepsilon^H(p)\cdot\varepsilon^G(n)) \\ &= (\varepsilon^H(p) + m)\cdot\varepsilon^G(n) - (\varepsilon^H(p)\cdot\varepsilon^G(n)) \\ &= \varepsilon^H(p)\cdot(m\cdot\varepsilon^G(n)) + \varepsilon^H(p)\cdot(-\varepsilon^G(n)) \\ &= \varepsilon^H(p)\cdot(m\cdot\varepsilon^G(n) - \varepsilon^G(n)) \\ &= p\cdot h(m,n) \end{aligned}$$

Now let $(f = (f_1, f_0), g = (G, g_0))\colon (G, H, \partial) \to (G', H', \partial')$ be a morphism in $\mathbf{XMod(GpGd)}$ then $\delta_1(f,g) = (f_L = f_1, f_M = G, f_N = f_0, f_P = g_0)$ is a morphism of crossed modules over group-groupoids.

Conversely, define a functor $\eta\colon \mathbf{X^2Mod(Gp)} \to \mathbf{XMod(GpGd)}$ by following way: Let $\mathcal{X} = (L, N, M, P)$ be a crossed square over groups. Let G and H be the corresponding group-groupoids to crossed modules (L, N, λ') and (M, P, μ), respectively. That is $G = L \rtimes N$, $G_0 = N$, $H = M \rtimes P$ and $H_0 = P$. Moreover, $\partial_1 = \lambda \times \nu$ and $\partial_0 = \nu$ whilst the action of H on G is given by

$$\begin{array}{ccl} (M \rtimes P) \times (L \rtimes N) & \to & (L \rtimes N) \\ ((m,p),(l.n)) & \mapsto & (m,p)\cdot(l,n) = (m\cdot(p\cdot l) + h(m, p\cdot n), p\cdot n). \end{array}$$

Here $(L \rtimes N, M \rtimes P, \lambda \times \nu)$ becomes a crossed module with the action given above. This crossed module is called the semi-direct product of crossed modules. Hence $\eta_0(\mathcal{X}) = (G, H, \partial)$ is a crossed module over group-groupoids. Now, let (f_L, f_M, f_N, f_P) be a morphism in $\mathbf{X^2Mod(Gp)}$. Then $\eta_1(f_L, f_M, f_N, f_P) = (f = (f_L \times f_N, f_N), g = (f_M \times f_P, f_P))$ is morphism in $\mathbf{XMod(GpGd)}$.

A natural equivalence $S\colon \eta\delta \Rightarrow 1_{\mathbf{XMod(GpGd)}}$ is given by the map $S_{(G,H,\partial)}\colon \eta\delta(G,H,\partial) \to (G,H,\partial)$ where $S_{(G,H,\partial)}$ is identity on both G_0 and H_0, and defined by $a \mapsto (a - \varepsilon^G(d_1^G(a)), d_1^G(a))$ on G and $b \mapsto (b - \varepsilon^H(d_1^H(b)), d_1^H(b))$ on H.

Conversely, for a crossed square over groups $\mathcal{X}$, a natural equivalence $T\colon 1_{\mathbf{X^2Mod(Gp)}} \Rightarrow \delta\eta$ is given by the map $T_{\mathcal{X}}\colon \mathcal{X} \to \delta\eta(\mathcal{X})$ where $T_{\mathcal{X}}$ is identity on P and M and and defined by $m \mapsto (m, 0)$ and $l \mapsto (l, 0)$.

Other details are straightforward and so is omitted. □

Since, by Loday [17], cat^2-groups are equivalent to crossed squares we can state that crossed modules over group-groupoids are equivalent to cat^2-groups.

6. Conclusion

Results obtained in this paper can be given in a more generic cases such as for an arbitrary category of groups with operations, or for an arbitrary modified category of interest etc. Moreover, notions of lifting [21, 27], covering [8] and actor crossed module [23] and homotopy of crossed module morphisms (particularly derivations) can be interpreted in the categories mentioned in this manuscript. As in the (co)homology theory of crossed modules over groups one can study on the (co)homology theory of crossed squares using the equivalence given in Theorem 5.4.

Acknowledgement

We would like to thank the referee for useful comments and bringing some related references to our attentions; and the editors for the editorial process.

References

[1] H.F. Akiz, N. Alemdar, O. Mucuk, T. Şahan, Coverings of internal groupoids and crossed modules in the category of groups with operations, Georgian Math. J. 20 (2013) 223–238.
[2] A. Aytekin, A note on simplicial groupoids, Erciyes Univ. J. Inst. Sci. Tech. 36 (2020) 82–87.
[3] J.C. Baez, A.D. Lauda, Higher-dimensional algebra V: 2-groups, Theory Appl. Categ. 12 (2004) 423–491.
[4] R. Brown, Computing homotopy types using crossed n-cubes of groups, In: N. Ray & G. Walker (Eds.), Adams Memorial Symposium on Algebraic Topology (London Mathematical Society Lecture Note Series, pp. 187-210). Cambridge: Cambridge University Press, 1992.
[5] R. Brown, Topology and Groupoids, BookSurge LLC, North Carolina, 2006.
[6] R. Brown,P.J. Higgins, R. Sivera, Nonabelian Algebraic Topology: filtered spaces, crossed complexes, cubical homotopy groupoids, European Mathematical Society Tracts in Mathematics 15, 2011.
[7] R. Brown, J.-L. Loday, Van Kampen theorems for diagrams of spaces, Topology 26 (1987) 311–335.
[8] R. Brown, O. Mucuk, Covering groups of non-connected topological groups revisited, Math. Proc. Camb. Phil. Soc. 115 (1994) 97–110.
[9] R. Brown, C.B. Spencer, Double groupoids and crossed modules, Cah. Topol. Géom. Différ. Categ. 17 (1976) 343–362.
[10] R. Brown, C.B. Spencer, G-groupoids, crossed modules and the fundamental groupoid of a topological group, Indagat. Math. 79 (1976) 296–302.
[11] T. Datuashvili, Categorical, homological, and homotopical properties of algebraic objects, J. Math. Sci. 225 (2017) 383–533.
[12] C. Ehresmann, Catégories doubles et catégories structurés, C. R. Acad. Sci. Paris 256 (1963) 1198–1201.
[13] C. Ehresmann, Catégories structurés, Ann. Sci. Ec. Norm. Super. 80 (1963) 349–425.
[14] G.J. Ellis, R. Steiner, Higher-dimensional crossed modules and the homotopy groups of $(n + 1)$-ads, J. Pure Appl. Algebra 46 (1987) 117–136.
[15] P.J. Higgins, Groups with multiple operators Proc. London Math. Soc. 3 (1956) 366–416.
[16] T. Kerler, V.V. Lyubashenko, Non-Semisimple Topological Quantum Field Theories for 3-Manifolds with Corners, Springer-Verlag, Berlin, Heidelberg, 2001.
[17] J.-L. Loday, Spaces with finitely many non-trivial homotopy groups, J. Pure Appl. Algebra 24 (1982) 179–202.
[18] O. Mucuk, H.F. Akiz, Monodromy groupoids of an internal groupoid in topological groups with operations, Filomat 29 (2015) 2355–2366.
[19] O. Mucuk, B. Kiliçarslan, T. Şahan, N. Alemdar, Group-groupoids and monodromy groupoids, Topology Appl. 158 (2011) 2034–2042.
[20] O. Mucuk, T. Şahan, Coverings and crossed modules of topological groups with operations, Turk. J. Math. 38(5) (2014) 833–845.
[21] O. Mucuk, T. Şahan, Group-groupoid actions and liftings of crossed modules, Georgian Math. J. 26 (2019) 437–447.
[22] O. Mucuk, T. Şahan, N. Alemdar, Normality and quotients in crossed modules and group-groupoids, Appl. Categor. Struct. 23 (2015) 415–428.
[23] K. Norrie, Actions and Automorphisms of crossed modules, Bull. Soc. Math. France 118 (1990) 129–146.
[24] G. Orzech, Obstruction theory in algebraic categories I, J. Pure Appl. Algebra 2 (1972) 287–314.
[25] G. Orzech, Obstruction theory in algebraic categories II, J. Pure Appl. Algebra 2 (1972) 315–340.
[26] T. Porter, Extensions, crossed modules and internal categories in categories of groups with operations, Proc. Edinb. Math. Soc. 30 (1987) 373–381.
[27] T. Şahan, Further remarks on liftings of crossed modules, Hacet. J. Math. Stat. 48 (2019) 743–752.
[28] S. Temel, Crossed squares, crossed modules over groupoids and cat$^{1-2}$-groupoids, Categ. Gen. Algebr. Struct. Appl. 13 (2020) 125–142.
[29] J.H.C. Whitehead, Note on a previous paper entitled "On adding relations to homotopy group", Ann. Math. 47 (1946) 806–810.
[30] J.H.C. Whitehead, Combinatorial homotopy II, Bull. Amer. Math. Soc. 55 (1949) 453–496.