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# On Weighted Generalized Composition Operators on Weighted Hardy Spaces

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**Abstract.** The present paper introduces the class of weighted generalized composition operators of higher order defined on the weighted Hardy spaces. The study of bounded operators belonging to this class is undertaken and an attempt is made to describe their structural and spectral properties.

## 1. Introduction

Let $\{\beta_n\}_{n=0}^{\infty}$ be a sequence of positive real numbers with $\beta_0 = 1$. The weighted sequence space $H^2(\beta)$ (also known as the weighted Hardy space) is the collection of all formal power series $\{f : f(z) = \sum_{n=0}^{\infty} f_n z^n$ such that $\sum_{n=0}^{\infty} |f_n|^2 \beta_n^2 < \infty\}$. $H^2(\beta)$ forms a Hilbert space with norm $\|.\|_\beta$ induced by the inner product defined as $\langle f, g\rangle = \sum_{n=0}^{\infty} f_n \overline{g}_n \beta_n^2$, where $f(z) = \sum_{n=0}^{\infty} f_n z^n$ and $g(z) = \sum_{n=0}^{\infty} g_n z^n$ belong to $H^2(\beta)$. Hence $\|f\|_\beta$ is given as

$$\|f\|_\beta^2 = \sum_{n=0}^{\infty} |f_n|^2 \beta_n^2.$$

The set $\{e_n(z) = z^n/\beta_n\}_{n\geq 0}$ forms an orthonormal basis for the space $H^2(\beta)$. These spaces were introduced by Kelley [9] in the year 1966 and have ever since occurred frequently in literature. Some well known special cases of these spaces are the Hardy space for $\beta_n = 1$ for each $n \geq 0$, the Bergman space for $\beta_n = (1/(n+1))^{1/2}$, the Dirichlet space for $\beta_n = (n+1)^{1/2}$ and the Fischer space for $\beta_n = n!^{1/2}$. The tendency of the weighted Hardy spaces to yield various known classes of function spaces for specific choices of the weight sequences makes the study over these spaces quite productive. A nice survey regarding the historical growth and applications of these spaces is provided in [13].

The multiplication operators and the composition operators form two important classes of operators which have been studied extensively over various function spaces ever since their inception. We refer to book by Cowen and MacCluer [3] for basic theory of multiplication and composition operators. The study of these operators has been lifted from the Lebesgue spaces to other function spaces, for instance to Lorentz space, Orlicz space, Lorentz-Zygmund space, to name a few, over the years (see [1, 4, 10] and the references

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therein). Their theory touches various branches of mathematics and physics such as entropy theory, ergodic theory and classical mechanics. The tendency of these operators to induce new classes of operators and their wide appearance have motivated mathematicians to come up with various generalizations of these operators. In the recent times, the generalized multiplication operators on weighted Hardy spaces have been introduced in [12] and further extended to $k^{th}$-order generalized multiplication operators in [5], where $k \geq 1$ is an integer. In a parallel manner, the study of generalized composition operators of higher order on weighted Hardy spaces $H^2(\beta)$ has been undertaken (see [4] and the references therein). Weighted composition operators on Hardy spaces are studied in [2] and [7]. Depending upon the nature of the fixed points of one of the inducing functions, spectra of the weighted composition operators on Hardy spaces are investigated by Gunatillake in [7]. In [6], the isometries between various Hardy spaces are obtained in terms of weighted composition operators.

Proceeding ahead in this direction, we introduce and describe the class of weighted generalized composition operators of higher order on weighted Hardy spaces. For specific choices of the inducing symbols, these operators coincide with the generalized multiplication and the generalized composition operators of higher orders. We discuss their boundedness, compactness, Fredholm behaviour and Hilbert-Schmidt behaviour amongst other structural properties. An attempt is also made to describe their spectral structure, specifically the point spectrum. Various examples have been provided to illustrate the results obtained during the course of study.

Before we proceed ahead, it is imperative that we set up the necessary terminology required for the subsequent study. We recall from [4] that for a natural number $k$ and an element $f$ of $H^2(\beta)$ with expression $f(z) = \sum_{n=0}^{\infty} f_n z^n$, $f^{(k)}$ is defined as

$$f^{(k)}(z) = \sum_{n=k}^{\infty} \alpha_n f_n z^{n-k},$$

where $\alpha_n = n!/(n-k)!,\ n \geq k$ and is called the $k^{th}$-derivative of $f$. If $\phi \in H^2(\beta)$ is such that the mapping

$$f \mapsto \phi . f^{(k)}$$

for each $f \in H^2(\beta)$ is continuous on $H^2(\beta)$, then this mapping is denoted by $M_{\phi,k}$ and is called as $k^{th}$-order generalized multiplication operator on $H^2(\beta)$ induced by $\phi$ (see [5]). Analogously, the continuous mapping on $H^2(\beta)$ defined as

$$f \mapsto f^{(k)} \circ \phi$$

for each $f \in H^2(\beta)$, is denoted by $C_{\phi,k}$ and called a generalized composition operator of $k^{th}$-order induced by the symbol $\phi$ (see [4]).

The symbols $\mathbb{N}$ and $\mathbb{C}$ denote respectively the set of all natural numbers and the set of all complex numbers. Throughout the paper, $k$ refers to a fixed natural number. The symbol $\Pi_0(T)$ denotes the point spectrum of the operator $T$. The symbols $Ker(T)$ and $R(T)$ are respectively used to denote the kernel and range space of an operator $T$. The set of all bounded linear operators on a Hilbert space $H$ is denoted by $\mathfrak{B}(H)$. The symbol $M^{\perp}$ denotes the orthogonal complement of the subspace $M$ of $H$.

## 2. The spectral structure

In this section, we shall describe the set of all eigen values, namely the point spectrum, of generalized composition and generalized multiplication operators of $k^{th}$-order induced by some specific symbols. Recall that a complex number $\lambda$ is said to be an eigen value of an operator $T$ defined on a Hilbert space $H$ if there exists a non zero element $f \in H$ such that $Tf = \lambda f$ (see [11]).

We first describe the case for $C_{\phi,k}$, the generalized composition operator of $k^{th}$-order, induced by the symbol $\phi(z) = az \in H^2(\beta)$, where $0 \neq a \in \mathbb{C}$. It is evident from the structure of $C_{\phi,k}$ that 0 always belongs to $\Pi_0(C_{\phi,k})$, so it is interesting to find the non-zero eigen values of $C_{\phi,k}$. Observe that if a non-zero complex number

$\lambda \in \Pi_0(C_{\phi,k})$, then there exists $0 \neq f(z) = \sum_{n=0}^{\infty} f_n z^n \in H^2(\beta)$ satisfying $C_{\phi,k} f = \lambda f$. This provides that for each $i \in \mathbb{N}$ and for each $0 \leq j \leq k-1$,

$$f_{ik+j} = \frac{\lambda^i}{a^{\frac{i}{2}(2j+(i-1)k)}} \frac{j! f_j}{(ik+j)!}.$$

Since $f \in H^2(\beta)$, therefore $\|f\|_\beta^2 = \sum_{n=0}^{\infty} |f_n|^2 \beta_n^2 < \infty$. Hence the above computation and the rearrangement phenomenon provides that

$$\sum_{j=0}^{k-1} \Big( \sum_{i=1}^{\infty} \frac{|\lambda|^{2i}}{|a|^{i(2j+(i-1)k)}} \frac{\beta_{ik+j}^2}{(ik+j)!^2} \Big) j!^2 |f_j|^2 < \infty.$$

Since each of the $k$-components in the left hand side of the above inequality comprises of a series of positive terms, the above inequality holds true if either each of the series $\sum_{i=1}^{\infty} \frac{|\lambda|^{2i}}{|a|^{i(2j+(i-1)k)}} \frac{\beta_{ik+j}^2}{(ik+j)!^2}$ converges, if $f_j \neq 0$ for each $0 \leq j \leq k-1$ or the Fourier coefficient $f_j$, $0 \leq j \leq k-1$, vanishes whenever the corresponding series diverges. With these observations, we can now state the following.

**Proposition 2.1.** *Let $\beta$ be such that $\phi(z) = az$, where $0 \neq a \in \mathbb{C}$, induces the bounded operator $C_{\phi,k}$ on $H^2(\beta)$ and $\lambda$ be a non-zero complex number. Then we have the following:*

1. *If $\sum_{i=1}^{\infty} \frac{|\lambda|^{2i}}{|a|^{i(2j+(i-1)k)}} \frac{\beta_{ik+j}^2}{(ik+j)!^2} < \infty$ for each $0 \leq j \leq k-1$, then $\lambda \in \Pi_0(C_{\phi,k})$.*
2. *If each of the series $\sum_{i=1}^{\infty} \frac{|\lambda|^{2i}}{|a|^{i(2j+(i-1)k)}} \frac{\beta_{ik+j}^2}{(ik+j)!^2}$ diverges for $0 \leq j \leq k-1$, then $\lambda$ cannot be an eigen value of $C_{\phi,k}$.*

In fact, the condition (1) in the above proposition can be relaxed and stated as: If $\sum_{i=1}^{\infty} \frac{|\lambda|^{2i}}{|a|^{i(2j+(i-1)k)}} \frac{\beta_{ik+j}^2}{(ik+j)!^2} < \infty$ for some $0 \leq j \leq k-1$, then $\lambda \in \Pi_0(C_{\phi,k})$. This is because if $0 \neq \lambda \in \mathbb{C}$ is such that the series $\sum_{i=1}^{\infty} \frac{|\lambda|^{2i}}{|a|^{i(2j_0+(i-1)k)}} \frac{\beta_{ik+j_0}^2}{(ik+j_0)!^2}$ converges, where $0 \leq j_0 \leq k-1$, then $\lambda$ is an eigen value and a corresponding eigen vector $0 \neq f(z) = \sum_{n=0}^{\infty} f_n z^n \in H^2(\beta)$ of $C_{\phi,k}$ is given by

$$f_n = \begin{cases} 0 & \text{if } n \neq ik + j_0 \text{ for any } i \geq 0 \\ \alpha & \text{if } n = j_0 \\ \frac{\alpha \lambda^i}{a^{\frac{i}{2}(2j_0+(i-1)k)}} \frac{j_0!}{(ik+j_0)!} & \text{if } n = ik + j_0 \text{ for some } i \in \mathbb{N}, \end{cases}$$

where $\alpha$ is a non-zero complex number.
For specific choices of the weight sequence $\beta$, we are able to determine completely the point spectrum of $C_{\phi,k}$ for some symbols $\phi \in H^2(\beta)$. However, we must ensure in such cases that $C_{\phi,k} \in \mathfrak{B}(H^2(\beta))$. For this purpose, we shall utilize the boundedness criterion for $C_{\phi,k}$ discussed in [4].

**Proposition 2.2.** *The point spectrum of the operator $C_{\phi,k}$, induced by $\phi(z) = z$, on the space $H^2(\beta)$ with $\beta$ defined as $\beta_n = n!$ for each $n \geq 0$, is the open unit disk in the complex plane.*

*Proof.* Since $\{\phi^n : n \geq 0\}$ is an orthogonal family in $H^2(\beta)$ and $\frac{\alpha_{n+k}}{\beta_{n+k}} \|\phi^n\|_\beta = \frac{\alpha_{n+k}}{\beta_{n+k}} \beta_n = 1$ for each $n \geq 0$, we have that $C_{\phi,k} \in \mathfrak{B}(H^2(\beta))$ (Theorem 2.7, [4]). Utilizing Proposition 2.1(1), we obtain that each $\lambda \in \mathbb{C}$, with $|\lambda| < 1$, is an eigen value. Further, if $|\lambda| \geq 1$, then Proposition 2.1(2) provides that $\lambda \notin \Pi_0(C_{\phi,k})$. Compiling this information yields the desired result. □

We recall that for $\beta_n = 1$ for each $n \geq 0$, the sequence space $H^2(\beta)$ coincides with the classical Hardy space $H^2$ of the unit circle. Straightforward computations provide us that the symbol $\phi(z) = z/2$ induces a bounded operator $C_{\phi,k}$ on this space (Theorem 2.7, [4]). In this setting, for $0 \neq \lambda \in \mathbb{C}$, each series

$$\sum_{i=1}^{\infty} \frac{|\lambda|^{2i}}{|a|^{i(2j+(i-1)k)}} \frac{\beta_{ik+j}^2}{(ik+j)!^2} = \sum_{i=1}^{\infty} \frac{|\lambda|^{2i}\, 2^{i(2j+(i-1)k)}}{(ik+j)!^2}$$

diverges for each $0 \leq j \leq k-1$ and therefore, Proposition 2.1(2) leads us into the next result.

**Proposition 2.3.** *On the Hardy space $H^2$, $\Pi_0(C_{\phi,k}) = \{0\}$, where $\phi(z) = z/2$.*

We shall now discuss the point spectrum of the $k^{th}$-order generalized multiplication operator $M_{\phi,k}$ when it is induced by the symbol $\phi(z) = az^p$, where $0 \neq a \in \mathbb{C}$ and $p \geq 0$ is an integer.

**Proposition 2.4.** *Let $\phi(z) = az^p$, where $0 \neq a \in \mathbb{C}$ and $p \geq 0$ is an integer, induces the bounded operator $M_{\phi,k}$ on $H^2(\beta)$. Then*

1. *If $p = k$, then $\Pi_0(M_{\phi,k}) = \{0\} \cup \{an(n-1)\cdots(n-k+1) : n \geq k\}$.*
2. *If $p > k$, then $\Pi_0(M_{\phi,k}) = \{0\}$.*

*Proof.* Since $k \geq 1$, $M_{\phi,k}(e_0) = 0 = 0e_0$ and thus $0 \in \Pi_0(M_{\phi,k})$. Let $0 \neq \lambda \in \Pi_0(M_{\phi,k})$. Then there exists $0 \neq f \in H^2(\beta)$, say $f(z) = \sum_{n=0}^{\infty} f_n z^n$, satisfying

$$\sum_{n=0}^{\infty} a\alpha_{n+k} f_{n+k} z^{n+p} = \sum_{n=0}^{\infty} \lambda f_n z^n.$$

This provides that $f_n = 0$ for $n = 0, 1, \cdots, p-1$ and for each $n \geq 0$,

$$a\alpha_{n+k} f_{n+k} = \lambda f_{n+p}.$$

Now consider the following cases:
(1) If $p = k$, we obtain that $f_{n+k}$ is non-zero for exactly one value of $n$. This is because $f$, being an eigen vector, is a non-zero element of $H^2(\beta)$. Also, since $f_n = 0$ for $0 \leq n \leq k-1$, $f_n \neq 0$ for atleast one $n \geq k$. However, if $f_n$ is non-zero for more than one value of $n \geq k$, say for $n_0$ and $n_1$ ($n_0 \neq n_1$), then $\alpha_{n_0} = \alpha_{n_1}$, which is not feasible. Therefore, $\lambda = a\alpha_{n+k}$ for exactly one $n \geq 0$. This yields that $\Pi_0(M_{\phi,k}) \subseteq \{0\} \cup \{an(n-1)\cdots(n-k+1) : n \geq k\}$. Also, the structure of $M_{\phi,k}$ provides that for each $n \geq k$, $M_{\phi,k}(z^n) = (a\alpha_n)z^n$, so that for each $n \geq k$, $a\alpha_n = an(n-1)\cdots(n-k+1)$ is an eigen value of $M_{\phi,k}$. Thus $\{0\} \cup \{an(n-1)\cdots(n-k+1) : n \geq k\} \subseteq \Pi_0(M_{\phi,k})$ and we attain the desired.
(2) If $p > k$, we obtain that $f_n = 0$ for each $n \geq 0$ and thus $\Pi_0(M_{\phi,k}) = \{0\}$. □

We shall now describe the case for $M_{\phi,k}$ which is induced by $\phi(z) = az^p$ with $p < k$.

**Proposition 2.5.** *Let $\phi(z) = az^p$, where $0 \leq p < k$ and $0 \neq a \in \mathbb{C}$, induces the bounded $k^{th}$-order generalized multiplication operator $M_{\phi,k}$ on $H^2(\beta)$ and let $m = k - p$. For $\lambda \in \mathbb{C}$ and each $0 \leq l \leq m-1$, consider the series $\sum_{i=0}^{\infty} A_{l,\lambda,i}$, where $A_{l,\lambda,i} = |\frac{\lambda}{a}|^{2i+2}\Big(\frac{l!}{(l+k)!}\frac{(l+m)!}{(l+k+m)!}\cdots\frac{(l+mi)!}{(l+k+mi)!}\Big)^2\beta^2_{p+l+m(i+1)}$. Then $\{\lambda \in \mathbb{C} : \sum_{i=0}^{\infty} A_{l,\lambda,i}$ converges for some $0 \leq l \leq m-1\} \subseteq \Pi_0(M_{\phi,k})$.*

*Proof.* Each of the series $\sum_{i=0}^{\infty} A_{l,\lambda,i}$, $0 \leq l \leq m-1$ is convergent for $\lambda = 0$ and the structure of $M_{\phi,k}$ provides that 0 is an eigen value of this operator. So the set inclusion is evident. Let us assume $0 \neq \lambda \in \mathbb{C}$ be such that the series $\sum_{i=0}^{\infty} A_{l_0,\lambda,i}$ is convergent for some $0 \leq l_0 \leq m-1$. Now it is a matter of routine computations to obtain that $f(z) = \sum_{n=0}^{\infty} f_n z^n$, where the Fourier coefficients of $f$ are given as

$$f_n = \begin{cases} 0 & \text{if } n \neq p + l_0 + mi \text{ for any } i \geq 0 \\ \alpha & \text{if } n = p + l_0 \\ \alpha\Big(\frac{\lambda}{a}\Big)^i \frac{l_0!}{(l_0+k)!}\frac{(l_0+m)!}{(l_0+k+m)!}\cdots\frac{(l_0+m(i-1))!}{(l_0+k+m(i-1))!} & \text{if } n = p + l_0 + mi \text{ for some } i \in \mathbb{N}, \end{cases}$$

where $0 \neq \alpha \in \mathbb{C}$, satisfies that $M_{\phi,k}f = \lambda f$. Further, utilizing the hypothesis, we compute and obtain that

$$\begin{aligned}\|f\|_\beta^2 &= |f_{p+l_0}|^2\beta_{p+l_0}^2 + \sum_{i=1}^{\infty} |f_{p+l_0+mi}|^2\beta_{p+l_0+mi}^2 \\ &= |\alpha|^2\beta_{p+l_0}^2 + |\alpha|^2\sum_{i=0}^{\infty}|\frac{\lambda}{a}|^{2i+2}\Big(\frac{l_0!}{(l_0+k)!}\frac{(l_0+m)!}{(l_0+k+m)!}\cdots\frac{(l_0+mi))!}{(l_0+k+mi)!}\Big)^2\beta_{p+l_0+m(i+l)}^2 \\ &= |\alpha|^2\beta_{p+l_0}^2 + |\alpha|^2\sum_{i=0}^{\infty} A_{l_0,\lambda,i} \\ &< \infty.\end{aligned}$$

We are, therefore, assured of the existence of $0 \neq f \in H^2(\beta)$ satisfying $M_{\phi,k}f = \lambda f$. That is, $f$ is an eigen vector corresponding to the eigen value $\lambda$ of $M_{\phi,k}$. This completes the proof. □

We shall now focus our attention towards the Hilbert-Schmidt behaviour of the $k^{th}$-order generalized composition and multiplication operators on $H^2(\beta)$. Recall that an operator $T$ on a separable Hilbert space $H$ is said to be Hilbert-Schmidt if $\sum \|Te_m\| < \infty$ for some orthonormal basis $\{e_m\}$ of $H$ (see [3]).
Utilizing that the set $\{e_n(z) = z^n/\beta_n : n \geq 0\}$ forms an orthonormal basis of $H^2(\beta)$, we compute and obtain that

$$\sum_{n=0}^{\infty}\|C_{\phi,k}e_n(z)\|_\beta^2 = \sum_{n=k}^{\infty}\|\frac{\alpha_n}{\beta_n}\phi^{n-k}\|_\beta^2$$

and

$$\sum_{n=0}^{\infty}\|M_{\phi,k}e_n(z)\|_\beta^2 = \sum_{n=k}^{\infty}\|\frac{\alpha_n}{\beta_n}z^{n-k}\phi\|_\beta^2.$$

With these observations, we arrive at our next result.

**Proposition 2.6.** *For $C_{\phi,k}$ and $M_{\phi,k}$ in $\mathfrak{B}(H^2(\beta))$, induced by $\phi \in H^2(\beta)$, we have the following:*

1. *$C_{\phi,k}$ is a Hilbert-Schmidt operator if and only if $\sum_{n=0}^{\infty}(\frac{\alpha_{n+k}}{\beta_{n+k}})^2\|\phi^n\|_\beta^2 < \infty$.*
2. *$M_{\phi,k}$ is a Hilbert-Schmidt operator if and only if $\sum_{n=0}^{\infty}(\frac{\alpha_{n+k}}{\beta_{n+k}})^2\|z^n\phi\|_\beta^2 < \infty$.*

**Example 2.7.** *For specific choices of the inducing symbols and weight sequences, we obtain certain interesting examples.*

1. *Consider $\phi(z) = z \in H^2(\beta)$ with $\beta$ defined as $\beta_n = n!$ for each $n \geq 0$. Then $C_{\phi,k} \in \mathfrak{B}(H^2(\beta))$ and $\sum_{n=0}^{\infty}\|C_{\phi,k}e_n(z)\|_\beta^2 = \sum_{n=0}^{\infty}(\frac{\beta_n}{\beta_{n+k}})^2(\frac{(n+k)!}{n!})^2 = \sum_{n=0}^{\infty} 1$, thus yielding that $C_{\phi,k}$ cannot be Hilbert-Schmidt.*
2. *Let $\phi(z) = z \in H^2(\beta)$ with $\beta$ defined as $\beta_n = (n!)^{3/2}$ for each $n \geq 0$. Then $C_{\phi,k} \in \mathfrak{B}(H^2(\beta))$ and*

$$\sum_{n=0}^{\infty}\|C_{\phi,k}e_n(z)\|_\beta^2 = \sum_{n=0}^{\infty}\frac{n!}{(n+k)!} = \sum_{n=0}^{\infty}\frac{1}{(n+k)\cdots(n+1)}.$$

   *Therefore, the first order generalized composition operator $C_{\phi,1}$ cannot be Hilbert-Schmidt, while if $k > 1$, the operator $C_{\phi,k}$ turns out to be Hilbert-Schmidt.*
3. *Consider the weighted sequence space $H^2(\beta)$ with weight sequence $\beta$ defined as*

$$\beta_n = \begin{cases} 1 & \text{if } n=0 \\ \alpha & \text{if } n = 1 \\ (n-1)^3\beta_{n-1} & \text{if } n \geq 2, \end{cases}$$

*where $\alpha$ is a positive real number. Utilizing the boundedness criterion for generalized multiplication operators (discussed in [5], which we shall also obtain in the next section as Corollary 3.7), we obtain that the operator $M_{\phi,2}$ induced by the symbol $\phi(z) = az$, $0 \neq a \in \mathbb{C}$ is bounded on $H^2(\beta)$. Also,*

$$\sum_{n=0}^{\infty} \|M_{\phi,2}e_n(z)\|_\beta^2 = \sum_{n=0}^{\infty} \frac{(n+2)^2(n+1)^2|a|^2}{(n+1)^6},$$

*which is a convergent series and hence this operator is a Hilbert-Schmidt $2^{nd}$-order generalized multiplication operator on $H^2(\beta)$.*

4. *In a more general setting, we obtain that $M_{\phi,k}$ on $H^2(\beta)$ induced by $\phi(z) = az^p$, $p \neq k$, where the weight sequence $\beta$ is given as $\beta_0 = 1$; $\beta_1, \beta_2, \cdots, \beta_t$ are any positive real numbers, where $t = max\{p, k\} - 1$ and*

$$\frac{\beta_{n+p}}{\beta_{n+k}} = \frac{1}{(n+1)^{k+1}}, \text{ for each } n \geq 0,$$

*is a Hilbert-Schmidt operator on this underlying sequence space $H^2(\beta)$.*

## 3. Weighted generalized composition operators

In this section, we introduce and discuss the class of $k^{th}$-order weighted generalized composition operators defined on the weighted Hardy spaces $H^2(\beta)$. Let us begin by formally defining these operators.

**Definition 3.1.** *Let $k \geq 1$ be a fixed natural number and $\phi, \psi \in H^2(\beta)$ be such that the mapping*

$$f \mapsto \psi.(f^{(k)} \circ \phi)$$

*for each $f \in H^2(\beta)$ is a well-defined, linear and bounded mapping on $H^2(\beta)$. Then, this mapping is called a weighted generalized composition operator of $k^{th}$-order on $H^2(\beta)$ and is denoted by $W_{\psi,\phi,k}$.*

If the symbols $\psi$ and $\phi$, respectively, induce the multiplication operator $M_\psi$ and the $k^{th}$-order generalized composition operator $C_{\phi,k}$ on $H^2(\beta)$, then we may write $W_{\psi,\phi,k} = M_\psi C_{\phi,k}$. However, it is worth pointing out here that the operator $W_{\psi,\phi,k}$ can be a bounded operator without $M_\psi$ and $C_{\phi,k}$ being bounded. For instance, consider $H^2(\beta)$ with $\beta$ defined as $\beta_n = 1$ for each $n \geq 0$ and let $\psi(z) = 0$ and $\phi(z) = z$. In this setting, $M_\psi$ a bounded operator, while the mapping $f \mapsto f^{(k)} \circ \phi$ defines an unbounded operator on $H^2(\beta)$ (Theorem 2.7, [4]), even though $W_{\psi,\phi,k}$ ($f \mapsto \psi.(f^{(k)} \circ \phi)$) is a bounded operator on $H^2(\beta)$.
Also, we observe that if the symbol $\psi(z) = 1$, then $W_{\psi,\phi,k}$ coincides with $C_{\phi,k}$, while if $\phi(z) = z$, the operator $W_{\psi,\phi,k}$ is the same as the $k^{th}$-order generalized multiplication operator $M_{\psi,k}$ on $H^2(\beta)$.
Clearly then, every $k^{th}$-order generalized composition operator and $k^{th}$-order generalized multiplication operator are examples of $W_{\psi,\phi,k}$ and we refer to [4, 5] for various illustrations of these operators. Let us begin the study with a non-trivial example of a $k^{th}$-order weighted generalized composition operator on $H^2(\beta)$.

**Example 3.2.** *Consider the space $H^2(\beta)$, where $\beta$ is an increasing sequence of positive reals with $\beta_0 = 1$. Let $\psi(z) = az^m$ and $\phi(z) = bz$, where $a$ and $b$ are non-zero complex numbers such that $|b| < 1$. We claim that for $m \leq k$, the mapping $f \mapsto \psi.(f^{(k)} \circ \phi)$ on $H^2(\beta)$ defines a bounded operator $W_{\psi,\phi,k}$. For, we compute and obtain that for $f(z) = \sum_{n=0}^{\infty} f_n z^n \in H^2(\beta)$,*

$$\begin{aligned}
\|\psi.(f^{(k)} \circ \phi)\|_\beta^2 &\leq \sum_{n=0}^{\infty} |a|^2|b|^{2n}\alpha_{n+k}^2|f_{n+k}|^2\beta_{n+k}^2 \\
&= \sum_{n=0}^{\infty} A_n^2|f_{n+k}|^2\beta_{n+k}^2 \\
&\leq M^2\|f\|_\beta^2,
\end{aligned}$$

*where the sequence* $\{A_n\}_{n\geq 0}$*, given as* $A_n = \alpha_{n+k}|a||b|^n$ *for each* $n \geq 0$*, converges to* $0$ *and therefore* $A_n \leq M$ *for each* $n \geq 0$*, for some* $M > 0$*. This establishes the existence of a bounded weighted generalized composition operator* $W_{\psi,\phi,k}$*.*

**Example 3.3.** *Working along parallel lines, one can establish that on the sequence space* $H^2(\beta)$ *with decreasing weight sequence* $\beta$*, the operator* $W_{\psi,\phi,k}$ *induced by* $\psi(z) = az^m$ *with* $m > k$ *and* $\phi(z) = bz$*, where* $0 \neq a, b \in \mathbb{C}$ *with* $|b| < 1$*, is a bounded operator on* $H^2(\beta)$*.*

**Example 3.4.** *There also exist bounded operators* $W_{\psi,\phi,k}$ *on* $H^2(\beta)$*, where the weight sequence* $\beta$ *is neither an increasing nor a decreasing sequence. For instance, consider the weighted Hardy space* $H^2(\beta)$ *with the weight sequence* $\beta$ *defined as*

$$\beta_n = \begin{cases} 1 & \text{if } n = 0, 1 \\ \frac{n(n-1)}{2^{n-2}}\beta_{n-1} & \text{if } n \geq 2. \end{cases}$$

*Then with straightforward computations, we obtain that the symbols* $\psi(z) = z$ *and* $\phi(z) = z/2$ *induce a bounded* $2^{nd}$*-order weighted composition operator on sequence space* $H^2(\beta)$*.*

Since the existence of bounded operators $W_{\psi,\phi,k}$ on $H^2(\beta)$ is ensured, it is natural to look for and determine the symbols in $H^2(\beta)$ which induce the bounded mapping $f \mapsto \psi.(f^{(k)} \circ \phi)$ for each $f \in H^2(\beta)$. The following theorem provides a necessary and sufficient condition for the boundedness of this mapping. It is worth recalling here that the product of two formal power series $f$ and $g$ in $H^2(\beta)$ is defined as $(f.g)(z) = \sum_{n=0}^{\infty} h_n z^n$, where $h_n = \sum_{k=0}^{n} f_{n-k} g_n$, $f(z) = \sum_{n=0}^{\infty} f_n z^n$ and $g(z) = \sum_{n=0}^{\infty} g_n z^n$. If there is no confusion, we use the symbol $fg$ to denote the product $f.g$ of two formal power series $f$ and $g$ in $H^2(\beta)$.

**Theorem 3.5.** *Let* $\psi$ *and* $\phi$ *be two formal power series in* $H^2(\beta)$ *such that* $\{\psi\phi^n : n \geq 0\}$ *is an orthogonal family in* $H^2(\beta)$*. Then the mapping* $f \mapsto \psi.(f^{(k)} \circ \phi)$ *on* $H^2(\beta)$ *defines a bounded operator if and only if there exists* $M > 0$ *such that* $\alpha_{n+k}\|\psi\phi^n\|_\beta \leq M\beta_{n+k}$ *for each* $n \geq 0$*.*

*Proof.* If the mapping is bounded, we are assured of the existence of $M > 0$ such that $\|W_{\psi,\phi,k} f\|_\beta \leq M\|f\|_\beta$ for each $f \in H^2(\beta)$. In particular, for $f(z) = z^n$, where $n \geq k$, the above inequality yields the necessary condition. For sufficiency, consider any $f(z) = \sum_{n=0}^{\infty} f_n z^n \in H^2(\beta)$. Since the family $\{\psi\phi^n : n \geq 0\}$ is orthogonal in $H^2(\beta)$, we get that

$$\begin{aligned} \|\psi.(f^{(k)} \circ \phi)\|_\beta^2 = \|\sum_{n=k}^{\infty} \alpha_n f_n \psi\phi^{n-k}\|_\beta^2 &= \sum_{n=k}^{\infty} \alpha_n^2 |f_n|^2 \|\psi\phi^{n-k}\|_\beta^2 \\ &\leq M^2 \sum_{n=k}^{\infty} |f_n|^2 \beta_n^2 \leq M^2 \|f\|_\beta^2, \end{aligned}$$

thereby providing that the induced operator $W_{\psi,\phi,k}$ is bounded on $H^2(\beta)$. □

As immediate consequences of this theorem, we obtain boundedness criterion for the $k^{th}$-order generalized composition operators (by substituting $\psi(z) = 1$) and the $k^{th}$-order generalized multiplication operators (by substituting $\phi(z) = z$) on $H^2(\beta)$, which have also independently been obtained in [4, 5].

**Corollary 3.6.** *[4] Let* $\{\phi^n : n \geq 0\}$ *be an orthogonal family in* $H^2(\beta)$*. Then the* $k^{th}$*-order generalized composition operator* $C_{\phi,k}$ *on* $H^2(\beta)$ *is bounded if and only if there exists some* $M > 0$ *such that* $\alpha_{n+k}\|\phi^n\|_\beta \leq M\beta_{n+k}$ *for each* $n \geq 0$*.*

**Corollary 3.7.** *[5] The* $k^{th}$*-order generalized multiplication operator* $M_{\phi,k}$ *on* $H^2(\beta)$*, induced by the symbol* $\phi(z) = az^m$*, where* $0 \neq a \in \mathbb{C}$ *and* $m \geq 0$ *is an integer, is bounded if and only if there exists some* $M > 0$ *such that* $|a|\beta_{n+m}\alpha_{n+k} \leq M\beta_{n+k}$ *for each* $n \geq 0$*.*

In our pursuit to describe the structural properties of $W_{\psi,\phi,k}$, we begin with the study of compactness of these operators.

**Theorem 3.8.** *Let $\psi, \phi \in H^2(\beta)$ be such that $W_{\psi,\phi,k} \in \mathfrak{B}(H^2(\beta))$ and $\{\psi\phi^n : n \geq 0\}$ is an orthogonal family in $H^2(\beta)$. Then a necessary and sufficient condition for $W_{\psi,\phi,k}$ to be compact is that $\frac{\alpha_n}{\beta_n}\|\psi\phi^{n-k}\|_\beta \to 0$ as $n \to \infty$.*

*Proof.* For the necessary part, assume that $W_{\psi,\phi,k}$ is a compact operator on $H^2(\beta)$. Since the sequence $\{e_n\}$ converges weakly to zero and a compact operator maps a weakly convergent sequence to a strongly convergent sequence, we obtain for each $n \geq k$,

$$\begin{aligned}\|W_{\psi,\phi,k}e_n\|_\beta^2 = \|\frac{\alpha_n}{\beta_n}\psi.\phi^{n-k}\|_\beta^2 &= (\frac{\alpha_n}{\beta_n})^2\langle\psi\phi^{n-k}, \psi\phi^{n-k}\rangle \\ &= (\frac{\alpha_n}{\beta_n})^2\|\psi\phi^{n-k}\|_\beta^2 \to 0\end{aligned}$$

as $n \to \infty$. Hence, $\frac{\alpha_n}{\beta_n}\|\psi\phi^{n-k}\|_\beta \to 0$ as $n \to \infty$.
For the sufficient part, we define a sequence $\{W_m\}_{m\geq 0}$ of compact operators on $H^2(\beta)$ defined as

$$W_m f(z) = \sum_{n=0}^{m} f_n(W_{\psi,\phi,m}z^n),$$

where $f(z) = \sum_{n=0}^{\infty} f_n z^n \in H^2(\beta)$. Our hypothesis provides us for every $\epsilon > 0$, a $n_0 \in \mathbb{N}$ ($n_0 \geq k$) such that $\frac{\alpha_n}{\beta_n}\|\psi\phi^{n-k}\|_\beta < \epsilon$ for all $n \geq n_0$. Then for each $m \geq n_0$,

$$\|W_{\psi,\phi,k}f - W_m f\|_\beta^2 = \sum_{n=m+1}^{\infty} \alpha_n^2 |f_n|^2 \|\psi\phi^{n-k}\|_\beta^2 < \epsilon^2\|f\|_\beta^2.$$

Being the uniform limit of a sequence of compact operators, $W_{\psi,\phi,k}$ is compact and the proof is complete. □

Next, we focus our attention towards the Hilbert-Schmidt behaviour of $k^{th}$-order weighted generalized composition operators. It is trivial to obtain the following for the operator $W_{\psi,\phi,k}$ to be Hilbert-Schmidt.

**Proposition 3.9.** *A necessary and sufficient condition for the bounded operator $W_{\psi,\phi,k}$ on $H^2(\beta)$ to be Hilbert-Schmidt is that $\sum_{n=0}^{\infty}(\frac{\alpha_{n+k}}{\beta_{n+k}})^2\|\psi\phi^n\|_\beta^2 < \infty$.*

**Example 3.10.** *Some illustrations of compact and Hilbert-Schmidt weighted generalized composition operators are described here:*

1. *Consider the bounded operator $W_{\psi,\phi,k}$ on $H^2(\beta)$, where $\beta_n = 2^n$ for each $n \geq 0$, $\psi(z) = z$ and $\phi(z) = z/2$. We obtain that*
$$\frac{\alpha_n}{\beta_n}\|\psi\phi^{n-k}\|_\beta = \frac{(n+k)!}{n!}\frac{\beta_{n-k+1}}{2^n 2^{n-k}} = \frac{(n+k)(n+k-1)\cdots(n+1)}{2^{n-1}}$$
*which converges to 0 as $n \to \infty$. Thus $W_{\psi,\phi,k}$ is a compact operator (utilizing Theorem 3.8).*
2. *Consider the weighted Hardy space $H^2(\beta)$ with the weight sequence $\beta_n = n!$. Let $\psi(z) = az^p$ and $\phi(z) = bz$ belong to $H^2(\beta)$, where $p \geq 0$ is an integer and $0 \neq a, b \in \mathbb{C}$ such that $|b| < 1$. In light of Theorem 3.5, we obtain that the induced operator $W_{\psi,\phi,k}$ is bounded on $H^2(\beta)$. Further,*
$$\sum_{n=0}^{\infty}(\frac{\alpha_{n+k}}{\beta_{n+k}})^2\|\psi\phi^n\|_\beta^2 = |a|^2\sum_{n=0}^{\infty}|b|^{2n}\frac{(n+p)!^2}{n!^2},$$
*which is a convergent series, thereby providing that this operator is a Hilbert-Schmidt operator.*
3. *Let $\psi(z) = az^p$ ($p \geq 0$) and $\phi(z) = z$ be two elements of $H^2(\beta)$, where the sequence $\beta$ is defined as*
$$\beta_n = \begin{cases} 1 & \text{if } n=0 \\ \beta_p & \text{if } n < k \\ (n-k+1)^{k+1}\beta_{n-k+p} & \text{if } n \geq k \end{cases}$$

*and $0 \neq a \in \mathbb{C}$ and $k \in \mathbb{N}$ is such that $k \geq p$. Since $\frac{\alpha_{n+k}}{\beta_{n+k}}\|\psi\phi^n\|_\beta = |a|\frac{(n+k)\cdots(n+1)}{(n+1)^{k+1}} \to 0$ as $n \to \infty$, Theorem 3.5 provides that the operator $W_{\psi,\phi,k}$ is bounded on $H^2(\beta)$. Also, one can see that the series*

$$\sum_{n=0}^{\infty}(\frac{\alpha_{n+k}}{\beta_{n+k}})^2\|\psi\phi^n\|_\beta^2 = |a|^2\sum_{n=0}^{\infty}\Big(\frac{(n+k)\cdots(n+1)}{(n+1)^{k+1}}\Big)^2$$

*is convergent and hence the operator is Hilbert-Schmidt.*

We now proceed ahead to describe the normality and isometric behaviour of the weighted generalized composition operators. Now onwards, we assume the sequence $\beta$ to be such that the symbols $\psi(z) = az^m$ and $\phi(z) = bz^p$ ($m, p \geq 0$ are integers and $0 \neq a, b \in \mathbb{C}$) induce the bounded operator $W_{\psi,\phi,k}$ on $H^2(\beta)$. In our pursuance, firstly we compute the adjoint of this operator. We obtain that for each $m_0 \geq 0$ and for each $f(z) = \sum\limits_{n=0}^{\infty} f_n z^n \in H^2(\beta)$,

$$\begin{aligned}
\langle W^*_{\psi,\phi,k}z^{m_0}, f(z)\rangle &= \langle z^{m_0}, \sum_{n=0}^{\infty}\alpha_{n+k}f_{n+k}ab^nz^{np+m}\rangle \\
&= \begin{cases}\beta^2_{m_0}\alpha_{t+k}\overline{f}_{t+k}\overline{ab}^t & \text{if } m_0 = tp+m \text{ for some } t \geq 0\\ 0 & \text{otherwise.}\end{cases}\\
&= \begin{cases}\langle\frac{\beta^2_{m_0}}{\beta^2_{t+k}}\alpha_{t+k}\overline{ab}^t z^{t+k}, f(z)\rangle & \text{if } m_0 = tp+m \text{ for some } t \geq 0\\ 0 & \text{otherwise.}\end{cases}
\end{aligned}$$

Hence for $\psi(z) = az^m$ and $\phi(z) = bz^p$, the adjoint of $W_{\psi,\phi,k}$ is given as

$$W^*_{\psi,\phi,k}z^n = \begin{cases}\frac{\beta^2_n}{\beta^2_{t+k}}\alpha_{t+k}\overline{ab}^t\ z^{t+k} & \text{if } n = tp+m \text{ for some } t \geq 0\\ 0 & \text{otherwise.}\end{cases} \tag{1}$$

We shall now describe the isometric behaviour of $W_{\psi,\phi,k}$. The structure of this operator provides that $W_{\psi,\phi,k}e_n = 0$ for each $0 \leq n \leq k-1$. This leads us to the following.

**Proposition 3.11.** *A $k^{th}$-order weighted generalized composition operator on $H^2(\beta)$ cannot be an isometry.*

Recall that a bounded operator $T$ is said to be a co-isometry if the adjoint $T^*$ of $T$ is an isometry. We shall now describe the co-isometric nature of the operator $W_{\psi,\phi,k}$, where $\psi(z) = az^m$, $\phi(z) = bz^p$, $m$ and $p$ are non-negative integers and $0 \neq a, b \in \mathbb{C}$.

**Proposition 3.12.** *Let $m, p \geq 0$ be integers and $a, b$ be two non-zero complex numbers. The bounded operator $W_{\psi,\phi,k}$, induced by the symbols $\psi(z) = az^m$ and $\phi(z) = bz^p$ in $H^2(\beta)$ is a co-isometry on $H^2(\beta)$ if and only if $m = 0$, $p = 1$ and $\beta_n|a||b|^n\alpha_{n+k} = \beta_{n+k}$ for each $n \geq 0$.*

*Proof.* We have the following cases which collectively prove the result:
(1) Let $m = 0 = p$. In this case, $\|W^*_{\psi,\phi,k}e_n\|_\beta = 0 \neq 1 = \|e_n\|_\beta$ for every $n \neq 0$ and therefore, the operator is not a co-isometry.
(2) Let $m = 0$ and $p = 1$. In this case, we obtain that $W_{\psi,\phi,k}$ is a co-isometry if and only if $W_{\psi,\phi,k}W^*_{\psi,\phi,k}e_n = e_n$ for each $n \geq 0$ if and only if $\frac{\beta_n}{\beta_{n+k}}|a||b|^n\alpha_{n+k} = 1$ for each $n \geq 0$.
(3) Let $m = 0$ and $p \geq 2$. Then $\|W^*_{\psi,\phi,k}e_n\|_\beta = 0$ for all $n \neq pt$, where $t \geq 0$ is an integer and hence the operator is not co-isometric.
(4) Let $m \geq 1$. Then $W_{\psi,\phi,k}$ is not a co-isometry, for in this case $\|W^*_{\psi,\phi,k}e_0\|_\beta = 0$. □

We proceed ahead and describe the conditions under which the operators $W_{\psi,\phi,k}$ and $W^*_{\psi,\phi,k}$, induced by specific symbols in $H^2(\beta)$, become partial isometries. Recall that an operator $T$ is said to be a partial isometry if it is an isometry on the orthogonal complement of $Ker(T)$ (see [11]). An equivalent characterization for an operator $T$ to be a partial isometry is that $TT^*T = T$.

We obtain the following information for the operator $W_{\psi,\phi,k}$ and its adjoint, induced by the symbols $\psi(z) = az^m$ and $\phi(z) = bz^p$, where $m, p \geq 0$ are integers and $a, b$ are non-zero complex numbers.

$$W_{\psi,\phi,k}W^*_{\psi,\phi,k}W_{\psi,\phi,k}e_n = \begin{cases} \frac{\beta^3_{p(n-k)+m}}{\beta^3_n}\alpha^3_n|a|^2|b^2|^{n-k}ab^{n-k}e_{p(n-k)+m} & \text{if } n \geq k \\ 0 & \text{otherwise} \end{cases}$$

and

$$W^*_{\psi,\phi,k}W_{\psi,\phi,k}W^*_{\psi,\phi,k}e_n = \begin{cases} \frac{\beta^4_n}{\beta^3_{t+k}}\alpha^3_{t+k}|a|^2|b^2|^t\overline{a}\overline{b}^t e_{t+k} & \text{if } n = tp+m \text{ for some } t \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

These observations pave way for the next result.

**Proposition 3.13.** *Suppose $W_{\psi,\phi,k} \in \mathfrak{B}(H^2(\beta))$, where $\psi(z) = az^m$ and $\phi(z) = bz^p$, where $m, p \geq 0$ are integers and $0 \neq a, b \in \mathbb{C}$. We have the following:*

1. *$W_{\psi,\phi,k}$ is a partial isometry if and only if $\beta_n = \beta_{p(n-k)+m}\alpha_n|a||b|^{n-k}$ for each $n \geq k$.*
2. *$W^*_{\psi,\phi,k}$ is a partial isometry if and only if $\beta_{t+k} = \beta_{tp+m}\alpha_{t+k}|a||b|^t$ for each $t \geq 0$.*

**Example 3.14.** *We provide certain examples of co-isometric and partial isometric weighted generalized composition operators here below:*

1. *On the weighted sequence space $H^2(\beta)$, with $\beta$ defined as $\beta_n = n!$ for each $n \geq 0$, the symbols $\psi(z) = a$ and $\phi(z) = bz \in H^2(\beta)$, where $a, b$ are unimodular complex numbers, induce a co-isometric $W_{\psi,\phi,k}$.*
2. *Consider the operator $W_{\psi,\phi,k}$ as defined in Example 3.4. It is trivial to obtain that for each $n \geq 2$,*

$$\beta_{p(n-k)+m}\alpha_n|a||b|^{n-k} = \frac{n(n-1)}{2^{(n-2)}}\beta_{n-1} = \beta_n,$$

   *thereby providing that this operator is a partial isometry.*

3. *Consider the weighted sequence space $H^2(\beta)$ where $\beta_n = 1$ for all $n \notin \{3\} \cup \{4m+1 : m \geq 1\}$, $\beta_3 = 12$ and for each $n \geq 1$,*

$$\beta_{4n+1} = \frac{3^n}{2(n+1)(n+2)(n+3)}\beta_{n+3}.$$

   *Then the mapping $f \mapsto \psi.(f^{(3)} \circ \phi)$ on $H^2(\beta)$ induced by the symbols $\psi(z) = 2z$ and $\phi(z) = z^4/3$ is a bounded mapping such that for each $t \geq 0$, $\beta_{4t+1}\alpha_{t+3}|a||b|^t = 2\frac{(t+3)!}{3^t 3!}\beta_{4t+1} = \beta_{t+3}$. This yields that $W^*_{\psi,\phi,3}$ is a partial isometry.*

We pursue ahead and discuss the normality of the operator $W_{\psi,\phi,k}$. We denote by $\psi_n$ the $n^{th}$-Fourier coefficient of $\psi(z) = \sum_{n=0}^{\infty}\psi_n z^n \in H^2(\beta)$. We obtain the following.

**Proposition 3.15.** *A $k^{th}$-order weighted generalized composition operator $W_{\psi,\phi,k}$ induced by non-zero symbols $\psi$ and $\phi$ in $H^2(\beta)$, such that $\psi_n \neq 0$ for some $0 \leq n < k$, can not be hyponormal.*

*Proof.* On the contrary, assume that the operator $W_{\psi,\phi,k}$ is hyponormal. Let $\psi(z) = \sum_{n=0}^{\infty}\psi_n z^n \in H^2(\beta)$ be such that $\psi_j \neq 0$, where $0 \leq j \leq k-1$. The structure of this operator provides that $\|W_{\psi,\phi,k}e_j\|_\beta^2 = 0$ and

$$\begin{aligned}
\|W^*_{\psi,\phi,k}e_j\|_\beta^2 &= \sum_{n=0}^{\infty} |\langle e_j, W_{\psi,\phi,k}e_n\rangle|^2 \\
&= \frac{\alpha_k^2}{\beta_k^2}|\langle e_j, \psi\rangle|^2 + \sum_{n=k+1}^{\infty} \frac{\alpha_n^2}{\beta_n^2}|\langle e_j, \psi\phi^{n-k}\rangle|^2 \\
&= k!^2|\psi_j|^2\frac{\beta_j^2}{\beta_k^2} + \sum_{n=k+1}^{\infty} \frac{\alpha_n^2}{\beta_n^2}|\langle e_j, \psi\phi^{n-k}\rangle|^2.
\end{aligned}$$

Since the operator is hyponormal, $\|W^*_{\psi,\phi,k}e_j\|_\beta^2 \leq \|W_{\psi,\phi,k}e_j\|_\beta^2$, thereby providing that $\psi_j = 0$, which is a contradiction. This completes the proof. □

The above proposition also provides that $W_{\psi,\phi,k}$ is not a self-adjoint operator, for every self-adjoint operator is hyponormal.
Recall that an operator $T$ is said to be Fredholm [8] if dimension of $Ker(T)$ is finite, codimension of $R(T)$ is finite and $R(T)$ is closed. Towards the end, we shall describe the Fredholm behaviour of the $k^{th}$-order weighted composition operator $W_{\psi,\phi,k}$. As a pre-requisite, we shall require the next result.

**Lemma 3.16.** *Let $\psi, \phi \in H^2(\beta)$ be such that $W_{\psi,\phi,k} \in \mathfrak{B}(H^2(\beta))$ and $\{\psi\phi^n : n \geq 0\}$ is an orthogonal family in $H^2(\beta)$. Then, $Ker(W_{\psi,\phi,k})$ is the linear span of the finite set $\{e_0, e_1, e_2, ..., e_{k-1}\}$.*

*Proof.* The structure of $W_{\psi,\phi,k}$ together with its linearity immediately provides that the set spanned by $\{e_0, e_1, e_2, ..., e_{k-1}\}$ is a subset of $Ker(W_{\psi,\phi,k})$. Conversely, for any $f(z) = \sum_{n=0}^{\infty} f_n z^n \in Ker(W_{\psi,\phi,k})$, the orthogonality of the family $\{\psi\phi^n : n \geq 0\}$ provides that $0 = \|W_{\psi,\phi,k}\, f\|_\beta^2 = \sum_{n=k}^{\infty} \alpha_n^2|f_n|^2\|\psi\phi^{n-k}\|_\beta^2$, which yields that $f_n = 0$ for each $n \geq k$. Hence the result. □

The following proposition provides a necessary condition for $W_{\psi,\phi,k}$ to be a Fredholm operator.

**Proposition 3.17.** *Let $W_{\psi,\phi,k} \in \mathfrak{B}(H^2(\beta))$ be a Fredholm operator induced by the symbols $\psi$ and $\phi$ such that $\{\psi\phi^n : n \geq 0\}$ is an orthogonal family in $H^2(\beta)$. Then there exists $\epsilon > 0$ such that $\frac{\alpha_{n+k}}{\beta_{n+k}}\|\psi\phi^n\|_\beta \geq \epsilon$ for each $n \geq 0$.*

*Proof.* Since $W_{\psi,\phi,k}$ is Fredholm, $R(W_{\psi,\phi,k})$ is closed and hence $W_{\psi,\phi,k}$ is bounded away from zero on $Ker(W_{\psi,\phi,k})^\perp$. Thus there exists $\epsilon > 0$ such that $\|W_{\psi,\phi,k}e_n\|_\beta \geq \epsilon\|e_n\|_\beta$ for each $n \geq k$. That is, $\frac{\alpha_n}{\beta_n}\|\psi\phi^{n-k}\|_\beta \geq \epsilon$ for each $n \geq k$. This completes the proof. □

As a consequence to the above proposition, we obtain that if $\psi$ and $\phi$ in $H^2(\beta)$ are such that $W_{\psi,\phi,k} \in \mathfrak{B}(H^2(\beta))$, $\{\psi\phi^n : n \geq 0\}$ is an orthogonal family in $H^2(\beta)$ and $\frac{\alpha_{n+k}}{\beta_{n+k}}\|\psi\phi^n\|_\beta \to 0$ as $n \to \infty$, then $W_{\psi,\phi,k}$ can not be Fredholm. For instance, the operator $W_{\psi,\phi,k}$ as defined in Example 3.2 or the one given in Example 3.10 (1) cannot be Fredholm.
Under a stronger hypothesis, we obtain a necessary and sufficient condition for $W_{\psi,\phi,k}$ on $H^2(\beta)$ to be a Fredholm operator.

**Theorem 3.18.** *Suppose $\psi, \phi \in H^2(\beta)$ be such that $W_{\psi,\phi,k} \in \mathfrak{B}(H^2(\beta))$ and $\{\psi\phi^n : n \geq 0\}$ is an orthogonal family in $H^2(\beta)$ spanning $H^2(\beta)$. Then $W_{\psi,\phi,k}$ is Fredholm if and only if $W_{\psi,\phi,k}$ has closed range.*

*Proof.* Firstly, we claim that for the orthogonal family $\{\psi\phi^n : n \geq 0\}$ spanning $H^2(\beta)$, $Ker(W^*_{\psi,\phi,k})$ is finite dimensional. Since the family $\{\psi\phi^n/\|\psi\phi^n\|_\beta : n \geq 0\}$ forms an orthonormal basis of $H^2(\beta)$, every $f \in Ker(W^*_{\psi,\phi,k})$ can be expressed uniquely as $\sum_{m=0}^{\infty} f_m\psi\phi^m$, where $f_m \in \mathbb{C}$. Also, for each $n \geq k$,

$$0 = \langle W^*_{\psi,\phi,k}f, e_n\rangle = \langle \sum_{m=0}^{\infty} f_m\psi\phi^m, \frac{\alpha_n}{\beta_n}\psi\phi^{n-k}\rangle = f_{n-k}\frac{\alpha_n}{\beta_n}.$$

Thus $f_{n-k} = 0$ for each $n \geq k$ so that $f = 0$ and therefore $Ker(W^*_{\psi,\phi,k}) = \{0\}$. This information together with Lemma 3.16 yields the sufficient part of the theorem. Necessary part follows from the definition of a Fredholm operator. □

Utilizing Proposition 3.17 and Theorem 3.18, we obtain the following equivalent conditions for $W_{\psi,\phi,k}$ to be Fredholm.

**Theorem 3.19.** *Let $\psi, \phi \in H^2(\beta)$ be such that $W_{\psi,\phi,k} \in \mathfrak{B}(H^2(\beta))$ and $\{\psi\phi^n : n \geq 0\}$ is an orthogonal family in $H^2(\beta)$ spanning $H^2(\beta)$. The following are equivalent:*

1. *$W_{\psi,\phi,k}$ is a Fredholm operator.*
2. *$R(W_{\psi,\phi,k})$ is closed.*
3. *There exists $\epsilon > 0$ such that $\frac{\alpha_{n+k}}{\beta_{n+k}}||\psi\phi^n||_\beta \geq \epsilon$ for each $n \geq 0$.*

*Proof.* It suffices to prove that condition (2) is implied by (3). In fact, we prove that if (3) holds, then $W_{\psi,\phi,k}$ is bounded away from zero on $Ker(W_{\psi,\phi,k})^\perp$ and thus $R(W_{\psi,\phi,k})$ is closed. In this pursuance, let $f \in (Ker(W_{\psi,\phi,k}))^\perp$. Then $f(z) = \sum_{n=k}^\infty f_n z^n$ and therefore

$$||W_{\psi,\phi,k}f||^2_\beta = \sum_{n=k}^\infty \alpha_n^2 |f_n|^2 ||\psi\phi^n||^2_\beta \geq \epsilon^2 ||f||^2_\beta,$$

for some $\epsilon > 0$. Hence the claim. □

As an illustration, consider the operator $W_{\psi,\phi,k}$ on $H^2(\beta)$ where $\beta_n = n!$ for each $n \geq 0$, induced by the symbols $\psi(z) = a \neq 0$ and $\phi(z) = z$. Clearly, $\{\psi\phi^n : n \geq 0\}$ is an orthogonal family in $H^2(\beta)$ which spans $H^2(\beta)$ and $\frac{\alpha_{n+k}}{\beta_{n+k}}||\psi\phi^n||_\beta = |a|$. Choose $\epsilon = |a| > 0$. Then, Theorem 3.19 provides that $W_{\psi,\phi,k}$ is a Fredholm operator on $H^2(\beta)$.

## References


[1] S.C. Arora, G. Datt and S. Verma, Multiplication operators on Lorentz spaces, Indian J. Maths. 48(3), (2006), 317–329.
[2] M.D. Contreras and A.G. Hernàndez-Dìaz, Weighted composition operators on Hardy spaces, J. Math. Anal. Appl 263, (2001), 224–233.
[3] C. Cowen and B. MacCluer, Composition Operators on Spaces of Analytic Functions, CRC Press, Bota Racon, 1995.
[4] G. Datt and M. Jain, Generalized composition operators of higher order on weighted Hardy spaces, Chamchuri J. Math. 9, (2017), 52–60.
[5] G. Datt and M. Jain, Generalized multiplication operators of higher order on weighted Hardy spaces, Asian European Journal of Mathematics, 13(5), (2020), 2050102. doi.org/10.1142/S1793557120501028.
[6] F. Forelli, The isometries on $H^p$, Canad. J. Math. 16, (1964), 721–728.
[7] G. Gunatillake, Invertible weighted composition operators, J. Funct. Anal. 261, (2011), 831–860.
[8] P.R. Halmos, A Hilbert Space Problem Book, Van Nostrand, Princeton, 1967.
[9] R.L. Kelley, Weighted shifts on Hilbert space, Dissertation, University of Michigan, Ann Arbor, Mich., 1966.
[10] B.S. Komal and S. Gupta, Multiplication operators between Orlicz spaces, Dissertationes Math. (Rozprawy Mat.) 41(3), (2001), 324–330.
[11] W. Rudin, Functional Analysis, Second Edition, McGraw Hill Education (India) Edition, 2006.
[12] S.K. Sharma and B.S. Komal, Generalized multiplication operators on weighted Hardy Spaces, Lobachevskii J. Math. 32(4), (2011), 298–303.
[13] A.L. Shields, Weighted shift operators and analytic function theory, Topics in Operator Theory, Math. Surveys, No.13, American Mathematical Society, Rhode Ireland (1974) 49–128.