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Steepest-Descent Ishikawa Iterative Methods for a Class of Variational Inequalities in Banach Spaces

Nguyen Buong[a], Nguyen Quynh Anh[b], Khuat Thi Binh[c]

[a]*Duy Tan University, 3 Quang Trung, Da Nang, Viet Nam; Vietnam Academy of Science and Technology,Institute of Information Technology, 18, Hoang Quoc Viet, Hanoi, Vietnam*
[b]*The People's Police University of Technology and Logistics, Thuan Thanh, Bac Ninh, Viet Nam*
[c]*Graduate University of Science and Technology, 18, Hoang Quoc Viet; Banking Academy, Ha Noi, Viet Nam*

Abstract. In this paper, for finding a fixed point of a nonexpansive mapping in either uniformly smooth or reflexive and strictly convex Banach spaces with a uniformly Gâteaux differentiable norm, we present a new explicit iterative method, based on a combination of the steepest-descent method with the Ishikawa iterative one. We also show its several particular cases one of which is the composite Halpern iterative method in literature. The explicit iterative method is also extended to the case of infinite family of nonexpansive mappings. Numerical experiments are given for illustration.

1. Introduction and preliminaries

Let E be a Banach space with the dual space E^*. For the sake of simplicity, the norms of E and E^* are denoted by the symbol $\|.\|$. We use $\langle x, x^*\rangle$ instead of $x^*(x)$ for $x^* \in E^*$ and $x \in E$. Let Q be a closed convex subset in E and let T be a nonexpansive mapping on Q, i.e., $T : Q \to Q$ such that $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in Q$. The set of fixed points of T is denoted by $\mathrm{Fix}(T)$, i.e., $\mathrm{Fix}(T) = \{x \in Q : x = Tx\}$.

Construction of fixed points of nonexpansive mappings is an important subject in the theory of nonlinear analysis and its applications in a number of applied areas, in particular, in image recovery and signal processing [1],[7]. Fundamental methods to find a fixed point of a nonexpansive mapping T on a closed convex subset Q of a Hilbert space H are Krasnosel'skii-Mann method [21], [23],

$$x^{k+1} = (1 - \beta_k)x^k + \beta_k T x^k, \; k \geq 1, \tag{1}$$

Ishikawa method [18],

$$x^{k+1} = T^k x^k, \; T^k = (1 - \beta_k)I + \beta_k T\Big((1 - \alpha_k)I + \alpha_k T\Big), \; k \geq 1, \tag{2}$$

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Email addresses: nbuong@ioit.ac.vn (Nguyen Buong), namlinhtn@gmail.com (Nguyen Quynh Anh), binhc3st@gmail.com (Khuat Thi Binh)

where I denotes the identity mapping of H and Halpern method [12],

$$x^{k+1} = t_k u + (1 - t_k)Tx^k, \ k \geq 1, \tag{3}$$

with any $u, x^1 \in Q$ and $\alpha_k, \beta_k, t_k \in (0, 1)$. A modification of the Halpern method is the viscosity approximation one,

$$x^{k+1} = t_k f(x^k) + (1 - t_k)Tx^k, \ k \geq 1, \tag{4}$$

introduced by Moudafi in [25], by using a contraction f on Q instead of u in (3). Further, Kim and Xu [20] provided a combination of the Krasnosel'skii-Mann and Halpern methods such as

$$x^{k+1} = t_k u + (1 - t_k)((1 - \beta_k)x^k + \beta_k Tx^k), \ k \geq 1, \tag{5}$$

and proved its strong convergence under conditions:

(t) $t_k \in (0, 1)$, $\lim_{k\to\infty} t_k = 0$ and $\sum_{k=1}^{\infty} t_k = \infty$,

$$\sum_{k=1}^{\infty} |t_{k+1} - t_k| < \infty; \ \sum_{k=1}^{\infty} |\beta_{k+1} - \beta_k| < \infty,$$

and additional assumptions on β_k. Next, Yao et al. [37] proposed a modified Krasnosel'skii-Mann iterative method

$$x^{k+1} = \Big((1 - \beta_k)I + \beta_k T\Big)(1 - t_k)x^k, \ k \geq 1, \tag{6}$$

and proved that if $Q \equiv H$, $\mathrm{Fix}(T) \neq \emptyset$, the parameter t_k and β_k satisfy, respectively, conditions (t) and
(β) $\beta_k \in [a, b] \subset (0, 1)$ for all $k \geq 1$,
then the sequence $\{x^k\}$, generated by (6), converges strongly to a fixed point of T. Shehu [28] extended this result from the Hilbert space H onto a uniformly convex Banach space E, having a uniformly Gâteaux differentiable norm. We know that both methods (1) and (2) have only weak convergence, in general (see, [11], for example). Clearly, (2) is indeed more general than (1). But research has been concentrated on (1) due probably to the reasons that (1) is simpler than (2) and that a convergence theorem for (1) may possibly lead to a convergence theorem for (2) provided that the sequence $\{\beta_k\}$ satisfies certain appropriate conditions. However, method (2) has its own right. As a matter of fact, method (1) may fail to convergence while method (2) can still converge for a Lipschitz pseudocontractive mapping in a Hilbert space (see, [9]). Reich [27] showed that if E is a uniformly convex Banach space, having a Fréchet differentiable norm, and if the sequence $\{\beta_k\}$ in (1) is such that $\sum_{k=1}^{\infty} \beta_k(1-\beta_k) = \infty$, then the sequence $\{x^k\}$, generated by (1), converges weakly to a point in $\mathrm{Fix}(T)$. An extension of this result was presented in [32], where Tan and Xu proved weak convergence of (2) under conditions:

$$\sum_{k=1}^{\infty} \beta_k(1 - \beta_k) = \infty, \sum_{k=1}^{\infty} \beta_k(1 - \alpha_k) < \infty$$

and $\limsup_{k\to\infty} \alpha_k < 1$. Next, Qin et al. [26], by using T^k in (2), considered the following iterative method,

$$x^{k+1} = t_k u + (1 - t_k)T^k x^k, \ k \geq 1, \tag{7}$$

that is a combination of the Ishikawa method with the Halpern one. They proved that the sequence, generated by (7), converges strongly to a point in $\mathrm{Fix}(T)$ in uniformly smooth Banach spaces, when t_k, β_k and α_k satisfy the conditions: (t), $\beta_k \to 0$, $\alpha_k \leq \bar{a} \in (0, 1)$, i.e., $\limsup_{k\to\infty} \alpha_k < 1$, and

$$\sum_{k=1}^{\infty} |t_{k+1} - t_k| < \infty, \sum_{k=1}^{\infty} |\alpha_{k+1} - \alpha_k| < \infty, \sum_{k=1}^{\infty} |\beta_{k+1} - \beta_k| < \infty. \tag{8}$$

Li [22] proposed a modification of (7), that is the viscosity approximation Ishikawa method,

$$x^{k+1} = t_k f(x^k) + (1 - t_k)T^k x^k,\ k \geq 1, \tag{9}$$

and proved a strong convergence result of (9) under the conditions: (t), (β) and $|\alpha_{k+1} - \alpha_k| \to 0$.

In this paper, we will show that (7) is a special case of an explicit iterative method, based on a combination of the steepest-descent method with the Ishikawa one, for solving the variational inequality problem: find a point $p_* \in E$ such that

$$p_* \in C : \quad \langle Fp_*, j(p_* - p)\rangle \leq 0 \quad \forall p \in C = \mathrm{Fix}(T), \tag{10}$$

where T is a nonexpansive mapping and F is an η-strongly accretive and γ-strictly pseudocontractive mapping on E. We will give a strongly convergent modification for Ishikawa method (2), that is similar to (6) for Krasnoselskii-Mann method (1.1), and a new variant of the viscosity approximation Ishikawa method.

Variational inequalities over the fixed point set of nonexpansive mappings have an important role in solving practical problems such as the signal recovery problem, beamforming problem, power control problem, bandwidth allocation problem and finance problem (see, e.g., [[13]-[17]). In order to solve the class of variational inequalities, in 2001, Yamada [36] introduced the hybrid steepest-descent method,

$$x^{k+1} = \Big(I - t_{k+1}\mu F\Big)Tx^k,\ k \geq 1, \tag{11}$$

and proved a strong convergence theorem, when the parameter t_k satisfies (t) with additional assumptions, $\mu \in (0, 2\eta/L^2)$ and the mapping F in (11) is η-strongly monotone and L-Lipschitz continuous on a Hilbert space H.

Clearly, when $C \equiv E$ ($T \equiv I$, the identity mapping of E), (10) is the operator equation $Fx = 0$, in fact. In order to find a solution of an η-strongly accretive and Lipschitz continuous mapping F, whose domain of definition is whole a uniformly smooth Banach space E, we can use the steepest-descent method, $x^1 \in E$ any element and

$$x^{k+1} = (I - t_k F)x^k,\ k \geq 1, \tag{12}$$

where t_k satisfies the condition (t) (see, [33],[34],[38], for details). When E is an either uniformly smooth or strictly convex reflexive Banach space with a uniformly Gâteaux differentiable norm, a combination of the steepest-descent method with the Krasnoselskii-Mann one, for solving the class of variational inequalities with an η-strongly accretive and γ-strictly pseudocontractive mapping F, was given in [2]. Following the result, the explicit iterative method, investigated in this paper, is

$$x^{k+1} = (I - t_k F)T^k x^k,\ k \geq 1, \tag{13}$$

where T^k is defined in (2). We will prove a strong convergent result for (13) under conditions (t), (β) and (α) $\alpha_k \in [0, \bar{a}]$ for all $k \geq 1$ and $\alpha_k \to 0$ as $k \to \infty$.

Further, we will consider the case that $C = \cap_{i\geq 1}\mathrm{Fix}(T_i) \neq \emptyset$, where $\{T_i\}$ is an infinite family of nonexpansive mappings on E. In this case, T^k is defined by

$$T^k = (1 - \beta_k)I + \beta_k W^k\Big((1 - \alpha_k)I + \alpha_k W^k\Big), \tag{14}$$

where $\{W^k\}$ is a sequence, satisfying the following conditions:

(i) there exists $Wx := \lim_{k\to\infty} W^k x$ for any $x \in E$ and if $\cap_{i\geq 1}\mathrm{Fix}(T_i) \neq \emptyset$ then we have that $\mathrm{Fix}(W) = \cap_{i\geq 1}\mathrm{Fix}(T_i)$ and

(ii) $\lim_{k\to\infty} \sup_{x\in B} \|W^k x - Wx\| = 0$, for any bounded subset B.

As particular case, we obtain a steepest-descent Krasnoselskii-Mann method, an extension of the result in [2] to the case of infinite family of nonexpansive mappings.

Now, we list some facts that will be used in the proof of our result.

A mapping J from E into E^*, satisfying the condition,

$$J(x) = \{x^* \in E^* : \langle x, x^*\rangle = \|x\|\|x^*\| \quad \text{and} \quad \|x^*\| = \|x\|\},$$

is called a normalized duality mapping of E. It is well known that if $x \neq 0$, then $J(tx) = tJ(x)$, for all $t > 0$ and $x \in E$, and $J(-x) = -J(x)$. Let $F : E \to E$ be an η-strongly accretive and γ-strictly pseudocontractive mapping, i.e., F satisfies, respectively, the following conditions:

$$\langle Fx - Fy, j(x-y)\rangle \geq \eta\|x-y\|^2,$$

and

$$\langle Fx - Fy, j(x-y)\rangle \leq \|x-y\|^2 - \gamma\|(I-F)x - (I-F)y\|^2,$$

for all $x, y \in E$ and some element $j(x-y) \in J(x-y)$, where η and $\gamma \in (0,1)$ are some positive constants. Clearly, if F is γ-strictly pseudocontractive mapping, then $\|Fx - Fy\| \leq L\|x-y\|$ with $L = 1 + 1/\gamma$ and, in this case, F is called L-Lipschitz continuous. In addition, if $L \in [0,1)$, then F is called contractive.

Let $S_1(0) := \{x \in E : \|x\| = 1\}$ and $S(0,r) := \{x \in E : \|x\| \leq r\}$ for a positive constant r. The space E is said to have a Gâteaux differentiable norm if the limit

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t}$$

exists for each $x, y \in S_1(0)$. Such an E is called a smooth Banach space. The space E is said to have a uniformly Gâteaux differentiable norm if the limit is attained uniformly for $x \in S_1(0)$. The norm of E is called Fréchet differentiable, if for all $x \in S_1(0)$, the limit is attained uniformly for $y \in S_1(0)$. The norm of E is called uniformly Fréchet differentiable (and E is called uniformly smooth) if the limit is attained uniformly for all $x, y \in S_1(0)$. It is well known that every uniformly smooth real Banach space is reflexive and has a uniformly Gâteaux differentiable norm (see, [10]).

Recall that a Banach space E is said to be
(i) uniformly convex, if for any $\varepsilon \in (0,2]$, the inequalities $\|x\| \leq 1, \|y\| \leq 1$, and $\|x-y\| \geq \varepsilon$ imply that there exists a $\delta = \delta(\varepsilon) \geq 0$ such that $\|(x+y)/2\| \leq 1 - \delta$;
(ii) strictly convex, if for $x, y \in S_1(0)$ with $x \neq y$, then

$$\|(1-\lambda)x + \lambda y\| < 1 \quad \forall\lambda \in (0,1).$$

It is well known that each uniformly convex Banach space E is reflexive and strictly convex; If the norm of E is uniformly Gâteaux differentiable, then J is norm to weak star uniformly continuous on each bounded subset of E; and if E is smooth, then duality mapping is single valued. In the sequel, we shall denote the single valued normalized duality mapping by j.

Lemma 1.1. *([8]) Let E be a real smooth Banach space and $F : E \to E$ be an η-strongly accretive and γ-strictly pseudocontractive with $\eta + \gamma > 1$. Then, we have:*
(i) for any $t \in (0,1)$, $I - tF$ is a contraction with contractive constant $1 - \lambda\tau$, where $\tau = 1 - \sqrt{(1-\eta)/\gamma}$.
(ii) when $t = 1$, $I - F$ also is contractive with constant $\tau_1 = \sqrt{(1-\eta)/\gamma}$.

Lemma 1.2. *Let E be a real smooth Banach space. Then, the following inequality holds*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle, \ \forall x, y \in E.$$

Lemma 1.3. *([35]) Let $\{a_k\}$ be a sequence of nonnegative real numbers satisfying the following condition $a_{k+1} \leq (1-b_k)a_k + b_k c_k + d_k$, where $\{b_k\}, \{c_k\}$ and $\{d_k\}$ are sequences of real numbers such that*
(i) $b_k \in [0,1]$ and $\sum_{k=1}^{\infty} b_k = \infty$;
(ii) $\limsup_{k\to\infty} c_k \leq 0$;
(iii) $\sum_{k=1}^{\infty} d_k < \infty$.
Then, $lim_{k\to\infty} a_k = 0$.

Lemma 1.4. *([31]) Let $\{x^k\}$ and $\{w^k\}$ be bounded sequences in a Banach space E such that $x^{k+1} = h_k x^k + (1-h_k)w^k$ for $k \geq 1$, where $\{h_k\}$ satisfies the condition*

$$0 < \liminf_{k\to\infty} h_k \leq \limsup_{k\to\infty} h_k < 1.$$

Assume that

$$\limsup_{k\to\infty} \Big(\|w^{k+1} - w^k\| - \|x^{k+1} - x^k\| \Big) \leq 0.$$

Then, $\lim_{k\to\infty} \|x^k - w^k\| = 0$.

Lemma 1.5. *([2],[3]) Let F be an η-strongly accretive and γ-strictly pseudocontractive mapping on an either uniformly smooth or real reflexive and strictly convex Banach space E, having a uniformly Gâteaux differentiable norm, such that $\eta + \gamma > 1$ and let T be a nonexpansive mapping on E with $C := Fix(T) \neq \emptyset$. Then, for a bounded sequence $\{x^k\}$ in E with $\lim_{k\to\infty} \|x^k - Tx^k\| = 0$, we have*

$$\limsup_{k\to\infty} \langle Fp_*, j(p_* - x^k) \rangle \leq 0, \tag{15}$$

where p_* is the unique solution of (10).

The rest of the paper is organized as follows. In Section 2, we present the theoretical results. In Section 3, we give two numerical experiments for illustration.

2. Main results

First, we consider the case that $C = Fix(T)$, where T is a nonexpansive mapping on E such that $Fix(T) \neq \emptyset$. We have the following result.

Theorem 2.1. *Let E, F and T be as in Lemma 1.5. Assume that t_k, β_k and α_k satisfy conditions (t), (β) and (α), respectively. Then, the sequence $\{x^k\}$, defined by (13) with T^k in (2), converges strongly to p_*, solving (10).*

Proof. Since $T^k p = p$ for any point $p \in Fix(T)$ and $k \geq 1$, by Lemma 1.1,

$$\begin{aligned}\|x^{k+1} - p\| &= \|(1 - t_k F)T^k x^k - (1 - t_k F)T^k p - t_k Fp\| \\ &\leq (1 - t_k\tau)\|x^k - p\| + t_k\tau\|Fp\|/\tau \leq \max\{\|x^1 - p\|, \|Fp\|/\tau\}.\end{aligned}$$

Therefore, $\{x^k\}$ is bounded. So, are the sequences $\{Tx^k\}$, $\{Tx^{k+1}\}$, $\{T^k x^k\}$, $\{T^{k+1}x^k\}$, $\{FT^k x^k\}$ and $\{Ty^k\}$ where $y^k = (1-\alpha_k)x^k + \alpha_k Tx^k$. Without any loss of generality, we assume that they are bounded by a positive constant M_1. Further, it is easy to see that

$$\begin{aligned}x^{k+1} &= t_k(I - F)T^k x^k + (1 - t_k)T^k x^k \\ &= t_k(I - F)T^k x^k + (1 - t_k)\Big[(1 - \beta_k)x^k + \beta_k T y^k\Big] \\ &= h_k x^k + (1 - h_k)w^k,\end{aligned} \tag{16}$$

where $h_k = (1 - t_k)(1 - \beta_k)$ and

$$w^k = \frac{t_k(I - F)T^k x^k}{1 - h_k} + \frac{(1 - t_k)\beta_k T y^k}{1 - h_k}.$$

Clearly, from conditions (t) and (β) we have $0 < \liminf_{k\to\infty} h_k \leq \limsup_{k\to\infty} h_k < 1$. Next, we can write that

$$\begin{aligned}
&\frac{t_{k+1}(I - F)T^{k+1}x^{k+1}}{1 - h_{k+1}} - \frac{t_k(I - F)T^k x^k}{1 - h_k} \\
&= \frac{t_{k+1}}{1 - h_{k+1}}\Big[(I - F)T^{k+1}x^{k+1} - (I - F)T^{k+1}x^k\Big] \\
&\quad + \frac{t_{k+1}}{1 - h_{k+1}}\Big[(I - F)T^{k+1}x^k - (I - F)T^k x^k\Big] \\
&\quad + \Big[\frac{t_{k+1}}{1 - h_{k+1}} - \frac{t_k}{1 - h_k}\Big]\times(I - F)T^k x^k, \\
&\frac{(1 - t_{k+1})\beta_{k+1}Ty^{k+1}}{1 - h_{k+1}} - \frac{(1 - t_k)\beta_k T y^k}{1 - h_k} \\
&= \frac{(1 - t_{k+1})\beta_{k+1}}{1 - h_{k+1}}\Big[Ty^{k+1} - Ty^k\Big] \\
&\quad\quad + \Big[\frac{(1 - t_{k+1})\beta_{k+1}}{1 - h_{k+1}} - \frac{(1 - t_k)\beta_k}{1 - h_k}\Big]Ty^k.
\end{aligned}$$

Thus,

$$\begin{aligned}
\|w^{k+1} - w^k\| &\leq \frac{t_{k+1}(1 - \tau_1)}{1 - h_{k+1}}\Big[\|x^{k+1} - x^k\| + 2M_1\Big] + \Big|\frac{t_{k+1}}{1 - h_{k+1}} - \frac{t_k}{1 - h_k}\Big|2M_1 \\
&\quad + \frac{(1 - t_{k+1})\beta_{k+1}}{1 - h_{k+1}}\Big(\|x^{k+1} - x^k\| + M_1(\alpha_{k+1} + \alpha_k)\Big) \\
&\quad\quad + \Big|\frac{(1 - t_{k+1})\beta_{k+1}}{1 - h_{k+1}} - \frac{(1 - t_k)\beta_k}{1 - h_k}\Big|M_1 \\
&= \Big[\frac{t_{k+1}(1 - \tau_1)}{1 - h_{k+1}} + \frac{(1 - t_{k+1})\beta_{k+1}}{1 - h_{k+1}}\Big]\|x^{k+1} - x^k\| + \tilde{c}_k, \\
&= \frac{t_{k+1}(1 - \tau_1) + (1 - t_{k+1})\beta_{k+1}}{t_{k+1} + \beta_{k+1} - t_{k+1}\beta_{k+1}}\|x^{k+1} - x^k\| + \tilde{c}_k, \\
&\leq \|x^{k+1} - x^k\| + \tilde{c}_k,
\end{aligned}$$

where $\tilde{c}_k$ is the sum of the remain terms and, by conditions (t), (β) and (α), $\tilde{c}_k \to 0$ as $k \to \infty$. Therefore,

$$\limsup_{k\to\infty}\Big(\|w^{k+1} - w^k\| - \|x^{k+1} - x^k\|\Big) \leq 0.$$

By virtue of Lemma 1.4,

$$\lim_{k\to\infty} \|x^k - w^k\| = 0. \tag{17}$$

Noting (16) and (17),

$$\lim_{k\to\infty} \|x^{k+1} - x^k\| = \lim_{k\to\infty}(1 - h_k)\|x^k - w^k\| = 0. \tag{18}$$

According to (13), $\|x^{k+1} - T^k x^k\| \le t_k M_1 \to 0$, as $k \to \infty$. This together with (18) implies that

$$\lim_{k\to\infty} \|x^k - T^k x^k\| = 0. \tag{19}$$

Now, we prove that

$$\lim_{k\to\infty} \|x^k - Tx^k\| = 0. \tag{20}$$

To do this, first we proved that $\lim_{k\to\infty} \|x^k - Ty^k\| = 0$. Indeed, from the definition of T^k and y^k, we know that $x^k - T^k x^k = \beta_k(x^k - Ty^k)$, and hence, by virtue of condition (β),

1. Theorem 2.1 has still value for the following method: $y^1 \in E$ is any element and

$$y^{k+1} = T^k(I - t_k F)y^k, \; k \ge 1, \tag{21}$$

with the same conditions on E, F, T, t_k, β_k and α_k. Indeed, putting $y^k = T^k x^k$ in (13) we obtain that $y^{k+1} = T^{k+1}x^{k+1} = T^{k+1}(I - t_k F)y^k$. Re-denoting $\beta_k := \beta_{k+1}$ and $\alpha_k := \alpha_{k+1}$, we obtain (21). Moreover, if $t_k \to 0$ then $\{x^k\}$ is convergent if and only if $\{y^k\}$ is so and their limits coincide. Indeed, from (13), it follows that $\|x^{k+1} - y^k\| \le t_k \|Fy^k\|$. Therefore, when $\{x^k\}$ is convergent, $\{x^k\}$ is bounded, and hence $\{y^k\}$ is bounded. Consequently, $\{Fy^k\}$ is also bounded. Since $t_k \to 0$ as $k \to \infty$, from the last inequality and the convegence of $\{x^k\}$ it follows the convergence of $\{y^k\}$ and that their limits coincide. The case, when $\{y^k\}$ converges, is similar.

2. We take $F = I - f$ with $f = a'I$ for a fixed number $a' \in (0, 1)$. Then, F is an η-strongly accretive and γ-strictly pseudocontractive mapping on E with some positive numbers η and γ such that $\eta + \gamma > 1$. Indeed, since

$$\begin{aligned}\langle Fx - Fy, j(x - y)\rangle &= (1 - a')\|x - y\|^2 \\ &= \|x - y\|^2 - \frac{1}{a'}\|a'x - a'y\|^2 = \|x - y\|^2 - \frac{1}{a'}\|fx - fy\|^2 \\ &= \|x - y\|^2 - \frac{1}{a'}\|(I - F)x - (I - F)y\|^2 \\ &\le \|x - y\|^2 - \gamma\|(I - F)x - (I - F)y\|^2, \; \gamma \in [0, 1),\end{aligned}$$

a fixed number. Clearly, $\eta + \gamma > 1$ for $\eta = 1 - a'$ and any fixed $\gamma \in (a', 1)$. Now, replacing F by $I - f = (1 - a')I$ in (13), we obtain the following algorithm,

$$x^{k+1} = (1 - t'_k)T^k x^k, \; k \ge 1, \tag{22}$$

where $t'_k = t_k(1 - a')$, and have the following result.

Theorem 2.2. *Let T be a nonexpansive mapping on an either uniformly smooth or strictly convex reflexive Banach space E with a uniformly Gâteaux differentiable norm. Assume that t_k, β_k and α_k satisfy conditions (t), (β) and (α), respectively. Fix a real number $a' \in (0, 1)$. Then, the sequence $\{x^k\}$, generated by (22), converges strongly to a point in $Fix(T)$.*

3. Next, we consider the case, when T is a nonexpansive mapping on a closed and convex subset Q of E. Clearly, with the starting point $x^1 \in Q$, for any point $x^k \in Q$, $T^k x^k \in Q$. Thus, if the set Q contains the original point of E then $x^{k+1} \in Q$, because $x^{k+1} = \tau_k T^k x^k$ with $\tau_k = 1 - t'_k \in (0, 1)$. It means that method (22) is well defined for any $x^1 \in Q$, and hence, Theorem 2.2 has value in this case. In the case that the set Q does not contain the original point of E, we take $f = a'I + (1 - a')u$ with a fixed $u \in Q$. It is easy to see that $F = I - f$

is also η-strongly accretive and γ-strictly pseudocontractive such that $\eta + \gamma > 1$. Then, instead of (22), we obtain the Halpern Ishikawa method,

$$\begin{aligned} &x^1 \in Q, \text{ any element,} \\ &x^{k+1} = t'_k u + \left(1 - t'_k\right) T^k x^k,\ k \geq 1, \end{aligned} \tag{23}$$

that is method (7) with re-denoting $t_k := t'_k$. Clearly, t_k satisfies condition (t) if and only if t'_k is so. Method (23), by Theorem 2.2, converges strongly in a uniformly smooth or strictly convex reflexive Banach space E, meantime, method (7) needs stronger conditions on t_k, β_k and α_k (additional condition (8) than that in our method.

4. Let $\tilde{a} > 1$ and let f be an $\tilde{a}$-co-coercive accretive mapping on E, i.e.,

$$\langle fx - fy, j(x - y)\rangle \geq \tilde{a}\|fx - fy\|^2,\ \forall x, y \in E.$$

It is easily seen that f is a contraction with constant $1/\tilde{a} \in (0, 1)$, and hence, $F := I - f$ is an η-strongly accretive mapping with $\eta = 1 - (1/\tilde{a})$. Moreover,

$$\begin{aligned} \langle Fx - Fy, j(x - y)\rangle &= \|x - y\|^2 - \langle fx - fy, j(x - y)\rangle \\ &\leq \|x - y\|^2 - \tilde{a}\|fx - fy\|^2 \\ &\leq \|x - y\|^2 - \gamma\|(I - F)x - (I - F)y\|^2, \end{aligned}$$

for any $\gamma \in (0, \tilde{a}]$. Taking any fixed $\gamma \in ((1/\tilde{a}), \tilde{a}]$, we get that F is a γ-strictly pseudocontractive mapping with $\eta + \gamma > 1$. Next, by replacing F by $I - f$ in (21), we obtain a new viscosity approximation Ishikawa method,

$$y^{k+1} = T^k(t_k f y^k + (1 - t_k)y^k),\ y^1 \in E,\ k \geq 1, \tag{24}$$

that is an improved modification of (7) and different from (9). Obviously, if f is an $\tilde{a}$-co-coercive accretive mapping on Q, a closed convex subset of E, then method (24) is also well defined for any $y^1 \in Q$.

For a given α-co-coercive accretive mapping f, we can obtain an $\tilde{\alpha}$-co-coercive accretive mapping $\tilde{f}$ with $\tilde{\alpha} > 1$ by considering $\tilde{f} := \beta f$ with a positive number $\beta < \alpha$. Indeed, $\tilde{\alpha} = \alpha/\beta > 1$ and

$$\begin{aligned} \langle \tilde{f}x - \tilde{f}y, j(x - y)\rangle &= \langle \beta fx - \beta fy, j(x - y)\rangle \\ &\geq \beta\alpha\|fx - fy\|^2 = \tilde{\alpha}\|\tilde{f}x - \tilde{f}y\|^2. \end{aligned}$$

Now, in the case when $\cap_{i\geq 1} Fix(T_i) \neq \emptyset$, we have the following result.

Theorem 2.3. *Let E, F, t_k, β_k and α_k be as in Theorem 2.1. Let $\{T_i\}$ be an infinite family of nonexpansive mappings on E such that $C := \cap_{i\geq 1} Fix(T_i) \neq \emptyset$. Then, any sequence, generated by (13) and (14), converges strongly to the point p_* in (10).*

Proof. As in the proof for Theorem 2.1, the sequence $\{x^k\}$, generated by (13) and (14), is bounded. Therefore, there exists a positive constant M_2 such that the sequences $\{x^k\}$, $\{T^k x^k\}$,$\{T^{k+1} x^k\}$, $\{FT^k x^k\}$, $\{W^k x^k\}$ and $\{W^{k+1} x^k\}$ belong to $S(0, M_2)$. Moreover, we have equality (16) with the same h_k, $y^k = (1 - \alpha_k)x^k + \alpha_k W^k x^k$ and

$$w^k = \frac{t_k(I - F)T^k x^k}{1 - h_k} + \frac{(1 - t_k)\beta_k W^k y^k}{1 - h_k}.$$

In order to estimate the value $\|w^{k+1} - w^k\|$, first of all we need compute the value $\|T^{k+1}x - T^k x\|$ for any $x \in S(0, M_2)$. Set $\tilde{y}^k = (1 - \alpha_k)x + \alpha_k W^k x$. It is easy to verify that $\tilde{y}^k \in S(0, M_2)$ and $W^k \tilde{y}^k \in S(0, M_2)$ for any

$x \in S(0, M_2)$. Consequently,

$$\begin{aligned}\|T^{k+1}x - T^k x\| &= \|(1-\beta_{k+1})x + \beta_{k+1}W^{k+1}\tilde{y}^{k+1} - ((1-\beta_k)x + \beta_k W^k \tilde{y}^k)\| \\ &\le |\beta_{k+1} - \beta_k|\|x\| + \beta_{k+1}\Big(\|\tilde{y}^{k+1} - \tilde{y}^k\| \\ &\quad + \|W^{k+1}\tilde{y}^k - W^k\tilde{y}^k\|\Big) + |\beta_{k+1} - \beta_k|\|W^k\tilde{y}^k\|,\end{aligned}$$

where

$$\|\tilde{y}^{k+1} - \tilde{y}^k\| \le |\alpha_{k+1} - \alpha_k|\|x\| + \alpha_{k+1}\|W^{k+1}x - W^k x\| + |\alpha_{k+1} - \alpha_k| M_2.$$

Therefore,

$$\begin{aligned}\|T^{k+1}x - T^k x\| \le 2|\beta_{k+1} - \beta_k| M_2 + \beta_{k+1}\Big[&2|\alpha_{k+1} - \alpha_k| M_2 \\ &+ \alpha_{k+1}\|W^{k+1}x - W^k x\| + \|W^{k+1}\tilde{y}^k - W^k\tilde{y}^k\|\Big].\end{aligned} \tag{25}$$

From conditions (β) and (α), we can deduce that there exists a subsequence $\{k_m\}$ of $\{k\}$ such that $\beta_{k_m} \to \beta'$ as $m \to \infty$. Then, $|\beta_{k_m+1} - \beta_{k_m}| \to 0$ and $|\alpha_{k_m+1} - \alpha_{k_m}| \to 0$ as $m \to \infty$. Now, replacing x and k in 25) by x^{k_m} and k_m, respectively, and using condition (ii) with $B = S(0, M_2)$ for W^{k_m}, we obtain that

$$\lim_{m\to\infty} \|T^{k_m+1}x^{k_m} - T^{k_m}x^{k_m}\| = 0.$$

Consider the procedure

$$x^{k_m+1} = h_{k_m}x^{k_m} + (1 - h_{k_m})w^{k_m}, \tag{26}$$

where $h_{k_m} = (1 - t_{k_m})(1 - \beta_{k_m})$ and

$$w^{k_m} = \frac{t_{k_m}(I - F)T^{k_m}x^{k_m}}{1 - h_{k_m}} + \frac{(1 - t_{k_m})\beta_{k_m}W^{k_m}y^{k_m}}{1 - h_{k_m}}.$$

It is easily to see that

$$\begin{aligned}&\frac{(1 - t_{k_m+1})\beta_{k_m+1}W^{k_m+1}y^{k_m+1}}{1 - h_{k_m+1}} - \frac{(1 - t_{k_m})\beta_{k_m}W^{k_m}y^{k_m}}{1 - h_{k_m}} \\ &= \frac{(1 - t_{k_m+1})\beta_{k_m+1}}{1 - h_{k_m+1}}\Big[W^{k_m+1}y^{k_m+1} - W^{k_m+1}y^{k_m}\Big] \\ &\quad + \frac{(1 - t_{k_m+1})\beta_{k_m+1}}{1 - h_{k_m+1}}\Big[W^{k_m+1}y^{k_m} - W^k y^{k_m}\Big] \\ &\quad + \Big[\frac{(1 - t_{k_m+1})\beta_{k_m+1}}{1 - h_{k_m+1}} - \frac{(1 - t_{k_m})\beta_{k_m}}{1 - h_{k_m}}\Big]W^{k_m}y^{k_m}.\end{aligned}$$

Thus, as in the proof of Theorem 2.1,

$$\begin{aligned}\|w^{k_m+1} - w^{k_m}\| &\le \Big[\frac{t_{k_m+1}(1 - \tau_1)}{1 - h_{k_m+1}} + \frac{(1 - t_{k_m+1})\beta_{k_m+1}}{1 - h_{k_m+1}}\Big]\|x^{k_m+1} - x^{k_m}\| + \bar{c}_{k_m}, \\ &= \frac{t_{k_m+1}(1 - \tau_1) + (1 - t_{k_m+1})\beta_{k_m+1}}{1 - \beta_{k_m+1} + t_{k_m+1}\beta_{k_m+1}}\|x^{k_m+1} - x^{k_m}\| + \bar{c}_{k_m}, \\ &\le \|x^{k_m+1} - x^{k_m}\| + \bar{c}_{k_m},\end{aligned}$$

$\bar{c}_{k_m} \to 0$ as $m \to \infty$. Therefore, we have the same equality (17) with k replaced by k_m, i.e., $\|x^{k_m} - w^{k_m}\| \to 0$, and hence, by Lemma **??** and (26), we get that $\|x^{k_m+1} - x^{k_m}\| \to 0$, which together with $\|x^{k_m+1} - T^{k_m}x^{k_m}\| \le t_{k_m}M_1 \to 0$ as $m \to \infty$ implies that

$$\lim_{m\to\infty} \|x^{k_m} - T^{k_m}x^{k_m}\| = 0. \tag{27}$$

Now, we prove that

$$\lim_{m\to\infty} \|x^{k_m} - W^{k_m}x^{k_m}\| = 0. \tag{28}$$

For this purpose, first, we prove that $\lim_{m\to\infty} \|x^{k_m} - W^{k_m}y^{k_m}\| = 0$, where the point $y^{k_m} = (1-\alpha_{k_m})x^{k_m} + \alpha_{k_m}W^{k_m}x^{k_m}$. Since $x^{k_m} - T^{k_m}x^{k_m} = \beta_{k_m}(x^{k_m} - W^{k_m}y^{k_m})$, and hence, by virtue of condition (β),

$$\|x^{k_m} - W^{k_m}y^{k_m}\| \le \|x^{k_m} - T^{k_m}x^{k_m}\|/a,$$

which together with (27) implies the last limit. On the other hand,

$$\begin{aligned}
\|x^{k_m} - W^{k_m}x^{k_m}\| &\le \|x^{k_m} - W^{k_m}y^{k_m}\| + \|W^{k_m}y^{k_m} - W^{k_m}x^{k_m}\| \\
&\le \|x^{k_m} - W^{k_m}y^{k_m}\| + \|y^{k_m} - x^{k_m}\| \\
&= \|x^{k_m} - W^{k_m}y^{k_m}\| + \|(1-\alpha_{k_m})x^{k_m} + \alpha_{k_m}W^{k_m}x^{k_m} - x^{k_m}\| \\
&= \|x^{k_m} - W^{k_m}y^{k_m}\| + \alpha_{k_m}\|x^{k_m} - W^{k_m}x^{k_m}\|
\end{aligned}$$

we obtain the inequality $\|x^{k_m} - W^{k_m}x^{k_m}\| \le \|x^{k_m} - W^{k_m}y^{k_m}\|/(1-\bar{a})$, from which and the last limit, we get (28). Now, from (28), the following inequality,

$$\|x^{k_m} - Wx^{k_m}\| \le \|x^{k_m} - W^{k_m}x^{k_m}\| + \sup_{x\in S(0,M_2)} \|W^{k_m}x - Wx\|,$$

and again condition (ii) for W^{k_m}, we have that $\lim_{m\to\infty} \|x^{k_m} - Wx^{k_m}\| = 0$. As in the proof of Theorem 2.1, the sequence $\{x^{k_m}\}$ converges strongly to p_* in (10) as $m \to \infty$. By the similar argument, any convergent subsequence of $\{x^k\}$ converges to p_*. As the point p_* in (10) is unique, all the sequence $\{x^k\}$ converges to p_*. This completes the proof. □

Remarks

5. All remarks 1-4 have still a value, when T^k is defined by (14).

6. Taking $\alpha_k = 0$ in (13) and (14), we obtain the steepest-descent Krasnoselskii-Mann method in [26] and its extension to an infinite family of nonexpansive mappings T_i on E, that is the method

$$x^{k+1} = (I - t_kF)((1-\beta_k)I + \beta_kW^k)x^k,\ k \ge 1,$$

and its equivalent formula is

$$x^{k+1} = \Big((1-\beta_k)I + \beta_kW^k\Big)(I - t_kF)x^k,\ k \ge 1, \tag{29}$$

(see, remark 1). Replacing F in (29) by $(1-a')I$, we get the method

$$y^{k+1} = \Big((1-\beta_k)I + \beta_kW^k\Big)(1 - t_k')y^k,\ k \ge 1,$$

strong convergence of which was proved in [29] in uniformly convex and uniformly smooth Banach spaces under conditions (t), (β),

$$\sum_{k=1}^{\infty} \lim_{k\to\infty} \sup_{x\in B} \|W^{k+1}x - W^kx\| = 0$$

Table 1: Computational results by (23) and (14) with $W^k = T_k$.

k	x_1^{k+1}	x_2^{k+1}	k	x_1^{k+1}	x_2^{k+1}
10	1.1363636364	0.6411155490	100	1.0148514851	0.9431215161
20	1.0714285714	0.7700827178	200	1.0074626866	0.9707901594
30	1.0483870968	0.8326554114	300	1.0049833887	0.9803526365
40	1.0365853659	0.8687796127	400	1.0037406484	0.9851987678
50	1.0294117647	0.8921748170	500	1.0029940120	0.9881273689

and (i) in the definition of W^k. Marino and Muglia [24], replacing (ii) in the definition of W^k by $\lim_{k\to\infty} \|W^{k+1}x - W^k x\| = 0$ uniformly in $x \in B$ and combining the steepest-descent method with the Krasnosel'skii-Mann one, studied the methods

$$x^{k+1} = \beta_k x^k + (1-\beta_k)(I - t_k D)W^k x^k \text{ and}$$
$$x^{k+1} = \beta_k (I - t_k D)x^k + (1-\beta_k)W^k x^k, \ k \geq 1, \tag{30}$$

in a setting Hilbert space H, where D is η-strongly monotone and L-Lipschitz continuous. Strong convergence of (30) is proved under conditions (t) with $\lim_{k\to\infty} |t_k - t_{k+1}|/t_{k+1} = 0$, $\beta_k \in (0, \overline{a}]$ with $\lim_{k\to\infty} |\beta_k - \beta_{k+1}|/\beta_{k+1} = 0$ and additional condition on constructing W^k from the given family $\{T_i\}$. We note that the mappings $V^k = T_1' \cdots T_k'$ where $T_i' = \gamma_i I + (1-\gamma_i)T_i$ with $\gamma_i \in (0,\infty)$ such that $\sum_{i=1}^{\infty} \gamma_i = \tilde{\gamma} < \infty$ and $S^k = \sum_{i=1}^{k} \gamma_i T_i / \tilde{\gamma}_k$ with $\tilde{\gamma}_k = \gamma_1 + \cdots + \gamma_k$ also satisfy conditions (i) and (ii) in the definition of W^k (see, [3]- [6]). In [3], the first author et al. introduced the methods,

$$x^{k+1} = (1-\beta_k)x^k + \beta_k S^k (I - t_k F)x^k \text{ and}$$
$$x^{k+1} = (1-\beta_k)S^k x^k + \beta_k (I - t_k F)x^k,$$

strong convergence of which have been investigated in strictly convex reflexive Banach spaces with a Gâteaux differentiable norm under conditions (t) and (β).
7. Li [22] studied also method (9) where T^k is defined in (14) with Shimoji and Takahashi's W^k-mapping (see, [30]). Katchang and Kumam [19] proposed the method,

$$x^{k+1} = t_k \gamma f(x^k) + (I - t_k A)T^k x^k, \ k \geq 1,$$

a modification of (9), and proved that it converges in the Banach space with a weak continuous duality mapping j under conditions (t), $\lim_{k\to\infty} \beta_k = 0$ and $\lim_{k\to\infty} \alpha_k = 0$, where A is a strongly positive bounded linear mapping on E and γ is a some positive constant.

3. Numerical experiments

Obviously, for the family of nonexpansive mappings $T_i = (1 - 1/(i+1))I$ with $E = \mathbb{R}^1$, we have that $\cap_{i\geq 1} Fix(T_i) = \{0\}$ and $\lim_{k\to\infty} T_k x = Ix$ for each $x \in \mathbb{R}^1$. Thus, condition (i) in the definition of W^k is not satisfied, because $Fix(I) = \mathbb{R}^1$.

It is easy to see that the family $\{T_i = P_{C_i}\}$, where P_{C_i} is the metric projection of $H = \mathbb{E}^2$, an Euclidian space, onto the set $C_i = \{x = (x_1, x_2) \in H : a_i \leq x_2 \leq b_i\}$ with $a_i = 1 - 1/(i+1)$ and $b_i = 2 + 1/(i+1)$ for all $i \geq 1$, satisfies conditions (i) and (ii) in the definition of W^k. In this case, we have that $C = \cap_{i=1}^{\infty} C_i = \{x \in \mathbb{E}^2 : 1 \leq x_2 \leq 2\}$ and we can take $W^k = T_k$ for all $k \geq 1$. Taking $u = (1.0; 0.0)$, we have that the solution of (1.10) $p_* = (1.0; 1.0)$. The computational results by method (23) and T^k in (14) with starting point $x^1 = (2.5; 2.5)$, $t_k = 1/(k+1)$, $\beta_k = 0.2 + 1/(k+1)$ and $\alpha_k = 1/(k+1)$ are given in Table 1.

Table 2: Computational results by (23) and (14) with $W^k = S^k$.

k	x_1^{k+1}	x_2^{k+1}	k	x_1^{k+1}	x_2^{k+1}
10	0.8226906920	0.9967100188	100	0.8216765320	1.3503455533
20	0.8116106625	1.1196844726	200	0.8261485102	1.4207098495
30	0.8123975068	1.1852032060	300	0.8280615950	1.4464230799
40	0.8142620005	1.2298614455	400	0.8291386059	1.4595495405
50	0.8160321266	1.2628985966	500	0.8298349294	1.4675113528

In the case that $a_i = 1 + 1/(i+1)$, we have $C = \{x \in \mathbb{E}^2 : 1.5 \leq x_2 \leq 2\}$ and $p_* = (1.0; 1.5)$. Moreover, condition (i) in the definition of W^k for T_k, i.e. $W^k = T_k$, does not hold. For computation by (23), we use $W^k = S^k$ in (14) where S^k is defined in Remark 6 with $\gamma_i = 1/i(i+1)$. The results of computation are given in Table 2.

The numerical results show the effectiveness of the method.

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