



Otopy Classification of Gradient Compact Perturbations of Identity in Hilbert Space

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Abstract. We prove that the inclusion of the space of gradient local maps into the space of all local maps from Hilbert space to itself induces a bijection between the sets of the respective otopy classes of these maps, where by a local map we mean a compact perturbation of identity with a compact preimage of zero.

1. Introduction

In 1985 E. N. Dancer ([10]) discovered that there is a better topological invariant than the equivariant degree for gradient maps in the case of S^1 group action, which means that in that case there are more equivariant gradient homotopy classes than equivariant homotopy ones. A few years later A. Parusiński ([13]) showed that for a disc without group action there is no better invariant for gradient maps than the usual topological degree. In other words, there is a bijection between sets of gradient and continuous homotopy classes. In 2005 E. N. Dancer, K. Gęba, S. Rybicki ([11]) provided the homotopy classification of equivariant gradient maps on the disc in the case of a compact Lie group action. In their proof the authors used the notion of otopy introduced in the 1990's by J. C. Becker and D. H. Gottlieb ([8, 9]). Later, in [2, 12] the equivariant and equivariant gradient otopy classifications instead of homotopy ones were studied.

The investigations mentioned above suggest the following general approach. Let us consider vector fields on open domains contained in some Riemannian manifold X (in some cases equipped with an action of a compact Lie group). If the set of zeros of such a vector field is compact, we call them *local maps*. We introduce the following notation. Let

- $C(X)$ ($\mathcal{G}(X)$) be the set of continuous (gradient) local maps,
- $C[X]$ ($\mathcal{G}[X]$) be the set of usual (gradient) otopy classes of continuous (gradient) local maps,
- $\iota: \mathcal{G}[X] \rightarrow C[X]$ be the function between the respective otopy classes induced by the inclusion $\mathcal{G}(X) \hookrightarrow C(X)$.

We will say that the inclusion $\mathcal{G}(X) \hookrightarrow C(X)$ has the *Parusiński property* if $\iota: \mathcal{G}[X] \rightarrow C[X]$ is bijective. In a series of papers [3–6] we proved that the respective inclusions have the Parusiński property if X is an open

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subset of \mathbb{R}^n or, more generally, a Riemannian manifold (not necessarily compact) without boundary. On the other hand, in [7] we showed that for an open invariant subset of a finite-dimensional representation of a compact Lie group the inclusion ι does not have the Parusiński property in general. Moreover, we gave in that case necessary and sufficient conditions for the Parusiński property in terms of Weyl group dimensions, which explains the phenomenon discovered by E. N. Dancer in 1985.

The presented paper is a natural continuation of our previous work. Namely, the main aim of this article is to prove that the inclusion of the space of gradient local maps into the space of all local maps has the Parusiński property if X is an open subset of a real separable Hilbert space. By *local map* we mean here a compact perturbation of identity with a compact preimage of zero. It is worth pointing out that in the proof of our main theorem we use a topological invariant, which is a version of the classical Leray-Schauder degree. But in our construction we manage to guarantee that finite-dimensional approximations of gradient maps are gradient, which is crucial for the proof of Theorem A. The results presented here may also be treated as an introduction to the study of Parusiński property for a representation of a compact Lie group G in a Hilbert space.

The organization of the paper is as follows. Section 2 contains some preliminaries. In Section 3 we describe a construction of the topological degree used in the proof of Theorem A. Our main results are stated in Section 4 and proved in Section 5. Final remarks are contained in Section 6. Finally, Appendix A presents technical facts used in Section 5.

2. Basic definitions

Assume that

- E is an infinite-dimensional real separable Hilbert space,
- Ω is an open connected subset of E .

Recall that a continuous map from a metric space A into a metric space B is called *compact* if it takes bounded subsets of A into relatively compact ones of B . Some authors use the term *completely continuous* instead of compact.

2.1. Local maps in Hilbert space

We write $f \in C(\Omega)$ if

- $f: D_f \subset \Omega \rightarrow E$,
- D_f is an open subset of Ω ,
- $f(x) = x - F(x)$, where $F: D_f \rightarrow E$ is compact,
- $f^{-1}(0)$ is compact.

Elements of $C(\Omega)$ are called *local maps*.

It is easy to check that the compactness of F implies that in the above definition the last condition that $f^{-1}(0)$ is compact can be equivalently replaced by the assumption that $f^{-1}(0)$ is bounded and closed in E . From this observation follows that if f is defined on $\text{cl } U$, where U is open and bounded, and f does not vanish on the boundary then $f \upharpoonright_U$ is a local map.

Moreover, we write $f \in \mathcal{G}(\Omega)$ if

- $f \in C(\Omega)$,
- f is gradient i.e. there is a C^1 -function $\varphi: D_f \rightarrow \mathbb{R}$ such that $f = \nabla\varphi$.

Elements of $\mathcal{G}(\Omega)$ are called *gradient local maps*.

2.2. Sets of otopy classes in Hilbert space

A map $h: \Lambda \subset I \times \Omega \rightarrow E$ is called an *otopy* if

- Λ is an open subset of $I \times \Omega$,
- $h(t, x) = x - F(t, x)$, where $F: \Lambda \rightarrow E$ is compact,
- $h^{-1}(0)$ is compact.

Similarly as in the case of local maps, the assumption that the set $h^{-1}(0)$ is bounded and closed in $I \times E$ implies its compactness. In particular, if $\Lambda \subset I \times \Omega$ is open and bounded, h is defined on $\text{cl } \Lambda$ (not only on Λ) and does not vanish on $\partial \Lambda$ then $h \upharpoonright_{\Lambda}$ is an otopy.

From the above and an easy to check fact that a straight-line homotopy between two compact maps is compact we obtain the following result.

Lemma 2.1 *Assume that $U \subset E$ is open and bounded and $h: I \times \text{cl } U \rightarrow E$ is a straight-line homotopy. If $h(t, x) \neq 0$ for $t \in I$ and $x \in \partial U$ and $h_0 \upharpoonright_U$ and $h_1 \upharpoonright_U$ are local maps, then $h \upharpoonright_{I \times U}$ is an otopy.*

An otopy is called *gradient*, if additionally

$$F(t, x) = \nabla_x \eta(t, x),$$

where $\eta: \Lambda \rightarrow \mathbb{R}$ is C^1 with respect to x .

Given an otopy $h: \Lambda \subset I \times \Omega \rightarrow E$ we can define for each $t \in I$:

- sets $\Lambda_t = \{x \in \Omega \mid (t, x) \in \Lambda\}$,
- maps $h_t: \Lambda_t \rightarrow E$ with $h_t(x) = h(t, x)$.

If h is a (gradient) otopy, we can say that h_0 and h_1 are (gradient) otopic. Observe that (gradient) otopy establishes an equivalence relation in $C(\Omega)$ ($\mathcal{G}(\Omega)$). Sets of otopy classes of the respective relation will be denoted by $C[\Omega]$ and $\mathcal{G}[\Omega]$.

Observe that if f is a (gradient) local map and U is an open subset of D_f such that $f^{-1}(0) \subset U$, then f and $f \upharpoonright_U$ are (gradient) otopic. This property of (gradient) local maps will be called *restriction property*. In particular, if $f^{-1}(0) = \emptyset$ then f is (gradient) otopic to the empty map.

Remark 2.2. It is worth pointing out that in [3, 5] we consider local maps and otopies in finite dimensional spaces. Unlike as in the case of Hilbert space we assume in the definition of both a local map and an otopy only the condition that the preimage of zero is compact. There is no need to assume the form $\text{Id} - F$ with F compact. However, subsequently in the proof of the main result of this paper we will need the form *identity minus compact* in a finite dimensional case. This will be guaranteed by boundedness of a domain of a map.

3. Definition of degree Deg

In this section we give a definition of the degree $\text{Deg}: C(\Omega) \rightarrow \mathbb{Z}$ and prove its correctness and otopy invariance.

3.1. Preparatory lemmas

Let us start with the following lemma concerning $f \in C(\Omega)$.

Lemma 3.1 *Assume that $X \subset D_f$ is closed in E and bounded. If $X \cap f^{-1}(0) = \emptyset$ then there is $\epsilon > 0$ such that $|f(x)| \geq 2\epsilon$ for all $x \in X$.*

Proof. Suppose that there is a sequence $\{x_n\} \subset X$ such that $\lim f(x_n) = 0$. By compactness of F there is a subsequence $\{x_{k_n}\}$ of $\{x_n\}$ such that $\lim F(x_{k_n}) = y$ and therefore $\lim x_{k_n} = y$. Since X is closed, we have $y \in X$ and, in consequence, $f(y) \neq 0$. But $f(y) = \lim f(x_{k_n}) = 0$, a contradiction. \square

Observe that there is an open bounded set U such that

$$f^{-1}(0) \subset U \subset \text{cl } U \subset D_f.$$

Corollary 3.2 *There is $\epsilon > 0$ such that $|f(x)| \geq 2\epsilon$ for all $x \in \partial U$.*

Let $\{e_i \mid i \in \mathbb{N}\}$ be an orthonormal basis in E . Let us introduce the following notation for $n \in \mathbb{N}$:

- $V_n = \text{span}\{e_1, \dots, e_n\}$,
- $U_n = U \cap V_n$ for any $U \subset E$,
- $f_n(x) = x - P_n F(x)$, where $P_n: E \rightarrow V_n$ is an orthogonal projection.

Throughout the paper we will make use of the following well-known characterization of relatively compact sets in Hilbert space.

Proposition 3.3 *A set $X \subset E$ is relatively compact iff it is bounded and*

$$\forall \delta > 0 \exists n_0 \forall n \geq n_0 \forall x \in X |x - P_n x| < \delta.$$

Now we are in position to show that for n large enough f and f_n are close to each other on $\text{cl } U$. Next from this observation we conclude that f_n are uniformly separated from 0 on ∂U .

Lemma 3.4 *There is N such that for all $n \geq N$ and all $x \in \text{cl } U$ we have:*

$$|f(x) - f_n(x)| < \epsilon \quad \text{and consequently} \quad |f_{n+1}(x) - f_n(x)| < \epsilon.$$

Proof. Since $\text{cl } U$ is bounded, $F(\text{cl } U)$ is relatively compact. By Proposition 3.3 there is N such that for all $n \geq N$ and all $x \in \partial U$ we have $|F(x) - P_n F(x)| < \epsilon$. Since $|f(x) - f_n(x)| = |F(x) - P_n F(x)|$, we obtain our assertion. \square

From now on let N be chosen as in the previous lemma.

Lemma 3.5 $|f_n(x)| \geq \epsilon$ for $x \in \partial U$ and $n \geq N$.

Proof. It is an easy consequence of Corollary 3.2 and Lemma 3.4. \square

3.2. Definition of Deg

In what follows, deg denotes the classical Brouwer degree. The infinite-dimensional degree that we are going to define in this paper will be denoted by Deg .

Since $\partial U_n \subset \partial U$ for any n , the next result follows from Lemma 3.5.

Corollary 3.6 $\text{deg}(f_n, U_n)$ is well-defined for $n \geq N$.

The following fact shows that the sequence $\{\text{deg}(f_n, U_n)\}_{n \geq N}$ is constant.

Lemma 3.7 $\text{deg}(f_{n+1}, U_{n+1}) = \text{deg}(f_n, U_n)$ for $n \geq N$.

Proof. Since $f_n^{-1}(0) \subset U_n$, there is an open subset $W \subset V_n$ such that

$$f_n^{-1}(0) \subset W \subset \text{cl } W \subset U_n$$

and there is $\delta > 0$ such that $W_\delta := W \times (-\delta, \delta) \subset U_{n+1}$. Let $g_n: W_\delta \rightarrow E$ be given by $g_n(x) = x - P_n F(P_n x)$ (in other words g_n is a suspension of $f_n \upharpoonright_W$). By definition, $g_n \upharpoonright_W = f_n \upharpoonright_W$ and $g_n^{-1}(0) = f_n^{-1}(0)$. Let us check the following sequence of equalities

$$\text{deg}(f_{n+1}, U_{n+1}) \stackrel{(1)}{=} \text{deg}(f_n, U_{n+1}) \stackrel{(2)}{=} \text{deg}(f_n, W_\delta) \stackrel{(3)}{=} \text{deg}(g_n, W_\delta) \stackrel{(4)}{=} \text{deg}(f_n, W) \stackrel{(5)}{=} \text{deg}(f_n, U_n).$$

The equalities (1) and (3) can be obtained using straight-line homotopies, which are otopies by Lemmas 3.4 and 3.5. In turn (2) and (5) are based on the restriction property of the degree and, finally, (4) follows from the fact that g_n is a suspension of f_n over W . This completes the proof. \square

Lemma 3.7 guarantees that the following definition does not depend on the choice of admissible N .

Definition 3.8. Define $\text{Deg } f = \text{Deg}(f, U) = \text{deg}(f_N, U_N)$.

Remark 3.9. Our degree gives the same values as the classical Leray-Schauder degree, which follows easily from the comparison of our construction and the definition and the proof of the well-definedness of the Leray-Schauder degree. However, in the proof of bijectivity of our degree in the gradient case we use the fact that finite-dimensional approximations of a gradient map appearing in our construction are gradient, which is not guaranteed by the original Leray-Schauder construction.

3.3. Correctness

Let us note that for the above construction we have chosen a neighbourhood U of $f^{-1}(0)$ and an orthonormal basis of E . Now we are going to prove that our definition of $\text{Deg } f$ does not depend on the choice of these both elements.

Proposition 3.10 *Let W and U be open bounded such that*

$$f^{-1}(0) \subset W \subset U \subset \text{cl } U \subset D_f.$$

Then $\text{Deg}(f, W) = \text{Deg}(f, U)$.

Proof. By Lemma 3.4, we have $|f(x) - f_n(x)| < \epsilon$ for $x \in \text{cl } U$ and, consequently, $f_n(x) \neq 0$ for $x \in \text{cl } U_n \setminus W_n \subset \text{cl } U \setminus W$ for sufficiently large n . Hence

$$\text{Deg}(f, W) = \text{deg}(f_n, W_n) = \text{deg}(f_n, U_n) = \text{Deg}(f, U). \quad \square$$

Corollary 3.11 *Let U and U' be open bounded subsets of D_f such that*

$$f^{-1}(0) \subset U \cap U' \subset \text{cl}(U \cap U') \subset \text{cl}(U \cup U') \subset D_f.$$

Then $\text{Deg}(f, U) = \text{Deg}(f, U \cap U') = \text{Deg}(f, U')$.

In this way we have proved that $\text{Deg } f$ does not depend on the choice of U .

In the remainder of this subsection we show that $\text{deg } f$ does not depend on the choice of an orthonormal basis in E . The reasoning requires some additional notation. Let V be a finite dimensional linear subspace of E . Set

- $U_V = U \cap V$,
- $P_V: E \rightarrow V$ — an orthogonal projection,
- $f_V(x) = x - P_V f(x)$.

Analogously to Corollary 3.6 and Lemma 3.7 one can prove the following result.

Lemma 3.12 *If V is a finite dimensional linear subspace of E such that $V_N \subset V$ then $\text{deg}(f_V, U_V)$ is well defined and $\text{deg}(f_N, U_N) = \text{deg}(f_V, U_V)$.*

Corollary 3.13 *$\text{Deg } f$ does not depend on the choice of an orthonormal basis in E .*

Proof. Let $\{e_i\}$ and $\{e'_i\}$ be two orthonormal bases in E . We will use analogous notation for them both writing prime where needed. For example, $V_n = \text{span}\{e_1, \dots, e_n\}$ and $V'_n = \text{span}\{e'_1, \dots, e'_n\}$. Let us choose N and N' for $\{e_i\}$ and $\{e'_i\}$ respectively as in Lemma 3.4. Put $V = V_N + V'_{N'}$. By Lemma 3.12,

$$\text{deg}(f_N, U_N) = \text{deg}(f_V, U_V) = \text{deg}(f'_{N'}, U'_{N'}),$$

which is our assertion. \square

3.4. Otopy invariance of degree

Let the map $h: \Lambda \subset I \times \Omega \rightarrow E$ given by $h(t, x) = x - F(t, x)$ be an otopy. We introduce the following notation:

$$\begin{aligned} \Lambda^t &= \{x \in \Omega \mid (t, x) \in \Lambda\}, & h^t: \Lambda^t &\rightarrow E, & h^t(x) &= h(t, x), \\ \Lambda_n &= \Lambda \cap (I \times V_n), & h_n: \Lambda &\rightarrow V_n, & h_n(t, x) &= x - P_n F(t, x), \\ \Lambda_n^t &= \Lambda^t \cap V_n, & h_n^t: \Lambda^t &\rightarrow V_n, & h_n^t(x) &= h_n(t, x). \end{aligned}$$

Note that for the needs of this subsection the time parameter t of otopy is a superscript, not a subscript.

Proposition 3.14 (otopy invariance) *If $h: \Lambda \subset I \times \Omega \rightarrow E$ is an otopy then $\text{Deg}(h^0, \Lambda^0) = \text{Deg}(h^1, \Lambda^1)$.*

Proof. Since $h^{-1}(0)$ is compact, there is an open bounded set $W \subset I \times E$ such that

$$h^{-1}(0) \subset W \subset \text{cl } W \subset \Lambda. \tag{3.1}$$

Hence for $i = 0, 1$ we have

$$(h^i)^{-1}(0) \subset W^i \subset \text{cl } W^i \subset \Lambda^i, \tag{3.2}$$

where $W^i = \{(i, x) \in W\}$. Analogously, as in Lemma 3.1, from (3.1) there is $\epsilon > 0$ such that $|h(z)| \geq 2\epsilon$ for $z \in \partial W$ and, as in Lemma 3.4, there is N such that $|h(z) - h_n(z)| < \epsilon$ for $n \geq N$ and $z \in \partial W$. Hence $|h_n(z)| \geq \epsilon$ for $z \in \partial W_n \subset \partial W$ and, in consequence,

- $\text{Deg}(h^i, \Lambda^i) = \text{deg}(h_n^i, W_n^i)$ for $i = 0, 1$,
- $h_n \upharpoonright_{W_n}$ is a finite-dimensional otopy,

which gives

$$\text{Deg}(h^0, \Lambda^0) = \text{deg}(h_n^0, W_n^0) = \text{deg}(h_n^1, W_n^1) = \text{Deg}(h^1, \Lambda^1). \quad \square$$

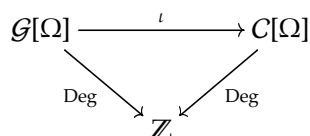
Remark 3.15. Since our degree is otopy invariant, it can be defined on the set of otopy class, i.e. $\text{Deg}: C[\Omega] \rightarrow \mathbb{Z}$. Moreover, any gradient otopy class (as a set of functions) is contained in a usual otopy class, and hence the degree makes sense as a function $\text{Deg}: \mathcal{G}[\Omega] \rightarrow \mathbb{Z}$. Without ambiguity we will use the symbol Deg in all the above cases.

4. Main results

Let us formulate the main results of our paper.

Theorem A *The functions $\text{Deg}: C[\Omega] \rightarrow \mathbb{Z}$ and $\text{Deg}: \mathcal{G}[\Omega] \rightarrow \mathbb{Z}$ are bijections.*

It is obvious that the inclusion $\mathcal{G}(\Omega) \hookrightarrow C(\Omega)$ induces a well-defined function $\iota: \mathcal{G}[\Omega] \rightarrow C[\Omega]$. The next result follows immediately from Theorem A and the commutativity of the diagram



Theorem B *The function $\iota: \mathcal{G}[\Omega] \rightarrow C[\Omega]$ is bijective.*

Remark 4.1. In other words, there is no better invariant than the Leray-Schauder degree that distinguishes between two gradient local maps which are not gradient otopic.

5. Proof of Theorem A

5.1. Injectivity of Deg: $C[\Omega] \rightarrow \mathbb{Z}$

Let $f: D_f \rightarrow E$ and $g: D_g \rightarrow E$ be local maps such that $\text{Deg } f = \text{Deg } g$. We show that f and g are otopic. The proof of that will be divided into two steps. In the first step we show that f is otopic to the suspension of its finite dimensional approximation and in the second step that suspensions of approximations for f and g are otopic one to another. We start with the observation that there exists an open bounded $U \subset E$ such that

- $f^{-1}(0) \subset U \subset \text{cl } U \subset D_f$,
- there is N such that $P_n(\text{cl } U) \subset D_f$ for all $n \geq N$.

The proof of this observation will be postponed to Appendix A (see Lemma A.1).

Step 1. For $n \geq N$ let $\Sigma f_n: \text{cl } U \cup P_n^{-1}(U_n) \rightarrow E$ be given by $\Sigma f_n(x) = x - P_n F(P_n x)$. Note that $\Sigma f_n \upharpoonright_{P_n^{-1}(U_n)}$ is a suspension of $f_n \upharpoonright_{U_n}$ (see Section 3). We prove the following sequence of otopy relations for n large enough:

$$f \stackrel{(1)}{\simeq} f \upharpoonright_U \stackrel{(2)}{\simeq} \Sigma f_n \upharpoonright_U \stackrel{(3)}{\simeq} \Sigma f_n \upharpoonright_{U \cup (P_n^{-1}(U_n) \cap \Omega)} \stackrel{(4)}{\simeq} \Sigma f_n \upharpoonright_{P_n^{-1}(U_n) \cap \Omega}. \tag{5.1}$$

The sets appearing in (5.1) are shown in Figure 1. First observe that all the maps in the above sequence are local, because $(\Sigma f_n)^{-1}(0) \subseteq U_n$ from Lemma 3.5. The relations (1), (3) and (4) follow immediately from the restriction property. To obtain (2) let us consider the straight-line homotopy $h_n: I \times \text{cl } U \rightarrow E$ given by $h_n(t, x) = (1 - t)f(x) + t\Sigma f_n(x)$. We show that there is $M \geq N$ such that $h_n(t, x) \neq 0$ for $t \in I, x \in \partial U$ and $n \geq M$. Thus $h_n \upharpoonright_{I \times U}$ is an otopy, which proves the relation (2). On the contrary, suppose that there is an increasing subsequence $\{n_k\}$ of natural numbers ($n_k \geq N$) and sequences $\{t_k\} \subset I$ and $\{x_k\} \subset \partial U$ such that $h_{n_k}(t_k, x_k) = 0$, i.e.,

$$x_k = F(x_k) + t_k(P_{n_k} F(P_{n_k} x_k) - F(x_k)).$$

By compactness of F and I , we can assume that sequences $t_k, F(x_k)$ and $F(P_{n_k} x_k)$ are convergent, so $\{x_k\}$ is also convergent to some point $x_0 \in \partial U$. Since $x_k \rightarrow x_0$ implies $P_{n_k} x_k \rightarrow x_0$, we obtain $f(x_0) = x_0 - F(x_0) = 0$, which contradicts the fact that f does not vanish on ∂U .

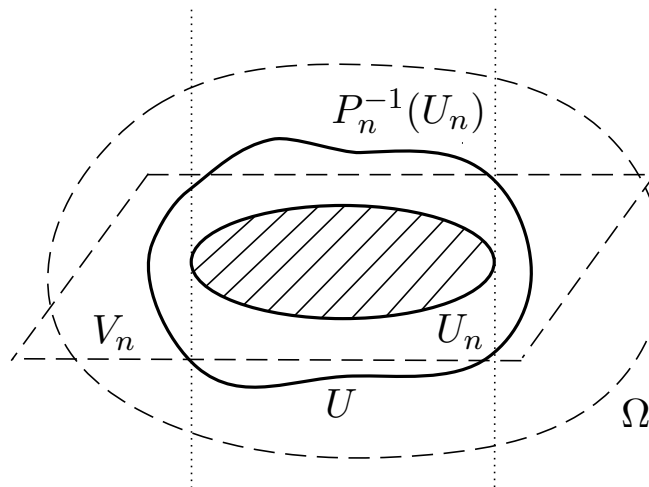


Figure 1: Domains in (5.1).

Step 2. The same reasoning can be applied to the map g . Similarly as for f let us introduce the notation Σg_n and a set $W \subset D_g$ (a counterpart of $U \subset D_f$). We obtain in this way an analogical sequence of relations, which gives $g \sim \Sigma g_n \upharpoonright_{P_n^{-1}(W_n) \cap \Omega}$. To finish the proof of injectivity it is enough to show that

$$\Sigma f_n \upharpoonright_{P_n^{-1}(U_n) \cap \Omega} \sim \Sigma g_n \upharpoonright_{P_n^{-1}(W_n) \cap \Omega}.$$

To do that we will use Lemma A.5, which shows how to suspend finite dimensional otopies.

By the definition of our degree, we have

$$\deg f_n \upharpoonright_{U_n} = \text{Deg } f = \text{Deg } g = \deg g_n \upharpoonright_{W_n}.$$

Unfortunately, this does not imply that $f_n \upharpoonright_{U_n}$ and $g_n \upharpoonright_{W_n}$ are finite dimensionally otopic, since U_n and W_n may not be contained in the same component of Ω_n . Therefore, using Lemma A.4, the problem will be lifted to a higher dimension, where the relation of otopy holds.

Precisely, note first that $f_n^{-1}(0) \subset U_n$, $g_n^{-1}(0) \subset W_n$ and $K = f_n^{-1}(0) \cup g_n^{-1}(0) \subset V_n$ is compact. By Lemma A.4, K is contained in one component of Ω_m for $m \geq n$ large enough. Let us denote this component by Ω'_m .

Set $X = P_n^{-1}(U_n) \cap U \cap \Omega'_m$ and $Y = P_n^{-1}(W_n) \cap W \cap \Omega'_m$. Observe that

- X and Y are open bounded,
- $\text{cl } X \subset \text{cl } U \subset D_f$ and $\text{cl } Y \subset \text{cl } W \subset D_g$,
- $X \cup Y \subset \Omega'_m \subset V_m$.

Since $\deg f_n \upharpoonright_{U_n} = \deg g_n \upharpoonright_{W_n}$, we have $\deg \Sigma f_n \upharpoonright_X = \deg \Sigma g_n \upharpoonright_Y$. Moreover, the maps $\Sigma f_n \upharpoonright_X$ and $\Sigma g_n \upharpoonright_Y$ are bounded, because Σf_n and Σg_n are defined on $\text{cl } X$ and $\text{cl } Y$ respectively. Since Ω'_m is connected there is a bounded finite dimensional otopy $k: \Gamma \subset I \times \Omega'_m \rightarrow V_m$ between $\Sigma f_n \upharpoonright_X$ and $\Sigma g_n \upharpoonright_Y$ (see [1, Rem. 2.3]). By Lemma A.5, there is an otopy in Ω between $\Sigma f_n \upharpoonright_{P_m^{-1}(X) \cap \Omega}$ and $\Sigma g_n \upharpoonright_{P_m^{-1}(Y) \cap \Omega}$. Finally, since $P_m^{-1}(X) \subset P_m^{-1}(P_n^{-1}(U_n)) = P_n^{-1}(U_n)$ and similarly $P_m^{-1}(Y) \subset P_n^{-1}(W_n)$, we obtain

$$\Sigma f_n \upharpoonright_{P_n^{-1}(U_n) \cap \Omega} \sim \Sigma f_n \upharpoonright_{P_m^{-1}(X) \cap \Omega} \sim \Sigma g_n \upharpoonright_{P_m^{-1}(Y) \cap \Omega} \sim \Sigma g_n \upharpoonright_{P_n^{-1}(W_n) \cap \Omega},$$

which completes the proof of injectivity of $\text{Deg}: C[\Omega] \rightarrow \mathbb{Z}$.

5.2. Injectivity of $\text{Deg}: \mathcal{G}[\Omega] \rightarrow \mathbb{Z}$

Let $f, g \in \mathcal{G}(\Omega)$ and $\text{Deg } f = \text{Deg } g$. To show that $[f] = [g]$ in $\mathcal{G}[\Omega]$ it is enough to observe that all otopies appearing in the sequence connecting f and g as in 5.1. are in fact gradient. Namely

1. otopies connecting gradient local maps with their restrictions are obviously gradient,
2. the straight-line homotopy $(1 - t)f + t\Sigma f_n$ is gradient, because f and Σf_n are gradient (note that if φ is a potential for F then $\varphi \circ P_n$ is a potential for $P_n F P_n$),
3. the otopy between $\Sigma f_n \upharpoonright_{P_m^{-1}(X) \cap \Omega}$ and $\Sigma g_n \upharpoonright_{P_m^{-1}(Y) \cap \Omega}$ appearing in Step 2 of 5.1 (see Lemma A.5) can be considered gradient, because by Main Theorem in [6, Sec. 2] $k(t, x)$ can be chosen gradient (if $\varphi(t, x)$ is a family of potentials for $k(t, x)$ then we can take $\varphi(t, x) + \frac{1}{2}|y|^2$ as a family of potentials for our otopy).

5.3. Surjectivity of $\text{Deg}: \mathcal{G}[\Omega] \rightarrow \mathbb{Z}$

Using standard local maps (see Section 3 in [3]) it is easy to construct for any $m \in \mathbb{Z}$ a gradient local map $f: D_f \subset V_n \cap \Omega \rightarrow V_n$ such that $\deg f = m$ ($V_n \cap \Omega$ is nonempty for n large enough). Since as we observed suspensions of gradient local maps are also gradient local, the map $\Sigma f: P_n^{-1}(D_f) \cap \Omega \rightarrow E$ is an element of $\mathcal{G}[\Omega]$ and $\text{Deg } \Sigma f = \deg f = m$.

5.4. Surjectivity of $\text{Deg}: C[\Omega] \rightarrow \mathbb{Z}$

Since any gradient local map is also a local map, it is an obvious consequence of 5.3. \square

6. Final remarks

This section is devoted to two possible directions of developments of subject presented here. Namely, we can additionally consider a group action and/or linear operators other than identity.

6.1. The case of a compact Lie group action

In [7] we proved that for a finite dimensional representation of a compact Lie group the function induced on the sets on otopy classes by the inclusion of the set of equivariant gradient local maps into the set of equivariant local maps is a bijection if and only if all Weyl groups appearing in the representation are finite. In consequence, contrary to our Theorem B the function ι need not be bijective. We expect that an analogical result holds for a Hilbert representation of a compact Lie group. Here we will just give an example of two equivariant gradient local maps in Hilbert space that are otopic but not gradient otopic, which illustrates that the function analogical to ι in Theorem B may not be bijective.

Example 6.1. Let E' be a Hilbert space and $E = \mathbb{C} \oplus E'$. Assume that S^1 acts on $\mathbb{C} \oplus E'$ by $g(z, x) = (gz, x)$. Consider for $i = 0, 1$ potentials $\varphi_i: \mathbb{C} \rightarrow \mathbb{R}$ given by

$$\varphi_i(z) = \begin{cases} (|z| - 1)^2 & \text{if } |z| \geq 1, \\ (1 - 2i)(|z| - 1)^2 & \text{if } |z| < 1. \end{cases}$$

Set $f_i = \nabla\varphi_i$ (see Figure 2) and $U = \{z \in \mathbb{C} \mid 1/2 < |z| < 3/2\}$. Let Id denote the identity on E' . Define $\tilde{f}_i: U \times E' \rightarrow E$ by $\tilde{f}_i = f_i \times \text{Id}$. It follows easily that \tilde{f}_0 and \tilde{f}_1 are equivariant otopic. We expect that it is possible to show that they are not equivariant gradient otopic.

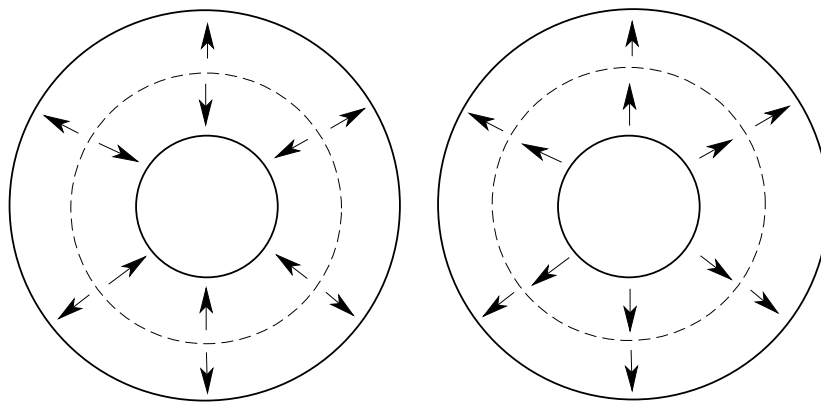


Figure 2: Maps f_0 and f_1 .

6.2. The case of an unbounded operator

In this paper we considered perturbations of the identity operator in Hilbert space. Possible applications in Hamiltonian systems and in the Seiberg-Witten theory suggest replacing the identity by an unbounded self-adjoint operator with a purely discrete spectrum. In that case in the absence of a group action we expect the result similar to Theorem B. However, if we take into account a group action similarly as in Subsection 6.1 we may obtain the function ι that is not bijective. This means that in the equivariant gradient case we may get an extra topological invariant.

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Appendix A

In this appendix we have collected some technical results needed in Section 5.

Lemma A.1 *Let Ω be an open subset of a separable Hilbert space E and K a compact subset of Ω . There exist an open bounded $U \subset E$ and natural number N such that*

- $K \subset U \subset \text{cl } U \subset \Omega$,
- $P_n(\text{cl } U) \subset \Omega$ for all $n \geq N$.

Proof. Let us denote by $B(x, R)$ the open ball and by $D(x, R)$ the closed ball in E of radius $R > 0$ centered at x . Note that for any $x \in \Omega$ there is $R_x > 0$ and $N_x \in \mathbb{N}$ such that $B(x, R_x) \subset \Omega$ and $|P_n x - x| < R_x/2$ for $n \geq N_x$. If $|y - x| \leq R_x/2$ then $|P_n y - x| \leq |P_n y - P_n x| + |P_n x - x| < |y - x| + R_x/2 \leq R_x$, and hence $P_n y \in B(x, R_x) \subset \Omega$. In other words $P_n(D(x, R_x/2)) \subset \Omega$ for $n \geq N_x$. Since K is compact, we can choose $x_1, \dots, x_m \in K$ such that $K \subset \bigcup_{i=1}^m B(x_i, R_{x_i}/2)$. Set $U = \bigcup_{i=1}^m B(x_i, R_{x_i}/2)$ and $N = \max\{N_{x_i} \mid i = 1, \dots, m\}$. It is easy to see that

$$P_n(\text{cl } U) = P_n\left(\bigcup_{i=1}^m D(x_i, R_{x_i}/2)\right) \subset \Omega$$

for $n \geq N$. \square

Corollary A.2 *With the same notation and assumptions as above, there is N such that $P_n(K) \subset \Omega$ for $n \geq N$.*

Remark A.3. The corollary is an immediate consequence of Lemma A.1, but it can also be easily concluded from the characterization of compact sets in E (Prop. 3.3).

Lemma A.4 *Let Ω be an open connected subset of E and $K \subset \Omega_n := \Omega \cap V_n$ be compact. Then K is contained in one component of Ω_m for m large enough.*

Proof. Since K is compact, it can be covered by a finite number of balls $B_i \subset \Omega_n$. Ω is connected, so there is a path $\omega_{ij} \subset \Omega$ from B_i to B_j for each pair i, j . By Corollary A.2, $P_{l_{ij}}(\omega_{ij}) \subset \Omega_{l_{ij}}$ for l_{ij} sufficiently large, so all balls B_i are contained in the same component of Ω_m , where $m := \max\{l_{ij} \mid i, j\}$. \square

Let $\Omega_n := \Omega \cap V_n$ and $\Gamma \subset I \times \Omega_n$ are open. Assume that $k: \Gamma \rightarrow V_n$ is a bounded finite dimensional otopy (see Remark 2.2). Let us define:

- $\Lambda = (\Gamma \times V_n^\perp) \cap (I \times \Omega)$
- $h: \Lambda \rightarrow E$ given by $h(t, x, y) = (k(t, x), y)$, where $t \in I, x \in V_n, y \in V_n^\perp$ (note that $(t, x, y) \in \Lambda$ implies $(t, x) \in \Gamma$).

Lemma A.5 *h is an otopy in Ω .*

Proof. Observe that

1. $h^{-1}(0) = k^{-1}(0) \times \{0\}$ is compact,
2. k is bounded and hence $\text{Id} \upharpoonright_{\Omega_n} - k$ is compact; in consequence $h(t, x, y) = (x, y) - (x - k(t, x), 0)$ is of the desired form ‘identity minus compact’.

Thus h is an otopy in Ω . \square

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