Filomat 34:5 (2020), 1629–1644
https://doi.org/10.2298/FIL2005629D

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Almost Periodic and Asymptotically Almost Periodic Type Functions in Lebesgue Spaces with Variable Exponents $L^{p(x)}$

Toka Diagana[a], Marko Kostić[b]

[a]Department of Mathematical Sciences, University of Alabama in Huntsville, 301 Sparkman Drive, Huntsville, AL 35899, USA
[b]Faculty of Technical Sciences, University of Novi Sad, Trg D. Obradovića 6, 21125 Novi Sad, Serbia

Abstract. In this paper we introduce and analyze an important class of (asymptotically) Stepanov almost periodic functions in the Lebesgue spaces with variable exponents, which generalizes in a natural fashion all the (asymptotically) almost periodic functions. We then make extensive use of these new functions to study some abstract Volterra integro-differential equations in Banach spaces including multi-valued ones.

1. Introduction

The notion of almost periodicity was introduced by Danish mathematician H. Bohr around 1924-1926 and later generalized by many other authors, see, e,g., [4], [12], [19] and references therein. Let I be either $\mathbb{R}$ or $\mathbb{R}_+$ and let $f : I \to X$ be a given continuous function. Given $\varepsilon > 0$, we call $\tau > 0$ an ε-period for $f(\cdot)$ iff $\|f(t+\tau) - f(t)\| \leq \varepsilon$, $t \in I$. The set consisting of all ε-periods for $f(\cdot)$ is denoted by $\vartheta(f, \varepsilon)$. The function f is said to be almost periodic, if and only if for each $\varepsilon > 0$ the set $\vartheta(f, \varepsilon)$ is relatively dense in I, which means that there exists $\ell > 0$ such that any subinterval of the interval I of length ℓ intersects $\vartheta(f, \varepsilon)$. The collection of all almost periodic functions will be denoted by $AP(I : X)$. Let us mention that a continuous periodic function is almost with the converse being false.

Similarly, a function $f \in C_b(\mathbb{R}_+ : X)$ ($C_b(\mathbb{R}_+ : X)$ being the collection of all bounded continuous functions which go from $\mathbb{R}_+$ to X) is said to be asymptotically almost periodic if and only if, for every $\varepsilon > 0$ we can find numbers $\ell > 0$ and $M > 0$ such that every subinterval of $\mathbb{R}_+$ of length ℓ contains at least one number τ such that $\|f(t+\tau) - f(t)\| \leq \varepsilon$ for all $t \geq M$. It is well known that a function $f \in C_b(\mathbb{R}_+ : X)$ is asymptotically almost periodic if and only if there exist functions $g \in AP(\mathbb{R}_+ : X)$ and $\phi \in C_0(\mathbb{R}_+ : X)$ ($C_0(\mathbb{R}_+ : X)$ being the collection of all continuous functions f which go from $\mathbb{R}_+$ to X such $f(t) \to 0$ as $t \to \infty$, in X) such that $f = g + \phi$. The collection of all asymptotically almost periodic functions will be denoted by $AAP(\mathbb{R}_+ : X)$.

The concept of almost periodicity (respectively, almost automorphy, pseudo-almost periodicity, and pseudo-almost automorphy) in the Lebesgue space with variable exponent $L^{p(x)}(I, X)$ was first introduced and studied by Diagana and Zitane [6, 7]. However, the translation-invariance of these newly introduced

2010 *Mathematics Subject Classification*. Primary 34C27; Secondary 35B15, 46E30
Keywords. Lebesgue spaces with variable exponents, generalized almost periodicity with variable exponents, generalized asymptotically almost periodicity with variable exponents, abstract Volterra integro-differential equations, abstract fractional differential equations.
Received: 08 October 2018; Accepted: 07 February 2020
Communicated by Dragan S. Djordjević
The second named author is partially supported by Grant no. 174024 of the Ministry of Science and Technological Development, Republic of Serbia.
Email addresses: toka.diagana@uah.edu (Toka Diagana), marco.s@verat.net (Marko Kostić)

spaces depends heavily upon the function $p \in C(\mathbb{R}_+)$. To remove such a restriction, we introduce some new concepts so that the obtained almost periodic (respectively, asymptotically almost periodic) in $L^{p(x)}(I, X)$ are automatically translation-invariant. Among other things, it will be shown that these new functions generalize in a natural fashion the classical notion of almost periodicity (respectively, asymptotic almost periodicity). Many properties of the new functions are analyzed including their compositions. Further, we will make extensive use of these new functions to study some abstract Volterra integro-differential equations in Banach spaces including multi-valued ones.

2. Preliminaries

Unless specified otherwise, we assume that $(X, \|\cdot\|)$ is a complex Banach space. If $(Y, \|\cdot\|_Y)$ is another Banach space, then we denote by $L(X, Y)$ the Banach algebra of all bounded linear operators from X into Y with $L(X, X)$ being denoted $L(X)$. If $A : D(A) \subset X \mapsto X$ is a closed linear operator, then its nullspace (or kernel) and range will be denoted respectively by $N(A)$ and $R(A)$. Further, we will identify A with its graph defined by $\{(a, Ax) : x \in X\}$. By $[D(A)]$ we denote the Banach space $(D(A), \|\cdot\|_{[D(A)]})$ where $\|\cdot\|_{[D(A)]}$ is the graph norm defined by, $\|x\|_{[D(A)]} := \|x\| + \|Ax\|$ for all $x \in D(A)$.

For given constants $s \in \mathbb{R}$ and $\theta \in (0, \pi]$, we set $\lceil s \rceil := \inf\{l \in \mathbb{Z} : s \leq l\}$ and $\Sigma_\theta := \{z \in \mathbb{C} \setminus \{0\} : |\arg(z)| < \theta\}$.

The symbol $C(I : X)$, where $I = \mathbb{R}$ or $I = \mathbb{R}_+$, stands for the space of all X-valued continuous functions on the interval I. By $C_b(I : X)$ (respectively, $BUC(I : X)$) we denote the subspaces of $C(I : X)$ consisting of all bounded (respectively, all bounded uniformly continuous functions). Both $C_b(I : X)$ and $BUC(I : X)$ are Banach spaces when they are equipped with the sup-norm.

The classical Gamma function is denoted by $\Gamma(\cdot)$. We also set $g_\zeta(t) := t^{\zeta-1}/\Gamma(\zeta)$, $\zeta > 0$. The convolution operator $*$ is defined by $f * g(t) := \int_0^t f(t-s)g(s)\, ds$.

2.1. Fractional Calculus

One of the first conferences on fractional calculus was held in New Haven (1974). Since then, fractional calculus has captured the attention of many mathematicians around the world. It has many applications in various fields such as mathematical physics, engineering, biology, aerodynamics, chemistry, economics etc. For more on fractional calculus and related issues, we refer to [1], [9], [16], [17] and the references therein.

The Mittag-Leffler function $E_{\alpha,\beta}(z)$, defined by

$$E_{\alpha,\beta}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \quad z \in \mathbb{C},$$

plays a crucial role in the analysis of fractional differential equations. Set, $E_\alpha(z) := E_{\alpha,1}(z)$, $z \in \mathbb{C}$.

Assuming that $\gamma \in (0, 1)$, then we define the Wright function $\Phi_\gamma(\cdot)$ by

$$\Phi_\gamma(t) := \mathcal{L}^{-1}\Big(E_\gamma(-\lambda)\Big)(t), \quad t \geq 0,$$

where $\mathcal{L}^{-1}$ denotes the inverse Laplace transform.

The Wright function $\Phi_\gamma(\cdot)$ is an entire function which can be equivalently introduced by the formula

$$\Phi_\gamma(z) = \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\Gamma(1 - \gamma - \gamma n)}, \quad z \in \mathbb{C}.$$

Let $\gamma \in (0, 1)$. If $u : [0, \infty) \to X$ satisfies, for every $T > 0$, $u \in C((0, T] : X)$, $u(\cdot) - u(0) \in L^1((0, T) : X)$ and $g_{1-\gamma} * (u(\cdot) - u(0)) \in W^{1,1}((0, T) : X)$, then we define its Caputo fractional derivative by,

$$\mathbf{D}_t^\gamma u(t) = \frac{d}{dt}\Big[g_{1-\gamma} * \Big(u(\cdot) - u(0)\Big)\Big](t), \quad t \in (0, T].$$

The Weyl-Liouville fractional derivative $D_{t,+}^{\gamma}u(t)$ of order γ is defined for those continuous functions $u : \mathbb{R} \to X$ such that $t \mapsto \int_{-\infty}^{t} g_{1-\gamma}(t-s)u(s)\,ds$, $t \in \mathbb{R}$ is a well-defined continuously differentiable mapping, by

$$D_{t,+}^{\gamma}u(t) := \frac{d}{dt}\int_{-\infty}^{t} g_{1-\gamma}(t-s)u(s)\,ds, \quad t \in \mathbb{R}.$$

Set $\mathbf{D}_t^1 u(t) := (d/dt)u(t)$ and $D_{t,+}^1 u(t) := -(d/dt)u(t)$.

2.2. Multivalued linear operators and degenerate resolvent operator families

Suppose that X and Y are two Banach spaces. A multivalued map (multimap) $\mathcal{A} : X \to P(Y)$ is said to be a multivalued linear operator, MLO for short, iff the following holds:

(i) $D(\mathcal{A}) := \{x \in X : \mathcal{A}x \neq \emptyset\}$ is a linear subspace of X;

(ii) $\mathcal{A}x + \mathcal{A}y \subseteq \mathcal{A}(x+y)$, $x,\ y \in D(\mathcal{A})$ and $\lambda\mathcal{A}x \subseteq \mathcal{A}(\lambda x)$, $\lambda \in \mathbb{C}$, $x \in D(\mathcal{A})$.

In the case that $X = Y$, then we say that $\mathcal{A}$ is an MLO in X. It is well known that for any $x,\ y \in D(\mathcal{A})$ and $\lambda,\ \eta \in \mathbb{C}$ with $|\lambda| + |\eta| \neq 0$, we have $\lambda\mathcal{A}x + \eta\mathcal{A}y = \mathcal{A}(\lambda x + \eta y)$. If $\mathcal{A}$ is an MLO, then $\mathcal{A}0$ is a linear manifold in Y and $\mathcal{A}x = f + \mathcal{A}0$ for any $x \in D(\mathcal{A})$ and $f \in \mathcal{A}x$. Define the kernel space $N(\mathcal{A})$ of $\mathcal{A}$ and the range $R(\mathcal{A})$ of $\mathcal{A}$ by $N(\mathcal{A}) := \{x \in D(\mathcal{A}) : 0 \in \mathcal{A}x\}$ and $R(\mathcal{A}) := \{\mathcal{A}x : x \in D(\mathcal{A})\}$, respectively. We write $\mathcal{A} \subseteq \mathcal{B}$ iff $D(\mathcal{A}) \subseteq D(\mathcal{B})$ and $\mathcal{A}x \subseteq \mathcal{B}x$ for all $x \in D(\mathcal{A})$.

Sums, mutual products, taking powers and products with complex scalars are standard operations for multivalued linear operators (see e.g. [3], [11] and [18]). It is said that an MLO operator $\mathcal{A} : X \to P(Y)$ is closed iff for any sequences (x_n) in $D(\mathcal{A})$ and (y_n) in Y such that $y_n \in \mathcal{A}x_n$ for all $n \in \mathbb{N}$ we have that the suppositions $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$ imply $x \in D(\mathcal{A})$ and $y \in \mathcal{A}x$.

Concerning the C-resolvent sets of MLOs in Banach spaces, our standing hypotheses will be that $\mathcal{A}$ is an MLO in X, as well as that $C \in L(X)$ is injective and $C\mathcal{A} \subseteq \mathcal{A}C$. The C-resolvent set of $\mathcal{A}$, $\rho_C(\mathcal{A})$ for short, is defined as the union of those complex numbers $\lambda \in \mathbb{C}$ for which

(i) $R(C) \subseteq R(\lambda - \mathcal{A})$;

(ii) $(\lambda - \mathcal{A})^{-1}C$ is a single-valued linear continuous operator on X.

The operator $\lambda \mapsto (\lambda - \mathcal{A})^{-1}C$ is called the C-resolvent of $\mathcal{A}$ ($\lambda \in \rho_C(\mathcal{A})$); the resolvent set of $\mathcal{A}$ is then defined by $\rho(\mathcal{A}) := \rho_I(\mathcal{A})$, where I denotes the identity operator on X. Set $R(\lambda : \mathcal{A}) \equiv (\lambda - \mathcal{A})^{-1}$ ($\lambda \in \rho(\mathcal{A})$). The basic properties of C-resolvent sets of single-valued linear operators continue to hold in our framework (cf. [18] for more details). For instance, $\rho(\mathcal{A})$ is always an open subset of $\mathbb{C}$ and $\rho(\mathcal{A}) \neq \emptyset$ implies that $\mathcal{A}$ is closed.

In the sequel, we will employ the following important condition:

(P) There exist finite constants $c,\ M > 0$ and $\beta \in (0,1]$ such that

$$\Psi := \Big\{\lambda \in \mathbb{C} : \mathbb{R}e\lambda \geq -c\Big(|\Im\lambda| + 1\Big)\Big\} \subseteq \rho(\mathcal{A})$$

and

$$\|R(\lambda : \mathcal{A})\| \leq M\Big(1 + |\lambda|\Big)^{-\beta}, \quad \lambda \in \Psi.$$

Then degenerate strongly continuous semigroup $(T(t))_{t>0} \subseteq L(X)$ generated by $\mathcal{A}$ satisfies estimate $\|T(t)\| \leq Me^{-ct}t^{\beta-1}$, $t > 0$ for some finite constant $M > 0$. Furthermore, we know that $(T(t))_{t>0}$ is given by the formula

$$T(t)x = \frac{1}{2\pi i}\int_{\Gamma} e^{\lambda t}\Big(\lambda - \mathcal{A}\Big)^{-1}x\,d\lambda, \quad t > 0,\ x \in X,$$

where Γ is the upwards oriented curve $\lambda = -c(|\eta| + 1) + i\eta$ ($\eta \in \mathbb{R}$). Assume that $0 < \gamma < 1$ and $\nu > -\beta$. Set

$$T_{\gamma,\nu}(t)x := t^{\gamma\nu} \int_0^\infty s^\nu \Phi_\gamma(s) T\Big(st^\gamma\Big)x\, ds, \quad t > 0,\ x \in X$$

and following E. Bazhlekova [1], R.-N. Wang, D.-H. Chen, T.-J. Xiao [24],

$$S_\gamma(t) := T_{\gamma,0}(t),\ P_\gamma(t) := \gamma T_{\gamma,1}(t)/t^\gamma, \quad t > 0.$$

Recall that $(S_\gamma(t))_{t>0}$ is a subordinated (g_γ, I)-regularized resolvent family generated by $\mathcal{A}$, which is not necessarily strongly continuous at zero. In [18], we have proved that there exists a finite constant $M_1 > 0$ such that

$$\Big\|S_\gamma(t)\Big\| + \Big\|P_\gamma(t)\Big\| \leq M_1 t^{\gamma(\beta-1)}, \quad t > 0. \tag{1}$$

Furthermore, in [19], we have proved that there exists a finite constant $M_2 > 0$ such that

$$\Big\|S_\gamma(t)\Big\| \leq M_2 t^{-\gamma},\ t \geq 1 \quad \text{and} \quad \Big\|P_\gamma(t)\Big\| \leq M_2 t^{-2\gamma},\ t \geq 1. \tag{2}$$

Set $R_\gamma(t) := t^{\gamma-1} P_\gamma(t)$, $t > 0$.

2.3. *Lebesgue spaces with variable exponents $L^{p(x)}$*

Let $\emptyset \neq \Omega \subseteq \mathbb{R}$ be a nonempty subset and let $M(\Omega : X)$ stand for we the collection of all measurable functions $f : \Omega \to X$; $M(\Omega) := M(\Omega : \mathbb{R})$. Furthermore, $\mathcal{P}(\Omega)$ denotes the vector space of all Lebesgue measurable functions $p : \Omega \to [1, \infty]$. For any $p \in \mathcal{P}(\Omega)$ and $f \in M(\Omega : X)$, set

$$\varphi_{p(x)}(t) := \begin{cases} t^{p(x)}, & t \geq 0,\ 1 \leq p(x) < \infty, \\ 0, & 0 \leq t \leq 1,\ p(x) = \infty, \\ \infty, & t > 1,\ p(x) = \infty \end{cases}$$

and

$$\rho(f) := \int_\Omega \varphi_{p(x)}(\|f(x)\|)\, dx.$$

We define the Lebesgue space $L^{p(x)}(\Omega : X)$ with variable exponent as follows,

$$L^{p(x)}(\Omega : X) := \Big\{f \in M(\Omega : X) : \lim_{\lambda \to 0+} \rho(\lambda f) = 0\Big\}$$

equivalently

$$L^{p(x)}(\Omega : X) = \Big\{f \in M(\Omega : X) : \text{ there exists } \lambda > 0 \text{ such that } \rho(\lambda f) < \infty\Big\};$$

see, e.g., [8, p. 73].

For every $u \in L^{p(x)}(\Omega : X)$, we introduce the Luxemburg norm of $u(\cdot)$ in the following manner:

$$\|u\|_{p(x)} := \|u\|_{L^{p(x)}(\Omega:X)} := \inf\Big\{\lambda > 0 : \rho(f/\lambda) \leq 1\Big\}.$$

Equipped with the above norm, the space $L^{p(x)}(\Omega : X)$ becomes a Banach space (see e.g. [8, Theorem 3.2.7] for scalar-valued case), coinciding with the usual Lebesgue space $L^p(\Omega : X)$ in the case that $p(x) = p \geq 1$ is a constant function. For any $p \in M(\Omega)$, we set

$$p^- := \operatorname{essinf}_{x\in\Omega} p(x) \quad \text{and} \quad p^+ := \operatorname{esssup}_{x\in\Omega} p(x).$$

Define

$$C_+(\Omega) := \Big\{p \in M(\Omega) : 1 < p^- \leq p(x) \leq p^+ < \infty \text{ for a.e. } x \in \Omega\Big\}$$

and

$$D_+(\Omega) := \Big\{p \in M(\Omega) : 1 \leq p^- \leq p(x) \leq p^+ < \infty \text{ for a.e. } x \in \Omega\Big\}.$$

For $p \in D_+([0,1])$, the space $L^{p(x)}(\Omega : X)$ behaves nicely, with almost all fundamental properties of the Lesbesgue space with constant exponent $L^p(\Omega : X)$ being retained; in this case, we know that

$$L^{p(x)}(\Omega : X) = \Big\{f \in M(\Omega : X) : \text{ for all } \lambda > 0 \text{ we have } \rho(\lambda f) < \infty\Big\}.$$

Furthermore, if $p \in C_+(\Omega)$, then $L^{p(x)}(\Omega : X)$ is uniformly convex and thus reflexive ([10]).

We will use the following lemma (see, e.g., [8, Lemma 3.2.20, (3.2.22); Corollary 3.3.4; p. 77] for scalar-valued case):

Lemma 2.1. *(i) Let $p, q, r \in \mathcal{P}(\Omega)$ such that*

$$\frac{1}{q(x)} = \frac{1}{p(x)} + \frac{1}{r(x)}, \quad x \in \Omega.$$

Then, for every $u \in L^{p(x)}(\Omega : X)$ and $v \in L^{r(x)}(\Omega)$, we have $uv \in L^{q(x)}(\Omega : X)$ and

$$\|uv\|_{q(x)} \leq 2\|u\|_{p(x)}\|v\|_{r(x)}.$$

(ii) Let Ω be of a finite Lebesgue's measure and let $p, q \in \mathcal{P}(\Omega)$ such $q \leq p$ a.e. on Ω. Then $L^{p(x)}(\Omega : X)$ is continuously embedded in $L^{q(x)}(\Omega : X)$.

(iii) Let $f \in L^{p(x)}(\Omega : X)$, $g \in M(\Omega : X)$ and $0 \leq \|g\| \leq \|f\|$ a.e. on Ω. Then $g \in L^{p(x)}(\Omega : X)$ and $\|g\|_{p(x)} \leq \|f\|_{p(x)}$.

For additional details upon Lebesgue spaces with variable exponents $L^{p(x)}$, we refer the reader to the following sources: [6], [7], [8], [10] and [23].

3. Stepanov generalizations of almost periodic and asymptotically almost periodic functions

Let $1 \leq p < \infty$, $l > 0$, and $f, g \in L^p_{loc}(I : X)$, where $I = \mathbb{R}$ or $I = [0, \infty)$. We define the Stepanov 'metric' by

$$D^p_{S_l}\big[f(\cdot), g(\cdot)\big] := \sup_{x \in I}\left[\frac{1}{l}\int_x^{x+l} \big\|f(t) - g(t)\big\|^p \, dt\right]^{1/p}.$$

Clearly, for every two numbers $l_1, l_2 > 0$, there exist two positive real constants $k_1, k_2 > 0$ independent of f, g, such that

$$k_1 D^p_{S_{l_1}}\big[f(\cdot), g(\cdot)\big] \leq D^p_{S_{l_2}}\big[f(\cdot), g(\cdot)\big] \leq k_2 D^p_{S_{l_1}}\big[f(\cdot), g(\cdot)\big].$$

The Stepanov norm of $f(\cdot)$ is introduced by setting

$$\|f\|_{S^p_l} := D^p_{S_l}[f(\cdot), 0].$$

In the sequel, we assume that $l_1 = l_2 = 1$.

Definition 3.1. *A function $f \in L^p_{loc}(I : X)$ is said to be Stepanov p-bounded (or S^p-bounded), if*

$$\|f\|_{S^p} := \sup_{t \in I}\left(\int_t^{t+1} \|f(s)\|^p \, ds\right)^{1/p} = \sup_{t \in I}\left(\int_0^1 \|f(s + t)\|^p \, ds\right)^{1/p} < \infty.$$

Endowed with the above norm, the space $L_S^p(I : X)$ consisting of all S^p-bounded functions, becomes a Banach space.

Definition 3.2. *A function $f \in L_S^p(I : X)$ is said to be Stepanov p-almost periodic or S^p-almost periodic shortly, if the function $\hat{f} : I \to L^p([0,1] : X)$, defined by*

$$\hat{f}(t)(s) := f(t+s), \quad t \in I,\ s \in [0,1]$$

is almost periodic.

Similarly,

Definition 3.3. *A function $f \in L_S^p([0,\infty) : X)$ is said to be asymptotically Stepanov p-almost periodic or asymptotically S^p-almost periodic, if the function $\hat{f}(\cdot)$ is asymptotically almost periodic.*

It is well known that the space of Stepanov almost periodic functions (respectively, asymptotically Stepanov almost periodic functions) denoted by $APS^p(I : X)$ (respectively, $AAPS^p([0,\infty) : X)$) is a closed linear subspace of $L_S^p(I : X)$ (respectively, $L_S^p([0,\infty) : X)$) and hence is a Banach space.

The symbol $S_0^p([0,\infty) : X)$ stands for the vector space consisting of all functions $q \in L_{loc}^p([0,\infty) : X)$ such that $\hat{q} \in C_0([0,\infty) : L^p([0,1] : X))$.

If $1 \leq p < q < \infty$ and $f(\cdot)$ is (asymptotically) Stepanov q-almost periodic, then $f(\cdot)$ is (asymptotically) Stepanov p-almost periodic. Therefore, the (asymptotic) Stepanov p-almost periodicity of $f(\cdot)$ for some $p \in [1,\infty)$ implies the (asymptotical) Stepanov p-almost periodicity of $f(\cdot)$. It is a well-known fact that if $f(\cdot)$ is an almost periodic (respectively, asymptotically almost periodic) function then $f(\cdot)$ is also S^p-almost periodic (respectively, asymptotically S^p-almost periodic) for $1 \leq p < \infty$. And in general, the converse statement is false.

4. Generalized almost periodic and generalized asymptotically almost periodic functions in Lebesgue spaces with variable exponents $L^{p(x)}$

The following notion of Stepanov $p(x)$-boundedness differs from the one introduced by Diagana and Zitane in [6, Definition 3.10] and [7, Definition 4.5], where the authors have used the condition $p \in C_+(\mathbb{R})$:

Definition 4.1. *Let $p \in \mathcal{P}([0,1])$ and let $I = \mathbb{R}$ or $I = [0,\infty)$. A function $f \in M(I : X)$ is said to be Stepanov $p(x)$-bounded (or $S^{p(x)}$-bounded), if $f(\cdot + t) \in L^{p(x)}([0,1] : X)$ for all $t \in I$, and $\sup_{t\in I} \|f(\cdot + t)\|_{p(x)} < \infty$, that is,*

$$\|f\|_{S^{p(x)}} := \sup_{t\in I} \inf\left\{\lambda > 0 : \int_0^1 \varphi_{p(x)}\left(\frac{\|f(x+t)\|}{\lambda}\right) dx \leq 1\right\} < \infty.$$

The collection of such functions will be denoted by $L_S^{p(x)}(I : X)$.

From Definition 4.1 it follows that the space $L_S^{p(x)}(I : X)$ is translation invariant in the sense that, for every $f \in L_S^{p(x)}(I : X)$ and $\tau \in I$, we have $f(\cdot + \tau) \in L_S^{p(x)}(I : X)$. This is not the case with the notion introduced by Diagana and Zitane [6]-[7], since there the space $L_S^{p(x)}(I : X)$ may or may not be translation invariant depending on $p(x)$. Furthermore, let us note that the notion introduced in these papers is meaningful even in the case that $p \in \mathcal{P}(\mathbb{R})$.

We introduce the concept of (asymptotic) $S^{p(x)}$-almost periodicity as follows:

Definition 4.2. *(i) Let $p \in \mathcal{P}([0,1])$ and let $I = \mathbb{R}$ or $I = [0,\infty)$. A function $f \in L_S^{p(x)}(I : X)$ is said to be Stepanov $p(x)$-almost periodic (or Stepanov $p(x)$-a.p.), if the function $\hat{f} : I \to L^{p(x)}([0,1] : X)$ is almost periodic. The collection of such functions will be denoted by $APS^{p(x)}(I : X)$.*

(ii) Let $p \in \mathcal{P}([0,1])$ and let $I = [0,\infty)$. A function $f \in L_S^{p(x)}(I : X)$ is said to be asymptotically Stepanov $p(x)$-almost periodic (or asymptotically Stepanov $p(x)$-a.p.), if the function $\hat{f} : I \to L^{p(x)}([0,1] : X)$ is asymptotically almost periodic. The collection of such functions will be denoted by $AAPS^{p(x)}(I : X)$. The abbreviation $S_0^{p(x)}([0,\infty) : X)$ will be used to denote the set of all functions $q \in L_S^{p(x)}([0,\infty) : X)$ such that $\hat{q} \in C_0([0,\infty) : L^{p(x)}([0,1] : X))$.

As in the case of Stepanov $p(x)$-boundedness, the space $APS^{p(x)}(I : X)$ is translation invariant in the sense that, for every $f \in APS^{p(x)}(I : X)$ and $\tau \in I$, we have $f(\cdot + \tau) \in APS^{p(x)}(I : X)$. A similar statement holds for the space $AAPS^{p(x)}([0,\infty) : X)$.

It can be easily checked that the notions of (asymptotic) Stepanov $p(x)$-boundedness and (asymptotic) Stepanov $p(x)$-almost periodicity are equivalent with those introduced in the previous section, provided that $p(x) \equiv p \geq 1$ is a constant function.

Equipped with the norm $\|\cdot\|_{S^{p(x)}}$, the space $L_S^{p(x)}(I : X)$ consisting of all S^p-bounded functions is a Banach space, which is continuously embedded in $L_S^1(I : X)$, for any $p \in \mathcal{P}([0,1])$. Furthermore, it can be easily shown that $APS^{p(x)}(I : X)$ ($AAPS^{p(x)}(I : X)$ with $I = [0,\infty)$) is a closed subspace of $L_S^{p(x)}(I : X)$ and therefore is Banach space itself, for any $p \in \mathcal{P}([0,1])$.

If $p \in \mathcal{P}([0,1])$, then Lemma 2.1(ii) yields $L^{p(x)}([0,1] : X) \hookrightarrow L^1([0,1] : X)$, where the symbol $\hookrightarrow$ stands for a "continuous embedding", so that $L_S^{p(x)}(I : X) \hookrightarrow L_S^1(I : X)$, as well.

We have

Proposition 4.3. *Suppose $p \in \mathcal{P}([0,1])$. Then the following continuous embedding hold,*

(i) $L_S^{p(x)}(I : X) \hookrightarrow L_S^1(I : X)$; and

(ii) $APS^{p(x)}(I : X) \hookrightarrow APS^1(I : X)$ and $AAPS^{p(x)}([0,\infty) : X) \hookrightarrow AAPS^1([0,\infty) : X)$.

Similarly,

Proposition 4.4. *Suppose $p \in D_+([0,1])$ and $1 \leq p^- \leq p(x) \leq p^+ < \infty$ for a.e. $x \in [0,1]$. Then the following continuous embedding hold,*

(i) $L_S^{p^+}(I : X) \hookrightarrow L_S^{p(x)}(I : X) \hookrightarrow L_S^{p^-}(I : X)$; and

(ii) $APS^{p^+}(I : X) \hookrightarrow APS^{p(x)}(I : X) \hookrightarrow APS^{p^-}(I : X)$ and $AAPS^{p^+}([0,\infty) : X) \hookrightarrow AAPS^{p(x)}([0,\infty) : X) \hookrightarrow AAPS^{p^-}([0,\infty) : X)$.

Now we will prove that any almost periodic function is $S^{p(x)}$-almost periodic, for any $p \in \mathcal{P}([0,1])$.

Proposition 4.5. *Let $p \in \mathcal{P}([0,1])$, and let $f : I \to X$ be almost periodic. Then $f(\cdot)$ is $S^{p(x)}$-almost periodic.*

Proof. To prove that $f(\cdot)$ is $S^{p(x)}$-bounded and $\|f\|_{L_S^{p(x)}} \leq \|f\|_\infty$, it suffices to show that, for every $t \in \mathbb{R}$, we have:

$$\Big[\|f\|_\infty, \infty\Big) \subseteq \Bigg\{\lambda > 0 : \int_0^1 \varphi_{p(x)}\Bigg(\frac{\|f(x+t)\|}{\lambda}\Bigg)dx \leq 1\Bigg\}. \tag{3}$$

For $\lambda \geq \|f\|_\infty$, we have $\|f(x+t)\|/\lambda \leq 1$, $t \in I$. It can be simply perceived that, in this case,

$$\varphi_{p(x)}\Bigg(\frac{\|f(x+t)\|}{\lambda}\Bigg) \leq 1, \quad t \in I,$$

so that the integrand does not exceed 1; as a matter of fact, by definition of $\varphi_{p(x)}(\cdot)$, we only need to observe that, for every $x \in [0,1]$ with $p(x) < \infty$, we have $(\|f(t+x)\|/\lambda)^{p(x)} \leq 1^{p(x)} = 1$, $t \in I$. Hence, (3) holds. Using the uniform continuity of $f(\cdot)$ and a similar argumentation, we can show that the function $\hat{f} : I \to L^{p(x)}([0,1] : X)$ is uniform continuous. For direct proof of almost periodicity of function $\hat{f} : I \to L^{p(x)}([0,1] : X)$, we can argue as follows. For $\epsilon > 0$ given as above, there is a finite number $l > 0$ such that any subinterval I' of I of length l contains a number $\tau \in I'$ such that $\|f(t+\tau) - f(t)\| \leq \epsilon$, $t \in I$. It suffices to observe that, for this $\epsilon > 0$, we can choose the same length $l > 0$ and the same ϵ-almost period τ from I' ensuring the validity of inequality $\|\hat{f}(t+\tau+\cdot) - \hat{f}(t+\cdot)\|_{L^{p(x)}([0,1]:X)} \leq \epsilon$, $t \in I$: in order to see that the last inequality holds true, we only need to prove that, for every $t \in I$, we have

$$[\epsilon, \infty) \subseteq \left\{ \lambda > 0 : \int_0^1 \varphi_{p(x)}\left(\frac{\|f(t+\tau+x) - f(t+x)\|}{\lambda} \right) dx \leq 1 \right\}.$$

Indeed, if $\lambda \geq \epsilon$, then $\|f(t+\tau+x) - f(t+x)\|/\lambda \leq 1$, $t \in I$ and the integrand cannot exceed 1 : this simply follows from definition of $\varphi_{p(x)}(\cdot)$ and observation that, for every $x \in [0,1]$ with $p(x) < \infty$, we have $(\|f(t+\tau+x) - f(t+x)\|/\lambda)^{p(x)} \leq 1^{p(x)} = 1$, $t \in I$. The proof of the proposition is thereby complete. □

We can similarly prove the following proposition:

Proposition 4.6. *Let $p \in \mathcal{P}([0,1])$, and let $f : [0,\infty) \to X$ be asymptotically almost periodic. Then $f(\cdot)$ is asymptotically $S^{p(x)}$-almost periodic.*

Taking into account Proposition 4.3(ii) and the method employed in the proof of Proposition 4.5, we can state the following:

Proposition 4.7. *Assume that $p \in \mathcal{P}([0,1])$ and $f \in L_S^{p(x)}(I : X)$. Then the following holds:*

(i) $L^\infty(I : X) \hookrightarrow L_S^{p(x)}(I : X) \hookrightarrow L_S^1(I : X)$.

(ii) $AP(I : X) \hookrightarrow APS^{p(x)}(I : X) \hookrightarrow APS^1(I : X)$ *and* $AAP([0,\infty) : X) \hookrightarrow AAPS^{p(x)}([0,\infty) : X) \hookrightarrow AAPS^1([0,\infty) : X)$.

(iii) The continuity (uniform continuity) of $f(\cdot)$ implies continuity (uniform continuity) of $\hat{f}(\cdot)$.

In general case, we have the following:

Proposition 4.8. *Assume that $p,\ q \in \mathcal{P}([0,1])$ and $p \leq q$ a.e. on $[0,1]$. Then we have:*

(i) $L_S^{q(x)}(I : X) \hookrightarrow L_S^{p(x)}(I : X)$.

(ii) $APS^{q(x)}(I : X) \hookrightarrow APS^{p(x)}(I : X)$ *and* $AAPS^{q(x)}([0,\infty) : X) \hookrightarrow AAPS^{p(x)}([0,\infty) : X)$.

(iii) If $p \in D_+([0,1])$, then

$$L^\infty(I : X) \cap APS^{p(x)}(I : X) = L^\infty(I : X) \cap APS^1(I : X)$$

and

$$L^\infty([0,\infty) : X) \cap AAPS^{p(x)}([0,\infty) : X) = L^\infty([0,\infty) : X) \cap AAPS^1([0,\infty) : X).$$

Proof. We will prove only (iii) for almost periodicity. Keeping in mind Proposition 4.4(ii), it suffices to assume that $p(x) \equiv p > 1$. Then, clearly, $L^\infty(I : X) \cap APS^p(I : X) \subseteq L^\infty(I : X) \cap APS^1(I : X)$ and it remains to be proved the opposite inclusion. So, let $f \in L^\infty(I : X) \cap APS^1(I : X)$. The required conclusion is a consequence of elementary definitions and following simple calculation, which is valid for any $t,\ \tau \in \mathbb{R}$:

$$\begin{aligned}\left[\int_t^{t+1} \left\|f(\tau+s)-f(s)\right\|^p ds\right]^{1/p} \\ &\leq \left[\int_t^{t+1} \left(2\|f\|_\infty\right)^{p-1} \left\|f(\tau+s)-f(s)\right\| ds\right]^{1/p} \\ &= \left(2\|f\|_\infty\right)^{(p-1)/p} \left[\int_t^{t+1} \left\|f(\tau+s)-f(s)\right\| ds\right]^{1/p}.\end{aligned}$$

□

Remark 4.9. *It is well known that $APS^{p(x)}(I : X)$ can be strictly contained in $APS^1(I : X)$, even in the case that $p(x) \equiv p > 1$ is a constant function. For example, H. Bohr and E. Følner have proved that, for any given number $p > 1$, we can construct a Stepanov almost periodic function defined on the whole real axis that is not Stepanov p-almost periodic (see [2, Example, p. 70]). The same example shows that $AAPS^p([0,\infty) : X)$ can be strictly contained in $AAPS^1([0,\infty) : X)$ for $p > 1$ (see e.g. [15, Lemma 1]).*

Remark 4.10. *Proposition 4.5 and Proposition 4.6 can be simply deduced by using Proposition 4.8(ii) and the equalities $AP(I : X) = APS^\infty(I : X) \cap C(I : X)$, $AAP([0,\infty) : X) = AAPS^\infty([0,\infty) : X) \cap C([0,\infty) : X)$, which can be proved almost trivially.*

Now we would like to present the following illustrative example:

Example 4.11. *Define $sign(0) := 0$. Then, for every almost periodic function $f : \mathbb{R} \to \mathbb{R}$, we have that the function $F(\cdot)$:=$sign(f(\cdot))$ is Stepanov 1-almost periodic ([21]). Since $F \in L^\infty(\mathbb{R})$, Proposition 4.8(iii) yields that the function $F(\cdot)$ is Stepanov p-almost periodic for any $p \geq 1$, while Proposition 4.7(i) yields that the function $F(\cdot)$ is Stepanov $p(x)$-bounded for any $p \in \mathcal{P}([0,1])$. Due to Proposition 4.4(ii), we have $F \in APS^{p(x)}(\mathbb{R} : \mathbb{C})$ for any $p \in D_+([0,1])$.*

Consider now the case that $f(x) := \sin x + \sin \sqrt{2}x$, $x \in \mathbb{R}$ and $p(x) := 1 - \ln x$, $x \in [0,1]$. We will prove that $F \notin APS^{p(x)}(\mathbb{R} : \mathbb{C})$. Speaking-matter-of-factly, it is sufficient to show that, for every $\lambda \in (0, 2/e)$ and for every $l > 0$, we can find an interval $I \subseteq \mathbb{R}$ of length $l > 0$ such that, for every $\tau \in I$, there exists $t \in \mathbb{R}$ such that

$$\begin{aligned}\int_0^1 \Big(\frac{1}{\lambda}\Big)^{1-\ln x} \Big| sign\Big[\sin(x+t+\tau) + \sin \sqrt{2}(x+t+\tau)\Big] \\ - sign\Big[\sin(x+t) + \sin \sqrt{2}(x+t)\Big]\Big|^{1-\ln x} dx = \infty. \qquad (4)\end{aligned}$$

Let $\lambda \in (0, 2/e)$ and $l > 0$ be given. Take arbitrarily any interval $I \subseteq \mathbb{R} \setminus \{0\}$ of length l and after that take arbitrarily any number $\tau \in I$. Since $(1/\lambda)^{1-\ln x} \geq 1/x$, $x \in [0,1]$ and $1 - \ln x \geq 1$, $x \in [0,1]$, a continuity argument shows that it is enough to prove the existence of a number $t \in \mathbb{R}$ such that

$$\Big[\sin(t+\tau) + \sin \sqrt{2}(t+\tau)\Big] \cdot \Big[\sin t + \sin \sqrt{2}t\Big] < 0. \qquad (5)$$

If $\sin \tau + \sin \sqrt{2}\tau > 0$ ($\sin \tau + \sin \sqrt{2}\tau < 0$), then we can take $t \sim 0-$ ($t \sim 0+$). Hence, we assume henceforward $\sin \tau + \sin \sqrt{2}\tau = 0$ and $\tau \neq 0$. There exist two possibilities:

$$\tau \in \frac{2\mathbb{Z}\pi}{1+\sqrt{2}} \setminus \{0\} \quad or \quad \tau \in \frac{(2\mathbb{Z}+1)\pi}{\sqrt{2}-1}.$$

In the first case, take $t_0 = \frac{\pi}{\sqrt{2}-1}$. Then an elementary argumentation shows that $\tau + t_0 \notin \frac{2\mathbb{Z}\pi}{1+\sqrt{2}} \cup \frac{(2\mathbb{Z}+1)\pi}{\sqrt{2}-1}$ so that $\sin(t_0+\tau) + \sin \sqrt{2}(t_0+\tau) \neq 0$. If $\sin(t_0+\tau) + \sin \sqrt{2}(t_0+\tau) > 0$ ($\sin(t_0+\tau) + \sin \sqrt{2}(t_0+\tau) < 0$), then for t

satisfying (5) *we can take any number belonging to a small left/right interval around t_0 for which* $\sin t + \sin \sqrt{2}t < 0$ ($\sin t + \sin \sqrt{2}t > 0$). *In the second case, there exists an integer $m \in \mathbb{Z}$ such that $\tau = \frac{(2m+1)\pi}{\sqrt{2}-1}$ and we can take $t_0 = \frac{(-2m+1)\pi}{\sqrt{2}-1}$. Then $\tau + t_0 = \frac{2\pi}{\sqrt{2}-1}$ and $\sin(t_0 + \tau) + \sin \sqrt{2}(t_0 + \tau) \neq 0$, so that we can use a trick similar to that used in the first case. Let us only mention in passing that, with the notion introduced in [5], the function $F(\cdot)$ cannot be $S^{p(x)}$-almost automorphic, as well.*

The situation is quite different if we consider the case that $f(x) := \sin x$, $x \in \mathbb{R}$. Then $F(\cdot)$ is Stepanov $p(x)$-almost periodic for any $p \in \mathcal{P}([0,1])$. Speaking-matter-of-factly, it can be easily shown that the mapping $\hat{F} : \mathbb{R} \to L^{p(x)}[0,1]$ is continuous and $\|F(t + \tau + \cdot) - F(t + \cdot)\|_{L^{p(x)}[0,1]} = 0$ for all $t \in \mathbb{R}$ and $\tau \in 2\pi\mathbb{Z}$. This, in turn, implies the claimed statement.

Keeping in mind the proofs of Proposition 4.5, [5, Proposition 3.5] and [15, Lemma 1], we can clarify the following result:

Proposition 4.12. *Suppose that $p \in \mathcal{P}([0,1])$ and $f : [0,\infty) \to X$ is an asymptotically $S^{p(x)}$-almost periodic function. Then there are uniquely determined $S^{p(x)}$-bounded functions $g : \mathbb{R} \to X$ and $q : [0,\infty) \to X$ satisfying the following conditions:*

(i) g is $S^{p(x)}$-almost periodic,

(ii) $\hat{q}$ belongs to the class $C_0([0,\infty) : L^{p(x)}([0,1] : X))$,

(iii) $f(t) = g(t) + q(t)$ for all $t \geq 0$.

Moreover, there exists an increasing sequence $(t_n)_{n\in\mathbb{N}}$ of positive reals such that $\lim_{n\to\infty} t_n = \infty$ and $g(t) = \lim_{n\to\infty} f(t + t_n)$ a.e. $t \geq 0$.

Remark 4.13. *The definition of an (equi-)Weyl $p(x)$-almost periodic function (see e.g. [19] for the case that $p(x) \equiv p \in [1,\infty)$) can be introduced as follows: Suppose $I = \mathbb{R}$ or $I = [0,\infty)$. Let $p \in \mathcal{P}(I)$ and $f(\cdot + \tau) \in L^{p(x)}(K : X)$ for any $\tau \in I$ and any compact subset K of I.*

(i) It is said that the function $f(\cdot)$ is equi-Weyl-$p(x)$-almost periodic, $f \in e - W_{ap}^{p(x)}(I : X)$ for short, iff for each $\epsilon > 0$ we can find two real numbers $l > 0$ and $L > 0$ such that any interval $I' \subseteq I$ of length L contains a point $\tau \in I'$ such that

$$\sup_{t\in I}\left[l^{(-1)/p(t)}\left\|f(\cdot + \tau) - f(\cdot)\right\|_{L^{p(x)}[t,t+l]}\right] \leq \epsilon.$$

(ii) It is said that the function $f(\cdot)$ is Weyl-$p(x)$-almost periodic, $f \in W_{ap}^{p(x)}(I : X)$ for short, iff for each $\epsilon > 0$ we can find a real number $L > 0$ such that any interval $I' \subseteq I$ of length L contains a point $\tau \in I'$ such that

$$\lim_{l\to\infty}\sup_{t\in I}\left[l^{(-1)/p(t)}\left\|f(\cdot + \tau) - f(\cdot)\right\|_{L^{p(x)}[t,t+l]}\right] \leq \epsilon.$$

The notion of (equi-)Weyl $p(x)$-almost periodicity as well as the corresponding notion for Besicovitch classes of almost periodic functions will not attract our attention here. We will also skip all details concerning asymptotical $p(x)$-almost periodicity for Weyl and Besicovitch classes.

5. Generalized two-parameter almost periodic type functions and composition principles

Assume that $(Y, \|\cdot\|_Y)$ is a complex Banach space, as well as that $I = \mathbb{R}$ or $I = [0,\infty)$. By $C_0([0,\infty) \times Y : X)$ we denote the space consisting of all continuous functions $h : [0,\infty) \times Y \to X$ such that $\lim_{t\to\infty} h(t, y) = 0$ uniformly for y in any compact subset of Y. A continuous function $f : I \times Y \to X$ is said to be uniformly continuous on bounded sets, uniformly for $t \in I$ iff for every $\epsilon > 0$ and every bounded subset K of Y there exists a number $\delta_{\epsilon,K} > 0$ such that $\|f(t, x) - f(t, y)\| \leq \epsilon$ for all $t \in I$ and all $x, y \in K$ satisfying that $\|x - y\| \leq \delta_{\epsilon,K}$. If $f : I \times Y \to X$, set $\hat{f}(t, y) := f(t + \cdot, y)$, $t \geq 0$, $y \in Y$.

We need to recall the following well-known definition (see e.g. [19] for more details):

Definition 5.1. *Let* $1 \leq p < \infty$.

(i) A function $f : I \times Y \to X$ *is said to be almost periodic iff* $f(\cdot, \cdot)$ *is bounded, continuous as well as for every* $\epsilon > 0$ *and every compact* $K \subseteq Y$ *there exists* $l(\epsilon, K) > 0$ *such that every subinterval* $J \subseteq I$ *of length* $l(\epsilon, K)$ *contains a number* τ *with the property that* $\|f(t + \tau, y) - f(t, y)\| \leq \epsilon$ *for all* $t \in I$, $y \in K$. *The collection of such functions will be denoted by* $AP(I \times Y : X)$.

(ii) A function $f : [0, \infty) \times Y \to X$ *is said to be asymptotically almost periodic iff it is bounded continuous and admits a decomposition* $f = g + q$, *where* $g \in AP([0, \infty) \times Y : X)$ *and* $q \in C_0([0, \infty) \times Y : X)$. *Denote by* $AAP([0, \infty) \times Y : X)$ *the vector space consisting of all such functions.*

The notion of (asymptotical) Stepanov $p(x)$-almost periodicity for the functions depending on two parameters is introduced as follows:

Definition 5.2. *Let* $p \in \mathcal{P}([0, 1])$.

(i) A function $f : I \times Y \to X$ *is called Stepanov* $p(x)$*-almost periodic,* $S^{p(x)}$*-almost periodic for short, iff* $\hat{f} : I \times Y \to L^{p(x)}([0, 1] : X)$ *is almost periodic. The vector space consisting of all such functions will be denoted by* $APS^{p(x)}(I \times Y : X)$.

(ii) A function $f : [0, \infty) \times Y \to X$ *is said to be asymptotically* $S^{p(x)}$*-almost periodic iff* $\hat{f} : [0, \infty) \times Y \to L^{p(x)}([0, 1] : X)$ *is asymptotically almost periodic. The vector space consisting of all such functions will be denoted by* $AAPS^{p(x)}([0, \infty) \times Y : X)$.

The proof of following proposition is very similar to the proof of [19, Lemma 2.2.6] and therefore omitted.

Proposition 5.3. *Let* $p \in \mathcal{P}([0, 1])$. *Suppose that* $f : [0, \infty) \times Y \to X$ *is an asymptotically* $S^{p(x)}$*-almost periodic function. Then there are two functions* $g : \mathbb{R} \times Y \to X$ *and* $q : [0, \infty) \times Y \to X$ *satisfying that for each* $y \in Y$ *the functions* $g(\cdot, y)$ *and* $q(\cdot, y)$ *are Stepanov* $p(x)$*-bounded, as well as that the following holds:*

(i) $\hat{g} : \mathbb{R} \times Y \to L^{p(x)}([0, 1] : X)$ *is almost periodic,*

(ii) $\hat{q} \in C_0([0, \infty) \times Y : L^{p(x)}([0, 1] : X))$,

(iii) $f(t, y) = g(t, y) + q(t, y)$ *for all* $t \geq 0$ *and* $y \in Y$.

Moreover, for every compact set $K \subseteq Y$, *there exists an increasing sequence* $(t_n)_{n \in \mathbb{N}}$ *of positive reals such that* $\lim_{n \to \infty} t_n = \infty$ *and* $g(t, y) = \lim_{n \to \infty} f(t + t_n, y)$ *for all* $y \in Y$ *and a.e.* $t \geq 0$.

In [19, Theorem 2.7.1, Theorem 2.7.2], we have slightly improved the important composition principle atributed to W. Long, S.-H. Ding [22, Theorem 2.2]. Further refinements for $S^{p(x)}$-almost periodicity can be deduced similarly, with appealing to Lemma 2.1(i)-(iii) and the arguments employed in the proof of [22, Theorem 2.2]:

Theorem 5.4. *Let* $I = \mathbb{R}$ *or* $I = [0, \infty)$, *and let* $p \in \mathcal{P}([0, 1])$. *Suppose that the following conditions hold:*

(i) $f \in APS^{p(x)}(I \times Y : X)$ *and there exist a function* $r \in \mathcal{P}([0, 1])$ *such that* $r(\cdot) \geq \max(p(\cdot), p(\cdot)/p(\cdot) - 1)$ *and a function* $L_f \in L_S^{r(x)}(I)$ *such that:*

$$\|f(t, x) - f(t, y)\| \leq L_f(t)\|x - y\|_Y, \quad t \in I, \ x, \ y \in Y; \tag{6}$$

(ii) $u \in APS^{p(x)}(I : Y)$, *and there exists a set* $\mathrm{E} \subseteq I$ *with* $m(\mathrm{E}) = 0$ *such that* $K := \{u(t) : t \in I \setminus \mathrm{E}\}$ *is relatively compact in* Y; *here,* $m(\cdot)$ *denotes the Lebesgue measure.*

Define $q \in \mathcal{P}([0, 1])$ *by* $q(x) := p(x)r(x)/p(x) + r(x)$, *if* $x \in [0, 1]$ *and* $r(x) < \infty$, $q(x) := p(x)$, *if* $x \in [0, 1]$ *and* $r(x) = \infty$. *Then* $q(x) \in [1, p(x))$ *for* $x \in [0, 1]$, $r(x) < \infty$ *and* $f(\cdot, u(\cdot)) \in APS^{q(x)}(I : X)$.

Concerning asymptotical two-parameter Stepanov $p(x)$-almost periodicity, we can deduce the following composition principles with $X = Y$; the proof is very similar to those of [19, Proposition 2.7.3, Proposition 2.7.4] established in the case of constant functions $p,\ q,\ r$:

Proposition 5.5. *Let $I = [0, \infty)$, and let $p \in \mathcal{P}([0, 1])$. Suppose that the following conditions hold:*

(i) *$g \in APS^{p(x)}(I \times X : X)$, there exist a function $r \in \mathcal{P}([0, 1])$ such that $r(\cdot) \geq \max(p(\cdot), p(\cdot)/p(\cdot) - 1)$ and a function $L_g \in L_S^{r(x)}(I)$ such that (6) holds with the function $f(\cdot, \cdot)$ replaced by the function $g(\cdot, \cdot)$ therein.*

(ii) *$v \in APS^{p(x)}(I : X)$, and there exists a set $\mathrm{E} \subseteq I$ with $m(\mathrm{E}) = 0$ such that $K = \{v(t) : t \in I \setminus \mathrm{E}\}$ is relatively compact in X.*

(iii) *$f(t, x) = g(t, x) + q(t, x)$ for all $t \geq 0$ and $x \in X$, where $\hat{q} \in C_0([0, \infty) \times X : L^{q(x)}([0, 1] : X))$ with $q(\cdot)$ defined as above;*

(iv) *$u(t) = v(t) + \omega(t)$ for all $t \geq 0$, where $\hat{\omega} \in C_0([0, \infty) : L^{p(x)}([0, 1] : X))$.*

(v) *There exists a set $E' \subseteq I$ with $m(E') = 0$ such that $K' = \{u(t) : t \in I \setminus E'\}$ is relatively compact in X.*

Then $f(\cdot, u(\cdot)) \in AAPS^{q(x)}(I : X)$.

6. Generalized (asymptotical) almost periodicity in Lebesgue spaces with variable exponents $L^{p(x)}$: action of convolution products

Throughout this section, we assume that $p \in \mathcal{P}([0, 1])$ and a multivalued linear operator $\mathcal{A}$ fulfills the condition (P). We will first investigate infinite convolution products. The results obtained can be simply incorporated in the study of existence and uniqueness of almost periodic solutions of the following abstract Cauchy differential inclusion of first order

$$u'(t) \in \mathcal{A}u(t) + g(t), \quad t \in \mathbb{R}$$

and the following abstract Cauchy relaxation differential inclusion

$$D_{t,+}^{\gamma} u(t) \in -\mathcal{A}u(t) + g(t),\ t \in \mathbb{R}, \tag{7}$$

where $D_{t,+}^{\gamma}$ denotes the Weyl-Liouville fractional derivative of order $\gamma \in (0, 1)$ and $g : \mathbb{R} \times X \to X$ satisfies certain assumptions; see [19] for further information in this direction. Keeping in mind composition principles clarified in the previous section, it is almost straightforward to reformulate some known results concerning semilinear analogues of the above inclusions (see e.g. [19, Theorem 2.7.6-Theorem 2.7.9; Theorem 2.9.10-Theorem 2.9.11; Theorem 2.9.17-Theorem 2.9.18]); because of that, this question will not be examined here for the sake of brevity.

We start by stating the following generalization of [20, Proposition 2.11] (the reflexion at zero keeps the spaces of Stepanov p-almost periodic functions unchanged, which may or may not be the case with the spaces of Stepanov $p(x)$-almost periodic functions):

Proposition 6.1. *Suppose that $q \in \mathcal{P}([0, 1])$, $1/p(x) + 1/q(x) = 1$ and $(R(t))_{t>0} \subseteq L(X, Y)$ is a strongly continuous operator family satisfying that $M := \sum_{k=0}^{\infty} \|R(\cdot + k)\|_{L^{q(x)}[0,1]} < \infty$. If $\check{g} : \mathbb{R} \to X$ is $S^{p(x)}$-almost periodic, then the function $G : \mathbb{R} \to Y$, given by*

$$G(t) := \int_{-\infty}^{t} R(t - s) g(s)\, ds, \quad t \in \mathbb{R}, \tag{8}$$

is well-defined and almost periodic.

Proof. Without loss of generality, we may assume that $X = Y$. It is clear that, for every $t \in \mathbb{R}$, we have that $G(t) = \int_0^\infty R(s)g(t-s)\,ds$ and that the last integral is absolutely convergent due to Lemma 2.1(i) and $S^{p(x)}$-boundedness of function $\check{g}(\cdot)$:

$$\begin{aligned}\int_0^\infty \|R(s)\|\|g(t-s)\|\,ds &= \sum_{k=0}^{\infty}\int_k^{k+1} \|R(s)\|\|g(t-s)\|\,ds\\ &= \sum_{k=0}^{\infty}\int_0^1 \|R(s+k)\|\|g(t-s-k)\|\,ds\\ &\leq 2\sum_{k=0}^{\infty}\|R(\cdot+k)\|_{L^{q(x)}([0,1]:X)}\|g(t-k-\cdot)\|_{L^{p(x)}([0,1]:X)}\\ &\leq 2M\sup_{t\in\mathbb{R}}\|\check{g}(\cdot-t)\|_{L^{p(x)}([0,1]:X)},\end{aligned}$$

for any $t \in \mathbb{R}$. Let a number $\epsilon > 0$ be fixed. Then there is a finite number $l > 0$ such that any subinterval I of $\mathbb{R}$ of length l contains a number $\tau \in I$ such that $\|\check{g}(t-\tau+\cdot) - \check{g}(t+\cdot)\|_{L^{p(x)}([0,1]:X)} \leq \epsilon$, $t \in \mathbb{R}$. Invoking Lemma 2.1(i) and this fact, we get

$$\begin{aligned}&\|G(t+\tau) - G(t)\|\\ &\quad\leq \int_0^\infty \|R(r)\|\cdot\|g(t+\tau-r) - g(t-r)\|\,dr\\ &\quad= \sum_{k=0}^{\infty}\int_k^{k+1} \|R(r)\|\cdot\|g(t+\tau-r) - g(t-r)\|\,dr\\ &\quad= \sum_{k=0}^{\infty}\int_0^1 \|R(r+k)\|\cdot\|g(t+\tau-r-k) - g(t-r-k)\|\,dr\\ &\quad\leq 2\sum_{k=0}^{\infty}\|R(\cdot+k)\|_{L^{q(x)}[0,1]}\|g(t+\tau-\cdot-k) - g(t-\cdot-k)\|_{L^{p(x)}[0,1]}\\ &\quad= 2\sum_{k=0}^{\infty}\|R(\cdot+k)\|_{L^{q(x)}[0,1]}\|\check{g}(\cdot-t-\tau+k) - \check{g}(\cdot-t+k)\|_{L^{p(x)}[0,1]}\\ &\quad\leq 2\epsilon\sum_{k=0}^{\infty}\|R(\cdot+k)\|_{L^{q(x)}[0,1]} = 2M\epsilon, \quad t \in \mathbb{R},\end{aligned}$$

which clearly implies that the set of all ϵ-periods of $G(\cdot)$ is relatively dense in $\mathbb{R}$. It remains to be proved the uniform continuity of $G(\cdot)$. Since $\hat{\check{g}}(\cdot)$ is uniformly continuous, we have the existence of a number $\delta \in (0,1)$ such that

$$\|\check{g}(\cdot-t') - \check{g}(\cdot-t)\|_{L^{p(x)}[0,1]} < \epsilon, \text{ provided } t,\ t' \in \mathbb{R} \text{ and } |t-t'| < \delta. \tag{9}$$

For any $\delta' \in (0,\delta)$, the above computation with $\tau = \delta' = t' - t$ and (9) together imply that, for every $t \in \mathbb{R}$,

$$\begin{aligned}&\left\|G(t+\delta') - G(t)\right\|\\ &\leq 2\sum_{k=0}^{\infty}\|R(\cdot+k)\|_{L^{q(x)}[0,1]}\|\check{g}(\cdot-t'+k) - \check{g}(\cdot-t+k)\|_{L^{p(x)}[0,1]}\\ &\leq 2\epsilon\sum_{k=0}^{\infty}\|R(\cdot+k)\|_{L^{q(x)}[0,1]} = 2M\epsilon.\end{aligned}$$

This completes the proof of proposition. □

Example 6.2. *(i) Suppose that $\beta \in (0,1)$ and $(R(t))_{t>0} = (T(t))_{t>0}$ is a degenerate semigroup generated by $\mathcal{A}$. Let us recall that there exists a finite constant $M > 0$ such that $\|T(t)\| \leq Mt^{\beta-1}$, $t \in (0,1]$ and $\|T(t)\| \leq Me^{-ct}$, $t \geq 1$. Let $p_0 > 1$ be such that*

$$\frac{p_0}{p_0-1}(\beta-1) \leq -1,$$

let $p \in \mathcal{P}([0,1])$, and let $\|T(\cdot)\|_{L^{q(x)}[0,1]} < \infty$. Assume that we have constructed a function $\breve{g} \in APS^{p(x)}(\mathbb{R} : X)$ such that $\breve{g} \notin APS^{p}(\mathbb{R} : X)$ for all $p \geq p_0$ (Question: Could we manipulate here somehow with the construction established in [2, Example, p. 70]?) Then, in our concrete situation, [20, Proposition 2.11] cannot be applied since

$$\frac{p}{p-1}(\beta-1) \leq -1, \quad p \in [1, p_0).$$

Now we will briefly explain that $\sum_{k=0}^{\infty} \|R(\cdot + k)\|_{L^{q(x)}[0,1]} < \infty$, showing that Proposition 6.1 is applicable. Strictly speaking, for $k = 0$, $\|T(\cdot)\|_{L^{q(x)}[0,1]} < \infty$ by our assumption, while, for $k \geq 1$, it can be simply shown that $\|R(\cdot + k)\|_{L^{q(x)}[0,1]} \leq Me^{-ck}$ so that $\sum_{k=0}^{\infty} \|R(\cdot + k)\|_{L^{q(x)}[0,1]} < \infty$, as claimed.

(ii) By a mild solution of (7), we mean the function $t \mapsto \int_{-\infty}^{t} R_\gamma(t-s)g(s)\,ds$, $t \in \mathbb{R}$. Let $p \in \mathcal{P}([0,1])$, and let $\|R_\gamma(\cdot)\|_{L^{q(x)}[0,1]} < \infty$. Then, for $k \geq 1$, we have $\|R_\gamma(\cdot + k)\|_{L^{q(x)}[0,1]} \leq M_2 k^{-1-\gamma}$. Hence, $\sum_{k=0}^{\infty} \|R_\gamma(\cdot + k)\|_{L^{q(x)}[0,1]} < \infty$ and we can apply Proposition 6.1.

In the following proposition, whose proof is very similar to that of [5, Proposition 3.12], we state some invariance properties of generalized asymptotical almost periodicity in Lebesgue spaces with variable exponents $L^{p(x)}$ under the action of finite convolution products (see also [19, Proposition 2.7.5, Lemma 2.9.3] for similar results). This proposition generalizes [20, Proposition 2.13] provided that $p > 1$ in its formulation.

Proposition 6.3. *Suppose that $p \in \mathcal{P}([0,1])$, $q \in D_+([0,1])$, $1/p(x) + 1/q(x) = 1$ and $(R(t))_{t>0} \subseteq L(X)$ is a strongly continuous operator family satisfying that, for every $t \geq 0$, we have that $m_t := \sum_{k=0}^{\infty} \|R(\cdot + t + k)\|_{L^{q(x)}[0,1]} < \infty$. Suppose, further, that $\breve{g} : \mathbb{R} \to X$ is $S^{p(x)}$-almost periodic, $q \in L_S^{p(x)}([0,\infty) : X)$ and $f(t) = g(t) + q(t)$, $t \geq 0$. Let r_1, $r_2 \in \mathcal{P}([0,1])$ and the following holds:*

(i) For every $t \geq 0$, the mapping $x \mapsto \int_0^{t+x} R(t+x-s)q(s)\,ds$, $x \in [0,1]$ belongs to the space $L^{r_1(x)}([0,1] : X)$ and we have

$$\lim_{t\to+\infty} \left\| \int_0^{t+x} R(t+x-s)q(s)\,ds \right\|_{L^{r_1(x)}[0,1]} = 0.$$

(ii) For every $t \geq 0$, the mapping $x \mapsto m_{t+x}$, $x \in [0,1]$ belongs to the space $L^{r_2(x)}[0,1]$ and we have

$$\lim_{t\to+\infty} \left\| m_{t+x} \right\|_{L^{r_2(x)}[0,1]} = 0.$$

Then the function $H(\cdot)$, given by

$$H(t) := \int_0^t R(t-s)f(s)\,ds, \quad t \geq 0,$$

is well-defined, bounded and belongs to the class $APS^{p(x)}(\mathbb{R} : X) + S_0^{r_1(x)}([0,\infty) : X) + S_0^{r_2(x)}([0,\infty) : X)$, with the meaning clear.

Remark 6.4. *In [20, Remark 2.14], we have examined the conditions under which the function $H(\cdot)$ defined above is asymptotically almost periodic, provided that the function $g(\cdot)$ is S^p-almost periodic for some $p \in [1,\infty)$. The interested reader may try to analyze similar problems with function $\breve{g}(\cdot)$ being $S^{p(x)}$-almost periodic for some $p \in \mathcal{P}([0,1])$.*

7. Applications

Let $\Omega \subseteq \mathbb{R}^n$ be an open bounded subset with smooth boundary $\partial\Omega$ and let $1 < p < \infty$. Among other things, one can make use of Proposition 6.3 to establish the existence and uniqueness of asymptotically $S^{p(x)}$-almost automorphic solutions to the damped Poisson-wave type equation, in the spaces $X := H^{-1}(\Omega)$ or $X := L^p(\Omega)$, given by

$$\begin{cases} \frac{\partial}{\partial t}\Big(m(x)\frac{\partial u}{\partial t}\Big) + \Big(2\omega m(x) - \Delta\Big)\frac{\partial u}{\partial t} + \Big(A(x;D) - \omega\Delta + \omega^2 m(x)\Big)u(x,t) = f(x,t), \\ t \geq 0,\ x \in \Omega\ ; \\ \\ u = \frac{\partial u}{\partial t} = 0, \quad (x,t) \in \partial\Omega \times [0,\infty), \\ \\ u(0,x) = u_0(x),\ m(x)\Big[(\frac{\partial u}{\partial t})(x,0) + \omega u_0\Big] = m(x)u_1(x), \quad x \in \Omega, \end{cases}$$

where $m(x) \in L^\infty(\Omega)$, $m(x) \geq 0$ a.e. $x \in \Omega$, Δ is the Dirichlet Laplacian in $L^2(\Omega)$, acting on its maximal domain, $H_0^1(\Omega) \cap H^2(\Omega)$, and $A(x;D)$ is a second-order linear differential operator on Ω with continuous coefficients on $\overline{\Omega}$, see, e.g., [11, Example 6.1] and [19] for further details.

Notice that we can also consider the existence and uniqueness of asymptotically $S^{p(x)}$-almost periodic solutions to the following fractional damped Poisson-wave type equation, in the spaces $X := H^{-1}(\Omega)$ or $X := L^p(\Omega)$, given by

$$\begin{cases} \mathbf{D}_t^\gamma\Big(m(x)\mathbf{D}_t^\gamma u\Big) + \Big(2\omega m(x) - \Delta\Big)\mathbf{D}_t^\gamma u + \Big(A(x;D) - \omega\Delta + \omega^2 m(x)\Big)u(x,t) = f(x,t), \\ \\ t \geq 0,\ x \in \Omega\ ; \\ \\ u = \mathbf{D}_t^\gamma u = 0, \quad (x,t) \in \partial\Omega \times [0,\infty), \\ \\ u(0,x) = u_0(x),\ m(x)\Big[\mathbf{D}_t^\gamma u(x,0) + \omega u_0\Big] = m(x)u_1(x), \quad x \in \Omega. \end{cases}$$

Additionally, it is also clear that Proposition 6.1 can be used to study the existence and uniqueness of almost periodic solutions of the following abstract integral inclusion

$$u(t) \in \mathcal{A}\int_{-\infty}^{t} a(t-s)u(s)\,ds + g(t),\ t \in \mathbb{R} \tag{10}$$

where $a \in L^1_{loc}([0,\infty))$, $a \neq 0$, $\check{g} : \mathbb{R} \to X$ is $S^{p(x)}$-almost periodic and $\mathcal{A}$ is a closed multivalued linear operator on X, see, e.g., [19].

References

[1] E. Bazhlekova, Fractional evolution equations in Banach spaces, PhD. Thesis, Eindhoven University of Technology, Eindhoven, 2001.

[2] H. Bohr, E. Følner, On some types of functional spaces: A contribution to the theory of almost periodic functions, Acta Math. 76 (1944) 31–155.

[3] R. Cross, Multivalued Linear Operators, Marcel Dekker Inc., New York, 1998.

[4] T. Diagana, Almost Automorphic Type and Almost Periodic Type Functions in Abstract Spaces, Springer, New York, 2013.

[5] T. Diagana, M. Kostić, Almost automorphic and asymptotically almost automorphic type functions in Lebesgue spaces with variable exponents $L^{p(x)}$, Book Chapter in: Recent Studies in Differential Equations, Nova Science Publishers Inc., 2020, pp. 1–28.

[6] T. Diagana, M. Zitane, Weighted Stepanov-like pseudo-almost periodic functions in Lebesgue space with variable exponents $L^{p(x)}$, Afr. Diaspora J. Math. 15 (2013) 56–75.

[7] T. Diagana, M. Zitane, Stepanov-like pseudo-almost automorphic functions in Lebesgue spaces with variable exponents $L^{p(x)}$, Electron. J. Differential Equations 2013, no. 188, 20 pp.

[8] L. Diening, P. Harjulehto, P. Hästüso, M. Ruzicka, Lebesgue and Sobolev Spaces with Variable Exponents, Lecture Notes in Mathematics, 2011. Springer, Heidelberg, 2011.

[9] K. Diethelm, The Analysis of Fractional Differential Equations: An Application-Oriented Exposition Using Differential Operators of Caputo Type, Springer-Verlag, Berlin, 2010.

[10] X. L. Fan, D. Zhao, On the spaces $L^{p(x)}(O)$ and $W^{m,p(x)}(O)$, J. Math. Anal. Appl. 263 (2001) 424–446.

[11] A. Favini, A. Yagi, Degenerate Differential Equations in Banach Spaces, Chapman and Hall/CRC Pure and Applied Mathematics, New York, 1998.

[12] G. M. N'Guérékata, Almost Automorphic and Almost Periodic Functions in Abstract Spaces, Kluwer Acad. Publ., Dordrecht, 2001.

[13] G. M. N'Guérékata, M. Kostić, Generalized asymptotically almost periodic and generalized asymptotically almost automorphic solutions of abstract multiterm fractional differential inclusions, Abstract Appl. Anal., volume 2018, Article ID 5947393, 17 pages https://doi.org/10.1155/2018/5947393.

[14] G. M. N'Guérékata, A. Pankov, Stepanov-like almost automorphic functions and monotone evolution equations, Nonlinear Anal. 68 (2008) 2658–2667.

[15] H. R. Henríquez, On Stepanov-almost periodic semigroups and cosine functions of operators, J. Math. Anal. Appl. 146 (1990) 420–433.

[16] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, Theory and Applications of Fractional Differential Equations, Elsevier Science B.V., Amsterdam, 2006.

[17] M. Kostić, Abstract Volterra Integro-Differential Equations, Taylor and Francis Group/CRC Press, Boca Raton, Fl., 2015.

[18] M. Kostić, Abstract Degenerate Volterra Integro-Differential Equations, Mathematical Institute SANU, Belgrade, 2020.

[19] M. Kostić, Almost Periodic and Almost Automorphic Solutions to Integro-Differential Equations, W. de Gruyter, Berlin, 2019.

[20] M. Kostić, Existence of generalized almost periodic and asymptotic almost periodic solutions to abstract Volterra integro-differential equations, Electron. J. Differential Equations, vol. 2017(239) (2017) 1–30.

[21] M. Levitan, Almost Periodic Functions, G.I.T.T.L., Moscow, 1959 (in Russian).

[22] W. Long, S.-H. Ding, Composition theorems of Stepanov almost periodic functions and Stepanov-like pseudo-almost periodic functions, Adv. Difference Equ., Vol. 2011, Article ID 654695, 12 pages, doi:10.1155/2011/654695.

[23] P. Q. H. Nguyen, On variable Lebesgue spaces, PhD.Thesis, Kansas State University. Pro- Quest LLC, Ann Arbor, MI, 2011. 63 pp.

[24] R.-N. Wang, D.-H. Chen, T.-J. Xiao, Abstract fractional Cauchy problems with almost sectorial operators, J. Differential Equations 252 (2012) 202–235.