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Unique Range Sets - A Further Study

Sanjay Mallick[a]

[a]*Department of Mathematics, Cooch Behar Panchanan Barma University, Cooch Behar - 736101, West Bengal, India.*

Abstract. The purpose of the paper is to investigate the problems of unique range sets in the most general setting. Accordingly, we have studied sufficient conditions for a general polynomial to generate a unique range set which put all the variants of unique range sets into one structure. Most importantly, as an application of the main result we have been able to accommodate not only examples of critically injective polynomials but also examples of non-critically injective polynomials generating unique range sets which are for the first time being exemplified in the literature. Furthermore, some of these examples show that characterization of unique range sets generated by non-critically injective polynomials does not always demand gap polynomials which also complements the recent results by An and Banerjee-Lahiri in this direction. Moreover, one of the lemmas proved in this paper improves and generalizes some results due to Frank-Reinders and Lahiri respectively.

1. Introduction, Definitions and Results

The problems on unique range sets and its allied notions like uniqueness polynomials, strong uniqueness polynomials have caused an increasing research interest during the last years in the literature of Value Distribution Theory. The journey was specifically started in 1977 after the introduction of Set Sharing by F. Gross [9] as the generalization of Nevanlinna's Value Sharing notion for the uniqueness of meromorphic functions.

For a non-constant meromorphic function f and a set $S \subset \mathbb{C} \cup \{\infty\}$, we define $E_f(S) = \bigcup_{a\in S}\{(z,p) \in \mathbb{C} \times \mathbb{N} : f(z) = a \textit{ with multiplicity } p\}$ and $\overline{E}_f(S) = \bigcup_{a\in S}\{(z,1) \in \mathbb{C}\times\mathbb{N} : f(z) = a\}$. If for any two non-constant meromorphic functions f and g, $E_f(S) = E_g(S)$ implies $f \equiv g$, then we say that S is a unique range set for meromorphic functions or URSM in short. If $\overline{E}_f(S) = \overline{E}_g(S)$ implies $f \equiv g$, then we say that S is a unique range set for meromorphic functions ignoring multiplicities or URSM-IM in short. Similar notions for entire functions are termed as URSE and URSE-IM respectively.

In 1982 Gross-Yang [10] considered the infinite set $S = \{z : e^z + z = 0\}$ and proved the existence of a URSE. Later many results concerning URSM or URSE have been obtained by many researchers [5, 6, 14, 16, ...] throughout the last few decades and the research mainly got attention towards finding finite unique range sets. In 1995, Li-Yang [14] unveiled that the base of these finite unique range sets are some suitable

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Email address: sanjay.mallick1986@gmail.com, smallick.ku@gmail.com, sanjay_jumath@yahoo.com (Sanjay Mallick)

polynomials. That is, finite unique range sets are nothing but the set of zeros of some suitable polynomials. These suitable polynomials are called as the polynomials of unique range sets or generating polynomials. So, natural queries arose "Which are these generating polynomials? In which class do they belong ? or How do we recognize them?" etc. These questions made the notion of unique range sets more interesting as well as challenging and hence the notion of unique range sets headed towards its characterization in terms of its generating polynomials. In 2000, H. Fujimoto [7] made the first attempt in this direction which led the ideas of uniqueness polynomials, strong uniqueness polynomials and on the basis of these fundamental notions, he opted a special class of polynomials whose zero sets are unique range sets under certain conditions. This special class of polynomials are termed as critically injective polynomials. On this occasion, let us shortly recall these definitions.

Definition 1.1. *[2, 7] A polynomial $P(z)$ in $\mathbb{C}$ is a uniqueness polynomial for meromorphic (entire) functions if for any two non-constant meromorphic (entire) functions f and g, $P(f) \equiv P(g)$ implies $f \equiv g$. In short, we call it UPM(UPE).*

Definition 1.2. *[2, 7] A polynomial $P(z)$ in $\mathbb{C}$ is a strong uniqueness polynomial for meromorphic (entire) functions if for any two non-constant meromorphic (entire) functions f, g and an arbitrary nonzero constant c, $P(f) \equiv cP(g)$ implies $c = 1$ and $f \equiv g$. In this case, we say $P(z)$ is an SUPM(SUPE) in brief.*

Since we are talking about finite unique range sets, so the definitions of URSM, SUPM and UPM clearly imply that every polynomial generating URSM is a SUPM and every SUPM is a UPM. Natural curiosity to know about the converse is of course justified but it is not always true. For example, $P(z) = z$; is a UPM but not a SUPM because for non-constant meromorphic functions g and f, where $f = 2g$; we have $(2g) = 2.(g)$; i.e., $P(2g) = 2P(g)$, which clearly implies $P(f) \equiv cP(g)$ but $c \neq 1$ and $f \not\equiv g$. Furthermore, $P(z) = z^5 + az^2 + b$, where $a, b \in \mathbb{C}$ be such that $P(z)$ has only simple zeros; is a SUPM [7, see Example 4.10, pp. 1192] but its zero set is not a URSM as the lowest cardinality of a URSM is at least 6 {see [14]}. So, under which condition a UPM(UPE) or a SUPM(SUPE) generates a URSM(URSE) has become the prime concern in the study of unique range sets.

Now we invoke the definition of critically injective polynomial.

Definition 1.3. *[7] Let $P(z)$ be a polynomial such that $P'(z)$ has mutually k distinct zeros given by $d_1, d_2, \ldots, d_k$ with multiplicities $q_1, q_2, \ldots, q_k$ respectively. Then $P(z)$ is called a critically injective polynomial or CIP in short, if $P(d_i) \neq P(d_j)$ for $i \neq j$, where $i, j \in \{1, 2, \cdots, k\}$.*

Any polynomial which is not CIP is called the non-critically injective polynomial or NCIP in short. Hence any polynomial is either CIP or NCIP. Now we state the result of Fujimoto [7].

Theorem 1.4. *[7] Let $P(z) = (z - \alpha_1)(z - \alpha_2) \ldots (z - \alpha_n)$ be a CIP of degree n in $\mathbb{C}$ having only simple zeros. Let $P'(z)$ has k distinct zeros with $k \geq 3$, or $k = 2$ and $P'(z)$ has no simple zero. Further suppose that $P(z)$ is an* SUPM (SUPE). *If S is the set of zeros of $P(z)$, then S is a* URSM (URSE) *whenever $n > 2k + 6$ ($n > 2k + 2$) while a* URSM-IM (URSE-IM) *whenever $n > 2k + 12$ ($n > 2k + 5$).*

After *Theorem 1.4*, all the existing results of unique range sets were found to be the special cases of this result. But note that, Fujimoto's characterization of unique range sets was completely based on CIP's. Since any polynomial is either a CIP or a NCIP, so we can say that Fujimoto's work [7] is no doubt a great achievement in characterizing the unique range sets but it is only a half characterization achieved in terms of its generating polynomials. Hence, natural question arises "What about the other half?" i.e., NCIP's. Under this circumstance, researchers are left with two options for the characterization of unique range sets in terms of its generating polynomials. That is, they need an answer to anyone of the following questions.

Question 1.5. *Do NCIP's generate unique range sets? If yes, then what are the criterions to be satisfied by those polynomials to generate unique range sets ?*

Question 1.6. *Can we have sufficient conditions for a general polynomial to generate a unique range set ?*

If the answer of *Question 1.5* is obtained, then we would have an answer to the counterpart of *Theorem 1.4* and that will significantly contribute to the characterization of unique range sets.

If the answer of *Question 1.6* is achieved, then that will automatically accommodate *Theorem 1.4* and the answer of *Question 1.5* in it.

Pertinent to this, we mention that recently An [1] and Banerjee-Lahiri [4] have separately made some efforts for the characterization of unique range sets towards *Question 1.6*. Following is the result of An [1].

Theorem 1.7. *[1] Let $P(z) = a_nz^n + a_mz^m + a_{m-1}z^{m-1} + \ldots + a_0$, $1 \le m < n$, $a_m \neq 0$ be a polynomial in $\mathbb{C}$ of degree n with only simple zeros and S be its zero set. Further suppose that $P'(z)$ has k distinct zeros and $I = \{i : a_i \neq 0\}$, $\lambda = \min\{i : i \in I\}$, $J = \{i - \lambda : i \in I\}$. If $n \ge \max\{m + 4, 2k + 7\}$, then the following statements are equivalent:*

(i) S is a URSM;

(ii) $P(z)$ is a SUPM;

(iii) S is not preserved by any non-trivial affine transformation of $\mathbb{C}$.

(iv) The greatest common divisors of the indices respectively in I and J are both 1.

Now we recall the result of Banerjee-Lahiri [4]. For that, we need to recall the notion of weighted sharing [12], [13] which has led the notion of unique range sets to a further refinement.

Definition 1.8. *[12, 13] Let r be a nonnegative integer or infinity. For $a \in \overline{\mathbb{C}}$ we denote by $E_r(a; f)$ the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m \le r$ and $r + 1$ times if $m > r$. If $E_r(a; f) = E_r(a; g)$, we say that f, g share the value a with weight r.*

We write f, g share (a, r) to mean that f, g share the value a with weight r. Clearly if f, g share (a, r), then f, g share (a, p) for any integer p, $0 \le p < r$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) respectively.

Definition 1.9. *[12] For $S \subset \overline{\mathbb{C}}$ we define $E_f(S, r) = \cup_{a \in S} E_r(a; f)$, where r is a non-negative integer or infinity. Clearly $E_f(S) = E_f(S, \infty)$ and $\overline{E}_f(S) = E_f(S, 0)$.*

Definition 1.10. *[4] A set $S \subset \overline{\mathbb{C}}$ is called a unique range set for meromorphic (entire) functions with weight r if for any two non-constant meromorphic (entire) functions f and g, $E_f(S, r) = E_g(S, r)$ implies $f \equiv g$. We write S is URSMr(URSEr) in short.*

From the *Definition 1.10*, it is clear that every URSMr(URSEr) is a URSM(URSE) but the converse may not be true. Following is the result of Banerjee-Lahiri [4].

Theorem 1.11. *[4] Let $P(z) = a_nz^n + \sum_{j=2}^{m} a_jz^j + a_0$ be a polynomial of degree n, where $n - m \ge 3$ and $a_pa_m \neq 0$ for some positive integer p with $2 \le p \le m$ and $\gcd(p, 3) = 1$. Suppose further that $S = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be the set of all distinct zeros of $P(z)$. Let k be the number of distinct zeros of the derivative $P'(z)$. If $n \ge 2k + 7$ ($n \ge 2k + 3$), then the following are equivalent:*

(i) $P(z)$ is a SUPM (SUPE);

(ii) S is a URSM2 (URSE2);

(iii) S is a URSM (URSE);

(iv) $P(z)$ is a UPM (UPE).

The position of UPM and SUPM should have been reversed in *Theorem 1.11* as it can be easily verified from the explanation made between *Definition 1.2* and *Definition 1.3*. However, the main observation comes out of *Theorem 1.7* and *Theorem 1.11* is that the authors always used some specific polynomials which have a gap after the first term (n-th degree term) i.e., gap polynomials, for the characterization of unique range sets and they tried to bring all the variants of unique range sets under a single umbrella through their characterization. It was really a novel attempt but we have instances of non-gap polynomials generating unique range sets [5, 6]. Also, the gap between the first two terms of the polynomials used in these theorems is at least 2 but we know that Yi's polynomial [16] generates URSM where the gap may be 1. Furthermore, there may be some other CIP's and most surprisingly some NCIP's having no gap or 1 gap between the first two terms and if those polynomials generate unique range sets, then obviously that sets can not be characterized by any of these theorems. So, the characterizations given by *Theorem 1.7* and *Theorem 1.11* provides partial answer to *Question 1.6*.

Another important observation from An [1] and Banerjee-Lahiri's [4] papers is that none of the authors provided any example of NCIP generating unique range set in their papers. As a result, it still remains uncertain whether the answer of *Question 1.5* is affirmative or negative, even after two decades of the publication of *Theorem 1.4*. So, again we find that the characterizations provided by An [1] and Banerjee-Lahiri [4] are inadequate with respect to *Question 1.6*.

In view of all the above discussions, a characterization of unique range sets in the most general setting with at least one example of NCIP generating unique range set have become indispensable. That is, we still need a proper answer of *Question 1.6*. In this paper, we try to provide the best possible answer of *Question 1.6* which is the prime objective of the paper. In fact, in our main result we consider a general polynomial irrespective of CIP or NCIP, gap or non-gap polynomial and provide some sufficient conditions for the same to generate a unique range set. As an application of the main result, besides providing examples of unique range sets generated by CIP's, we also provide examples of unique range sets generated by NCIP's which are for the first time being exemplified in the literature. Some of these examples further show that there exists NCIP generating unique range set having no gap or 1 gap between the first two terms of the polynomial. That is, our result also provides an answer to the counterpart of *Theorem 1.7* and *Theorem 1.11*. In a word, we obtain a general characterization of unique range sets unifying all its variants into our result.

Before going to our main result, we make a short discussion on the structure of a general polynomial as this will play an important role throughout the rest of this paper.

Let $P_1(z) = \sum_{i=0}^{n} a_i z^i$, where $a_n \neq 0$ and $a_i \in \mathbb{C}$ for $i \in \{0, 1, \ldots, n\}$ be such that $P_1(z)$ has only simple zeros. Now consider $P(z) = \sum_{i=0}^{n} \frac{a_i}{a_n} z^i$, where a_i's are same as in $P_1(z)$. Clearly $P_1(z)$ and $P(z)$ have exactly same zeros. Hence, if the set of zeros of $P(z)$ forms unique range set, so is the set of zeros of $P_1(z)$ and vice versa. So without loss of generality for a general polynomial of degree n having only simple zeros we may consider $P(z)$ instead of $P_1(z)$. Now we claim that $P(z)$ can be written in the form

$$P(z) = \prod_{i=1}^{k} (z - \alpha_i)^{m_i} + c;$$

with $k = s + t$, where $s(\geq 1)$ denotes the number of m_i's such that $m_i \geq 2$ and $t(\geq 0)$ denotes the number of m_i's such that $m_i = 1$ and $\sum\limits_{i=1}^{k} m_i = n$. The constants $c, \alpha_i \in \mathbb{C}$.

Since $P(z)$ has only simple zeros, so let us write

$$P(z) = \prod_{i=1}^{n} (z - \gamma_i)$$

and

$$P'(z) = n \prod_{i=1}^{q} (z - \sigma_i)^{r_i},$$

where γ_i for $i \in \{1, \ldots, n\}$ and σ_i for $i \in \{1, \ldots, q\}$ are the distinct zeros of $P(z)$ and $P'(z)$ respectively. Suppose that $P(z)$ is a NCIP. Then there exist at least two zeros of $P'(z)$ say σ_u, σ_v for $u, v \in \{1, 2, \ldots, q\}$ such that $P(\sigma_u) = P(\sigma_v)$. For the sake of convenience, we assume that σ_u, σ_v are the only zeros of $P'(z)$ to satisfy $P(\sigma_u) = P(\sigma_v)$. Later we shall show that considering more zeros of $P'(z)$, we shall have the same result as claimed. Suppose $P(\sigma_u) = P(\sigma_v) = c$. Then

$$P(z) = \prod_{i=1}^{n}(z-\gamma_i) - P(\sigma_u) + c$$
$$\implies P(z) = Q(z) + c,$$

where $Q(z) = \prod_{i=1}^{n}(z-\gamma_i) - P(\sigma_u)$. Now clearly $P^{'}(z) = Q^{'}(z)$. Observe that $Q(\sigma_u) = 0 = Q(\sigma_v)$ and at the same time $P^{'}(\sigma_u) = Q^{'}(\sigma_u) = P^{'}(\sigma_v) = Q^{'}(\sigma_v) = 0$. Therefore

$$Q(z) = (z-\sigma_u)^{r_u+1}(z-\sigma_v)^{r_v+1}R_{n-r_u-r_v-2}(z),$$

where $R_{n-r_u-r_v-2}(z)$ is a polynomial of degree $(n - r_u - r_v - 2)$ having only simple zeros. If in this case, we would have more than two zeros of $P^{'}(z)$ satisfying $P(\sigma_u) = P(\sigma_v)$, then also the above procedure would lead us to our claim. Hence it is clear that for every NCIP our claim is correct and $s \geq 2$.

Let $P(z)$ be a CIP. Since in this case $P(\sigma_u) \neq P(\sigma_v)$ for all $u, v \in \{1, 2, \ldots, q\}$, so only one σ_u can repeat at a time in the formation of $Q(z)$. Then by the same arguments as above we can get

$$P(z) = (z-\sigma_u)^{r_u+1}R_{n-r_u-1}(z) + c,$$

where $R_{n-r_u-1}(z)$ is a polynomial of degree $(n - r_u - 1)$ having only simple zeros. It is clear that in this case $s = 1$.
Since any polynomial is either CIP or NCIP, so our claim is established.

Henceforth, for any polynomial of degree n having only simple zeros we proceed with the following polynomial:

$$P(z) = \prod_{i=1}^{k}(z-\alpha_i)^{m_i} + c; \tag{1.1}$$

with $k = s + t$, where $s(\geq 1)$ denotes the number of m_i's such that $m_i \geq 2$ and $t(\geq 0)$ denotes the number of m_i's such that $m_i = 1$ and $\sum_{i=1}^{k} m_i = n$. The constants $c, \alpha_i \in \mathbb{C}$. Set

$$\lambda_j = -\prod_{i=1}^{k}(\beta_{j_m} - \alpha_i)^{m_i} \tag{1.2}$$

for β_{j_m} being the distinct zeros of $P^{'}(z)$ with $m = 1, 2, \ldots, d_j$, where $d_j \in \mathbb{N}$. The reason behind denoting different zeros of $P^{'}(z)$ by β_{j_m} is obvious because if $P(z)$ is a NCIP, then it must have at least one λ_j such that there exist 'd_j' number of different β_{j_m}'s producing it, where $d_j \geq 2$ and if $P(z)$ is a CIP, then of course $d_j = 1$ for each λ_j. Clearly the value of d_j depends upon the value of λ_j; i.e., for different λ_j's we would have different d_j's. Note that in a NCIP there may be some β_{j_m} with $d_j = 1$. By $p_{j_m}(\geq 1)$, we denote the multiplicity of $(z - \beta_{j_m})$ in $P^{'}(z)$ such that λ_j is non-zero and l being the number of those distinct non-zero λ_j's. Observe that $\lambda_j = c - P(\beta_{j_m}) \neq c$, otherwise $P(z)$ would have multiple zeros. Let us now place an example to clarify the above discussion.

Example 1.12. *Let $P_1(z) = z^2(z^2-2) + c$, where $c \neq 0, 1$. Now $P_1^{'}(z) = 4z(z+1)(z-1)$. Clearly only $z = \pm 1$ gives non-zero λ_j and each of these zeros of $P_1^{'}(z)$ produces same $\lambda_j(= -1)$; i.e., λ_1. So, here $l = 1$ and hence we have only one d_j which is d_1. The value of d_1 is 2. Corresponding p_{j_m}'s are P_{1_1}, P_{1_2} with $P_{1_1} = 1$, $P_{1_2} = 1$.*

Remark 1.13. *Note that in* Example 1.12, *$s = 1$ but $P_1(z)$ is a NCIP. So this example shows that $s = 1$ does not always imply the polynomial to be a CIP rather the converse is true; i.e., we can always write a CIP in such a format where $s = 1$. Similarly for any NCIP we can have a structure of the polynomial such that $s \geq 2$. As for example, we can write $P_1(z) = (z+1)^2(z-1)^2 + c - 1 = (z+1)^2(z-1)^2 + c_1$, where $c_1 \neq 0, -1$; i.e., $s \geq 2$. Another point we should keep in mind that in this new structure the value of non-zero λ_j, d_j, P_{j_m} etc would be changed.*

Remark 1.14. *Further note that we would always get at least one $\lambda_j \neq 0$ except when there exists one combination of s and t in $P(z)$ such that $s = 1$ and $t = 0$. It is clarified in [7, pp. 1183, l. 4-8] that $P(z)$ with $s = 1$ and $t = 0$ can never generate a unique range set.*

Now we state the main result of the paper.

Theorem 1.15. *Let $P(z)$ be defined by (1.1) satisfying the following conditions:*
(i) $k \geq 4$ or
(ii) $k = 3$ and $\gcd(m_i, n) = 1$ for at least one of the m_i's such that $m_i \geq 2$ or
(iii) $k = 3$ and $\gcd(m_1, n) \neq 1$, where $m_1 \geq 2$, $m_2 = m_3 = 1$ and $n = \sum_{i=1}^{3} m_i \geq 5$ or
(iv) $k = 2$ and $\gcd(m_i, n) = 1$ for at least one of the m_i's such that $n = \sum_{i=1}^{2} m_i \geq 5$ or
(v) $k = 2$ and $\gcd(m_i, n) \neq 1$ for each m_i such that $n \geq 2(b_1 + b_2) + 1$, where $b_1 = \gcd(m_1, n)$ and $b_2 = \gcd(m_2, n)$.
Let $S = \{z : P(z) = 0\}$. Then for $n \geq \max\{2(b_1 + b_2) + 1, 4s + 2t + 2\sum_{j=1}^{l} d_j - 2\sum_{j=1}^{l}\sum_{m=1}^{d_j} p_{j_m} + 5, 2s + 2t + 3\}$, the following are equivalent:
(i) $P(z)$ is a UPM;
(ii) S is a URSM2;
(iii) $P(z)$ is a SUPM.

Corollary 1.16. *Let $P(z)$ be defined by (1.1) with $k \geq 2$ and $S = \{z : P(z) = 0\}$. Then for $n \geq \max\{4s + 2t + 2\sum_{j=1}^{l} d_j - 2\sum_{j=1}^{l}\sum_{m=1}^{d_j} p_{j_m} + 1, 2s + 2t + 1\}$, the following are equivalent:*
(i) $P(z)$ is a UPE;
(ii) S is a URSE2;
(iii) $P(z)$ is a SUPE.

For standard notations of Nevanlinna Theory used in the paper we refer our readers to follow [3, 11]. We now prove some lemmas which will be needed on the way of proving the main theorem.

2. Lemma

For the sake of convenience, from now on, by f and g we shall mean two non-constant meromorphic functions defined on $\mathbb{C}$ unless otherwise stated. Let us denote by H the following function.

$$H = \left(\frac{f''}{f'} - \frac{2f'}{f-1}\right) - \left(\frac{g''}{g'} - \frac{2g'}{g-1}\right). \tag{2.1}$$

Lemma 2.1. *[17] Let f, g share $(1, 0)$ and $H \not\equiv 0$. Then*

$$N_E^{1)}(r, 1; f) = N_E^{1)}(r, 1; g) \leq N(r, H) + S(r, f) + S(r, g).$$

Lemma 2.2. *Let f, g share $(1, 0)$ and $H \not\equiv 0$. Further suppose that $a_j \in \mathbb{C} - \{1\}$ for $j = \{1, 2 \ldots, q\}$. Then*

$$\begin{aligned} N(r,H) \quad \leq \quad & \overline{N}(r,\infty; f \mid\geq 2) + \sum_{j=1}^{q} \overline{N}(r, a_j; f \mid\geq 2) \\ & + \overline{N}(r,\infty; g \mid\geq 2) + \sum_{j=1}^{q} \overline{N}(r, a_j; g \mid\geq 2) \\ & + \overline{N}_*(r, 1; f, g) + \overline{N}_0(r, 0; f') + \overline{N}_0(r, 0; g'), \end{aligned}$$

where $\overline{N}_0(r,0;f^{'})$ *denotes the reduced counting function of those zeros of* $f^{'}$ *which are not zeros of* $(f-1)\prod_{j=1}^{q}(f-a_j)$ *and* $\overline{N}_0(r,0;g^{'})$ *is defined similarly.*

Proof. From the construction of H, it is obvious that possible poles of H occur at (i) multiple poles of f and g; (ii) multiple a_j-points of f and g; (iii) 1-points of f and g having different multiplicities; (iv) zeros of $f^{'}$ which are not zeros of $(f-1)\prod_{j=1}^{q}(f-a_j)$; (v) zeros of $g^{'}$ which are not zeros of $(g-1)\prod_{j=1}^{q}(g-a_j)$. Since all poles of H are simple, so the lemma is obvious. □

Lemma 2.3. *[3] Let f, g share (1,m), where $0 \le m < \infty$. Then*

$$\overline{N}(r,1;f)+\overline{N}(r,1;g)-N_E^{1)}(r,1;f)+\left(m-\frac{1}{2}\right)\overline{N}_*(r,1;f,g)\le\frac{1}{2}[N(r,1;f)+N(r,1;g)].$$

Lemma 2.4. *Let f, g share $(1,2)$. Suppose that $a_j \in \mathbb{C}-\{1\}$ for $j=\{1,2,\ldots,q\}$. Then one of the following cases holds:*

(i)

$$\left(q-\frac{1}{2}\right)[T(r,f)+T(r,g)] \le N_2(r,\infty;f)+N_2(r,\infty;g)+\sum_{j=1}^{q}N_2(r,a_j;f)+\sum_{j=1}^{q}N_2(r,a_j;g)+S(r,f)+S(r,g);$$

(ii) $\dfrac{1}{f-1}=\dfrac{A}{g-1}+B$, *where* $A(\neq 0)$, $B\in\mathbb{C}$.

Proof. Suppose $H\not\equiv 0$. By the second fundamental theorem we get

$$qT(r,f)\le\overline{N}(r,\infty;f)+\sum_{j=1}^{q}\overline{N}(r,a_j;f)+\overline{N}(r,1;f)-N_0(r,0;f^{'})+S(r,f) \tag{2.2}$$

and

$$qT(r,g)\le\overline{N}(r,\infty;g)+\sum_{j=1}^{q}\overline{N}(r,a_j;g)+\overline{N}(r,1;g)-N_0(r,0;g^{'})+S(r,g). \tag{2.3}$$

From (2.2) and (2.3), we get

$$\begin{aligned} q[T(r,f)+T(r,g)] \le\ & \overline{N}(r,\infty;f)+\sum_{j=1}^{q}\overline{N}(r,a_j;f)-N_0(r,0;f^{'}) \\ & +\overline{N}(r,\infty;g)+\sum_{j=1}^{q}\overline{N}(r,a_j;g)-N_0(r,0;g^{'}) \\ & +\overline{N}(r,1;f)+\overline{N}(r,1;g)+S(r,f)+S(r,g). \end{aligned} \tag{2.4}$$

Now using *Lemma 2.3*, *Lemma 2.1* and *Lemma 2.2* for $m=2$, we get

$$\begin{aligned} q[T(r,f)+T(r,g)] \le\ & N_2(r,\infty;f)+\sum_{j=1}^{q}N_2(r,a_j;f) \\ & +N_2(r,\infty;g)+\sum_{j=1}^{q}N_2(r,a_j;g) \\ & +\frac{1}{2}[N(r,1;f)+N(r,1;g)]+S(r,f)+S(r,g), \end{aligned} \tag{2.5}$$

which implies (i).

Now suppose $H \equiv 0$. Then on integration we get $\dfrac{1}{f-1} = \dfrac{A}{g-1} + B$, where $A(\neq 0), B \in \mathbb{C}$. □

Remark 2.5. *Taking $q = 1$ and $a_1 = 0$ in this* lemma, *we would directly get* Theorem 1 *of [13]. So this* lemma *is clearly a generalization of* Theorem 1 *in [13].*

Remark 2.6. *Since $N_2(r,\infty; f) \leq T(r, f)$ and $N_2(r,\infty; g) \leq T(r, g)$, so part-i of this* lemma *can be written as*

$$\left(q - \frac{3}{2}\right)[T(r, f) + T(r, g)] \quad \leq \quad \sum_{j=1}^{q} N_2(r, a_j; f) + \sum_{j=1}^{q} N_2(r, a_j; g) + S(r, f) + S(r, g), \tag{2.6}$$

for $q \geq 2$. Now if we assume $f_1 = \dfrac{1}{f-1}$ and $g_1 = \dfrac{1}{g-1}$, then clearly f_1 and g_1 share $(\infty, 2)$. Also $f - a_j = \dfrac{f_1+1}{f_1} - a_j = (1-a_j)\left(\dfrac{f_1 - \frac{1}{a_j-1}}{f_1}\right)$; i.e., a_j-points of f are $\frac{1}{a_j-1}$-points of f_1. Since $a_j \neq 1$, so $\frac{1}{a_j-1} = c_j(say) \in \mathbb{C}$. Therefore (2.6) implies

$$\left(q - \frac{3}{2}\right)[T(r, f_1) + T(r, g_1)] \quad \leq \quad \sum_{j=1}^{q} N_2(r, c_j; f_1) + \sum_{j=1}^{q} N_2(r, c_j; g_1) + S(r, f_1) + S(r, g_1),$$

where $q \geq 2$ and f_1, g_1 share $(\infty, 2)$. Further in part-ii it remains nothing to show that $\dfrac{1}{f-1} = \dfrac{A}{g-1} + B$, where $A(\neq 0), B \in \mathbb{C}$, implies $f_1 = Ag_1 + B$ with $A(\neq 0), B \in \mathbb{C}$. Hence this lemma *is also a direct improvement of the* Lemma *used in [6].*

Lemma 2.7. *[15] Let $\dfrac{1}{f-1} = \dfrac{A}{g-1} + B$, where $A(\neq 0), B \in \mathbb{C}$. If*

$$\overline{N}(r, 0; f) + \overline{N}(r,\infty; f) + \overline{N}(r, 0; g) + \overline{N}(r,\infty; g) < T(r),$$

where $T(r) = \max\{T(r, f), T(r, g)\}$. Then either $fg = 1$ or $f \equiv g$.

Lemma 2.8. *Let $F = \prod_{i=1}^{k}(f - \alpha_i)^{m_i}$ and $G = \prod_{i=1}^{k}(g - \alpha_i)^{m_i}$, where $k \geq 2$, $\sum_{i=1}^{k} m_i = n$ and $\alpha_i \in \mathbb{C}$ for $i = 1, 2, \ldots, k$. If*

(i) $k \geq 4$ or

(ii) $k = 3$ and $\gcd(m_i, n) = 1$ for at least one of the m_i's such that $m_i \geq 2$ or

(iii) $k = 3$ and $\gcd(m_1, n) \neq 1$, where $m_1 \geq 2$, $m_2 = m_3 = 1$ and $n = \sum_{i=1}^{3} m_i \geq 5$ or

(iv) $k = 2$ and $\gcd(m_i, n) = 1$ for at least one of the m_i's such that $n = \sum_{i=1}^{2} m_i \geq 5$ or

(v) $k = 2$ and $\gcd(m_i, n) \neq 1$ for each m_i such that $n \geq 2(b_1 + b_2) + 1$, where $b_1 = \gcd(m_1, n)$ and $b_2 = \gcd(m_2, n)$, then $FG \neq a$, where a is any non-zero complex number.

Proof. If possible suppose that $FG = a$. Then

$$\prod_{i=1}^{k}(f - \alpha_i)^{m_i} \prod_{i=1}^{k}(g - \alpha_i)^{m_i} = a. \tag{2.7}$$

It is clear from (2.7) that each α_i-point of f is a pole of g.

(i) Suppose that z_0 be a α_i-point of f of multiplicity p and a pole of g of multiplicity q. Then $m_ip = nq$; i.e., $m_ip \geq n$; i.e., $\frac{1}{p} \leq \frac{m_i}{n}$. Since α_i-point of f is chosen arbitrarily, so we would get similar inequality for every α_i-point of f. Now using $\sum_{i=1}^{k} m_i = n$ and the second fundamental theorem for these α_i-points of f, we get

$$\begin{aligned}(k-2)T(r,f) &\leq \sum_{i=1}^{k} \overline{N}(r,\alpha_i;f) + S(r,f)\\ &\leq \sum_{i=1}^{k} \frac{m_i}{n} T(r,f) + S(r,f)\\ &\leq T(r,f) + S(r,f),\end{aligned}$$

which is a contradiction for $k \geq 4$.

(ii) In this part we have $k = 3$. First of all let us suppose that there exist only one m_i say $m_2 \geq 2$ such that $\gcd(m_2, n) = 1$. Suppose z_0 be any α_2-point of f of multiplicity r and a pole of g of multiplicity u. Then $m_2r = nu$. Since $\gcd(m_2, n) = 1$, so n divides r; i.e., $r = nv$ for some $v \in \mathbb{N}$. Hence $r \geq n$ i.e., $\frac{1}{r} \leq \frac{1}{n}$. Now proceeding as the above part we shall get

$$\begin{aligned}T(r,f) &\leq \sum_{i=1}^{3} \overline{N}(r,\alpha_i;f) + S(r,f)\\ &\leq \frac{m_1}{n}T(r,f) + \frac{1}{n}T(r,f) + \frac{m_3}{n}T(r,f) + S(r,f)\\ &\leq \frac{n-m_2+1}{n}T(r,f) + S(r,f)\\ &\leq (1 - \frac{m_2-1}{n})T(r,f) + S(r,f),\end{aligned}$$

which is a contradiction as $m_2 \geq 2$. Now if we have more than one m_i's such that $m_i \geq 2$ and $\gcd(m_i, n) = 1$, then proceeding in a similar fashion we arrive at a contradiction.

(iii) Let $k = 3$ and $\gcd(m_1, n) \neq 1$, where $m_1 \geq 2$, $m_2 = m_3 = 1$ and $n \geq 5$. Clearly $m_1 = n-2$. Since $\gcd(m_1, n) \neq 1$; i.e., $\gcd(n-2, n) \neq 1$, so $\gcd(n-2, n)$ would be precisely 2. Hence we write $n = 2n_1$ and $n - 2 = 2p_1$, where $\gcd(n_1, p_1) = 1$. Now if z_0 be any α_1-point of f of multiplicity r and a pole of g of multiplicity u, then $(n-2)r = nu$. This implies $2p_1r = 2n_1u$; i.e., $p_1r = n_1u$. Since $\gcd(n_1, p_1) = 1$, so n_1 divides r; i.e., $r \geq n_1$ or $\frac{1}{r} \leq \frac{1}{n_1} = \frac{2}{n}$. Now proceeding similarly like the above part we would get

$$\begin{aligned}T(r,f) &\leq \sum_{i=1}^{3} \overline{N}(r,\alpha_i;f) + S(r,f)\\ &\leq \frac{2}{n}T(r,f) + \frac{1}{n}T(r,f) + \frac{1}{n}T(r,f) + S(r,f)\\ &\leq \frac{4}{n}T(r,f) + S(r,f),\end{aligned}$$

which is a contradiction for $n \geq 5$.

(iv) Let $k = 2$ and $\gcd(m_1, n) = 1$. Since $m_1 + m_2 = n$, so $\gcd(m_2, n) = 1$. Now proceeding similarly as above, we would get $\overline{N}(r,\alpha_i;f) \leq \frac{1}{n}T(r,f)$ and $\overline{N}(r,\alpha_i;g) \leq \frac{1}{n}T(r,g)$ for $i = 1, 2$. Note that (2.7) clearly implies $\overline{N}(r,\infty;f) = \sum_{i=1}^{2} \overline{N}(r,\alpha_i;g)$. Also using the first fundamental theorem, we get

$$T(r,f) = T(r,g) + O(1) \tag{2.8}$$

from (2.7). So using the second fundamental theorem and (2.8), we get

$$\begin{aligned} T(r,f) &\leq \sum_{i=1}^{2}\overline{N}(r,\alpha_i;f)+\overline{N}(r,\infty;f)+S(r,f)\\ &\leq \frac{2}{n}T(r,f)+\frac{2}{n}T(r,g)+S(r,f)\\ &\leq \frac{4}{n}T(r,f)+S(r,f), \end{aligned}$$

which is a contradiction for $n \geq 5$.

(v) Let $k = 2$ and $\gcd(m_i, n) = b_i \neq 1$ for $i = 1, 2$. Now let us work with b_1. Here we shall again apply similar arguments as part (i) and get $m_1 p = nq$, where p is the multiplicity of an α_1-point of f and q is the multiplicity of the corresponding pole of g. Since $\gcd(m_1, n) = b_1$, so $m_1 = b_1 p_1$ and $n = b_1 n_1$ for some $p_1, n_1 \in \mathbb{N}$, where $\gcd(p_1, n_1) = 1$. Now $m_1 p = nq$ implies $b_1 p_1 p = b_1 n_1 q$; i.e., $p_1 p = n_1 q$. Since $\gcd(p_1, n_1) = 1$, so n_1 divides p; i.e., $n_1 r = p$ for some $r \in \mathbb{N}$. Therefore $\frac{1}{p} \leq \frac{1}{n_1} = \frac{b_1}{n}$. Proceeding in the same way we can show that for any α_2-point of f of multiplicity v, we have $\frac{1}{v} \leq \frac{b_2}{n}$. Similar inequalities hold for α_i-points of g also. Now in view of (2.8), using the second fundamental theorem, we get

$$\begin{aligned} T(r,f) &\leq \sum_{i=1}^{2}\overline{N}(r,\alpha_i;f)+\overline{N}(r,\infty;f)+S(r,f)\\ &\leq \frac{b_1+b_2}{n}T(r,f)+\frac{b_1+b_2}{n}T(r,g)+S(r,f)\\ &\leq \frac{2(b_1+b_2)}{n}T(r,f)+S(r,f), \end{aligned}$$

which is a contradiction for $n \geq 2(b_1 + b_2) + 1$. □

Remark 2.9. *Note that if f and g are non-constant entire functions, then conclusion of the above lemma is always true for $k \geq 2$. In that case, we do not need any restriction over m_i's and n.*

3. Proof of the Theorem

Proof. Since every URSMr is a URSM, every polynomial generating URSM is a SUPM and every SUPM is a UPM, so automatically (*ii*) $\Longrightarrow$ (*iii*) and (*iii*) $\Longrightarrow$ (*i*). Therefore to prove the theorem it is enough to show that (*i*) $\Longrightarrow$ (*ii*).

Let $P(z)$ be a UPM and $E_f(S, 2) = E_g(S, 2)$. We need to show that $E_f(S, 2) = E_g(S, 2)$ implies $f \equiv g$. Now suppose that

$$F=\frac{P(f)-c}{-c}=\frac{\prod_{i=1}^{k}(f-\alpha_i)^{m_i}}{-c}, \tag{3.1}$$

$$G=\frac{P(g)-c}{-c}=\frac{\prod_{i=1}^{k}(g-\alpha_i)^{m_i}}{-c}. \tag{3.2}$$

Then $T(r, F) = nT(r, f)$ and $T(r, G) = nT(r, g)$ and hence $S(r, F) + S(r, G) = S(r, f) + S(r, g)$. Clearly F and G share $(1, 2)$. Therefore in view of *Lemma 2.4*, one of the following cases holds.

Case-1. Let

$$\left(q-\frac{1}{2}\right)[T(r,F)+T(r,G)] \leq N_2(r,\infty;F)+N_2(r,\infty;G) + \sum_{j=1}^{q} N_2(r,a_j;F) + \sum_{j=1}^{q} N_2(r,a_j;G) + S(r,F)+S(r,G). \quad (3.3)$$

Now consider λ_j as defined by (1.2). The existence of non-zero λ_j for $P(z)$ is ensured from *Remark 1.14*. Now put $a_q = 0$ and $a_j = \frac{\lambda_j}{c}$ for those λ_j which are distinct and non-zero, where $j = 1, 2, \ldots, q-1$. Let us suppose $q = l+1$, where $l \geq 1$. So (3.3) reduces to

$$\left(l+\frac{1}{2}\right)[T(r,F)+T(r,G)] \leq N_2(r,\infty;F)+N_2(r,\infty;G) + N_2(r,0;F)+N_2(r,0;G) + \sum_{j=1}^{l} N_2(r,\frac{\lambda_j}{c};F) + \sum_{j=1}^{l} N_2(r,\frac{\lambda_j}{c};G) + S(r,F)+S(r,G). \quad (3.4)$$

Now for $p_{j_m}(\geq 1)$ being the multiplicity of $(z-\beta_{j_m})$ in $P'(z)$, we get

$$F - a_j = F + \frac{\lambda_j}{-c} = \frac{1}{-c}\prod_{m=1}^{d_j}(f-\beta_{j_m})^{(p_{j_m}+1)} Q_{n-\sum_{m=1}^{d_j}(p_{j_m}+1)}(f),$$

where $Q_{n-\sum_{m=1}^{d_j}(p_{j_m}+1)}(f)$ is a polynomial in f of degree $n - \sum_{m=1}^{d_j}(p_{j_m}+1)$. Therefore

$$N_2(r,a_j;F) = 2\sum_{m=1}^{d_j}\overline{N}(r,\beta_{j_m};f) + N_2\Big(r,0;Q_{n-\sum_{m=1}^{d_j}(p_{j_m}+1)}(f)\Big) \leq \Big(n+d_j-\sum_{m=1}^{d_j}p_{j_m}\Big)T(r,f).$$

Similarly, we would have

$$N_2(r,a_j;G) = 2\sum_{m=1}^{d_j}\overline{N}(r,\beta_{j_m};g) + N_2\Big(r,0;Q_{n-\sum_{m=1}^{d_j}(p_{j_m}+1)}(g)\Big) \leq \Big(n+d_j-\sum_{m=1}^{d_j}p_{j_m}\Big)T(r,g).$$

From (1.1), we know that $k = s+t$, where $s(\geq 1)$ denotes the number of m_i's such that $m_i \geq 2$ and $t(\geq 0)$ denotes the number of m_i's such that $m_i = 1$. So from (3.1) and (3.2), we get

$$N_2(r,0;F) = 2\sum_{i=1}^{s}\overline{N}(r,\alpha_i;f) + \sum_{i=s+1}^{k} N_2(r,\alpha_i;f),$$

when $t \geq 1$ and

$$N_2(r,0;F) = 2\sum_{i=1}^{s}\overline{N}(r,\alpha_i;f),$$

when $t = 0$ and

$$N_2(r,\infty;F) = 2\overline{N}(r,\infty;f).$$

Similarly,

$$N_2(r,0;G) = 2\sum_{i=1}^{s}\overline{N}(r,\alpha_i;g) + \sum_{i=s+1}^{k}N_2(r,\alpha_i;g),$$

when $t \geq 1$ and

$$N_2(r,0;G) = 2\sum_{i=1}^{s}\overline{N}(r,\alpha_i;g),$$

when $t = 0$ and

$$N_2(r,\infty;G) = 2\overline{N}(r,\infty;g).$$

So in view of (3.4) when $t \geq 1$, we have

$$\begin{aligned}
& n\left(\frac{2l+1}{2}\right)[T(r,f)+T(r,g)] \\
\leq\; & 2\overline{N}(r,\infty;f) + 2\sum_{i=1}^{s}\overline{N}(r,\alpha_i;f) + \sum_{i=s+1}^{k}N_2(r,\alpha_i;f) \\
& +2\overline{N}(r,\infty;g) + 2\sum_{i=1}^{s}\overline{N}(r,\alpha_i;g) + \sum_{i=s+1}^{k}N_2(r,\alpha_i;g) \\
& +\sum_{j=1}^{l}\Big(n+d_j-\sum_{m=1}^{d_j}p_{j_m}\Big)T(r,f) + \sum_{j=1}^{l}\Big(n+d_j-\sum_{m=1}^{d_j}p_{j_m}\Big)T(r,g) + S(r,f)+S(r,g) \\
\leq\; & \Big(2+2s+t+ln+\sum_{j=1}^{l}d_j - \sum_{j=1}^{l}\sum_{m=1}^{d_j}p_{j_m}\Big)\Big[T(r,f)+T(r,g)\Big] + S(r,f)+S(r,g),
\end{aligned}$$

which is a contradiction for $n \geq 4s + 2t + 2\sum_{j=1}^{l}d_j - 2\sum_{j=1}^{l}\sum_{m=1}^{d_j}p_{j_m} + 5$.

Similarly, when $t = 0$, we get contradiction for $n \geq 4s + 2\sum_{j=1}^{l}d_j - 2\sum_{j=1}^{l}\sum_{m=1}^{d_j}p_{j_m} + 5$.

Case-2. Let $\frac{1}{F-1} = \frac{A}{G-1} + B$, where $A(\neq 0), B \in \mathbb{C}$. Now

$$\begin{aligned}
& \overline{N}(r,0;F) + \overline{N}(r,\infty;F) + \overline{N}(r,0;G) + \overline{N}(r,\infty;G) \\
\leq\; & (s+t+1+s+t+1)T(r,f) \\
\leq\; & \left(\frac{2s+2t+2}{n}\right)T(r,F) \\
<\; & \left(\frac{2s+2t+3}{n}\right)T(r,F) \\
\leq\; & T(r,F),
\end{aligned}$$

as $n \geq 2s + 2t + 3$. So in view of *Lemma 2.7*, we have either $FG = 1$ or $F \equiv G$.

Let $FG = 1$. Therefore we have

$$\prod_{i=1}^{k}(f-\alpha_i)^{m_i}\prod_{i=1}^{k}(g-\alpha_i)^{m_i} = c^2. \tag{3.5}$$

In view of *Lemma 2.8*, this case is impossible. So $F \equiv G$. This implies

$$P(f) \equiv P(g). \tag{3.6}$$

Since $P(z)$ is a UPM, so $f \equiv g$. □

4. Applications

In this section, we discuss some examples which would help us to understand the far reaching applications of our results. At first, we discuss examples of CIP's in the direction of *Theorem 1.15*.

Example 4.1. *First consider Yi's polynomial [16].*

$$P_Y(z) = z^n + az^{n-r} + b = z^{n-r}(z^r + a) + b, \tag{4.1}$$

where n, r are two positive integers having no common factors, $r \geq 2$ and $a, b \in \mathbb{C}^*$ be such that $P_Y(z)$ does not have any multiple zeros. Now

$$P_Y'(z) = z^{n-r-1}\left(nz^r + (n-r)a\right). \tag{4.2}$$

According to Fujimoto's result for UPM in [8], $P_Y(z)$ is a UPM for $n \geq r+3$. Here $k \geq 3$ and $\gcd(n, n-r) = 1$ *from the (4.1). So for $n - r \geq 3$, $P_Y(z)$ satisfies condition (i) or (ii) of* Theorem 1.15 *and it is a UPM.*

Now let us count the cardinality of the generated URSM2. From (4.1) and (4.2), by simple calculation one can easily compute that '$nz^r + (n-r)a$' has r distinct zeros and each of them produces non-zero distinct λ_j. So in this case $l = r$, hence d_j's are $d_1, d_2, \ldots, d_r$ and $d_1 = d_2 = \ldots = d_r = 1$. Corresponding p_{j_m}'s are $p_{1_1}, p_{2_1}, \ldots, p_{r_1}$ and each of them is equal to 1. *So,*

$$2\sum_{j=1}^{l} d_j = 2\sum_{j=1}^{r} 1 = 2r \quad \text{and}$$

$$2\sum_{j=1}^{l}\sum_{m=1}^{d_j} p_{j_m} = 2\sum_{j=1}^{r}(p_{j_1} + \ldots + p_{j_{d_j}}) = 2[(p_{1_1} + \ldots + p_{1_{d_1}}) + (p_{2_1} + \ldots + p_{2_{d_2}}) + \ldots + (p_{r_1} + \ldots + p_{r_{d_r}})]$$

$= 2[p_{1_1} + p_{2_1} + \ldots + p_{r_1}] = 2r$, *as* $d_1 = d_2 = \ldots = d_r = 1$ *and* $p_{1_1} = p_{2_1} = \ldots = p_{r_1} = 1$.
Also we get $s = 1$ and $t = r$, hence the zero set of $P_Y(z)$ is a URSM2 for $n \geq max\{4+2r+2r-2r+5, 2+2r+3\} = 2r+9$.

Example 4.2. *Let us consider Frank-Reinder's polynomial [6].*

$$P_{FR}(z) = \frac{(n-1)(n-2)}{2}z^n - n(n-2)z^{n-1} + \frac{n(n-1)}{2}z^{n-2} - c, \quad c \neq 0, 1;$$

Suppose that

$$R(z) = \frac{2}{(n-1)(n-2)}P_{FR}(z) = z^{n-2}\left(z^2 - \frac{2n}{n-1}z + \frac{n}{n-2}\right) - c, \quad c \neq 0, 1.$$

i.e.,

$$R(z) = z^{n-2}\left(z^2 - \frac{2n}{n-1}z + \frac{n}{n-2}\right) - c, \quad c \neq 0, 1. \tag{4.3}$$

Now

$$R'(z) = nz^{n-3}(z-1)^2. \tag{4.4}$$

Again one can apply Fujimoto's result for UPM in [8] and show that $R(z)$ is a UPM for $n \geq 6$. From (4.3) we see that for $n \geq 6$ we have $k = 3$, $m_1 = n - 2 \geq 2$, $m_2 = m_3 = 1$. Now if n is odd, then clearly $\gcd(n-2, n) = 1$; *i.e., condition (ii) of* Theorem 1.15 *is satisfied and if n is even; i.e.,* $\gcd(n-2, n) \neq 1$, *then condition (iii) of* Theorem 1.15 *is satisfied as $m_2 = m_3 = 1$ and $n \geq 6$.*

From (4.3) and (4.4), we observe that only $z = 1$ produces nonzero λ_j. So, in this case, we get $l = 1$; i.e., $d_j = d_1$ and $d_1 = 1$. Corresponding p_{j_m} would be p_{1_1} and its value is equal to 2. So,

$$2\sum_{j=1}^{l} d_j = 2(1) = 2 \quad \textit{and}$$

$$2\sum_{j=1}^{l}\sum_{m=1}^{d_j} p_{j_m} = 2\sum_{j=1}^{l}(p_{j_1} + \ldots + p_{j_{d_j}}) = 2p_{1_1} = 4.$$

Also we have $s = 1$ and $t = 2$, hence the zero set of $R(z)$ is a URSM2 for $n \geq max\{4+4+2-4+5, 2+4+3\} = 11$. Since the zeros of $R(z)$ and $P_{FR}(z)$ are same, so the zero set of $P_{FR}(z)$ is a URSM2 for $n \geq 11$.

Now we construct a NCIP which generates URSM.

Example 4.3. *Consider the polynomial $R(z) = z^n + 2z^{n-2} + z^{n-4} + c$, where $n(\geq 7)$ is odd and $c \in \mathbb{C}$ be such that $R(z)$ does not have any multiple zero. Clearly $R(z) = z^{n-4}(z+i)^2(z-i)^2 + c$. Here $s = 3$, so $R(z)$ is a NCIP. Now we prove that $R(z)$ is a UPM.*

Let $R(f) \equiv R(g)$. So we have

$$f^n + 2f^{n-2} + f^{n-4} = g^n + 2g^{n-2} + g^{n-4}. \tag{4.5}$$

Now suppose that $h = \dfrac{f}{g}$ and putting this value in (4.5), we get

$$g^n(h^n - 1) + 2g^{n-2}(h^{n-2} - 1) + g^{n-4}(h^{n-4} - 1) = 0;$$

i.e.,

$$g^4(h^n - 1) + 2g^2(h^{n-2} - 1) + (h^{n-4} - 1) = 0. \tag{4.6}$$

Next let us consider the following subcases.
Subcase-1.1. *Suppose that h is non-constant. Then from (4.6), we get*

$$\begin{aligned} \left(g^2 + \frac{h^{n-2}-1}{h^n-1}\right)^2 &= \left(\frac{h^{n-2}-1}{h^n-1}\right)^2 - \left(\frac{h^{n-4}-1}{h^n-1}\right) \\ &= \frac{(h^{n-2}-1)^2 - (h^{n-4}-1)(h^n-1)}{(h^n-1)^2} \\ &= \frac{(h^n - 2h^{n-2} + h^{n-4})}{(h^n-1)^2} \\ &= \frac{h^{n-4}(h^2-1)^2}{(h^n-1)^2}. \end{aligned} \tag{4.7}$$

Since n is odd, then $n = 2p + 1$ for some positive integer p. Hence $n - 4 = 2(p-2) + 1$. Now it is to be noted that from (4.5), we get

$$f^{n-4}(f^2+1)^2 = g^{n-4}(g^2+1)^2$$

$$\Longrightarrow \frac{f^{n-4}}{g^{n-4}} = \left(\frac{g^2+1}{f^2+1}\right)^2$$

$$\Longrightarrow h^{n-4} = \left(\frac{g^2+1}{f^2+1}\right)^2$$

$$\implies h^{2(p-2)+1} = \left(\frac{g^2+1}{f^2+1}\right)^2$$

$$\implies h = \left(\frac{g^2+1}{h^{(p-2)}(f^2+1)}\right)^2.$$

Since f, g and h are meromorphic functions, so is f^2+1, g^2+1 and $h^{(p-2)}$ and so is $\left(\frac{g^2+1}{h^{(p-2)}(f^2+1)}\right)$. Now let us denote $\left(\frac{g^2+1}{h^{(p-2)}(f^2+1)}\right) = \vartheta$. Hence $h = \vartheta^2$ for a meromorphic function ϑ.
Now putting $h = \vartheta^2$ in (4.7), we get

$$\left(g^2 + \frac{(\vartheta^2)^{n-2}-1}{(\vartheta^2)^n - 1}\right)^2 = \frac{[\vartheta^{(n-4)}(\vartheta^4-1)]^2}{((\vartheta^2)^n-1)^2}.$$

Then we have

$$g^2 = -\frac{(\vartheta^{2n-6}+\vartheta^{2n-8}+\ldots+\vartheta^2+1) \mp \vartheta^{n-4}(\vartheta^2+1)}{\vartheta^{2n-2}+\vartheta^{2n-4}+\vartheta^{2n-6}+\ldots+\vartheta^6+\vartheta^4+\vartheta^2+1};$$

i.e.,

$$g^2 = -\frac{(\vartheta^{2n-6}+\vartheta^{2n-8}+\ldots+\vartheta^2+1) + \vartheta^{n-4}(\vartheta^2+1)}{\vartheta^{2n-2}+\vartheta^{2n-4}+\vartheta^{2n-6}+\ldots+\vartheta^6+\vartheta^4+\vartheta^2+1} \tag{4.8}$$

or,

$$g^2 = -\frac{(\vartheta^{2n-6}+\vartheta^{2n-8}+\ldots+\vartheta^2+1) - \vartheta^{n-4}(\vartheta^2+1)}{\vartheta^{2n-2}+\vartheta^{2n-4}+\vartheta^{2n-6}+\ldots+\vartheta^6+\vartheta^4+\vartheta^2+1}. \tag{4.9}$$

From (4.8), we observe that the numerator and denominator in the right hand side may have common factors say $(\vartheta-\delta_i)$'s for $i = 1, 2, \ldots, p$ iff

$$\delta_i^{n-4}(\delta_i^2+1) = \delta_i^{2n-2} + \delta_i^{2n-4};$$

i.e.,

$$\delta_i^{n-4}(\delta_i^2+1)(\delta_i^n-1) = 0.$$

Since n is odd, so $\delta_i = 0, i, -i$ are neither the zeros of

$$(z^{2n-6}+\ldots+z^2+1) + z^{n-4}(z^2+1)$$

nor the zeros of

$$z^{2n-2}+z^{2n-4}+z^{2n-6}+\ldots+z^6+z^4+z^2+1.$$

So the value of p is at most $(2n-2) - \{(n-4)+2\} = n$; i.e., the numerator and the denominator may have at most n factors in common. It is easy to show that the denominator has only simple factors, so we do not give the calculation here. Hence the denominator has at least $n-2$ distinct non-common factors. Suppose $(\vartheta-\gamma_i)$'s for $i = \{1, 2, \ldots, n-2\}$ are the distinct non-common factors of the denominator. Clearly each γ_i-point of ϑ is a pole of g of multiplicity at least 2. So using second fundamental theorem on these points, we get

$$\begin{aligned}
(n-4)T(r,\vartheta) &\leq \sum_{i=1}^{n-2} \overline{N}(r,\gamma_i;\vartheta) + S(r,\vartheta)\\
&\leq \frac{1}{2}\sum_{i=1}^{n-2} N(r,\gamma_i;\vartheta) + S(r,\vartheta)\\
&\leq \frac{n-2}{2}T(r,\vartheta) + S(r,\vartheta),
\end{aligned}$$

which is a contradiction for $n \geq 7$.
We can similarly arrive at a contradiction for equation (4.9). So we deal the next subcase.
Subcase-1.2. *Suppose that h is constant. Since g is non-constant, so from (4.6), we get $h^n = h^{n-2} = h^{n-4} = 1$. Since n is odd, so $h = 1$; i.e., $f \equiv g$.*

We also find that in this case $k = 3$ and $\gcd(n, 2) = 1$. Hence, $R(z)$ satisfies condition (ii) of Theorem 1.15 *and it is also a UPM for $n \geq 7$. Now we count the cardinality of the generated URSM2.*

Note that $R'(z) = z^{n-5}\left(z^2+1\right)\left(nz^2+(n-4)\right)$. Since n is odd, one can easily verify that for this polynomial we have only two β_{j_m} namely '$\pm\sqrt{\frac{4-n}{n}}$' producing non-zero λ_j and each of these λ_j is distinct; i.e, here in the formula of cardinality we would have $l = 2$ and $d_1 = 1 = d_2$. Clearly p_{j_m}'s are P_{1_1}, P_{2_1} with $P_{1_1} = 1 = P_{2_1}$. So,

$$2\sum_{j=1}^{l} d_j = 2\sum_{j=1}^{2} d_j = 2(d_1+d_2) = 4 \quad \text{and}$$

$$2\sum_{j=1}^{l}\sum_{m=1}^{d_j} p_{j_m} = 2\sum_{j=1}^{2}(p_{j_1}+\ldots+p_{j_{d_j}}) = 2[(p_{1_1}+\ldots+p_{1_{d_1}})+(p_{2_1}+\ldots+p_{2_{d_2}})] = 2[p_{1_1}+p_{2_1}] = 4.$$

Also we have $s = 3$ and $t = 0$, hence the zero set of $R(z)$ is a URSM2 for $n \geq max\{12+4-4+5, 6+0+3\} = 17$.

The above NCIP also generates URSE of cardinality 13. Since the gap between the first two terms of $R(z)$ is 1, so we provide the following example to show that there also exists NCIP having no gap between the first two terms which generates URSE of cardinality 9. This example is the direct application of *Corollary 1.16*.

Example 4.4. *Let $R_1(z) = z^n + 2z^{n-1} + z^{n-2} + c$, where $n(\geq 5)$ is odd and $c \in \mathbb{C}$ be such that $R_1(z)$ does not have any multiple zero. Clearly $R_1(z) = z^{n-2}(z+1)^2 + c$. Here $s = 2$, so $R_1(z)$ is a NCIP. Now we prove that $R_1(z)$ is a UPE.*

Let f and g be two non-constant entire functions such that $R(f) \equiv R(g)$. So we have

$$f^n + 2f^{n-1} + f^{n-2} = g^n + 2g^{n-1} + g^{n-2}. \tag{4.10}$$

Now suppose that $h = \dfrac{f}{g}$ and putting this value in (4.10), we get

$$g^n(h^n-1) + 2g^{n-1}(h^{n-1}-1) + g^{n-2}(h^{n-2}-1) = 0;$$

i.e.,

$$g^2(h^n-1) + 2g(h^{n-1}-1) + (h^{n-2}-1) = 0. \tag{4.11}$$

If h is non-constant, then proceeding similarly like Subcase-1.1 of Example 4.3, *we get*

$$g = -\frac{(\vartheta^{2n-4}+\ldots+\vartheta^2+1)+\vartheta^{n-2}}{\vartheta^{2n-2}+\vartheta^{2n-4}+\ldots+\vartheta^2+1} \tag{4.12}$$

or,

$$g = -\frac{(\vartheta^{2n-4}+\ldots+\vartheta^2+1)-\vartheta^{n-2}}{\vartheta^{2n-2}+\vartheta^{2n-4}+\ldots+\vartheta^2+1}, \tag{4.13}$$

where ϑ is a non-constant meromorphic function with $\vartheta^2 = \dfrac{f}{g}$. Also using the similar arguments we would get that in (4.12), there are $(n-1)$ distinct non-common factors in the numerator with respect to the denominator. Suppose these non-common factors are $(\vartheta - \zeta_i)$ for $i = 1, 2, \ldots, n-1$. Clearly each ζ_i-point of ϑ is a pole of g. Since g does not *have any pole, so ϑ omits $n-1$ points which contradicts the fact that ϑ is non-constant as $n \geq 5$. We would also have contradiction for $n \geq 5$ if we consider (4.13) instead of (4.12).*

For h to be constant, we obviously get $h^n = h^{n-1} = h^{n-2} = 1$ which implies $h = 1$ and hence $f \equiv g$.

Also for $R_1(z)$ we have $k = 2$, hence $R_1(z)$ satisfies the conditions of Corollary 1.16. *Now we count the cardinality of the generated URSE2.*

In this case, $R_1'(z) = z^{n-3}(z+1)(nz + (n-2))$, so we have only $z = \frac{2-n}{n}$ producing non-zero λ_j; i.e, here in the formula of cardinality we would have $l = 1$ and $d_j = d_1 = 1$. So, we have only one p_{j_m} namely p_{1_1} with $p_{1_1} = 1$. Hence,

$$2\sum_{j=1}^{l} d_j = 2(d_1) = 2 \quad \textit{and}$$

$$2\sum_{j=1}^{l}\sum_{m=1}^{d_j} p_{j_m} = 2\sum_{j=1}^{l}(p_{j_1} + \ldots + p_{j_{d_j}}) = 2[p_{1_1}] = 2.$$

Also we have $s = 2$ and $t = 0$, hence the zero set of $R(z)$ is a URSE2 for $n \geq max\{8 + 2 - 2 + 1, 4 + 0 + 1\} = 9$.

Remark 4.5. *Example 4.3-4.4 prove the existence of NCIP's having gap between the first two terms less than* 3*, which can generate unique range sets. In particular, Example 4.4 prove that non-gap polynomials can be NCIP's and at the same time they can generate unique range sets. So, authors choice of gap polynomials in [1, 4] for constructing unique range sets without considering the injectivity hypothesis was good but it lacked putting all the variants of unique range sets under a single umbrella.*

5. Concluding Remark and Some Open Questions

In [8], Fujimoto provided a necessary sufficient condition for a CIP to be a uniqueness polynomial. In [2], An-Wang-Wong provided a necessary sufficient condition for a general polynomial having gap between the first two terms at least 3, to be a strong uniqueness polynomial. In the same paper, An-Wang-Wong also provided a necessary sufficient condition for a CIP to be a uniqueness polynomial. But still no theory have been obtained for a NCIP or a general polynomial which includes both CIP and NCIP, to be a uniqueness polynomial. In Examples 4.1-4.4, we have seen that in case of CIP's we are very much comfortable to verify whether it is a uniqueness polynomial or not as we have the established theory of Fujimoto but in case of NCIP's we need to prove it separately whenever the gap between the first two terms of the polynomial is less than 3. Since the characterization of a general polynomial to generate a unique range set has already been made in this paper and a necessary sufficient condition has also been found between a uniqueness polynomial and the corresponding unique range set, so if one can find the answer of anyone of the following questions, then that will ease the effort of finding uniqueness polynomials which are non-critically injective and at the same time that will at instant help us to exemplify any polynomial to generate a unique range set.

Question 5.1. *What is the necessary sufficient condition for a NCIP to be a uniqueness polynomial?*

Question 5.2. *What is the necessary sufficient condition for a general polynomial to be a uniqueness polynomial?*

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