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Hyperbolization of the Limit Sets of Some Geometric Constructions

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Abstract. Inspired by the construction of Sierpiński carpets, we introduce a new class of fractal sets. For a such fractal set K, we construct a Gromov hyperbolic space X (which is also a strongly hyperbolic space) and show that K is isometric to the Gromov hyperbolic boundary of X. Moreover, under some conditions, we show that $Con(K)$ and X are roughly isometric, where $Con(K)$ is the hyperbolic cone of K.

1. The first section

During the past several years, the hyperbolic construction of all kinds of fractal sets has been considered by many authors. For examples, in [6, 7, 9] the authors proved that for an iterated function system $\{S_j\}_{j=1}^N$ of similitudes, there is a natural graph structure in the representing symbolic space to make it a hyperbolic graph in the sense of Gromov, and the Gromov hyperbolic boundary at infinity is Hölder equivalent to the self-similar set generated by $\{S_j\}_{j=1}^N$. Under this framework they studied the Lipschitz equivalence of self-similar sets and the topological properties of the attractors. In [8], the author obtained that the Julia sets of postcritically finite rational maps arise as Gromov hyperbolic boundaries at infinity. In [4], the author established connections between a metric space X and the large-scale geometry of the hyperspace $\mathcal{H}(X)$ of its nondegenerate closed bounded subsets, and studied mappings on X in terms of the induced mappings on $\mathcal{H}(X)$. In [5], Z. Ibragimov and J. Simanyi considered the hyperbolization of the ternary Cantor set. They introduced a construction of the ternary Cantor set within the context of Gromov hyperbolic geometry and proved that the ternary Cantor set is isometric to the hyperbolic boundary of some Gromov hyperbolic space. Their results have been generalized to the uniform Cantor sets case in [10]. From the construction of Cantor sets or uniform Cantor sets, we know that the gaps, which were removed from the origin interval are still similar to the origin interval. Many fractal sets have the same properties. For examples, the Sierpiński gasket and Sierpiński carpet. Let us recall the construction of Sierpiński carpet. The construction of the Sierpiński carpet begins with a square Δ. The square is cut into 9 congruent subsquares in a 3-by-3 grid, and the central subsquare is removed. The same procedure is then applied recursively to the remaining 8 subsquares, ad infinitum. From the construction, we can see that the removed central subsquare in each step are still similar to the origin square Δ. Based on this intuition, in this paper, we introduction a new class of fractal sets (see Definition 2.1) and consider its hyperbolic construction.

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The paper is organized as follows. In section 2, we define a fractal set K and give some background knowledge about Gromov hyperbolic space; in section 3, we construct a Gromov hyperbolic space X and under some conditions, show that $Con(K)$ and X are roughly isometric, where $Con(K)$ is the hyperbolic cone of K; in section 4, we prove the fractal set K is isometric to the Gromov hyperbolic boundary of X.

2. Basic Concepts

In order to introduce the set K we talk about in this paper, we first present some notations and definitions. Let

$$\Gamma_0 = \{\emptyset\}, \quad \Gamma_k = \{i_1 i_2 \cdots i_k : i_j \in \mathbb{N},\ j = 1, 2, \ldots, k\},$$

$$\Gamma_\infty = \{i_1 i_2 i_3 \cdots : i_j \in \mathbb{N},\ j \in \mathbb{N}\}, \quad \Gamma = \bigcup_{k=0}^{\infty} \Gamma_k.$$

Let us fix two maps $\mathbf{n} : \Gamma \to \mathbb{N}$ and $\mathbf{m} : \Gamma \to \mathbb{N}$. After that, let

$$\Lambda_0 = \{\emptyset\}, \quad \Lambda_k = \{i_1 i_2 \cdots i_k \in \Gamma_k : 1 \le i_1 \le \mathbf{n}(\emptyset),\ 1 \le i_2 \le \mathbf{n}(i_1), \ldots,\ 1 \le i_k \le \mathbf{n}(i_1 i_2 \cdots i_{k-1})\},$$

$$\Lambda_\infty = \{i_1 i_2 i_3 \cdots : i_1 i_2 \cdots i_k \in \Lambda_k,\ k \in \mathbb{N}\}, \quad \Lambda = \bigcup_{k=0}^{\infty} \Lambda_k.$$

For $k \in \mathbb{N}$ we let $S_k = \{i_1 i_2 \cdots i_k : i_1 i_2 \cdots i_{k-1} \in \Lambda_{k-1},\ 1 \le i_k \le \mathbf{m}(i_1 i_2 \cdots i_{k-1})\}$ and $S = \bigcup_{k=1}^{\infty} S_k$.

For $\alpha = i_1 i_2 \cdots i_k \in \Gamma$, we denote its length by $|\alpha|$, i.e. $|\alpha| = k$. For $\alpha = i_1 i_2 i_3 \ldots \in \Gamma_\infty$ and $k \in \mathbb{N}$, let $(\alpha)_k$ be the initial k characters of α, i.e. $(\alpha)_k = i_1 i_2 \cdots i_k$.

$(\mathbb{R}^d, \rho)$ is d-dimensional Euclidean space with the usual metric. $\forall\, x, y \in \mathbb{R}^d$, we denote $\rho(x, y)$ by $|xy|$ for convenience. $\forall\, A \subseteq \mathbb{R}^d$, we denote $\mathrm{diam}(A)$ by $|A|$. For $A, B \subseteq \mathbb{R}^d$, if there exists a similitude T such that $A = TB$, then we write $A \simeq B$ (recall that a map $T : \mathbb{R}^d \to \mathbb{R}^d$ is a similitude if and only if there exists $r > 0$ such that $\forall\, x, y \in \mathbb{R}^d$, $|xy| = r|T(x)T(y)|$). The interior of a set $A \subseteq \mathbb{R}^d$ is written A^o, and the closure of A is written $\overline{A}$. For $x, y \in \mathbb{R}$, we denote $\max\{x, y\}$ by $x \vee y$ and $\min\{x, y\}$ by $x \wedge y$.

Definition 2.1. *Suppose V is a nonempty bounded open set on $\mathbb{R}^d$ with $\left(\overline{V}\right)^o = V$. Let $\mathbf{n} : \Gamma \to \mathbb{N}$ and $\mathbf{m} : \Gamma \to \mathbb{N}$ be two maps. According to above introductions, we obtain Λ and S decided by $\mathbf{n}$ and $\mathbf{m}$. We declare that the compact set K fulfill the structure (V, Λ, S) if*

(1) *For any $\alpha \in \Lambda$, there exist two classes of open sets, $\{V_{\alpha i}\}_{i=1}^{\mathbf{n}(\alpha)}$ and $\{W_{\alpha j}\}_{j=1}^{\mathbf{m}(\alpha)}$, such that*

$$\overline{V}_\alpha = \Big(\bigcup_{i=1}^{\mathbf{n}(\alpha)} \overline{V}_{\alpha i}\Big) \cup \Big(\bigcup_{j=1}^{\mathbf{m}(\alpha)} \overline{W}_{\alpha j}\Big).$$

Besides $V_{\alpha i} \simeq V_\alpha$, $W_{\alpha j} \simeq V_\alpha$ for all i, j, and they are disjoint pairwise ($V_\emptyset = V$ for convention);

(2) $\lim_{|\alpha| \to \infty} |V_\alpha| = 0$, *that is, $\forall\, \varepsilon > 0$, there exists $k \in \mathbb{N}$ such that $|V_\alpha| < \varepsilon$ for any $|\alpha| \ge k$ and $\alpha \in \Lambda$;*

(3) *K is the compact set satisfying $K = \bigcap_{k=0}^{\infty} \bigcup_{\alpha \in \Lambda_k} \overline{V}_\alpha$.*

From the construction, we know, every open subset $W_{\alpha j}$ removed from V_α is still similar to V_α, thus similar to the origin open set V. In the rest of the paper, when we claim that K fulfills the structure (V, Λ, S), we always suppose that V is a open bounded set on $\mathbb{R}^d$ with $\left(\overline{V}\right)^o = V$ and $V \neq \emptyset$. Λ and S are decided by some maps $\mathbf{n}$, $\mathbf{m}$ following the rules we described above.

Example 2.2. ***(Moran set)*** *Let us fix a sequence of positive integers $\{n_k\}_{k \ge 1}$. Let $\{c_{k,j}\}_{1 \le k, 1 \le j \le n_k}$ be a sequence of positive numbers satisfying $\Sigma_{j=1}^{n_k} c_{k,j} < 1$. Set $D_k = \max_{1 \le j \le n_k} c_{k,j}$ and assume $\lim_{k \to \infty} \prod_{s=1}^{k} D_s = 0$. Let $\mathbf{n} : \Gamma \to \mathbb{N}$ be the map such that $\mathbf{n}(\alpha) = n_{|\alpha|+1}$ for $\alpha \in \Gamma$, and $\mathbf{m} : \Gamma \to \mathbb{N}$ some map satisfying $\mathbf{m}(\alpha) \le n_{|\alpha|+1} + 1$ for $\alpha \in \Gamma$. Fix an open interval $V = V_\emptyset \subseteq \mathbb{R}$. For any $\alpha \in \Lambda_{k-1}$, we can find open intervals $V_{\alpha j}$ with $1 \le j \le n_k$ belonging to V_α, and they are disjoint pairwise. Besides $|V_{\alpha j}|/|V_\alpha| = c_{k,j}$. Then $\{W_{\alpha j}\}_{j=1}^{\mathbf{m}(\alpha)}$ consists of component intervals of $\overline{V}_\alpha \setminus \bigcup_{j=1}^{n_k} \overline{V}_{\alpha j}$. There is a compact set K fulfill the structure $(V,\ \Lambda,\ S)$ and K is indeed a Moran set. See Fig. 1.*

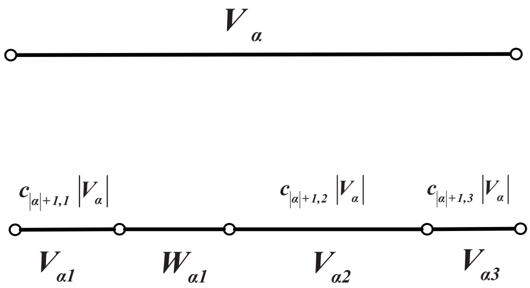

Figure 1: Construction of the open interval V_α with $n_{|\alpha|+1} = 3$ and $\mathbf{m}(\alpha) = 1$

Example 2.3. ***(Sierpiński carpet)*** *Let* $\mathbf{n} : \Gamma \to \mathbb{N}$ *be the map such that* $\mathbf{n}(\alpha) \equiv 8$ *for* $\alpha \in \Gamma$*, and* $\mathbf{m} : \Gamma \to \mathbb{N}$ *the map satisfying* $\mathbf{m}(\alpha) \equiv 1$ *for* $\alpha \in \Gamma$*. Fix an open square* $V \subseteq \mathbb{R}^2$*. For all* $\alpha \in \Lambda_{k-1}$*, let* V_α *be an open square and it could be divided into* 9 *squares with the same volume. Let* $W_{\alpha 1}$ *denote the central open square and* $V_{\alpha 1}, V_{\alpha 2}, \ldots, V_{\alpha 8}$ *denote the others. There is a unique set K fulfill the structure* (V, Λ, S) *and K is the Sierpiński carpet. See Fig. 2.*

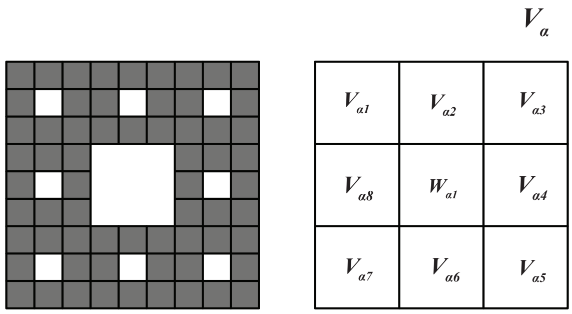

Figure 2: Left: the first two stages of the construction of Sierpiński carpet; Right: construction of V_α

Example 2.4. ***(Sierpiński gasket)*** *Let* $\mathbf{n} : \Gamma \to \mathbb{N}$ *be the map such that* $\mathbf{n}(\alpha) \equiv 3$ *for* $\alpha \in \Gamma$*, and* $\mathbf{m} : \Gamma \to \mathbb{N}$ *the map satisfying* $\mathbf{m}(\alpha) \equiv 1$ *for* $\alpha \in \Gamma$*. Fix an open equilateral triangle* $V \subseteq \mathbb{R}^2$*. For all* $\alpha \in \Lambda_{k-1}$*, let* V_α *be an open equilateral triangle and it could be divided into* 4 *equilateral triangles with the same volume. Let* $W_{\alpha 1}$ *denote the central open equilateral triangle and* $V_{\alpha 1}, V_{\alpha 2}, V_{\alpha 3}$ *denote the others. There is a unique set K fulfill the structure* (V, Λ, S) *and K is the Sierpiński gasket. See Fig. 3.*

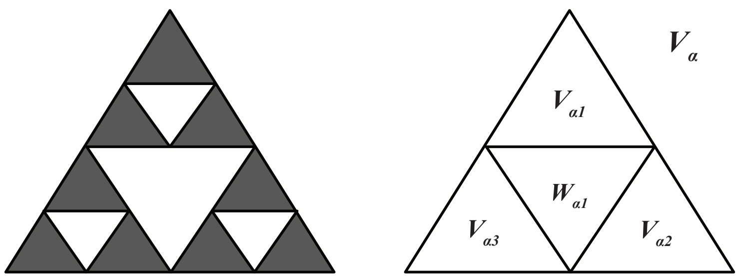

Figure 3: Left: the first two stages of the construction of Sierpiński gasket; Right: construction of V_α

Now we turn to a brief discussion of Gromov hyperbolic spaces. A metric space (X, d) is called Gromov δ-hyperbolic (or δ-hyperbolic) if there exists $\delta \geq 0$ such that for all $x, y, z, w \in X$,

$$d(x, y) + d(z, w) \leq (d(x, z) + d(y, w)) \vee (d(x, w) + d(y, z)) + 2\delta.$$

For $x, y, z \in X$, the Gromov product of x and y with respect to z is defined by

$$(x|y)_z = \frac{1}{2}\Big(d(x,z) + d(y,z) - d(x,y)\Big).$$

Alternatively, the space (X, d) is δ-hyperbolic if

$$(x|y)_v \geq (x|z)_v \wedge (z|y)_v - \delta$$

for all $x, y, z, v \in X$.

To each Gromov hyperbolic space X, we associate a boundary at infinity ∂X (also called the Gromov boundary). Let us fix a base point $v \in X$. A sequence $\{a_i\}$ in X is said to converge at infinity if $\lim_{i,j\to\infty}(a_i|a_j)_v = \infty$. Two such sequences $\{a_i\}$ and $\{b_i\}$ are equivalent if $\lim_{i\to\infty}(a_i|b_i)_v = \infty$. The boundary at infinity is defined to be the equivalence classes of sequences converging at infinity. Obviously, the boundary at infinity is independent of the base point. A metric d on ∂X is called a visual metric if there exist $v \in X$, $C \geq 1$ and $\epsilon > 0$ such that for all $x, y \in \partial X$,

$$\frac{1}{C}\rho_{\epsilon,v}(x,y) \leq d(x,y) \leq C\rho_{\epsilon,v}(x,y), \quad \textit{where} \quad \rho_{\epsilon,v}(x,y) = e^{-\epsilon(x|y)_v}.$$

Here $(x|y)_v$ is the Gromov product on ∂X defined by

$$(x|y)_v = \inf\{\liminf_{i\to\infty}(a_i|b_i)_v : \{a_i\} \in x,\ \{b_i\} \in y\}.$$

Definition 2.5. *We say that a metric space is strongly hyperbolic with parameter $\varepsilon > 0$ if*

$$\exp(-\varepsilon(x|y)_o) \leq \exp(-\varepsilon(x|z)_o) + \exp(-\varepsilon(z|y)_o)$$

for all $x, y, z, o \in X$; equivalently, the four-point condition

$$\exp\Big(\frac{\varepsilon}{2}(|xy| + |zt|)\Big) \leq \exp\Big(\frac{\varepsilon}{2}(|xz| + |yt|)\Big) + \exp\Big(\frac{\varepsilon}{2}(|xt| + |zy|)\Big)$$

holds for all $x, y, z, t \in X$.

A strongly hyperbolic space is a Gromov hyperbolic space with better properties. For example, the strongly hyperbolic space is boundary continuous and its Gromov hyperbolic boundary is a Ptolemy space under the visual metric (refer to [1, 3, 11]).

Recall the hyperbolic cone construction [2]. Let (X, d) be a bounded metric space, and let $Con(X) = X \times (0, \mathrm{diam}(X)]$. The metric ρ_C on $Con(X)$ is defined by

$$\rho_C((x,r),(y,s)) = 2\log\left(\frac{d(x,y) + r \vee s}{\sqrt{rs}}\right).$$

The space $(Con(X), \rho_C)$ is δ-hyperbolic with well properties (k-visual and k-roughly geodesic for some $k \geq 0$).

In the end of this section, we recall the definition of k-rough isometry:

Definition 2.6. *Let (X, d_1) and (Y, d_2) be arbitrary metric spaces. Suppose that $f : X \to Y$ is a map such that $f(X)$ is k-cobounded in Y for some $k \geq 0$, that is* $\mathrm{dist}(y, f(X)) \leq k$ *for all $y \in Y$. We say that f is a k-rough isometry if*

$$|d_2(f(a), f(b)) - d_1(a,b)| \leq k$$

for all $a, b \in X$. In this case, we also say that X and Y are k-roughly isometric.

3. Hyperbolization

In the rest of this paper, we fix a compact set K that fulfills the structure (V, Λ, S). Λ and S are decided by some maps $\mathbf{n}, \mathbf{m} : \Gamma \to \mathbb{N}$. For all $\alpha \in \Lambda$ and $\beta \in S$, V_α and W_β are given. **SET** $X = \{W_\beta\}_{\beta \in S}$. We are in the place to find a metric for X under which X is hyperbolic. In the final section, we establish an isometry between K and ∂X.

According to the assumptions in Definition 2.1, we have $V_\alpha \simeq V$ for all $\alpha \in \Lambda$ and $W_\beta \simeq V$ for all $\beta \in S$. Let $\{T_\alpha\}_{\alpha\in\Lambda}, \{R_\alpha\}_{\beta\in S}$ be two classes of similitudes such that $T_\alpha V = V_\alpha$ and $R_\beta V = W_\beta$. Fix a point x in V such that $B(x, |V|/M) = \{y : |xy| < |V|/M\} \subseteq V$, where $M > 1$ is some positive number. Set $x_\beta = R_\beta(x)$ for all $\beta \in S$.

For $\alpha, \beta \in S$, we define a map $u : X \times X \to \mathbb{R}$ by

$$u(W_\alpha, W_\beta) = \frac{|W_\alpha| + |W_\beta|}{2} + 2M|x_\alpha x_\beta|$$

and set

$$h(W_\alpha, W_\beta) = 2\log\frac{u(W_\alpha, W_\beta)}{\sqrt{|W_\alpha||W_\beta|}}.$$

In order to prove that (X, h) is a Gromov hyperbolic space, we derive some properties of $\{W_\beta\}_{\beta\in S}$ first.

Lemma 3.1. (1) *For any $\alpha \in \Lambda$, $\forall\, 1 \le i \le \mathbf{n}(\alpha)$ and $\forall\, 1 \le j \le \mathbf{m}(\alpha)$, we have $V_{\alpha i} \subseteq V_\alpha$ and $W_{\alpha j} \subseteq V_\alpha$;*

(2) *$\forall\, \alpha, \beta \in S$, if $\alpha \ne \beta$, then $W_\alpha \bigcap W_\beta = \emptyset$. Furthermore we have $|x_\alpha x_\beta| \ge \frac{|W_\alpha|+|W_\beta|}{M}$.*

Proof. (1) For any $\alpha \in \Lambda$, $T_\alpha V = V_\alpha$, so $\left(\overline{T_\alpha V}\right)^o = T_\alpha\left(\overline{V}\right)^o = T_\alpha V$, that is $\left(\overline{V}_\alpha\right)^o = V_\alpha$. By Definition 2.1, $\overline{V}_{\alpha i} \subseteq \overline{V}_\alpha$, then $\left(\overline{V_{\alpha i}}\right)^o \subseteq \left(\overline{V_\alpha}\right)^o$, that is $V_{\alpha i} \subseteq V_\alpha$. Analogously, we obtain $W_{\alpha j} \subseteq V_\alpha$.

(2) Suppose that $\alpha, \beta \in S$ and $\alpha \ne \beta$. Write $\alpha = \hat{\alpha} i (\hat{\alpha} \in \Lambda_{|\alpha|-1})$ and $\beta = \hat{\beta} j\, (\hat{\beta} \in \Lambda_{|\beta|-1})$. If $\hat{\alpha} = \hat{\beta}$, the conclusion is obvious. If $\hat{\alpha} \ne \hat{\beta}$, without loss of generality we assume $|\hat{\alpha}| \ge |\hat{\beta}|$. If $(\hat{\alpha})_{|\hat{\beta}|} = \hat{\beta}$, we write $\hat{\alpha}$ as $\hat{\beta} i_1 i_2 \ldots i_t$. Since $V_{\hat{\beta} i_1} \bigcap W_{\hat{\beta} j} = \emptyset$ and $W_\alpha \subseteq V_{\hat{\beta} i_1}$, we have the desired result. If $(\hat{\alpha})_{|\hat{\beta}|} \ne \hat{\beta}$, we write $\hat{\alpha} = i_1 i_2 \ldots i_s j_1 j_2 \ldots j_p$ and $\hat{\beta} = i_1 i_2 \ldots i_s l_1 l_2 \ldots j_q$ with $j_1 \ne l_1$. Since $V_{i_1 i_2 \ldots i_s j_1} \bigcap V_{i_1 i_2 \ldots i_s l_1} = \emptyset$, the conclusion is obvious now.

For $\alpha, \beta \in S$ with $\alpha \ne \beta$, we obtain $W_\alpha \bigcap W_\beta = \emptyset$. One can verify that $B(x_\alpha, |W_\alpha|/M) \bigcap B(x_\beta, |W_\beta|/M) = \emptyset$. Hence $|x_\alpha x_\beta| \ge \frac{|W_\alpha|+|W_\beta|}{M}$. □

Theorem 3.2. *(X, h) is a metric space.*

Proof. It suffices to prove h satisfies the triangle inequality, that is for any $\alpha, \beta, \gamma \in S$, we have

$$h(W_\alpha, W_\beta) = 2\log\frac{u(W_\alpha, W_\beta)}{\sqrt{|W_\alpha||W_\beta|}} \le 2\log\frac{u(W_\alpha, W_\gamma)u(W_\gamma, W_\beta)}{\sqrt{|W_\alpha||W_\beta|}|W_\gamma|} = h(W_\alpha, W_\gamma) + h(W_\gamma, W_\beta).$$

Without loss of generality, assume α, β and γ are different pairwise. It suffices to show

$$|W_\gamma| u(W_\alpha, W_\beta) \le u(W_\alpha, W_\gamma) u(W_\gamma, W_\beta).$$

But

$$|W_\gamma| u(W_\alpha, W_\beta) = 2M|x_\alpha x_\beta||W_\gamma| + \frac{1}{2}|W_\gamma|\Big(|W_\alpha| + |W_\beta|\Big)$$

and

$$\begin{aligned} u(W_\alpha, W_\gamma)u(W_\gamma, W_\beta) = {} & \frac{1}{4}\Big(|W_\alpha| + |W_\gamma|\Big)\Big(|W_\gamma| + |W_\beta|\Big) + 4M^2|x_\alpha x_\gamma||x_\gamma x_\beta| \\ & + M|x_\alpha x_\gamma|\Big(|W_\gamma| + |W_\beta|\Big) + M|x_\gamma x_\beta|\Big(|W_\alpha| + |W_\gamma|\Big). \end{aligned}$$

By Lemma 3.1, we obtain $|x_\alpha x_\gamma| \wedge |x_\gamma x_\beta| \ge |W_\gamma|/M$. Besides $|x_\alpha x_\gamma| \vee |x_\gamma x_\beta| \ge |x_\alpha x_\beta|/2$. Then $4M^2|x_\alpha x_\gamma||x_\gamma x_\beta| \ge 2M|x_\alpha x_\beta||W_\gamma|$. According to Lemma 3.1, one can easily verify that $M|x_\alpha x_\gamma|(|W_\gamma| + |W_\beta|) \ge |W_\gamma||W_\beta|$ and $M|x_\gamma x_\beta|(|W_\alpha| + |W_\gamma|) \ge |W_\gamma||W_\alpha|$. In conclusion, the triangle inequality holds, thus h is a metric on X. □

Theorem 3.3. *(X, h) is a strongly hyperbolic space with parameter* 1.

Proof. First of all, we prove that for any $\alpha, \beta, \gamma, \eta \in S$, the following inequality

$$u(W_\alpha, W_\beta)u(W_\gamma, W_\eta) \leq u(W_\alpha, W_\gamma)u(W_\beta, W_\eta) + u(W_\alpha, W_\eta)u(W_\beta, W_\gamma) \tag{1}$$

holds.

According to the definition of u, we have

$$\begin{aligned} u(W_\alpha, W_\beta)u(W_\gamma, W_\eta) = &\frac{1}{4}\Big(|W_\alpha| + |W_\beta|\Big)\Big(|W_\gamma| + |W_\eta|\Big) + 4M^2|x_\alpha x_\beta||x_\gamma x_\eta| \\ &+ M|x_\alpha x_\beta|\Big(|W_\gamma| + |W_\eta|\Big) + M|x_\gamma x_\eta|\Big(|W_\alpha| + |W_\beta|\Big), \end{aligned}$$

$$\begin{aligned} u(W_\alpha, W_\gamma)u(W_\beta, W_\eta) = &\frac{1}{4}\Big(|W_\alpha| + |W_\gamma|\Big)\Big(|W_\beta| + |W_\eta|\Big) + 4M^2|x_\alpha x_\gamma||x_\beta x_\eta| \\ &+ M|x_\alpha x_\gamma|\Big(|W_\beta| + |W_\eta|\Big) + M|x_\beta x_\eta|\Big(|W_\alpha| + |W_\gamma|\Big), \end{aligned}$$

$$\begin{aligned} u(W_\alpha, W_\eta)u(W_\beta, W_\gamma) = &\frac{1}{4}\Big(|W_\alpha| + |W_\eta|\Big)\Big(|W_\beta| + |W_\gamma|\Big) + 4M^2|x_\alpha x_\eta||x_\beta x_\gamma| \\ &+ M|x_\alpha x_\eta|\Big(|W_\beta| + |W_\gamma|\Big) + M|x_\beta x_\gamma|\Big(|W_\alpha| + |W_\eta|\Big). \end{aligned}$$

Since $(\mathbb{R}^d, \rho)$ is Euclidean space, ρ is ptolemaic. Thus

$$|x_\alpha x_\beta||x_\gamma x_\eta| \leq |x_\alpha x_\gamma||x_\beta x_\eta| + |x_\alpha x_\eta||x_\beta x_\gamma|. \tag{2}$$

It is obvious that

$$\Big(|W_\alpha| + |W_\beta|\Big)\Big(|W_\gamma| + |W_\eta|\Big) \leq \Big(|W_\alpha| + |W_\gamma|\Big)\Big(|W_\beta| + |W_\eta|\Big) + \Big(|W_\alpha| + |W_\eta|\Big)\Big(|W_\beta| + |W_\gamma|\Big). \tag{3}$$

Since ρ satisfies the triangle inequality, we also have

$$\begin{aligned} M|x_\alpha x_\beta|\Big(|W_\gamma| + |W_\eta|\Big) + M|x_\gamma x_\eta|\Big(|W_\alpha| + |W_\beta|\Big) \leq &M|x_\alpha x_\gamma|\Big(|W_\beta| + |W_\eta|\Big) + M|x_\beta x_\eta|\Big(|W_\alpha| + |W_\gamma|\Big) \\ &+ M|x_\alpha x_\eta|\Big(|W_\beta| + |W_\gamma|\Big) + M|x_\beta x_\gamma|\Big(|W_\alpha| + |W_\eta|\Big). \end{aligned} \tag{4}$$

Combining the equations (2), (3), (4), it is easy to see equation (1) is valid.

Secondly, note that

$$\exp\left\{\frac{1}{2}\Big(h(W_\alpha, W_\beta) + h(W_\gamma, W_\eta)\Big)\right\} = u(W_\alpha, W_\beta)u(W_\gamma, W_\eta)/\sqrt{|W_\alpha||W_\beta||W_\gamma||W_\eta|}.$$

We could obtain other equations analogously. Combining these equations and using equation (1), we have

$$\exp\left\{\frac{1}{2}\Big(h(W_\alpha, W_\beta) + h(W_\gamma, W_\eta)\Big)\right\} \leq \exp\left\{\frac{1}{2}\Big(h(W_\alpha, W_\gamma) + h(W_\beta, W_\eta)\Big)\right\} + \exp\left\{\frac{1}{2}\Big(h(W_\alpha, W_\eta) + h(W_\beta, W_\gamma)\Big)\right\}.$$

In conclusion, (X, h) is a strongly hyperbolic space with parameter 1. □

Next we present some connections between the hyperbolic cone $Con(K)$ and the Gromov hyperbolic space (X, h). First of all, we prove the following two lemmas, which are also important in the final section.

For each $k \in \mathbb{N}$, let D_k be a nonempty finite subset of Γ_k. We say that the sequence $\{D_k\}_{k=1}^\infty$ has the *property* **A**, if for any $\alpha \in D_k$, we have $(\alpha)_s \in D_s$ for all $1 \leq s \leq k$.

Lemma 3.4. *Suppose the sequence* $\{D_k\}_{k=1}^{\infty}$ *has the property* **A**. *Then there is* $\beta \in \Gamma_\infty$ *such that* $(\beta)_k \in D_k$ *for all* $k \in \mathbb{N}$.

Proof. We claim that there exists $i_1 \in D_1$ such that for any $k \in \mathbb{N}$, we could find $\zeta \in D_k$ with $(\zeta)_1 = i_1$. If this claim is false, that is for any $j \in D_1$, there exists $N(j) \in \mathbb{N}$ such that $(\zeta)_1 \neq j$ for any $\zeta \in D_{N(j)}$. Moreover $\{\zeta \in D_k : (\zeta)_1 = j\} = \emptyset$ for any $k \geq N(j)$. Since D_1 is finite, $\max\{N(j) : j \in D_1\} < \infty$. Set $N_1 = \max\{N(j) : j \in D_1\}$, it is easy to see $D_{N_1} = \emptyset$, which is a contradiction. Let $D_k^1 = \{\alpha \in D_k : (\alpha)_1 = i_1\}$, D_k^1 is a nonempty finite subset of D_k. Furthermore, if $\alpha \in D_{k+1}^1$, then $(\alpha)_1 = i_1$. Thus $(\alpha)_k \in D_k^1$, which implies that $\{D_k^1\}$ has the property **A**.

Similarly, there is an index i_2 such that $i_1 i_2 \in D_2^1$ and for any $k \geq 3$, there is $\alpha \in D_k^1$ such that $(\alpha)_2 = i_1 i_2$. Set

$$D_k^2 = \begin{cases} D_k^1, & \text{for } k = 1, \\ \{\alpha \in D_k^1 : (\alpha)_2 = i_1 i_2\}, & \text{for } k \geq 2. \end{cases}$$

Obviously, D_k^2 is a nonempty finite subset of D_k^1 and $\{D_k^2\}$ has the property **A**.

Inductively, for any $l \in \mathbb{N}$, we can find $i_1 i_2 \cdots i_l \in D_l^{l-1}$ such that for any $k > l$, there is $\alpha \in D_k^{l-1}$ such that $(\alpha)_l = i_1 i_2 \cdots i_l$. Let

$$D_k^l = \begin{cases} \{i_1 i_2 \cdots i_k\}, & \text{for } 1 \leq k \leq l, \\ \{\alpha \in D_k^{l-1} : (\alpha)_l = i_1 \cdots i_l\}, & \text{for } k > l. \end{cases}$$

Obviously, D_k^l is a nonempty finite subset of D_k^{l-1} and $\{D_k^l\}_{k=1}$ has the property **A**.

Thus, for any $l \geq 1$, we obtain $i_1 i_2 \cdots i_l \in D_l^l \subset D_l$. Put $\beta = i_1 i_2 i_3 \ldots$, we have $(\beta)_k = i_1 i_2 \ldots i_k \in D_k$ for all $k \in \mathbb{N}$. □

Lemma 3.5. $\forall\, x \in K$, *there exists* $\alpha \in \Lambda_\infty$ *such that* $\{x\} = \bigcap_{k=0}^{\infty} \overline{V}_{(\alpha)_k}$.

Proof. Let us fix a point $x \in K$. Since $K = \bigcap_{k=0}^{\infty} \bigcup_{\alpha \in \Lambda_k} \overline{V}_\alpha$, we have $x \in \bigcup_{\alpha \in \Lambda_k} \overline{V}_\alpha$ for each $k \in \mathbb{N}$. Put $D_k = \{\alpha \in \Lambda_k : x \in \overline{V}_\alpha\}$. D_k is a nonempty finite set. $\forall\, \alpha \in D_k$ and $1 \leq s \leq k$, since $\overline{V}_\alpha \subseteq \overline{V}_{(\alpha)_s}$, we have $(\alpha)_s \in D_s$. Hence by Lemma 3.4, there exists $\alpha \in \Lambda_\infty$ such that $(\alpha)_k \in D_k$, that is $x \in \overline{V}_{(\alpha)_k}$. Besides, by Definition 2.1, $\lim_{k\to\infty} |\overline{V}_{(\alpha)_k}| = 0$. Hence $\bigcap_{k=0}^{\infty} \overline{V}_{(\alpha)_k}$ contains only one point, i.e. x. □

Theorem 3.6. *Let* (K, ρ) *be a metric space with* $\mathrm{diam}(K) > 0$ *(recall that* ρ *is the usual metric on* $\mathbb{R}^d$*). If there exists a constant* $C \geq 1$ *such that for any* $\alpha \in \Lambda$,

$$\min\left\{\frac{|V_{\alpha i}|}{|V_\alpha|}, \frac{|W_{\alpha j}|}{|V_\alpha|} : 1 \leq i \leq \mathbf{n}(\alpha), 1 \leq j \leq \mathbf{m}(\alpha)\right\} \geq \frac{1}{C},$$

then $(Con(K), \rho_C)$ *and* (X, h) *are roughly isometric.*

Proof. Let $\tilde{C}$ be a constant such that $\tilde{C}|K| > |V|$. By Lemma 3.5, for every $x \in K$, there exists $\alpha \in \Lambda_\infty$ such that $\{x\} = \bigcap_{k=0}^{\infty} \overline{V}_{(\alpha)_k}$. Now define the map $g : K \to \Lambda_\infty$ by $g(x) = \alpha$.

Given a point $(x, r) \in Con(K)$ with $g(x) = \alpha$, we could find a unique k such that $|\overline{V}_{(\alpha)_{k+1}}| < r \leq |\overline{V}_{(\alpha)_k}|$. Define $f : Con(K) \to X$ by $f(x, r) = W_{(\alpha)_k 1}$, and we claim that f is a rough isometry.

First of all, we show that $f(Con(K))$ is cobounded in X.

For any $\alpha \in \Lambda$ and any $1 \leq j \leq \mathbf{m}(\alpha)$,

$$h(W_{\alpha 1}, W_{\alpha j}) = 2 \log \frac{2M|x_{\alpha 1} x_{\alpha j}| + \big(|W_{\alpha 1}| + |W_{\alpha j}|\big)/2}{\sqrt{|W_{\alpha 1}||W_{\alpha j}|}} \leq 2 \log \frac{2M|V_\alpha| + |V_\alpha|}{|V_\alpha|/C} = 2 \log \big((2M+1)C\big).$$

For $\alpha \in \Lambda$ with $|V_\alpha| > |K|$, we take $\beta = \alpha 11 \ldots 1$ such that $|V_\beta| \leq |K|$ and $|\beta|$ is minimum. We have

$$h(W_{\alpha 1}, W_{\beta 1}) = 2 \log \frac{2M|x_{\alpha 1} x_{\beta 1}| + \big(|W_{\alpha 1}| + |W_{\beta 1}|\big)/2}{\sqrt{|W_{\alpha 1}||W_{\beta 1}|}} \leq 2 \log \frac{2M|V_\alpha| + |V_\alpha|}{\sqrt{|V_\alpha|/C}\,\sqrt{|V|/(C^2\tilde{C})}} \leq 2 \log \Big((2M+1)\sqrt{C^3\tilde{C}}\Big).$$

By the above arguments, when talking about the distance from any point $W_{\alpha i} \in X$ to $f(Con(K))$, we could assume that $|V_\alpha| \leq |K|$ and $i = 1$. Take a point $y \in \overline{V}_\alpha \bigcap K$, then $(y, |V_\alpha|) \in Con(K)$ since $|V_\alpha| \leq |K|$. Suppose $g(y) = \beta$, that is $\{y\} = \bigcap_{k=0}^{\infty} \overline{V}_{(\beta)_k}$, then there exists $s \in \mathbb{N}$ such that $|V_{(\beta)_s}| < |V_\alpha| \leq |V_{(\beta)_{s-1}}|$. Let $\tilde{\beta} = (\beta)_s$, we have $f(y, |V_{(\beta)_s}|) = W_{\tilde{\beta}1} \in f(Con(K))$. But

$$h(W_{\tilde{\beta}1}, W_{\alpha 1}) = 2\log \frac{2M|x_{\tilde{\beta}1}x_{\alpha 1}| + \left(|W_{\tilde{\beta}1}| + |W_{\alpha 1}|\right)/2}{\sqrt{|W_{\tilde{\beta}1}||W_{\alpha 1}|}} \leq 2\log \frac{4M|V_\alpha| + |V_\alpha|}{\sqrt{|V_\alpha|/C^2}\sqrt{|V_\alpha|/C}} = 2\log\left((4M+1)\sqrt{C^3}\right),$$

which implies that $f(Con(K))$ is cobounded in X.

Secondly, take two points $(x, r), (y, s) \in Con(K)$. Suppose $f(x, r) = W_{(\alpha)_k 1}$ and $f(y, s) = W_{(\beta)_l 1}$, we put

$$A = \frac{|xy| + r \vee s}{2M|x_{(\alpha)_k 1}x_{(\beta)_l 1}| + \left(|W_{(\alpha)_k 1}| + |W_{(\beta)_l 1}|\right)/2} \quad \text{and} \quad B = \frac{\sqrt{|W_{(\alpha)_k 1}||W_{(\beta)_l 1}|}}{\sqrt{rs}}.$$

Since $|W_{(\alpha)_k 1}| \geq |V_{(\alpha)_k}|/C$, $|W_{(\alpha)_l 1}| \geq |V_{(\alpha)_l}|/C$,

$$|V_{(\alpha)_k}| \geq r > |V_{(\alpha)_{k+1}}| \geq \frac{|V_{(\alpha)_k}|}{C} \quad \text{and} \quad |V_{(\alpha)_l}| \geq s > |V_{(\alpha)_{l+1}}| \geq \frac{|V_{(\alpha)_l}|}{C},$$

we have

$$\frac{1}{C}\sqrt{rs} \leq \sqrt{|W_{(\alpha)_k 1}||W_{(\beta)_l 1}|} \leq C\sqrt{rs},$$

thus $\frac{1}{C} \leq B \leq C$.

By Lemma 3.1, we have $|x_{(\alpha)_k 1}x_{(\beta)_l 1}| \geq \left(|W_{(\alpha)_k 1}| + |W_{(\beta)_l 1}|\right)/M$. Since $x \in \overline{V}_{(\alpha)_k}, y \in \overline{V}_{(\beta)_l}$, and $W_{(\alpha)_k 1} \subseteq V_{(\alpha)_k}, W_{(\beta)_l 1} \subseteq V_{(\beta)_l}$, we have

$$\begin{aligned}
|xy| + r \vee s &\leq |x_{(\alpha)_k 1}x_{(\beta)_l 1}| + |x_{(\alpha)_k 1}x| + |yx_{(\beta)_l 1}| + r \vee s \\
&\leq |x_{(\alpha)_k 1}x_{(\beta)_l 1}| + C\left(|W_{(\alpha)_k 1}| + |W_{(\beta)_l 1}|\right) + C\left(|W_{(\alpha)_k 1}| \vee |W_{(\beta)_l 1}|\right) \\
&\leq |x_{(\alpha)_k 1}x_{(\beta)_l 1}| + C\left(|W_{(\alpha)_k 1}| + |W_{(\beta)_l 1}|\right) + C\left(|W_{(\alpha)_k 1}| + |W_{(\beta)_l 1}|\right) \\
&\leq 4C\left(2M|x_{(\alpha)_k 1}x_{(\beta)_l 1}| + \frac{1}{2}\left(|W_{(\alpha)_k 1}| + |W_{(\beta)_l 1}|\right)\right).
\end{aligned}$$

Hence $A \leq 4C$. Besides, since

$$\begin{aligned}
2M|x_{\alpha 1}x_{\beta 1}| + \frac{1}{2}\left(|W_{\alpha 1}| + |W_{\beta 1}|\right) &\leq 2M(|xy| + C(r+s)) + \frac{C}{2}(r+s) \\
&\leq 2M\left(|xy| + 2C(r \vee s)\right) + C(r \vee s) \\
&\leq \max\{2M, 4MC + C\}\left(|xy| + r \vee s\right) \\
&\leq (4MC + C)(|xy| + r \vee s),
\end{aligned}$$

we have $A \geq \frac{1}{(4M+1)C}$.

In conclusion, one can easily verify that

$$|h(f(x,r), f(y,s)) - \rho_C((x,r),(y,s))| = |\log(AB)| \leq 2\log\left((4M+1)C^2\right),$$

which implies f is a rough-isometric map. Obviously, $(Con(K), \rho_C)$ and (X, h) are roughly isometric. □

4. Boundary at infinity

In this section, we are in the place to establish an isometry between K and ∂X. We derive some properties of ∂X first. Let $W_1 \in X$ be the base point and denote it by o. For any $\{W_{\alpha_n}\}_{n=1}^{\infty} \in a \in \partial X$, we have

$$(W_{\alpha_n}|W_{\alpha_m})_o = \log \frac{u(W_{\alpha_n}, W_1)u(W_{\alpha_m}, W_1)}{|W_1|u(W_{\alpha_n}, W_{\alpha_m})}.$$

But for all $\alpha \in S$,

$$\frac{|W_1|}{2} \leq u(W_\alpha, W_1) \leq (2M+1)|V|,$$

thus

$$\log \frac{|W_1|}{4u(W_{\alpha_n}, W_{\alpha_m})} \leq (W_{\alpha_n}|W_{\alpha_m})_o \leq \log \frac{(2M+1)^2|V|^2}{|W_1|u(W_{\alpha_n}, W_{\alpha_m})},$$

Because of this argument, the following lemma is apparent.

Lemma 4.1. *$\{W_{\alpha_n}\}_{n=1}^{\infty}$ converge at infinity if and only if $\lim_{n,m\to\infty} u(W_{\alpha_n}, W_{\alpha_m}) = 0$. We also have $\lim_{n\to\infty} |W_{\alpha_n}| = 0$ and $\{x_{\alpha_n}\}_{n=1}^{\infty}$ is a Cauchy sequence.*

By similar arguments, we could observe that two sequences $\{W_{\alpha_n}\}$ and $\{W_{\beta_n}\}$ converging at infinity are equivalent if and only if $\lim_{n\to\infty} u(W_{\alpha_n}, W_{\beta_n}) = 0$. We also need the following lemma which presents some properties of $\{x_\alpha\}_{\alpha\in S}$.

Lemma 4.2. *For $\{W_{\alpha_n}\}_{n=1}^{\infty} \in a \in \partial X$, if the members of $\{W_{\alpha_n}\}_{n=1}^{\infty}$ are different pairwise, then the limit of the Cauchy sequence $\{x_{\alpha_n}\}_{n=1}^{\infty}$ belongs to the compact K.*

Proof. Without loss of generality, we assume that $\{|\alpha_n|\}_{n=1}^{\infty}$ is strictly increasing. Let $\beta_n = (\alpha_n)_{|\alpha_n|-1}$ and take a point $y_{\beta_n} \in \overline{V}_{\beta_n} \bigcap K$. It is obvious that $|x_{\alpha_n} y_{\beta_n}| \leq |V_{\beta_n}|$. For any $\varepsilon > 0$, there exists N_1 such that $|x_{\alpha_i} x_{\alpha_j}| < \frac{\varepsilon}{3}$ for $i, j > N_1$. By Definition 2.1, there exists N_2 such that $|x_{\alpha_i} y_{\beta_i}| < \frac{\varepsilon}{3}$ for $i > N_2$. So for $i, j > \max\{N_1, N_2\}$, we have $|y_{\beta_i} y_{\beta_j}| < \varepsilon$, that is $\{y_{\beta_n}\}_{n=1}^{\infty}$ is a cauchy sequence. Finally, we obtain $\lim_{n\to\infty} x_{\alpha_n} = \lim_{n\to\infty} y_{\beta_n} \in K$ since K is compact. □

After all these preparations, we turn to our main theorem:

Theorem 4.3. *There is an isometry $\phi : \partial X \to K$.*

Proof. For any $a \in \partial X$, take $\{W_{\alpha_n}\}_{n=1}^{\infty} \in a$. $\{W_{\alpha_n}\}_{n=1}^{\infty}$ has infinite different members since $\lim_{n\to\infty} |W_{\alpha_n}| = 0$. Find a subsequence $\{W_{\alpha_{n_k}}\}_{k=1}^{\infty}$ whose members are different pairwise. By Lemma 4.2, $\lim_{n\to\infty} x_{\alpha_n} = \lim_{k\to\infty} x_{\alpha_{n_k}} \in K$ and we denote it by x_a. Take $\{W_{\beta_n}\}_{n=1}^{\infty} \in a$, since $\lim_{n\to\infty} |x_{\alpha_n} x_{\beta_n}| = 0$, we have $\lim_{n\to\infty} x_{\beta_n} = x_a$. Therefore, x_a is well defined, and we define $\phi : \partial X \to K$ by $\phi(a) = x_a$.

For $a, b \in \partial X$, if $x_a = x_b$, take any $\{W_{\alpha_n}\}_{n=1}^{\infty} \in a$ and any $\{W_{\beta_n}\}_{n=1}^{\infty} \in b$. It is obvious that $\lim_{n\to\infty} x_{\alpha_n} = x_a = x_b = \lim_{n\to\infty} x_{\beta_n}$, that is $\lim_{n\to\infty} |x_{\alpha_n} x_{\beta_n}| = 0$. Besides we have $\lim_{n\to\infty}(|W_{\alpha_n}| + |W_{\beta_n}|) = 0$. So we get $\lim_{n\to\infty} u(W_{\alpha_n}, W_{\beta_n}) = 0$, that is $a = b$. Hence ϕ is injective.

By Lemma 3.5, for any $x \in K$, we find $\alpha \in \Lambda_\infty$ such that $x \in \bigcap_{k=0}^{\infty} \overline{V}_{(\alpha)_k}$. $\{W_{(\alpha)_k 1}\}_{k=0}^{\infty}$ converges at infinity and $\lim_{k\to\infty} x_{(\alpha)_k 1} = x$, that is $\phi(\{\{W_{(\alpha)_k 1}\}_{k=0}^{\infty}\}) = x$ which shows ϕ is surjective.

Define $d : \partial X \times \partial X \to \mathbb{R}$ by $d(a, b) = |x_a x_b|$ for any $a, b \in \partial X$. It suffices to show d is a visual metric.

For $a, b \in \partial X$, $a \neq b$, take any $\{W_{\alpha_n}\}_{n=1}^{\infty} \in a$ and any $\{W_{\beta_n}\}_{n=1}^{\infty} \in b$, we have

$$(W_{\alpha_n}|W_{\beta_n})_o = \log \frac{u(W_{\alpha_n}, W_1)u(W_{\beta_n}, W_1)}{|W_1|u(W_{\alpha_n}, W_{\beta_n})}.$$

We can also obtain

$$\lim_{n\to\infty} u(W_{\alpha_n}, W_1) = \frac{|W_1|}{2} + 2M|x_1 x_a|, \tag{5}$$

$$\lim_{n\to\infty} u(W_{\beta_n}, W_1) = \frac{|W_1|}{2} + 2M|x_1 x_b|, \tag{6}$$

and

$$\lim_{n\to\infty} u(W_{\alpha_n}, W_{\beta_n}) = 2M|x_a x_b|. \tag{7}$$

By equations (5), (6) and (7), we have

$$e^{-(a|b)_o} = \frac{2M|W_1|}{\left(\frac{|W_1|}{2} + 2M|x_1 x_a|\right)\left(\frac{|W_1|}{2} + 2M|x_1 x_b|\right)}|x_a x_b|.$$

But

$$\frac{2M|W_1|}{(2M+\frac{1}{2})^2|V|^2} \le \frac{2M|W_1|}{\left(\frac{|W_1|}{2} + 2M|x_1 x_a|\right)\left(\frac{|W_1|}{2} + 2M|x_1 x_b|\right)} \le \frac{8M}{|W_1|}.$$

Take a constant C such that $C > \frac{8M}{|W_1|} \vee \frac{(2M+\frac{1}{2})^2|V|^2}{2M|W_1|}$, we get

$$\frac{1}{C}e^{-(a|b)_o} \le d(a,b) = |x_a x_b| \le Ce^{-(a|b)_o},$$

which shows that d is a visual metric and $\phi : \partial X \to K$ is an isometric map. □

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