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Fixed Points of Relaxed (ψ, φ)-Weakly N-Contraction Mappings in Modular Spaces

Getahun B. Wega[a], Habtu Zegeye[a], Oganeditse A. Boikanyo[a]

[a]*Department of Mathematics and Statistical Sciences, Botswana International University of Science and Technology, Pvt Bag 16, Palapye, Botswana*

Abstract. The purpose of this paper is to study the existence and approximation of a common fixed point of a pair of mappings satisfying a relaxed (ψ, φ)-weakly N-contractive condition and existence and approximation of a fixed point of a relaxed (ψ, φ)-weakly N-contraction mapping in the setting of modular spaces. Our theorems improve and generalize the results in Mongkolkeha and Kumam [23] and Öztürk et. al [26]. To validate our results numerical examples are provided.

1. Introduction

Let X be an arbitrary vector space over the set of real numbers or complex numbers. A functional $\rho : X \to [0, \infty)$ is called a modular if for arbitrary $x, y \in X$,

i) $\rho(x) = 0$ if and only if $x = 0$;
ii) $\rho(\alpha x) = \rho(x)$ for every scalar α with $|\alpha| = 1$;
iii) $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ if and only if $\alpha + \beta = 1$ and $\alpha \geq 0, \beta \geq 0$.

If we replace (iii) with

(iii)′ $\rho(\alpha x + \beta y) \leq \alpha\rho(x) + \beta\rho(y)$ if and only if $\alpha + \beta = 1$ and $\alpha \geq 0, \beta \geq 0$, we say that ρ is a convex modular.

If ρ is a modular on X, then the set defined by

$$X_\rho = \{x \in X : \rho(\lambda x) \to 0 \text{ as } \lambda \to 0\},$$

is called a modular space. X_ρ is a vector subspace of X.

The concept of a modular space was introduced by Nakano [25] which was again redefined and generalized by Musielak and Orlize [24]. For a current review of the theory of Musielak-Orlicz spaces and modular spaces, the reader is referred to the books of Musielak and Orlicz [24] and Kozlowski [9]. The study of

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Email addresses: getahunbekele2012@gmail.com (Getahun B. Wega), habtuzh@yahoo.com (Habtu Zegeye), boikanyoa@gmail.com (Oganeditse A. Boikanyo)

fixed point theory in the context of this space was introduced by Khamsi et al. [14] and studied by several authors (see, eg, [1–3, 9, 13, 15–17, 19, 20]).

A fundamental result in the theory of fixed points is the classical Banach contraction principle which was established by Stefan Banach in 1922 [5]. It provides the existence and uniqueness of the fixed point of contraction mapping in complete metric spaces. A mapping $T : X \to X$, where (X, d) a metric space, is called a contraction if there exists $k \in [0, 1)$ such that $d(Tx, Ty) \leq kd(x, y)$, for all $x, y \in X$. We denote the set of fixed point of $T : C \subseteq X \to X$ by $F(T) = \{x \in C : x = Tx\}$.

The generalization of this principle has been obtained either by relaxing the property of the mapping or by generalizing the domain of the mapping (see, for example, [4, 6–8, 10–12, 21, 22]).

One of the generalizations of the Banach contraction principle is the fixed point theorem for weakly contraction mappings in Modular spaces. In 2012, Mongkolkeha and Kumam [23] introduced φ-weakly contraction mapping and proved the existence and uniqueness of a fixed point of φ-weakly contraction mappings in Modular spaces. A mapping $T : X_\rho \to X_\rho$, is said to be a φ-weakly contraction if

$$\rho(Tx - Ty) \leq M(x, y) - \varphi(M(x, y)),$$

where $\varphi : [0, \infty) \to [0, \infty)$ is a continuous and monotone nondecreasing function such that $\varphi(t) = 0$ if and only if $t = 0$ and $M(x, y) = \max\left\{\frac{\rho(\frac{1}{2}(y-Tx)+\rho(\frac{1}{2}(x-Ty))}{2}, \rho(Tx - x), \rho(Ty - y), \rho(x - y)\right\}$. In addition, they proved the existence of a unique fixed point of (ψ, φ)-weak contraction self mapping in Modular spaces. A mapping $T : X_\rho \to X_\rho$, is said to be a (ψ, φ)-weak contraction if

$$\psi(\rho(Tx - Ty)) \leq \psi(M(x, y)) - \varphi(M(x, y)),$$

where $M(x, y) = \max\left\{\frac{\rho(\frac{1}{2}(y-Tx)+\rho(\frac{1}{2}(x-Ty))}{2}, \rho(Tx - x), \rho(Ty - y), \rho(x - y)\right\}$ and $\psi, \varphi : [0, \infty) \to [0, \infty)$ are both continuous and monotone nondecreasing functions with $\psi(t) = 0 = \varphi(t)$ if and only if $t = 0$.

On the other hand, existence results of a common fixed point of mappings have been studied by different authors. For instance, in 2012, Mongkolkeha and Kumam [23] established the existence of a common fixed point of a pair mappings $S, T : X_\rho \to X_\rho$, satisfying the following condition for any $x, y \in X_\rho$

$$\rho(Sx - Ty) \leq M(x, y) - \varphi(M(x, y)),$$

where $\varphi : [0, \infty) \to [0, \infty)$ is a continuous and monotone nondecreasing function such that $\varphi(t) = 0$ if and only if $t = 0$ and $M(x, y) = \max\left\{\frac{\rho(\frac{1}{2}(y-Sx)+\rho(\frac{1}{2}(x-Ty))}{2}, \rho(Sx - x), \rho(Ty - y), \rho(x - y)\right\}$. Furthermore, they proved the existence of a common fixed point of a pair of mappings $S, T : X_\rho \to X_\rho$, satisfying (ψ, φ)-weakly contractive condition. A pair of mappings $S, T : X_\rho \to X_\rho$, is said to satisfy a (ψ, φ)-weakly contractive condition if for any $x, y \in X_\rho$, we have

$$\psi(\rho(Sx - Ty)) \leq \psi(M(x, y)) - \varphi(M(x, y)),$$

where $\psi, \varphi : [0, \infty) \to [0, \infty)$ are both continuous and monotone nondecreasing functions with $\psi(t) = 0 = \varphi(t)$ if and only if $t = 0$ and $M(x, y) = \max\left\{\frac{\rho(\frac{1}{2}(y-Sx)+\rho(\frac{1}{2}(x-Ty))}{2}, \rho(Sx - x), \rho(Ty - y), \rho(x - y)\right\}$.

Recently, Öztürk et. al [26] introduced and proved the existence of a common fixed point of a pair of mappings in modular spaces satisfying the generalized (ψ, φ)-weakly contractive condition provided that one of the mapping is ρ-continuous. A pair of mappings $S, T : X_\rho \to X_\rho$, is said to satisfy a generalized (ψ, φ)-weakly contractive condition if for any $x, y \in X_\rho$, we have

$$\psi(\rho(Sx - Ty)) \leq \psi(M(x, y)) - \varphi(M(x, y)) + L\psi(N(x, y)),$$

where $L \geq 0$, $\psi \in \Psi := \{\psi : [0, \infty) \to [0, \infty) : \psi$ is a continuous nondecreasing function and $\psi(t) = 0$ if and only if $t = 0\}$, $\varphi \in \Phi := \{\varphi : [0, \infty) \to [0, \infty) : \varphi$ is a lower-semi continuous function and

$\varphi(t) = 0$ if and only if $t = 0\}$, $M(x,y) = \max\left\{\frac{\rho(\frac{1}{2}(y-Sx))+\rho(\frac{1}{2}(x-Ty))}{2}, \rho(Sx-x), \rho(Ty-y), \rho(x-y)\right\}$ and $N(x,y) = \min\left\{\rho(y-Sx), \rho(x-Ty), \rho(Ty-y), \rho(x-Sx)\right\}$.

Motivated and inspired by the above results, it is our purpose in this paper to

i) introduce the concepts of a *relaxed* (ψ,φ)*-weakly N-contractive* condition and a *relaxed* (ψ,φ)*-weakly N-contraction* mapping;
ii) study the existence and approximation of a common fixed point of a pair of mappings satisfying a *relaxed* (ψ,φ)*-weakly N-contractive* condition and existence and approximation of fixed points of a *relaxed* (ψ,φ)*-weakly N-contraction* mappings in the setting of modular spaces.

Furthermore, we provide examples to show that our results apply to the class of mappings more general than the class of mappings considered by other authors, for instance, Mongkolkeha and Kumam [23] and Öztürk et. al [26]. Finally, a numerical examples are provided to validate our results.

2. Preliminaries

Now, we recall some basic notations and facts about modular spaces as formulated in [24].

Definition 2.1. *Let X_ρ be a modular space.*

a) A sequence $\{x_n\}$ in X_ρ is said to be
  i) ρ-convergent to $x \in X_\rho$ if $\rho(x_n - x) \to 0$ as $n \to \infty$.
  ii) ρ-Cauchy if $\rho(x_n - x_m) \to 0$ as $n, m \to \infty$.

b) A subset C of X_ρ is said to be ρ-closed if the ρ-limit of a ρ-convergent sequence of C always belongs to C.

c) A subset C of X_ρ is said to be ρ-complete if any ρ-Cauchy sequence in C is a ρ-convergent sequence and its limit is in C.

Observe that ρ-convergence does not imply ρ-Cauchy, since ρ does not satisfy triangular inequality. But one can easily show that this will happen if and only if ρ satisfy the $\triangle_2$-condition.

Definition 2.2. *A modular ρ is said to satisfy the $\triangle_2$-condition if $\rho(2x_n) \to 0$ as $n \to \infty$, whenever $\rho(x_n) \to 0$ as $n \to \infty$.*

Definition 2.3. *Let $T : C \to C$ be a map, where C is a subset of a modular space X_ρ. We say that T is ρ-continuous if $\rho(x_n - x) \to 0$ implies $\rho(Tx_n - Tx) \to 0$, as $n \to \infty$.*

Definition 2.4. *The modular function ρ is uniformly continuous if for every $\epsilon > 0$ and $L > 0$, there exists $\delta > 0$ such that*

$$|\rho(g) - \rho(h+g)| < \epsilon; \text{ if } \rho(h) < \delta \text{ and } \rho(g) \le L.$$

3. Main result

3.1. Existence and approximation of a common fixed point of mappings satisfying relaxed (ψ,φ)-weakly N-contractive condition in modular spaces

We set $\Psi := \{\psi : [0,\infty) \to [0,\infty) : \psi$ is a continuous nondecreasing function and $\psi(t) = 0$ if and only if $t = 0\}$ and $\Phi := \{\varphi : [0,\infty) \to [0,\infty) : \varphi$ is a lower-semi continuous function and $\varphi(t) = 0$ if and only if $t = 0\}$.

First, we introduce the following definition.

Definition 3.1. *Let C be a nonempty subset of a modular space X_ρ and $N \in \mathbb{N}$. Two mappings $T, S : C \to C$ are said to satisfy a relaxed (ψ, φ)-weakly N-contractive condition if for any $x, y \in C$ the following holds:*

$$\psi\Big(\rho(S^N x - T^N y)\Big) \le \psi\Big(M(x,y)\Big) - \varphi\Big(M(x,y)\Big) + L\psi\Big(N(x,y)\Big), \tag{1}$$

where $\psi \in \Psi, \varphi \in \Phi, L \ge 0$, $T^N = T(T^{N-1}), N = 1, 2, ...,$ with $T^0 = I$, the identity map,
$M(x,y) = \max\Big\{\frac{\rho(y-T^N y)(1+\rho(y-S^N x))}{1+\rho(S^N x - T^N y)}, \frac{\rho(y-T^N y)(1+\rho(S^N x-T^N y))}{1+\rho(y-T^N y)}, \frac{\rho(\frac{1}{2}(y-S^N x))+\rho(\frac{1}{2}(x-T^N y))}{2}, \rho(S^N x - x), \rho(T^N y - y), \rho(x-y)\Big\}$
and $N(x,y) = \min\Big\{\rho(y - S^N x), \rho(S^N x - x), \rho(T^N y - y), \rho(T^N y - x)\Big\}$.

Remark 3.2. *The following example shows that the class of pair of mappings satisfying a relaxed (ψ, φ)-weakly N-contractive condition contains mappings which are not in the class of mappings satisfying a generalized (ψ, φ)-weakly contractive condition.*

Example 3.3. *Let $X_\rho = \mathbb{R}$, the real number system $\mathbb{R}$, be the space modular with $\rho(x) = x^2$. Let $C = \{x \in X_\rho : 0 \le x \le 1\}$. Define $T, S : C \to C$ by*

$$Tx = \begin{cases} \frac{3}{4} \ for\ 0 \le x \le \frac{1}{2} \\ 1 \ for\ \frac{1}{2} < x \le 1 \end{cases} \quad and\ Sx = 1.$$

Clearly, ρ is a uniformly continuous and satisfies Δ_2-condition. Moreover, we have

$$\begin{aligned}
\rho(Sx - Ty) &= \frac{1}{16}\ for\ (x,y) \in [0, \frac{1}{2}] \times [0, \frac{1}{2}], \\
\rho(Sx - Ty) &= 0\ for\ (x,y) \in [0, \frac{1}{2}] \times (\frac{1}{2}, 1], \\
\rho(Sx - Ty) &= \frac{1}{16}\ for\ (x,y) \in (\frac{1}{2}, 1] \times [0, \frac{1}{2}], \\
\rho(Sx - Ty) &= 0\ for\ (x,y) \in (\frac{1}{2}, 1] \times (\frac{1}{2}, 1],
\end{aligned}$$

and hence $\rho(Sx - Ty) \le \frac{1}{16}$ for all $x, y \in C$. Now, define the functions $\psi, \varphi : [0, \infty) \to [0, \infty)$ by $\psi(t) = t$ and $\varphi(t) = \frac{t}{16}$ for all $t \ge 0$. Then, we show that

$$\psi\Big(\rho(Sx - Ty)\Big) \le \psi\Big(M(x,y)\Big) - \varphi\Big(M(x,y)\Big) + L\psi\Big(N(x,y)\Big),\ \ for\ \ all\ \ x, y \in C,$$

where $L \ge 0$, $M(x,y) = \max\Big\{\frac{\rho(y-Ty)(1+\rho(y-Sx))}{1+\rho(Sx-Ty)}, \frac{\rho(y-Ty)(1+\rho(Sx-Ty))}{1+\rho(y-Ty)}, \frac{\rho(\frac{1}{2}(y-Sx))+\rho(\frac{1}{2}(x-Ty))}{2}, \rho(Sx-x), \rho(Ty-y), \rho(x-y)\Big\}$
and $N(x,y) = \min\Big\{\rho(y - Sx), \rho(Sx - x), \rho(Ty - y), \rho(Ty - x)\Big\}$. *To show this we consider the following four cases.*
Case 1. *Let $(x,y) \in [0, \frac{1}{2}] \times [0, \frac{1}{2}]$. Then,*

$$\begin{aligned}
M(x,y) \ \ = \ \ \max\Big\{&\frac{16(y-\frac{3}{4})^2(1+(y-1)^2)}{17}, \frac{17(y-\frac{3}{4})^2}{16(1+(y-\frac{3}{4})^2)}, \\
&\frac{16(1-y)^2+(4x-3)^2}{128}, (1-x)^2, (\frac{3}{4}-y)^2, (x-y)^2\Big\}.
\end{aligned}$$

i) If $M(x,y) = (1-x)^2$, then

$$\rho(Sx - Ty) = \frac{1}{16} \le (1-x)^2 - \frac{1}{16}(1-x)^2 = \frac{15}{16}(1-x)^2,$$

and hence

$$\psi\Big(\rho(Sx - Ty)\Big) \le \psi\Big(M(x,y)\Big) - \varphi\Big(M(x,y)\Big) + L\psi\Big(N(x,y)\Big),$$

for all $(x,y) \in [0, \frac{1}{2}] \times [0, \frac{1}{2}]$ and $L \ge 0$.

ii) If $M(x,y) = \frac{16(y-\frac{3}{4})^2(1+(y-1)^2)}{17}$, *then*

$$\begin{aligned}\rho(Sx-Ty) = \frac{1}{16} &\leq \frac{16(y-\frac{3}{4})^2(1+(y-1)^2)}{17} - \frac{1}{16}\frac{16(y-\frac{3}{4})^2(1+(y-1)^2)}{17}\\ &= \frac{15}{16}\frac{16(y-\frac{3}{4})^2(1+(y-1)^2)}{17},\end{aligned}$$

which implies

$$\psi\Big(\rho(Sx-Ty)\Big) \leq \psi\Big(M(x,y)\Big) - \varphi\Big(M(x,y)\Big) + L\psi\Big(N(x,y)\Big),$$

for all $(x,y) \in [0,\frac{1}{2}] \times [0,\frac{1}{2}]$ *and* $L \geq 0$.
Similarly, if we assume the other options for $M(x,y)$ *we obtain that*

$$\psi\Big(\rho(Sx-Ty)\Big) \leq \psi\Big(M(x,y)\Big) - \varphi\Big(M(x,y)\Big) + L\psi\Big(N(x,y)\Big),$$

for all $(x,y) \in [0,\frac{1}{2}] \times [0,\frac{1}{2}]$ *and* $L \geq 0$.

Case 2. *Let* $(x,y) \in [0,\frac{1}{2}] \times (\frac{1}{2},1]$. *Then, we get*

$$\begin{aligned}\rho(Sx-Ty) &= 0 \leq M(x,y) = \max\Big\{(y-1)^2(1+(y-1)^2),\\ &\frac{(y-1)^2}{1+(1-y)^2}, \frac{(1-y)^2+(1-x)^2}{8}, (1-x)^2, (1-y)^2, (x-y)^2\Big\}.\end{aligned}$$

This gives

$$\psi\Big(\rho(Sx-Ty)\Big) \leq \psi\Big(M(x,y)\Big) - \varphi\Big(M(x,y)\Big) + L\psi\Big(N(x,y)\Big) \ for\ any\ L \ \geq \ 0.$$

Case 3. *Let* $(x,y) \in (\frac{1}{2},1] \times [0,\frac{1}{2}]$. *Then,*

$$\begin{aligned}M(x,y) &= \max\Big\{\frac{16(y-\frac{3}{4})^2(1+(y-1)^2)}{17}, \frac{17(y-\frac{3}{4})^2}{16(1+(y-\frac{3}{4})^2)},\\ &\frac{16(1-y)^2+(4x-3)^2}{128}, (1-x)^2, (\frac{3}{4}-y)^2, (x-y)^2\Big\},\end{aligned}$$

i) If $M(x,y) = \frac{16(y-\frac{3}{4})^2(1+(y-1)^2)}{17}$, *then*

$$\begin{aligned}\rho(Sx-Ty) = \frac{1}{16} &\leq \frac{16(y-\frac{3}{4})^2(1+(y-1)^2)}{17} - \frac{1}{16}\frac{16(y-\frac{3}{4})^2(1+(y-1)^2)}{17}\\ &= \frac{15}{16}\frac{16(y-\frac{3}{4})^2(1+(y-1)^2)}{17},\end{aligned}$$

which implies

$$\psi\Big(\rho(Sx-Ty)\Big) \leq \psi\Big(M(x,y)\Big) - \varphi\Big(M(x,y)\Big) + L\psi\Big(N(x,y)\Big),$$

for all $(x,y) \in (\frac{1}{2},1] \times [0,\frac{1}{2}]$ *and* $L \geq 0$.

ii) If $M(x,y) = \frac{17(y-\frac{3}{4})^2}{16(1+(y-\frac{3}{4})^2)}$*, then*

$$\begin{aligned}\rho(Sx-Ty) = \frac{1}{16} &\leq \frac{17(y-\frac{3}{4})^2}{16(1+(y-\frac{3}{4})^2)} - \frac{1}{16}\frac{17(y-\frac{3}{4})^2}{16(1+(y-\frac{3}{4})^2)}\\ &= \frac{15}{16}\frac{17(y-\frac{3}{4})^2}{16(1+(y-\frac{3}{4})^2)},\end{aligned}$$

which yields

$$\psi\Big(\rho(Sx-Ty)\Big) \leq \psi\Big(M(x,y)\Big) - \varphi\Big(M(x,y)\Big) + L\psi\Big(N(x,y)\Big),$$

for all $(x,y) \in (\frac{1}{2},1]\times[0,\frac{1}{2}]$ *and* $L \geq 0$.
Similarly, if we assume the other options for $M(x,y)$ *we obtain that*

$$\psi\Big(\rho(Sx-Ty)\Big) \leq \psi\Big(M(x,y)\Big) - \varphi\Big(M(x,y)\Big) + L\psi\Big(N(x,y)\Big),$$

for all $(x,y) \in (\frac{1}{2},1]\times[0,\frac{1}{2}]$ *and* $L \geq 0$.

Case 4. *If* $(x,y) \in (\frac{1}{2},1]\times(\frac{1}{2},1]$*. It follows that*

$$\begin{aligned}\rho(Sx-Ty) &= 0 \leq M(x,y) = \max\Big\{(y-1)^2(1+(y-1)^2), \frac{(y-1)^2}{1+(1-y)^2},\\ &\quad \frac{(1-y)^2+(1-x)^2}{8}, (1-x)^2, (1-y)^2, (x-y)^2\Big\},\end{aligned}$$

which yields

$$\psi\Big(\rho(Sx-Ty)\Big) \leq \psi\Big(M(x,y)\Big) - \varphi\Big(M(x,y)\Big) + L\psi\Big(N(x,y)\Big),$$

for all $(x,y) \in (\frac{1}{2},1]\times(\frac{1}{2},1]$ *and* $L \geq 0$*. Therefore,* T *and* S *are pairs of mappings satisfying the relaxed* (ψ,φ)*-weakly* 1*-contractive condition. However, if we take* $x = \frac{3}{4}$ *and* $y = \frac{1}{2}$*, then we have*

$$\begin{aligned}\rho(Sx-Ty) &= (1-\frac{3}{4})^2 = \frac{1}{16}, & \rho(x-y) &= (\frac{3}{4}-\frac{1}{2})^2 = \frac{1}{16},\\ \rho(x-Sx) &= (\frac{3}{4}-1)^2 = \frac{1}{16}, & \rho(y-Ty) &= (\frac{1}{2}-\frac{3}{4})^2 = \frac{1}{16},\\ \rho(y-Sx) &= (\frac{1}{2}-1)^2 = \frac{1}{4}, & \rho(x-Ty) &= (\frac{3}{4}-\frac{3}{4})^2 = 0,\\ \rho(\frac{1}{2}(y-Sx)) &= (\frac{1}{2}(\frac{1}{2}-1))^2 = \frac{1}{16}, & \rho(\frac{1}{2}(x-Ty)) &= (\frac{1}{2}(\frac{3}{4}-\frac{3}{4}))^2 = 0,\end{aligned}$$

with $M(x,y) = \frac{1}{16}$ *and* $N(x,y) = 0$*. This implies that*

$$\psi(\frac{1}{16}) \leq \psi(\frac{1}{16}) - \varphi(\frac{1}{16}),$$

which is impossible. Therefore, the pair of mappings do not satisfy generalized (ψ,φ)*-weakly contractive condition.*

Now, we prove our main Theorem.

Theorem 3.4. *Let* C *be a nonempty* ρ*-closed subset of a complete modular space* X_ρ*, where* ρ *is a uniformly continuous modular that satisfies the* Δ_2*-condition. Let* $S,T : C \to C$ *be a pair of mappings satisfying a relaxed* (ψ,φ)*-weakly* N*-contractive condition. Then,* S *and* T *have a unique common fixed point. Moreover, for every* $x_0 \in C$*, the sequence of iterates* $x_{2n+2} = T^N x_{2n+1}, \quad x_{2n+1} = S^N x_{2n}$*, for* $n \geq 0$*,* ρ*-converges to the unique common fixed point of* S *and* T*.*

Proof. Let x_0 be a given point in C. We construct a sequence $\{x_n\}$ in C for $n \geq 0$ by the following two steps iterative process:

$$x_{2n+2} = T^N x_{2n+1}, \qquad x_{2n+1} = S^N x_{2n}. \tag{2}$$

Now, we divide the proof into five steps.

Step 1. We prove that $\rho(x_n - x_{n+1}) \to 0$ as $n \to \infty$.
If there exists $n_0 > 0$ such that $\rho(x_{n_0} - x_{n_0+1}) = 0$, then we get that $\rho(x_n - x_{n+1}) = 0$, for all $n > n_0$ and hence the assumption holds. Now, assume that $\rho(x_n - x_{n+1}) \neq 0$ for all $n \geq 0$. Then from the properties of the functions ψ and φ and substituting $x = x_{2n}$ and $y = x_{2n+1}$ in (1), we obtain

$$\begin{aligned}\psi\Big(\rho(x_{2n+1} - x_{2n+2})\Big) &= \psi\Big(\rho(S^N x_{2n} - T^N x_{2n+1})\Big)\\ &\leq \psi\Big(M(x_{2n}, x_{2n+1})\Big) - \varphi\Big(M(x_{2n}, x_{2n+1})\Big)\\ &\quad + L\psi\Big(N(x_{2n}, x_{2n+1})\Big),\end{aligned} \tag{3}$$

where

$$\begin{aligned}M(x_{2n}, x_{2n+1}) &= \max\Big\{\frac{\rho(x_{2n+1} - T^N x_{2n+1})(1 + \rho(x_{2n+1} - S^N x_{2n}))}{1 + \rho(S^N x_{2n} - T^N x_{2n+1})},\\ &\quad \frac{\rho(x_{2n+1} - T^N x_{2n+1})(1 + \rho(S^N x_{2n} - T^N x_{2n+1}))}{1 + \rho(x_{2n+1} - T^N x_{2n+1})},\\ &\quad \frac{\rho(\frac{1}{2}(x_{2n+1} - S^N x_{2n})) + \rho(\frac{1}{2}(x_{2n} - T^N x_{2n+1}))}{2},\\ &\quad \rho(S^N x_{2n} - x_{2n}), \rho(T^N x_{2n+1} - x_{2n+1}), \rho(x_{2n} - x_{2n+1})\Big\}\\ &= \max\Big\{\frac{\rho(x_{2n+1} - x_{2n+2})(1 + \rho(x_{2n+1} - x_{2n+1}))}{1 + \rho(x_{2n+1} - x_{2n+2})},\\ &\quad \frac{\rho(x_{2n+1} - x_{2n+2})(1 + \rho(x_{2n+1} - x_{2n+2}))}{1 + \rho(x_{2n+1} - x_{2n+2})},\\ &\quad \frac{\rho(\frac{1}{2}(x_{2n+1} - x_{2n+1})) + \rho(\frac{1}{2}(x_{2n} - x_{2n+2}))}{2},\\ &\quad \rho(x_{2n+1} - x_{2n}), \rho(x_{2n+2} - x_{2n+1}), \rho(x_{2n} - x_{2n+1})\Big\},\end{aligned}$$

which implies

$$\begin{aligned}M(x_{2n}, x_{2n+1}) &\leq \max\Big\{\frac{\rho(x_{2n+1} - x_{2n+2})}{1 + \rho(x_{2n+1} - x_{2n+2})}, \rho(x_{2n+1} - x_{2n+2}),\\ &\quad \frac{\rho(x_{2n} - x_{2n+1}) + \rho(x_{2n+1} - x_{2n+2})}{2},\\ &\quad \rho(x_{2n+1} - x_{2n}), \rho(x_{2n+2} - x_{2n+1}), \rho(x_{2n} - x_{2n+1})\Big\}\\ &= \max\{\rho(x_{2n+1} - x_{2n+2}), \rho(x_{2n} - x_{2n+1})\},\end{aligned}$$

and

$$\begin{aligned}
N(x_{2n}, x_{2n+1}) &= \min\Big\{\rho(x_{2n+1} - S^N x_{2n}), \rho(S^N x_{2n} - x_{2n}), \\
&\quad \rho(T^N x_{2n+1} - x_{2n+1}), \rho(x_{2n} - T^N x_{2n+1})\Big\} \\
&= \min\Big\{\rho(x_{2n+1} - x_{2n+1}), \rho(x_{2n+1} - x_{2n}), \\
&\quad \rho(x_{2n+2} - x_{2n+1}), \rho(x_{2n} - x_{2n+2})\Big\} \\
&= \min\{0, \rho(x_{2n+1} - x_{2n}), \rho(x_{2n+2} - x_{2n+1}), \rho(x_{2n} - x_{2n+2})\} = 0.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
\psi\Big(\rho(x_{2n+1} - x_{2n+2})\Big) &\leq \psi\Big(\max\{\rho(x_{2n+1} - x_{2n+2}), \rho(x_{2n} - x_{2n+1})\}\Big) \\
&\quad -\varphi\Big(\max\{\rho(x_{2n+1} - x_{2n+2}), \rho(x_{2n} - x_{2n+1})\}\Big). \qquad (4)
\end{aligned}$$

Now, we consider the following cases.
If $\max\Big\{\rho(x_{2n+1} - x_{2n+2}), \rho(x_{2n} - x_{2n+1})\Big\} = \rho(x_{2n+1} - x_{2n+2})$ for some n, then inequality (4) implies that

$$\begin{aligned}
\psi\Big(\rho(x_{2n+1} - x_{2n+2})\Big) &\leq \psi\Big(\rho(x_{2n+1} - x_{2n+2})\Big) - \varphi\Big(\rho(x_{2n+1} - x_{2n+2})\Big) \\
&< \psi\Big(\rho(x_{2n+1} - x_{2n+2})\Big), \qquad (5)
\end{aligned}$$

a contradiction. Consequently, $\max\Big\{\rho(x_{2n+1} - x_{2n+2}), \rho(x_{2n} - x_{2n+1})\Big\} = \rho(x_{2n} - x_{2n+1})$. Thus, from (4) we have

$$\begin{aligned}
\psi\Big(\rho(x_{2n+1} - x_{2n+2})\Big) &\leq \psi\Big(\rho(x_{2n} - x_{2n+1})\Big) - \varphi\Big(\rho(x_{2n} - x_{2n+1})\Big) \\
&< \psi\Big(\rho(x_{2n} - x_{2n+1})\Big). \qquad (6)
\end{aligned}$$

By the same arguments, we obtain

$$\begin{aligned}
\psi\Big(\rho(x_{2n} - x_{2n+1})\Big) &\leq \psi\Big(\rho(x_{2n-1} - x_{2n})\Big) - \varphi\Big(\rho(x_{2n-1} - x_{2n})\Big) \\
&< \psi\Big(\rho(x_{2n-1} - x_{2n})\Big). \qquad (7)
\end{aligned}$$

From (6) and (7), it follows that $\{\rho(x_n - x_{n+1})\}$ is decreasing and bounded from below. Hence, there exists $r \geq 0$ such that $\lim_{n\to\infty} \rho(x_n - x_{n+1}) = r$. Taking the limits as $n \to \infty$ on both sides of the inequality (6) we have

$$\psi(r) \leq \psi(r) - \varphi(r). \qquad (8)$$

Now, if $r > 0$ then we get $\psi(r) \leq \psi(r) - \varphi(r) < \psi(r)$ a contradiction. Therefore, we have $r = 0$ and hence

$$\lim_{n\to\infty} \rho(x_n - x_{n+1}) = 0. \qquad (9)$$

Step 2. We show that $\{x_n\}$ is a ρ-Cauchy sequence.
It is sufficient to show that $\{x_{2n}\}$ is a ρ-Cauchy sequence. Assume the contrary. Then there exists $\epsilon > 0$ and subsequences $\{m_k\}$ and $\{n_k\}$ of positive integers satisfying $m_k > n_k \geq k$ such that the following inequalities hold:

$$\rho(x_{2n_k} - x_{2m_k}) \geq \epsilon, \qquad \rho(2(x_{2n_k-1} - x_{2m_k})) < \epsilon. \qquad (10)$$

Using (10) we obtain the following.

$$\begin{aligned}
\epsilon &\leq \rho(x_{2n_k} - x_{2m_k}) \\
&= \rho(x_{2n_k} - x_{2n_k-1} + x_{2n_k-1} - x_{2m_k}) \\
&\leq \rho(2(x_{2n_k} - x_{2n_k-1})) + \rho(2(x_{2n_k-1} - x_{2m_k})) \\
&< \rho(2(x_{2n_k} - x_{2n_k-1})) + \epsilon. \qquad (11)
\end{aligned}$$

From (9), (11) and by the fact ρ satisfy Δ_2-condition, we get

$$\lim_{k\to\infty} \rho(x_{2n_k} - x_{2m_k}) = \epsilon, \tag{12}$$

and by uniform continuity of ρ, we obtain

$$\lim_{k\to\infty} \rho(x_{2n_k+1} - x_{2m_k}) = \lim_{k\to\infty} \rho(x_{2n_k} - x_{2m_k-1}) = \epsilon. \tag{13}$$

Furthermore, putting $x = x_{2n_k}$ and $y = x_{2m_k-1}$ in (1), we have

$$\begin{aligned}
\psi\Big(\rho(x_{2n_k+1} - x_{2m_k})\Big) &= \psi\Big(\rho(S^N x_{2n_k} - T^N x_{2m_k-1})\Big) \\
&\leq \psi\Big(M(x_{2n_k}, x_{2m_k-1})\Big) - \varphi\Big(M(x_{2n_k}, x_{2m_k-1})\Big) \\
&\quad + L\psi\Big(N(x_{2n_k}, x_{2m_k-1})\Big),
\end{aligned} \tag{14}$$

where

$$\begin{aligned}
M(x_{2n_k}, x_{2m_k-1}) &= \max\Big\{\frac{\rho(x_{2m_k-1} - T^N x_{2m_k-1})(1 + \rho(x_{2m_k-1} - S^N x_{2n_k}))}{1 + \rho(S^N x_{2n_k} - T^N x_{2m_k-1})}, \\
&\quad \frac{\rho(x_{2m_k-1} - T^N x_{2m_k-1})(1 + \rho(S^N x_{2n_k} - T^N x_{2m_k-1}))}{1 + \rho(x_{2m_k-1} - T^N x_{2m_k-1})}, \\
&\quad \frac{\rho(\frac{1}{2}(x_{2m_k-1} - S^N x_{2n_k})) + \rho(\frac{1}{2}(x_{2n_k} - T^N x_{2m_k-1}))}{2}, \\
&\quad \rho(S^N x_{2n_k} - x_{2n_k}), \rho(T^N x_{2m_k-1} - x_{2m_k-1}), \rho(x_{2n_k} - x_{2m_k-1})\Big\}, \\
&= \max\Big\{\frac{\rho(x_{2m_k-1} - x_{2m_k})(1 + \rho(x_{2m_k-1} - x_{2n_k+1}))}{1 + \rho(x_{2n_k+1} - x_{2m_k})}, \\
&\quad \frac{\rho(x_{2m_k-1} - x_{2m_k})(1 + \rho(x_{2n_k+1} - x_{2m_k}))}{1 + \rho(x_{2m_k-1} - x_{2m_k})}, \\
&\quad \frac{\rho(\frac{1}{2}(x_{2m_k-1} - x_{2n_k+1})) + \rho(\frac{1}{2}(x_{2n_k} - x_{2m_k}))}{2}, \\
&\quad \rho(x_{2n_k+1} - x_{2n_k}), \rho(x_{2m_k} - x_{2m_k-1}), \rho(x_{2n_k} - x_{2m_k-1})\Big\},
\end{aligned}$$

and hence by definition of $M(x_{2n_k}, x_{2m_k-1})$ and property of ρ we obtain

$$\begin{aligned}
\rho(x_{2n_k} - x_{2m_k-1}) &\leq M(x_{2n_k}, x_{2m_k-1}) \\
&\leq \max\Big\{\frac{\rho(x_{2m_k-1} - x_{2m_k})(1 + \rho(x_{2m_k-1} - x_{2n_k+1}))}{1 + \rho(x_{2n_k+1} - x_{2m_k})}, \\
&\quad \frac{\rho(x_{2m_k-1} - x_{2m_k})(1 + \rho(x_{2m_k} - x_{2n_k+1}))}{1 + \rho(x_{2m_k-1} - x_{2m_k})}, \\
&\quad \frac{\rho(x_{2m_k-1} - x_{2m_k}) + \rho(x_{2m_k} - x_{2n_k+1})}{2} \\
&\quad + \frac{\rho(x_{2n_k} - x_{2n_k+1}) + \rho(x_{2n_k+1} - x_{2m_k})}{2}, \\
&\quad \rho(x_{2n_k+1} - x_{2n_k}), \rho(x_{2m_k} - x_{2m_k-1}), \rho(x_{2n_k} - x_{2m_k-1})\Big\},
\end{aligned} \tag{15}$$

and

$$\begin{aligned} N(x_{2n_k}, x_{2m_k-1}) &= \min\Big\{\rho(x_{2m_k-1} - S^N x_{2n_k}), \rho(S^N x_{2n_k} - x_{2n_k}), \\ &\quad \rho(T^N x_{2m_k-1} - x_{2m_k-1}), \rho(x_{2n_k} - T^N x_{2m_k-1})\Big\} \\ &= \min\Big\{\rho(x_{2m_k-1} - x_{2n_k+1}), \rho(x_{2n_k+1} - x_{2n_k}), \\ &\quad \rho(x_{2m_k} - x_{2m_k-1}), \rho(x_{2n_k} - x_{2m_k})\Big\}. \end{aligned} \tag{16}$$

Taking the limit as $k \to \infty$ on both sides of (14) and using (11),(13), (15) and (16), we obtain $\psi(\epsilon) \leq \psi(\epsilon) - \varphi(\epsilon)$ a contradiction. Hence, $\{x_n\}$ is a ρ-Cauchy sequence.

Step 3. We prove the existence of a common fixed point of S^N and T^N.
As X_ρ is a ρ-complete modular space and C is a ρ-closed subset of X_ρ there exists a $u \in C$ such that $\rho(x_n - u) \to 0$ as $n \to \infty$. Moreover, we have $\rho(x_{2n} - u) \to 0$ and $\rho(x_{2n+1} - u) \to 0$ as $n \to \infty$. Now, we prove that $S^N u = u$. Suppose $S^N u \neq u$. Then we have

$$\begin{aligned} \rho(u - S^N u) &\leq M(u, x_{2n+1}) = \max\Big\{\frac{\rho(x_{2n+1} - T^N x_{2n+1})(1 + \rho(x_{2n+1} - S^N u))}{1 + \rho(S^N u - T^N x_{2n+1})}, \\ &\quad \frac{\rho(x_{2n+1} - T^N x_{2n+1})(1 + \rho(S^N u - T^N x_{2n+1}))}{1 + \rho(x_{2n+1} - T^N x_{2n+1})}, \\ &\quad \frac{\rho(\frac{1}{2}(x_{2n+1} - S^N u)) + \rho(\frac{1}{2}(u - T^N x_{2n+1}))}{2}, \\ &\quad \rho(x_{2n+1} - T^N x_{2n+1}), \rho(S^N u - u), \rho(u - x_{2n+1})\Big\} \\ &\leq \max\Big\{\frac{\rho(x_{2n+1} - x_{2n+2})(1 + \rho(x_{2n+1} - S^N u))}{1 + \rho(S^N u - x_{2n+2})}, \\ &\quad \frac{\rho(x_{2n+1} - x_{2n+2})(1 + \rho(S^N u - x_{2n+2}))}{1 + \rho(x_{2n+1} - x_{2n+2})}, \\ &\quad \frac{\rho(x_{2n+1} - u) + \rho(u - S^N u) + \rho(u - x_{2n+1}) + \rho(x_{2n+1} - x_{2n+2})}{2}, \\ &\quad \rho(x_{2n+1} - x_{2n+2}), \rho(S^N u - u), \rho(u - x_{2n+1})\Big\}, \end{aligned}$$

and hence taking the limit as $n \to \infty$ we obtain

$$\lim_{n\to\infty} M(u, x_{2n+1}) = \rho(S^N u - u). \tag{17}$$

Moreover, from the definition of $N(x, y)$ we have

$$\begin{aligned} 0 \leq N(u, x_{2n+1}) &= \min\Big\{\rho(S^N u - x_{2n+1}), \rho(S^N u - u), \rho(T^N x_{2n+1} - x_{2n+1}), \\ &\quad \rho(u - T^N x_{2n+1})\Big\} \\ &= \min\Big\{\rho(S^N u - x_{2n+1}), \rho(S^N u - u), \rho(x_{2n+2} - x_{2n+1}), \\ &\quad \rho(u - x_{2n+2})\Big\} \\ &\leq \rho(x_{2n+2} - x_{2n+1}), \end{aligned}$$

which implies that

$$\lim_{n\to\infty} N(u, x_{2n+1}) = 0. \tag{18}$$

Thus, from (1) we obtain

$$\begin{aligned}\psi(\rho(S^N u - x_{2n+2})) &= \psi(\rho(S^N u - T^N x_{2n+1})) \\ &\leq \psi(M(u, x_{2n+1})) - \varphi(M(u, x_{2n+1})) \\ &\quad + L\psi(N(u, x_{2n+1}).\end{aligned} \tag{19}$$

Taking $n \to \infty$ on both sides of the inequality (19) and making use of (17), (18) and uniform continuity of ρ we obtain

$$\begin{aligned}\psi(\rho(S^N u - u) &\leq \psi(\rho(S^N u - u)) - \varphi(\rho(S^N u - u)) \\ &< \psi(\rho(S^N u - u)),\end{aligned} \tag{20}$$

which is a contradiction. Therefore, $S^N u = u$. Similarly, we obtain that $T^N u = u$ and hence $T^N u = S^N u = u$.

Step 4. We prove the uniqueness of the common fixed point of T^N and S^N.
Assume that w is another common fixed point of T^N and S^N, that is, $T^N w = S^N w = w$ such that $w \neq u$. Then, by (1), we have

$$\begin{aligned}\psi\Big(\rho(u - w)\Big) &= \psi\Big(\rho(S^N u - T^N w)\Big) \\ &\leq \psi\Big(M(u, w)\Big) - \varphi\Big(M(u, w)\Big) + L\psi\Big(N(u, w)\Big),\end{aligned} \tag{21}$$

where

$$\begin{aligned}M(u, w) &= \max\Big\{\frac{\rho(w - T^N w)(1 + \rho(w - S^N u))}{1 + \rho(S^N u - T^N w)}, \frac{\rho(w - T^N w)(1 + \rho(S^N u - T^N w))}{1 + \rho(w - T^N w))}, \\ &\quad \frac{\rho(\frac{1}{2}(w - S^N u)) + \rho(\frac{1}{2}(u - T^N w))}{2}, \rho(S^N u - u), \rho(T^N w - w), \rho(u - w)\Big\} \\ &= \rho(u - w),\end{aligned} \tag{22}$$

and

$$\begin{aligned}N(u, w) &= \min\Big\{\rho(w - S^N u), \rho(S^N u - u), \rho(T^N w - w), \rho(u - T^N w)\Big\} \\ &= 0.\end{aligned} \tag{23}$$

From (21), (22) and (23), we obtain

$$\psi\Big(\rho(u - w)\Big) \leq \psi\Big(\rho(u - w)\Big) - \varphi\Big(\rho(u - w)\Big) < \psi\Big(\rho(u - w)\Big), \tag{24}$$

a contradiction. Thus, $u = w$ and hence the common fixed point is unique. That is, T^N and S^N have a unique common fixed point.

Step 5. We prove that u is the unique common fixed point of T and S. Since $u \in F(T^N)$ and $u \in F(S^N)$, we have $T^N(Tu) = T(T^N u) = Tu$ and $S^N(Su) = S(S^N u) = Su$, which imply that $Tu \in F(T^N)$ and $Su \in F(S^N)$. Now, assume that $Tu \neq Su$. Then, by (1), we have

$$\begin{aligned}\psi\Big(\rho(Su - Tu)\Big) &= \psi\Big(\rho(S^N(Su) - T^N(Tu))\Big) \leq \psi\Big(M(Su, Tu)\Big) \\ &- \varphi\Big(M(Su, Tu)\Big) + L\psi\Big(N(Su, Tu)\Big),\end{aligned} \tag{25}$$

where

$$
\begin{aligned}
M(Su,Tu) &= \max\Big\{\frac{\rho(Tu-T^N(Tu))(1+\rho(Tu-S^N(Su)))}{1+\rho(S^N(Su)-T^N(Tu))},\\
&\quad \frac{\rho(Tu-T^N(Tu))(1+\rho(S^N(Su)-T^N(Tu))}{1+\rho(Tu-T^N(Tu)))},\\
&\quad \frac{\rho(\frac{1}{2}(Tu-S^N(Su)))+\rho(\frac{1}{2}(Su-T^N(Tu)))}{2},\\
&\quad \rho(S^N(Su)-Su),\rho(T^N(Tu)-Tu),\rho(Su-Tu)\Big\}\\
&= \rho(Su-Tu), \qquad (26)
\end{aligned}
$$

and

$$
\begin{aligned}
N(Su,Tu) &= \min\Big\{\rho(Tu-S^N(Su)),\rho(S^N(Su)-Su),\rho(T^N(Tu)-Tu),\\
&\quad \rho(Su-T^N(Tu))\Big\} = 0. \qquad (27)
\end{aligned}
$$

From (25), (26) and (27), we obtain

$$\psi\Big(\rho(Su-Tu)\Big) \leq \psi\Big(\rho(Su-Tu)\Big) - \varphi\Big(\rho(Su-Tu)\Big) < \psi\Big(\rho(Su-Tu)\Big), \qquad (28)$$

a contradiction. Hence, $Su = Tu$ and is a common fixed point of S^N and T^N. By uniqueness of a common fixed point of S^N and T^N, we get $Su = Tu = u$. Therefore, u is a unique common fixed point of S and T. □

If, in Theorem 3.4, we remove the fractions $\frac{\rho(y-T^Ny)(1+\rho(y-S^Nx))}{1+\rho(S^Nx-T^Ny)}$ and $\frac{\rho(y-T^Ny)(1+\rho(S^Nx-T^Ny))}{1+\rho(y-T^Ny)}$ from the set where $M(x,y)$ is chosen, then we obtain the following theorem.

Theorem 3.5. *Let C be a nonempty ρ-closed subset of a complete modular space X_ρ, where ρ is a uniformly continuous modular that satisfies the Δ_2-condition. Let $S,T : C \to C$ be a pair of mappings satisfying the following condition:*

$$\psi\Big(\rho(S^Nx - T^Ny)\Big) \leq \psi\Big(M(x,y)\Big) - \varphi\Big(M(x,y)\Big) + L\psi\Big(N(x,y)\Big), \qquad (29)$$

for any $x,y \in C$, where $\psi \in \Psi, \varphi \in \Phi, L \geq 0, M(x,y) = \max\Big\{\frac{\rho(\frac{1}{2}(y-S^Nx))+\rho(\frac{1}{2}(x-T^Ny))}{2}, \rho(S^Nx-x), \rho(T^Ny-y), \rho(x-y)\Big\}$ and $N(x,y) = \min\Big\{\rho(y-S^Nx), \rho(S^Nx-x), \rho(T^Ny-y), \rho(x-T^Ny)\Big\}$. Then, S and T have a unique common fixed point. Moreover, for every $x_0 \in C$, the sequence of iterates $x_{2n+2} = T^Nx_{2n+1}, \quad x_{2n+1} = S^Nx_{2n}$, for $n \geq 0$, ρ-converges to the unique common fixed point of S and T.

Proof. The method of proof of Theorem 3.4, provides the required assertion. □

If, in Theorem 3.5, we assume that $\psi(t) = t$ and $L = 0$ we obtain the following corollary.

Corollary 3.6. *Let C be a nonempty ρ-closed subset of a complete modular space X_ρ, where ρ is a uniformly continuous modular that satisfies the Δ_2-condition. Let $S,T : C \to C$ be a pair of mappings satisfying a relaxed (ψ,φ)-weakly N-contractive condition with $\psi(t) = t$ and $L = 0$. Then, S and T have a unique common fixed point. Moreover, for every $x_0 \in C$, the sequence of iterates $x_{2n+2} = T^Nx_{2n+1}, \quad x_{2n+1} = S^Nx_{2n}$, for $n \geq 0$, ρ-converges to the unique common fixed point of S and T.*

Corollary 3.7. *Let C be a nonempty ρ-closed subset of a complete modular space X_ρ, where ρ is a uniformly continuous modular that satisfies the Δ_2-condition. Let $S,T : C \to C$ be a pair of mappings satisfying the following condition:*

$$\rho(S^Nx - T^Ny) \leq M(x,y) - \varphi\Big(M(x,y)\Big), \qquad (30)$$

for any $x, y \in C$, where $\varphi \in \Phi$ and $M(x,y) = \max\left\{\frac{\rho(\frac{1}{2}(y-S^Nx))+\rho(\frac{1}{2}(x-T^Ny))}{2}, \rho(S^Nx-x), \rho(T^Ny-y), \rho(x-y)\right\}$. Then, S and T have a unique common fixed point. Moreover, for every $x_0 \in C$, the sequence of iterates $x_{2n+2} = T^Nx_{2n+1}$, $x_{2n+1} = S^Nx_{2n}$, for $n \geq 0$, ρ-converges to the unique common fixed point of S and T.

If, in Theorems 3.4 and 3.5, we consider $N = 1$, then we get the following corollaries.

Corollary 3.8. *Let C be a nonempty ρ-closed subset of a complete modular space X_ρ, where ρ is a uniformly continuous modular that satisfies the Δ_2-condition. Let $S, T : C \to C$ be a pair of mappings satisfying a relaxed (ψ, φ)-weakly 1-contractive condition. Then, S and T have a unique common fixed point. Moreover, for every $x_0 \in C$, the sequence of iterates $x_{2n+2} = Tx_{2n+1}$, $x_{2n+1} = Sx_{2n}$, for $n \geq 0$, ρ-converges to the unique common fixed point of S and T.*

Corollary 3.9. *Let C be a nonempty ρ-closed subset of a complete modular space X_ρ, where ρ is a uniformly continuous modular that satisfies the Δ_2-condition. Let $S, T : C \to C$ be a pair of mappings satisfying the following condition:*

$$\psi\Big(\rho(Sx - Ty)\Big) \leq \psi\Big(M(x,y)\Big) - \varphi\Big(M(x,y)\Big) + L\psi\Big(N(x,y)\Big), \tag{31}$$

for any $x, y \in C$, where $\psi \in \Psi, \varphi \in \Phi, L \geq 0$, $M(x,y) = \max\left\{\frac{\rho(\frac{1}{2}(y-Sx))+\rho(\frac{1}{2}(x-Ty))}{2}, \rho(Sx-x), \rho(Ty-y), \rho(x-y)\right\}$ and $N(x,y) = \min\left\{\rho(y-Sx), \rho(Sx-x), \rho(Ty-y), \rho(x-Ty)\right\}$. Then, S and T have a unique common fixed point. Moreover, for every $x_0 \in C$, the sequence of iterates $x_{2n+2} = Tx_{2n+1}$, $x_{2n+1} = Sx_{2n}$, for $n \geq 0$, ρ-converges to the unique common fixed point of S and T.

Remark 3.10. *If, in Theorem 3.5, we consider $N = 1$, then we obtain Theorem 3.1 of Öztürk et.al [26]. We remark that the proof of Theorem 3.1 of Öztürk et.al [26] has a gap in concluding the Cauchy nature of the sequence considered. To complete the gap one may impose the assumption that ρ is uniformly continuous and show how it holds.*

Remark 3.11. *If, in Corollary 3.9, we assume $L = 0$, then we obtain Theorem 3.3 of Mongkolkeha and Kumam [23]. In fact, the proof of Theorem 3.3 of Mongkolkeha and Kumam [23] has a gap in concluding the Cauchy nature of the sequence considered. To fill the gap one may impose the assumption that ρ is uniformly continuous which satisfies $\triangle_2$-condition and show how it holds.*

3.2. Existence and approximation of fixed points of a relaxed (ψ, φ)-weakly N-contraction mappings in modular spaces

Definition 3.12. *Let C be a nonempty subset of a modular space X_ρ and $N \in \mathbb{N}$. A mapping $T : C \to C$ is called a relaxed (ψ, φ)-weakly N-contraction mapping if for any $x, y \in C$ the following condition holds:*

$$\psi\Big(\rho(T^Nx - T^Ny)\Big) \leq \psi\Big(M(x,y)\Big) - \varphi\Big(M(x,y)\Big) + L\psi\Big(N(x,y)\Big), \tag{32}$$

where $\psi \in \Psi, \varphi \in \Phi, L \geq 0, T^N = T(T^{N-1}), N = 1, 2, ...,$ with $T^0 = I$, the identity map,
$M(x,y) = \max\left\{\frac{\rho(y-T^Ny)(1+\rho(y-T^Nx))}{1+\rho(T^Nx-T^Ny)}, \frac{\rho(y-T^Ny)(1+\rho(T^Ny-T^Nx))}{1+\rho(y-T^Ny)}, \frac{\rho(\frac{1}{2}(y-T^Nx)+\rho(\frac{1}{2}(x-T^Ny))}{2}, \rho(T^Nx-x), \rho(T^Ny-y), \rho(x-y)\right\}$ *and* $N(x,y) = \min\left\{\rho(y-T^Nx), \rho(T^Nx-x), \rho(T^Ny-y), \rho(x-T^Ny)\right\}$.

Remark 3.13. *The following example shows that the class of relaxed (ψ, φ)-weakly N-contraction mappings includes mappings which are not in a class of (ψ, φ)-weak contraction mappings.*

Example 3.14. *Let $X_\rho = \mathbb{R}$, the real number system $\mathbb{R}$, be a space modular with $\rho(x) = |x|$. Let $C = \{x \in X_\rho : 0 \leq x \leq 2\}$. Define $T : C \to C$ as:*

$$Tx = \begin{cases} 0 \; for \; 0 \leq x \leq 1 \\ x - 1 \; for \; 1 < x \leq 2. \end{cases}$$

Define the functions $\psi, \varphi : [0, \infty) \to [0, \infty)$ as $\psi(t) = t$ and $\varphi(t) = \frac{1}{2}t$ for all $t \geq 0$. Now, we show that

$$\psi\Big(\rho(T^2x - T^2y)\Big) \leq \psi\Big(M(x, y)\Big) - \varphi\Big(M(x, y)\Big) + L\psi\Big(N(x, y)\Big), \quad for \;\; any \;\; x, y \in C,$$

where $M(x, y) = \max\left\{\frac{\rho(y-T^2y)(1+\rho(y-T^2x))}{1+\rho(T^2x-T^2y)}, \frac{\rho(y-T^2y)(1+\rho(T^2y-T^2x))}{1+\rho(y-T^2y)}, \frac{\rho(\frac{1}{2}(y-T^2x)+\rho(\frac{1}{2}(x-T^2y))}{2}, \rho(T^2x - x), \rho(T^2y - y), \rho(x - y)\right\}$ and $N(x, y) = \min\left\{\rho(y - T^2x), \rho(T^2x - x), \rho(T^2y - y), \rho(x - T^2y)\right\}$ with $L \geq 0$. Since $T^2x = 0$ for all $x \in C$, we get

$$\begin{aligned} |T^2x - T^2y| &= 0 \leq \max\{y(y + 1), \frac{x + y}{4}, \frac{y}{y + 1}, x, y, |x - y|\} \\ &\quad -\frac{1}{2}\max\{y(y + 1), \frac{x + y}{4}, \frac{y}{y + 1}, x, y, |x - y|\} \\ &\quad +L(\min\{y, x\}) \; for \; any \; x, \; y \in C \; and \; L \geq 0. \end{aligned}$$

Hence, T is relaxed (ψ, φ)-weakly 2-contraction mapping with 0 as a unique fixed point. However, if we consider $x = 1$ and $y = 2$, we have

$$\begin{aligned} \rho(T1 - T2) &= |0 - 1| = 1, \qquad \rho(T1 - 1) = |0 - 1| = 1, \\ \rho(T2 - 2) &= |1 - 2| = 1, \qquad \rho(\frac{1}{2}(T2 - 1)) = \frac{1}{2}|1 - 1| = 0, \\ \rho(\frac{1}{2}(T1 - 2)) &= \frac{1}{2}|0 - 2| = 1, \qquad \rho(2 - 1) = |2 - 1| = 1, \end{aligned}$$

and hence $M(x, y) = 1$ and $N(x, y) = 0$, which yields that

$$\psi(1) \leq \psi(1) - \varphi(1),$$

which is a contradiction. Therefore, T is not a (ψ, φ)-weak contraction mapping.

Now, we prove the following theorems.

Theorem 3.15. *Let C be a nonempty ρ-closed subset of a complete modular space X_ρ, where ρ is a uniformly continuous modular that satisfies the Δ_2-condition. Let $T : C \to C$ be a relaxed (ψ, φ)-weakly N-contraction mapping. Then T has a unique fixed point. Moreover, for every $x_0 \in C$, the sequence of iterates $\{T^N x_n\}$ ρ-converges to the unique fixed point of T.*

Proof. The proof of the theorem follows from the proof of Theorem 3.4 with $S = T$. □

Theorem 3.16. *Let C be a nonempty ρ-closed subset of a complete modular space X_ρ, where ρ is a uniformly continuous modular that satisfies the Δ_2-condition. Let T be a self mapping on C such that for some positive integer N, T^N satisfying the following condition:*

$$\psi\Big(\rho(T^Nx - T^Ny)\Big) \leq \psi\Big(M(x, y)\Big) - \varphi\Big(M(x, y)\Big) + L\psi\Big(N(x, y)\Big), \tag{33}$$

for any $x, y \in C$, where $\psi \in \Psi, \varphi \in \Phi, L \geq 0, M(x, y) = \max\left\{\frac{\rho(\frac{1}{2}(y-T^Nx))+\rho(\frac{1}{2}(x-T^Ny))}{2}, \rho(T^Nx-x), \rho(T^Ny-y), \rho(x-y)\right\}$ and $N(x, y) = \min\left\{\rho(y - T^Nx), \rho(T^Nx - x), \rho(T^Ny - y), \rho(x - T^Ny)\right\}$. Then, T has a unique fixed point. Moreover, for every $x_0 \in C$, the sequence of iterates $\{T^N x_n\}$ ρ-converges to the unique fixed point of T.

Proof. It follows from the proof of Theorem 3.5 with $S = T$. □

If, in Theorem 3.16, we assume that $\psi(t) = t$ and $L = 0$ we obtain the following corollary.

Corollary 3.17. *Let C be a nonempty ρ-closed subset of a complete modular space X_ρ, where ρ is a uniformly continuous modular that satisfies the Δ_2-condition. Let T be a self mapping on C such that for some positive integer N, T^N satisfying the following condition:*

$$\rho(T^N x - T^N y) \leq M(x,y) - \varphi\Big(M(x,y)\Big), \tag{34}$$

for any $x, y \in C$ where, $\varphi \in \Phi$ and $M(x,y) = \max\Big\{\frac{\rho(\frac{1}{2}(y-T^N x))+\rho(\frac{1}{2}(x-T^N y))}{2}, \rho(T^N x - x), \rho(T^N y - y), \rho(x-y)\Big\}$. Then, T has a unique fixed point. Moreover, for every $x_0 \in C$, the sequence of iterates $\{T^N x_n\}$ ρ-converges to the unique fixed point of T.

If, in Theorems 3.15 and 3.16, we consider $N = 1$, then we get the following corollaries.

Corollary 3.18. *Let C be ρ-closed subset of X_ρ. Let T be a relaxed (ψ, φ)-weakly 1-contraction self mapping on C. Then T has a unique fixed point. Moreover, for every $x_0 \in C$, the sequence of iterates $\{Tx_n\}$ ρ-converges to the unique fixed point of T.*

Corollary 3.19. *Let C be ρ-closed subset of X_ρ. Let T be a self mapping on C satisfying the following condition:*

$$\psi\Big(\rho(Tx - Ty)\Big) \leq \psi\Big(M(x,y)\Big) - \varphi\Big(M(x,y)\Big) + L\psi\Big(N(x,y)\Big), \tag{35}$$

for any $x, y \in C$, where $\psi \in \Psi, \varphi \in \Phi, L \geq 0, M(x,y) = \max\Big\{\frac{\rho(\frac{1}{2}(y-Tx))+\rho(\frac{1}{2}(x-Ty))}{2}, \rho(Tx - x), \rho(Ty - y), \rho(x-y)\Big\}$ and $N(x,y) = \min\Big\{\rho(y - Tx), \rho(Tx - x), \rho(Ty - y), \rho(x - Ty)\Big\}$. Then, T has a unique fixed point. Moreover, for every $x_0 \in C$, the sequence of iterates $\{Tx_n\}$ ρ-converges to the unique fixed point of T.

Remark 3.20. *If, in Corollary 3.17, we assume that $N = 1$, then we obtain Theorem 2.3 of Öztürk et.al [26]. We note that the proof of Theorem 2.3 of Öztürk et.al [26] has a gap in concluding the Cauchy nature of the sequence considered. To fill the gap one may impose the assumption that ρ is uniformly continuous and show how it holds.*

Remark 3.21. *If, in Corollary 3.19, we consider $L = 0$, then we obtain the result obtained by Mongkolkeha and Kumam [23] which requires the assumption that ρ is uniformly continuous which satisfies $\triangle_2$-condition.*

4. Numerical example

In this section, we present some numerical experiment results to explain the conclusion of Theorem 3.15.

Example 4.1. *Let $X_\rho = \mathbb{R}$, the real number system $\mathbb{R}$, be a space modular with $\rho(x) = x^2$. Let $C = \{x \in X_\rho : 0 \leq x \leq 1\}$. Define $T : C \to C$ by:*

$$Tx = \begin{cases} \frac{x+1}{3} \ for \ 0 \leq x \leq \frac{1}{2}, \\ 1 - x \ for \ \frac{1}{2} \leq x \leq 1, \end{cases}$$

which implies that

$$T^2 x = \begin{cases} \frac{x+4}{9} \ for \ 0 \leq x \leq \frac{1}{2}, \\ \frac{2-x}{3} \ for \ \frac{1}{2} \leq x \leq 1. \end{cases}$$

Clearly, C is ρ-closed and ρ is uniformly continuous with Δ_2-condition. Moreover,

$$\begin{aligned}
\rho(T^2x - T^2y) &= \frac{(x-y)^2}{81}, \text{ for } (x,y) \in [0,\tfrac{1}{2}] \times [0,\tfrac{1}{2}],\\
\rho(T^2x - T^2y) &= \frac{(2-3x-y)^2}{81} \le \frac{25(x-y)^2}{81}, \text{ for } (x,y) \in [0,\tfrac{1}{2}] \times (\tfrac{1}{2},1],\\
\rho(T^2x - T^2y) &= \frac{(x-2+3y)^2}{81} \le \frac{25(y-x)^2}{81}, \text{ for } (x,y) \in (\tfrac{1}{2},1] \times [0,\tfrac{1}{2}],\\
\rho(T^2x - T^2y) &= \frac{(x-y)^2}{9}, \text{ for } (x,y) \in (\tfrac{1}{2},1] \times (\tfrac{1}{2},1],
\end{aligned}$$

and hence

$$\rho(T^2x - T^2y) \le \frac{25(x-y)^2}{81}, \text{ for all } x,y \in C.$$

Define the functions $\psi, \varphi : [0,\infty) \to [0,\infty)$ by $\psi(t) = t$ and $\varphi(t) = \frac{2}{9}t$ for all $t \ge 0$. Now, we show that

$$\psi\Big(\rho(T^2x - T^2y)\Big) \le \psi\Big(M(x,y)\Big) - \varphi\Big(M(x,y)\Big) + L\psi\Big(N(x,y)\Big),$$

for any $x, y \in C$ where, $L \ge 0$,

$$\begin{aligned}
M(x,y) &= max\Big\{\frac{\rho(y-T^2y)(1+\rho(y-T^2x))}{1+\rho(T^2x-T^2y)}, \frac{\rho(y-T^2y)(1+\rho(T^2y-T^2x))}{1+\rho(y-T^2y)},\\
&\quad \frac{\rho(\frac{1}{2}(y-T^2x)) + \rho(\frac{1}{2}(x-T^2y))}{2}, \rho(T^2x-x), \rho(T^2y-y), \rho(x-y)\Big\}\\
&= max\Big\{\frac{(8y-4)^2(9+(3y+x-2)^2)}{9(81+(2-3x-y)^2)}, \frac{(8y-4)^2(81+(2-3x-y)^2)}{81(81+(8y-4)^2)},\\
&\quad \frac{(y+4-9x)^2+9(2-x-3y)^2}{648}, \frac{(8y-4)^2}{81}, \frac{(2-4x)^2}{9}, (x-y)^2\Big\},
\end{aligned}$$

and

$$\begin{aligned}
N(x,y) &= min\Big\{\rho(y-T^2x), \rho(T^2x-x), \rho(T^2y-y), \rho(x-T^2y)\Big\}\\
&= min\Big\{\frac{(3y-2+x)^2}{9}, \frac{(8y-4)^2}{81}, \frac{(2-4x)^2}{9}, \frac{(9x-y-4)^2}{81}\Big\}.
\end{aligned}$$

For this we will consider the following six cases.

Case 1. *Assume $M(x,y) = (x-y)^2$. Then*

$$\rho(T^2x - T^2y) \le \frac{25(x-y)^2}{81} \le (x-y)^2 - \frac{2}{9}(x-y)^2 = \frac{7}{9}(x-y)^2,$$

and hence

$$\psi\Big(\rho(T^2x - T^2y)\Big) \le \psi\Big(M(x,y)\Big) - \varphi\Big(M(x,y)\Big) + L\psi\Big(N(x,y)\Big),$$

for any $(x,y) \in (\frac{1}{2},1] \times [0,\frac{1}{2}]$ and $L \ge 0$.

Case 2. *Assume $M(x,y) = \frac{(8y-4)^2(9+(3y+x-2)^2)}{9(81+(2-3x-y)^2)}$. Then*

$$\begin{aligned}
\rho(T^2x - T^2y) &\le \frac{25(x-y)^2}{81} \le \frac{(8y-4)^2(9+(3y+x-2)^2)}{9(81+(2-3x-y)^2)}\\
&\quad -\frac{2}{9}\frac{(8y-4)^2(9+(3y+x-2)^2)}{9(81+(2-3x-y)^2)}\\
&= \frac{7}{9}\frac{(8y-4)^2(9+(3y+x-2)^2)}{9(81+(2-3x-y)^2)},
\end{aligned}$$

and hence

$$\psi\Big(\rho(T^2x - T^2y)\Big) \leq \psi\Big(M(x,y)\Big) - \varphi\Big(M(x,y)\Big) + L\psi\Big(N(x,y)\Big),$$

for any $(x,y) \in (\frac{1}{2},1] \times [0,\frac{1}{2}]$ *and* $L \geq 0$.

Similarly, if we assume the other options for $M(x,y)$ *we obtain that*

$$\psi\Big(\rho(T^2x - T^2y)\Big) \leq \psi\Big(M(x,y)\Big) - \varphi\Big(M(x,y)\Big) + L\psi\Big(N(x,y)\Big),$$

for any $x,y \in C$ *and* $L \geq 0$. *Hence, all conditions of Theorem 3.15 are satisfied. Moreover, the sequence of iterates* $\{T^2x_n\}$ ρ*-converges to* $\frac{1}{2} \in F(T) = \{\frac{1}{2}\}$. *To indicate this for different initial points such as* $x_0 = 0$, $x_0 = 0.2$, $x_0 = 0.5$, $x_0 = 0.8$ *and* $x_0 = 1$, *the numerical experiment result using MATLAB is given in Figure 1 below. From this we obtain that in all cases the Algorithm* $\{T^2x_n\}$ ρ*-converges to* $\frac{1}{2}$.

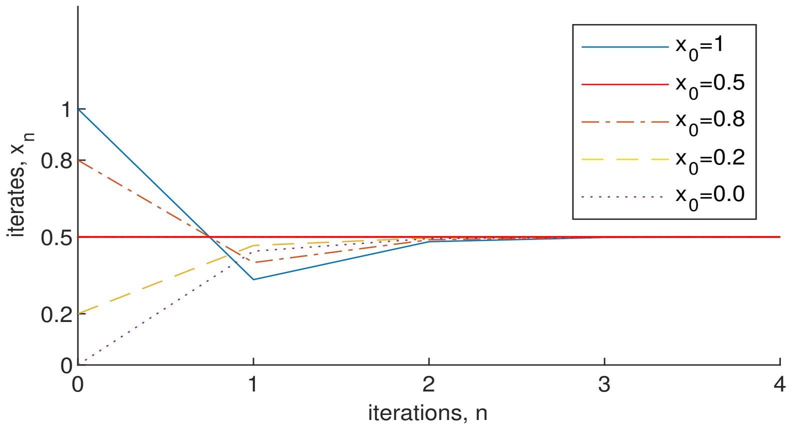

Figure 1

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