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An Interior Point Algorithm for Solving Linear Optimization Problems Using a New Trigonometric Kernel Function

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Abstract. In this paper, a primal-dual interior point algorithm for solving linear optimization problems based on a new kernel function with a trigonometric barrier term which is not only used for determining the search directions but also for measuring the distance between the given iterate and the μ-center for the algorithm is proposed. Using some simple analysis tools and prove that our algorithm based on the new proposed trigonometric kernel function meets $O\left(\sqrt{n}\log n \log \frac{n}{\varepsilon}\right)$ and $O\left(\sqrt{n}\log \frac{n}{\varepsilon}\right)$ as the worst case complexity bounds for large and small-update methods. Finally, some numerical results of performing our algorithm are presented.

1. Introduction

The main concern of this paper is to propose a primal-dual interior point algorithm for solving Linear Optimization (LO) problem:

$$(P) \qquad \min\{c^T x : Ax = b, x \geq 0\},$$

and its dual as:

$$(D) \qquad \max\{b^T y : A^T y + s = c, s \geq 0\},$$

where $A \in \mathbb{R}^{m\times n}$, $x, c, s \in \mathbb{R}^n$ and $y, b \in \mathbb{R}^m$.

In recent years, many researchers have attempted to find methods that have the best theoretical and practical results in solving mathematical programming problems. The class of Interior Point Methods (IPMs) is one of these kind of methods that received a lot of attention by the researchers. An important and pioneer work in this direction goes back to the landmark paper proposed by Karmarkar in [14]. He introduced the the so-called polynomial time IPMs for solving LO problems. Some later, the concept of primal-dual IPMs was suggested by Kojima et al. [16] and Megiddo [20]. Nesterov and Nemirovskii in [22] extended IPMs from LO to more general convex optimization problems such as Convex Quadratic Optimization (CQO), Semidefinite Optimization (SDO), Second Order Cone Optimization (SOCO), Nonlinear Complementarity Problem (NCP), Linear Complementarity Problem (LCP), and Convex Quadratic Semidefinite Optimization

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(CQSDO).
In addition to practical performance of IPMs, iteration complexity bound is one of the crucial problems in IPMs. Although, Karmarkar [14] showed that their algorithm has a polynomial complexity $O(nL)$ iteration with $O(n^{\frac{7}{2}}L)$ bit operations, Nesterov and Nemirovskii proved that their algorithms for solving convex problems in large neighborhood of the central path has $O(n \log \frac{n}{\varepsilon})$ iteration complexity bound.
The concept of *kernel-based interior point methods* was first introduced by Peng in [23]. Kernel functions play an important role in the design and analysis of primal-dual IPMs. They are not only used to measure the distance between the given iterate and the μ-center but also to determine the search directions. Peng et al. in [24] proved that their algorithm for solving LO problems based on the so-called Self-Regular (SR) kernel functions has the best known iteration complexity bounds for large and small-update methods, namely, $O(\sqrt{n} \log n \log \frac{n}{\varepsilon})$ and $O(\sqrt{n} \log \frac{n}{\varepsilon})$, respectively. Since then, several attempts for introducing non-SR kernel functions in order to at least meeting the complexity results of SR barrier functions have been started. A comparative study on the kernel functions was provided in [1–3, 9, 17, 24, 25, 32].

Due to literature, nowadays, it seems that primal-dual IPMs based on trigonometric kernel functions received a great of interest by the researchers in this field. For more information on new interior point algorithms based on the trigonometric kernel function, we refer the interested reader to the works proposed in [5–8, 10–13, 15, 18, 19, 26–29]. Some of these functions obtained the so far best known iteration complexity, see e.g. [5, 10–13, 26, 27]. It has to be noted that, due to [1], all the researches in introducing new kernel functions are basically focused on finding a kernel function for which the complexity of large-update methods is equal to (or even better than) $O(\sqrt{n} \log \frac{n}{\varepsilon})$, or show that such a kernel function does not exist. This motivates us to work on theoretical complexity aspects of several kernel functions which are not self-regular. The research line of this paper coincides to this fact. Indeed, our main concern is on deriving the theoretical complexity of a new proposed kernel function in the way of verifying the aforementioned question.

Motivate by these works, in this paper we introduce a new kernel function with a new trigonometric barrier term. By means of some simple analysis tools, we analyze the large-update primal dual IPM based on the new proposed kernel function and show that the algorithm enjoys $O(\sqrt{n} \log n \log \frac{n}{\varepsilon})$ as the worst case iteration complexity bound. Finally, we present some numerical results. The results are obtained by performing interior point algorithm based on the new proposed kernel function and six kernel functions defined in literature. Comparison of the obtained results shows that the new proposed kernel function outperforms the other considered kernel functions.

The paper is organized as follows: In Section 2, we recall some concepts of IPMs LO problems. In Section 3, we introduce the new kernel function and some of its properties. Section 4 is devoted to describe the proximity reduction during an inner iteration. We also obtain a default value for the step size in this section. The worst case iteration bound for the large update primal-dual IPMs based on the new kernel function is provided in Section 5. In section 6, we present some numerical results. Finally, some concluding remarks are given in Section 7.

We use the following notational conventions: Throughout the paper, the Euclidian norm of a vector is denoted by $\|.\|$. We denote the nonnegative and positive orthants by $\mathbb{R}^n_+$ and $\mathbb{R}^n_{++}$, respectively. For a vector $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, x_* is the minimum component of x. For given vectors x and s, the vectors xs and $\frac{x}{s}$ denote the coordinate-wise operations on the vectors, i.e., whose components are $x_i s_i$ and $\frac{x_i}{s_i}$, respectively. We say that $f(t) = \Theta(g(t))$, if there exist positive constants ω_1 and ω_2 so that $\omega_1 g(t) \leq f(t) \leq \omega_2 g(t)$, for all $t \in \mathbb{R}_{++}$. We also say that $f(t) = O(g(t))$, if there exists a positive constant ω so that $f(t) \leq \omega g(t)$, for all $t \in \mathbb{R}_{++}$.

2. Preliminaries

In this section, we recall some concepts of the IPMs such as central path, search direction and large- and small-update methods. Moreover, a generic interior point algorithm for LO problems is presented.

Throughout the paper, we assume that the following assumptions hold:

$A1$: Interior Point Condition (IPC) holds, i.e., there exists a strictly feasible point, namely, $x^0 > 0$ and (y^0, s^0) with $s^0 > 0$, so that:

$$\begin{aligned} Ax^0 &= b, \qquad x^0 > 0, \\ A^T y^0 + s^0 &= c, \qquad s^0 > 0. \end{aligned}$$

$A2$: The matrix A has full row rank, i.e., $rank(A) = m \leq n$.

The optimality conditions for problems (P) and (D) are given by:

$$\begin{aligned} Ax &= b, \qquad x \geq 0; \\ A^T y + s &= c, \qquad s \geq 0; \\ xs &= 0. \end{aligned} \tag{1}$$

The key idea behind the primal-dual IPMs for solving LO problems is to replace the last equation in (1), the so called complementarity condition, with the nonlinear parametric equation $xs = \mu\mathbf{e}$, where μ is a real positive parameter and $\mathbf{e} = (1, 1, \ldots, 1)^T$. This leads us to the following parametric system:

$$\begin{aligned} Ax &= b, \qquad x \geq 0; \\ A^T y + s &= c, \qquad s \geq 0; \\ xs &= \mu\mathbf{e}. \end{aligned} \tag{2}$$

Using Assumptions $A1$ and $A2$, one can deduce that system (2) has a unique solution for any $\mu > 0$, see e.g. [24]. Let $(x(\mu), y(\mu), s(\mu))$ be the unique solution of system (2) for $\mu > 0$. We call $x(\mu)$ and $(y(\mu), s(\mu))$ the μ-centers of (P) and (D), respectively. The set of μ-centers, for all $\mu > 0$, defines a homotopy path which is called the central path of (2) [21]. As $\mu \to 0$, the limit of the central path exists and converges to the analytic center of the optimal solution set of (P) and (D), see e.g. [20, 31].
For fixed $\mu > 0$, applying the Newton method to the parameterized system (2) implies the following system for the search direction $(\Delta x, \Delta y, \Delta s)$:

$$\begin{aligned} A\Delta x &= 0; \\ A^T \Delta y + \Delta s &= 0; \\ x\Delta s + s\Delta x &= \mu\mathbf{e} - xs. \end{aligned} \tag{3}$$

To simplify, we define the scaled vector v as:

$$v := \sqrt{\frac{xs}{\mu}}.$$

Now, let us further define, the new search directions d_x and d_s as:

$$d_x = \frac{v\Delta x}{x}, \qquad d_s = \frac{v\Delta s}{s}. \tag{4}$$

By using the above notation and some simple calculus, the search direction $(d_x, \Delta y, d_s)$ is obtained by solving the following system:

$$\begin{aligned} \bar{A} d_x &= 0; \\ \bar{A}^T \Delta y + d_s &= 0; \\ d_x + d_s &= v^{-1} - v, \end{aligned} \tag{5}$$

where

$$\begin{aligned} \bar{A} &:= \frac{1}{\mu} A V^{-1} X = A S^{-1} V; \\ V &:= diag(v), X := diag(x), S := diag(s). \end{aligned} \tag{6}$$

Note that, system (5) has a unique solution and we can get the search direction $(\Delta x, \Delta y, \Delta s)$ easily by using (4). Now, consider the univariate kernel function $\psi_c(t)$ as follows:

$$\begin{aligned}
\psi_c(t) &: \mathbb{R}_{++} \to \mathbb{R}_+,\\
\psi_c(t) &= \frac{t^2-1}{2} - \log t.
\end{aligned}$$

One can simply see that the right-hand side of the last equation in (5) is equal to the negative gradient of the proximity function $\Psi_c(v) = \sum_{i=1}^{n} \psi_c(v_i)$, induced from $\psi_c(t)$. Note that $\psi_c(t)$ is a strictly convex function on $\mathbb{R}_{++}$ and satisfies:

$$\psi_c(1) = \psi_c'(1) = 0, \tag{7}$$

$$\lim_{t\to\infty} \psi_c(t) = \lim_{t\to 0} \psi_c(t) = +\infty. \tag{8}$$

Now, by replacing the right-hand side of the last equation in (5) by $-\nabla\Psi(v)$, one can get the following system for $(d_x, \Delta y, d_s)$:

$$\begin{aligned}
\bar{A} d_x &= 0;\\
\bar{A}^T \Delta y + d_s &= 0;\\
d_x + d_s &= -\nabla\Psi(v).
\end{aligned} \tag{9}$$

This system has a unique solution [24]. From discussion above, we conclude that:

$$d_x = d_s = 0 \Leftrightarrow \psi'(v) = 0 \Leftrightarrow v = \mathbf{e} \Leftrightarrow \Psi(v) = 0,$$

namely, if and only if $xs = \mu\mathbf{e}$, i.e. if and only if $x = x(\mu)$ and $s = s(\mu)$. Otherwise, we have $\Psi(v) > 0$. Hence, if $(x, y, s) \neq (x(\mu), y(\mu), s(\mu))$, then $(\Delta x, \Delta y, \Delta s) \neq 0$ which implies that we can compute the step size α by some line search rules to obtain a new triple (x_+, y_+, s_+) as below:

$$x_+ = x + \alpha\Delta x, \qquad y_+ = y + \alpha\Delta y, \qquad s_+ = s + \alpha\Delta s. \tag{10}$$

Summarizing the above argument, we can outline this procedure in the following primal-dual interior point scheme [24].

Algorithm 1. Generic Primal-dual IPM for LO

Input
  a proximity function $\Psi(v)$
  a threshold parameter $\tau > 0$
  an accuracy parameter $\varepsilon > 0$
  a barrier update parameter θ, $0 < \theta < 1$
begin
  $x := \mathbf{e}; s := \mathbf{e}; \mu := 1; v := \mathbf{e};$
  while $n\mu > \varepsilon$ do
  begin
    $\mu := (1-\theta)\mu;$
    while $\Psi(v) > \tau$ do
    begin
      $x := x + \alpha\Delta x$
      $s := s + \alpha\Delta s$
      $y = y + \alpha\Delta y$
      $v := \sqrt{\frac{xs}{\mu}}$
    end
  end
end

Now, we illustrate an iteration of Algorithm 1. Starting a strictly feasible point (x^0, y^0, s^0), assume that, for given $\mu > 0$, an approximation of the μ-center $(x(\mu), y(\mu), s(\mu))$ is at hand. If $n\mu \leq \varepsilon$, then the algorithm is terminated. Otherwise, the parameter μ is decreased by the factor $1-\theta$, where $\theta \in (0,1)$. This part of the algorithm is known as outer iteration loop. While the value of $\Psi(v)$ is greater than the threshold τ, the new iterate is computed by taking Newton steps. In fact, this part of the algorithm constitutes the inner iterations loop. This procedure is repeated until we get to the point in which $n\mu \leq \varepsilon$. We note that the total number of iterations is given by multiplication of the inner and outer iterations.

3. The new kernel function

In this section, we introduce a new kernel function with trigonometric barrier term and investigate its properties. We define the new kernel function:

$$\psi(t) = \frac{t^2-1}{2} - \int_1^t e^{5p\tan(h(x))}dx, \qquad p \geq 1, \tag{11}$$

where

$$h(x) = \frac{1-x}{2+4x}\pi. \tag{12}$$

It can be easily seen that, when t goes to zero, then $h(t) \to \frac{\pi}{2}$ and therefore $\psi(t) \to +\infty$. On the other hand, when $t \to +\infty$, we can conclude that $h(t) \to 0$, which in turn implies that $\psi(t) \to +\infty$. These relations show that $\psi(t)$ is a barrier (kernel) function [1].

The first three derivatives of the proposed kernel function are given by:

$$\psi'(t) = t - e^{5p\tan(h(t))} \tag{13}$$

$$\psi''(t) = 1 + \frac{30p\pi}{(2+4t)^2}\left(1+\tan^2(h(t))\right)e^{5p\tan(h(t))} \tag{14}$$

$$\psi'''(t) = \left(1+\tan^2(h(t))\right)e^{5p\tan(h(t))}k(t) \tag{15}$$

where

$$k(t) = -\frac{240p\pi}{(2+4t)^3} - \frac{360p\pi^2}{(2+4t)^4}\tan(h(t)) - \frac{900p^2\pi^2}{(2+4t)^4}\left(1+\tan^2(h(t))\right) \tag{16}$$

Obviously, $\psi(1) = \psi'(1) = 0$. Therefore, one can easily describe the function $\psi(t)$ by its second derivative according to:

$$\psi(t) = \int_1^t \int_1^\xi \psi''(\zeta)d\zeta d\xi. \tag{17}$$

For the kernel function $\psi(t)$, given by (11), we have the following results:

Lemma 3.1. *(Lemma 2.1 in [7]) For the function $h(t)$ given by (12), one has:*

$$\tan(h(t)) - \frac{1}{3\pi t} > 0, \qquad 0 < t \leq \frac{1}{2}.$$

Lemma 3.2. *For the kernel function defined by (11), we have:*

i) $\psi''(t) > 1, \qquad \forall\ t > 0,$

ii) $t\psi''(t) - \psi'(t) > 0, \qquad \forall\ t > 1,$

iii) $t\psi''(t) + \psi'(t) > 0, \quad \forall\ t > 0,$

iv) $\psi'''(t) < 0, \quad \forall\ t > 0.$

Proof. First of all, it has to be noted that, for the function $h(t)$, given by (12), we have:

$$\tan(h(t)) \geq 0, \quad \text{for all } t \in (0,1],$$
$$\tan(h(t)) \in [-1,0), \quad \text{for all } t > 1.$$

To prove (i), for all $t > 0$, we have:

$$\psi''(t) = 1 + \frac{30p\pi}{(2+4t)^2}\left(1+\tan^2(h(t))\right)e^{5p\tan(h(t))} \geq 1.$$

To prove (ii), we have:

$$t\psi''(t) - \psi'(t) = \left(\frac{30p\pi t}{(2+4t)^2}(1+\tan^2(h(t))) + 1\right)e^{5p\tan(h(t))} > 0.$$

For proving (iii), first suppose that $t \in (0, \frac{1}{4}]$. Using Lemma 3.1, and the fact that $\tan(h(t)) \geq 1$, we have:

$$\begin{aligned}
t\psi''(t) + \psi'(t) &= 2t + \left(\frac{30p\pi t}{(2+4t)^2}(1+\tan^2(h(t))) - 1\right)e^{5p\tan(h(t))} \\
&> 2t + \left(\frac{30p\pi t}{(2+4t)^2}(1+\frac{\tan(h(t))}{3\pi t}) - 1\right)e^{5p\tan(h(t))} \\
&= 2t + \left(\frac{30p\pi t}{(2+4t)^2} + \frac{10p}{(2+4t)^2}\tan(h(t)) - 1\right)e^{5p\tan(h(t))} \\
&\geq 2t + \left(\frac{30p\pi t}{(2+4t)^2} + \frac{10p}{(2+4t)^2} - 1\right)e^{5p\tan(h(t))} > 0.
\end{aligned}$$

If $t \in (\frac{1}{4}, \frac{1}{2}]$, then

$$\begin{aligned}
t\psi''(t) + \psi'(t) &\geq 2t + \left(\frac{15p\pi}{2(2+4t)^2}(1+\tan^2(h(t))) - 1\right)e^{5p\tan(h(t))} \\
&\geq 2t + \left(\frac{15p\pi}{2(2+4t)^2} - 1\right)e^{5p\tan(h(t))} > 0.
\end{aligned}$$

If $t \in (\frac{1}{2}, 1]$, then

$$\begin{aligned}
t\psi''(t) + \psi'(t) &\geq 2t + \left(\frac{15p\pi}{(2+4t)^2}(1+\tan^2(h(t))) - 1\right)e^{5p\tan(h(t))} \\
&\geq 2t + \left(\frac{15p\pi}{(2+4t)^2} - 1\right)e^{5p\tan(h(t))} > 0.
\end{aligned}$$

For case $t > 1$, using the fact that $\psi'(1) = 0$ and $\psi''(t) > 1$, the function $\psi'(t)$ is an increasing and non-negative function, for all $t > 0$. This implies that $t\psi''(t) + \psi'(t) > 0$, for all $t > 1$.
In order to prove (iv), we note that, for $t \in (0,1]$, it can be easily seen that $\psi'''(t) < 0$. Now, let $t > 1$. To

prove the statement in this case, it is sufficient to show that $k(t) < 0$. To do so, we have:

$$\begin{aligned}
k(t) &= -\frac{240p\pi}{(2+4t)^3} - \frac{360p\pi^2}{(2+4t)^4}\tan(h(t)) - \frac{900p^2\pi^2}{(2+4t)^4}\left(1+\tan^2(h(t))\right) \\
&\leq -\frac{240p\pi}{(2+4t)^3} - \frac{360p\pi^2}{(2+4t)^4}\tan(h(t)) - \frac{900p^2\pi^2}{(2+4t)^4} \\
&\leq -\frac{240p\pi}{(2+4t)^3} + \frac{360p\pi^2}{(2+4t)^4} - \frac{900p^2\pi^2}{(2+4t)^4} \\
&= \frac{-240p\pi(2+4t) + 360p\pi^2 - 900p^2\pi^2}{(2+4t)^4} < 0.
\end{aligned}$$

This completes the proof of the lemma. □

As a consequence of Lemma 3.2, we conclude that the new proposed kernel function is an eligible kernel function [1].
The so called exponential convexity (e-convexity) property of the kernel functions plays an important role in the complexity analysis of the primal-dual IPMs based on these functions. The following technical lemma provides equivalent statements for the e-convexity property of a function.

Lemma 3.3. *(Lemma 2.1.2 in [24]) Suppose that $\psi(t)$, for $t > 0$, is a twice continuously differentiable function. Then, the following statements are equivalent:*

i) $\psi(\sqrt{t_1 t_2}) \leq \frac{1}{2}(\psi(t_1) + \psi(t_2))$, *for* $t_1, t_2 > 0$

ii) $\psi'(t) + t\psi''(t) \geq 0$ *for* $t > 0$

iii) $\psi(e^{\xi})$ *is a convex function.*

Using the third part of Lemma 3.2, the kernel function proposed as (11) has the e-convexity property.
Now, let us define the norm based proximity measure $\delta(v)$ as follows:

$$\delta(v) := \frac{1}{2}\|\psi'(v)\| = \frac{1}{2}\sqrt{\sum_{i=1}^{n}(\psi'(v))^2}. \tag{18}$$

Thus, we have:

$$\Psi(v) = 0 \Leftrightarrow \delta(v) = 0 \Leftrightarrow v = \mathbf{e}.$$

Using (17) and the super convexity property of ψ, i.e. $\psi''(t) \geq 1$, for all $t > 0$, one can easily obtain the following properties for the proximity function $\Psi(v)$ that is induced from ψ.
The following lemma gives some other properties of the new proposed kernel function [25].

Lemma 3.4. *Suppose that the kernel function $\psi(t)$ is given by (11). Then, we have:*

i) $\frac{1}{2}(t-1)^2 \leq \psi(t) \leq \frac{1}{2}\psi'(t)^2$, *for all* $t > 0$.

ii) $\Psi(v) \leq 2\delta(v)^2$, *for any* $v > 0$.

iii) $\|v\| \leq \sqrt{n} + \sqrt{2\Psi(v)}$, *for any* $v > 0$.

Corollary 3.5. *If $\Psi(v) \geq 1$, then we have:*

$$\delta(v) \geq \sqrt{\frac{1}{2}}. \tag{19}$$

Proof. It is easily followed from the second item of Lemma 3.4. □

In the sequel, we investigate the growth behavior of the new kernel function and its related real value matrix function.

Lemma 3.6. *Assume that $\beta \geq 1$ and the function $\psi(t)$ is given by (11). Then, we have*

$$\psi(\beta t) \leq \psi(t) + \frac{1}{2}(\beta^2 - 1)t^2$$

Proof. The proof is similar to the proof of Lemma 4.1 in [28]. Therefore, we omit it here. □

As a consequence of Lemma 3.6, we have the following lemma.

Lemma 3.7. *Let $v > 0$ and $\beta \geq 1$. Then, one has:*

$$\Psi(\beta v) \leq \Psi(v) + \frac{\beta^2 - 1}{2}\left(2\Psi(v) + 2\sqrt{2n\Psi(v)} + n\right).$$

4. An estimation for the step size

In this section, we focus on providing a default value for the step size during an inner iteration of Algorithm 1. To do so, we first note that, after an inner iteration, the new point is given by:

$$x_+ = x + \alpha\Delta x, \quad y_+ = y + \alpha\Delta y, \quad s_+ = s + \alpha\Delta s,$$

where, α is the so called the step size. Due to (4), the new iterate can be rewritten as:

$$x_+ = \frac{x}{v}(v + \alpha d_x), \quad y_+ = y + \alpha\Delta y, \quad s_+ = \frac{s}{v}(v + \alpha d_s).$$

By defining vector $v_+ := \sqrt{\frac{x_+ s_+}{\mu}}$, we conclude that:

$$v_+^2 = \frac{x_+ s_+}{\mu} = (v + \alpha d_x)(v + \alpha d_s). \tag{20}$$

As a consequence of *e*-convexity property, we have:

$$\Psi(v_+) = \Psi(\sqrt{(v + \alpha d_x)(v + \alpha d_s)} \leq \frac{1}{2}\left(\Psi(v + \alpha d_x) + \Psi(v + \alpha d_s)\right).$$

We denote the gap between the proximity function before and after one step below as a function with respect α:

$$f(\alpha) \quad := \quad \Psi(v_+) - \Psi(v). \tag{21}$$

Now, similar to [1], we define the default step size α as:

$$\tilde{\alpha} = \frac{1}{\psi''(\rho(2\delta))}, \tag{22}$$

where $\rho : [0, \infty) \to (0, 1]$ is the inverse of the function $-\frac{1}{2}\psi'(t)$ in the interval $(0, 1]$.
According to [1], the eligibility of the kernel function implies the following lemma.

Lemma 4.1. *(Lemma 4.5 in [1]) For any α satisfying $\alpha \leq \bar{\alpha}$, one has:*

$$f(\alpha) \leq -\alpha\delta^2 \tag{23}$$

Lemma 4.2. *Assume that* $\Psi(v) \geq 1$, ρ *is the inverse of the function* $-\frac{1}{2}\psi'(t)$ *in the interval* $(0, 1]$ *and* $\tilde{\alpha}$ *is defined by (22). Thus, we have*

$$f(\tilde{\alpha}) \leq -\frac{\delta^2}{\psi''(\rho(2\delta))} \leq \Theta\left(-\frac{\delta}{p\left(1 + (\frac{1}{5p}\log(4\delta+1))^2\right)}\right). \tag{24}$$

Proof. Using Lemma 4.1 and the fact that $\tilde{\alpha} \leq \overline{\alpha}$, we have $f(\tilde{\alpha}) \leq -\tilde{\alpha}\delta^2$. Now, we compute the inverse function $-\frac{1}{2}\psi'(t)$, for $0 < t \leq 1$. Using $-\frac{1}{2}\psi'(t) = s$, we obtain t as a function of s. For this purpose, we have:

$$-\left(t - e^{5p\tan(h(t))}\right) = 2s.$$

This implies that,

$$e^{5p\tan(h(t))} \leq 2s + 1,$$

where the last inequality is obtained from the fact that $t \in (0, 1]$. Now, letting $t = \rho(2\delta)$, we have $4\delta = -\psi'(t)$ which implies that

$$e^{5p\tan(h(t))} \leq 4\delta + 1 \tag{25}$$
$$e^{5p\tan(h(t))} \leq 4\delta + 1 \tag{26}$$
$$5p\tan(h(t)) \leq \log(4\delta + 1) \tag{27}$$
$$\tan(h(t)) \leq \frac{1}{5p}\log(4\delta + 1). \tag{28}$$

Now, using (25)–(28) and the fact that $t \in (0, 1]$, we obtain the following relations:

$$\begin{aligned}
\tilde{\alpha} = \frac{1}{\psi''(t)} &= \frac{1}{1 + \frac{30p\pi}{(2+4t)^2}\left(1 + \tan^2(h(t))\right)e^{5p\tan(h(t))}} \\
&\geq \frac{1}{1 + \frac{30p\pi}{(2+4t)^2}\left(1 + (\frac{1}{5p}\log(4\delta+1))^2\right)(4\delta+1)} \\
&\geq \frac{1}{1 + \frac{15p\delta\pi}{2}\left(1 + (\frac{1}{5p}\log(4\delta+1))^2\right)} \\
&\geq \frac{1}{2\delta + 45p\pi\delta\left(1 + (\frac{1}{5p}\log(4\delta+1))^2\right)} \\
&= \Theta\left(\frac{1}{\delta p\left(1 + (\frac{1}{5p}\log(4\delta+1))^2\right)}\right),
\end{aligned}$$

where the last inequality is obtained from the fact that $\Psi(v) \geq 1$ and $\delta \geq \sqrt{\frac{1}{2}}$ by Corollary 3.5. Thus, we have:

$$f(\tilde{\alpha}) \leq -\frac{\delta^2}{\psi''(\rho(2\delta))} \leq \Theta\left(-\frac{\delta}{p\left(1 + (\frac{1}{5p}\log(4\delta+1))^2\right)}\right),$$

which completes the proof of the lemma. □

Corollary 4.3. *From Lemma 4.2 and the second part of Lemma 3.4, one can easily see that:*

$$\begin{aligned}
f(\tilde{\alpha}) &\leq \Theta\left(\frac{-\delta}{p\left(1 + (\frac{1}{5p}\log(4\delta+1))^2\right)}\right) \\
&\leq \Theta\left(\frac{-\Psi^{\frac{1}{2}}(v)}{p\left(1 + (\frac{\log\Psi}{5p})^2\right)}\right).
\end{aligned} \tag{29}$$

5. Iteration complexity

In this section, we focus on the iteration complexity of Algorithm 1 based on the proximity function $\Psi(v)$ induced by ψ, defined by (11). During an inner iteration, using $\tilde{\alpha}$, defined by (22), as a default value for the step size. Therefore, updating the parameter μ to $(1-\theta)\mu$, for $\theta \in (0,1)$, implies that $v_+ = \frac{1}{\sqrt{1-\theta}}v$. Thus, using Lemma 3.7 with $\beta = \frac{1}{\sqrt{1-\theta}}$, one can easily see that:

$$\Psi(v_+) \leq \Psi(v) + \frac{\theta}{2(1-\theta)}(2\Psi(v) + 2\sqrt{2n\Psi(v)} + n). \tag{30}$$

right after updating the parameter μ to $(1-\theta)\mu$, for some $\theta \in (0,1)$. Note that at the start of each outer iteration of the algorithm and just before updating of the parameter μ, we have $\Psi(v) \leq \tau$. From (30), one can easily see that the proximity function $\Psi(v)$ exceeds the threshold τ after updating of μ. So, we need to compute the number of inner iterations required to return the iterations back to the situation where $\Psi(v) \leq \tau$. First, we represent the value of proximity function $\Psi(v)$ after μ-update by Ψ_0, and the subsequent values by Ψ_j, for $j = 1, \ldots, L-1$, where L is the total number of inner iterations in an outer iteration.
Therefore

$$\Psi_0 \leq \tau + \frac{\theta}{2(1-\theta)}(2\tau + 2\sqrt{2n\tau} + n) \tag{31}$$

As we are working on the large neighborhood of the central path, we assume that $\tau = O(n) \geq 1$. This fact together with (31) imply that $\Psi_0 = O(n)$. Moreover, in the all inner iterations, we have $\Psi_j > \tau \geq 1$. Therefore, from Corollary 4.3, the decrease of Ψ in any inner iteration is then given as:

$$\Psi_{j+1} \leq \Psi_j - \kappa\Delta\Psi_j, \qquad j = 0, 1, \ldots, L-1, \tag{32}$$

where κ is some positive constant and $\Delta\Psi_j$ is defined by

$$\Delta\Psi_j = \frac{\Psi_j^{\frac{1}{2}}}{p\left(1 + (\frac{\log\Psi_j}{5p})^2\right)}. \tag{33}$$

To proceed, we need the following technical lemma.

Lemma 5.1. *Given $\alpha \in [0,1]$ and $t \geq -1$, one has*

$$(1+t)^{\alpha} \leq 1 + \alpha t.$$

Using Lemma 5.1, we can provide the worst case iteration bound for the total number of inner iterations in an outer iteration as follows:

Theorem 5.2. *Let $\tau \geq 1$. Then, using (32), one has*

$$L \leq 1 + \frac{2p\left(1 + (\frac{\log\Psi_0}{5p})^2\right)}{\kappa}\Psi_0^{\frac{1}{2}}. \tag{34}$$

Proof. Using (32), for all $j = 0, 1, \ldots, L-1$, we have

$$\begin{aligned}
0 \le \Psi_{j+1}^{\frac{1}{2}} &\le \left(\Psi_j - \frac{\kappa}{p\left(1+(\frac{\log\Psi_j}{5p})^2\right)}\Psi_j^{\frac{1}{2}}\right)^{\frac{1}{2}} \\
&= \Psi_j^{\frac{1}{2}}\left(1 - \frac{\kappa}{p\left(1+(\frac{\log\Psi_j}{5p})^2\right)}\Psi_j^{-\frac{1}{2}}\right)^{\frac{1}{2}} \\
&\le \Psi_j^{\frac{1}{2}}\left(1 - \frac{\kappa}{2p\left(1+(\frac{\log\Psi_j}{5p})^2\right)}\Psi_j^{-\frac{1}{2}}\right) \\
&= \Psi_j^{\frac{1}{2}} - \frac{\kappa}{2p\left(1+(\frac{\log\Psi_j}{5p})^2\right)},
\end{aligned} \tag{35}$$

where the last inequality is obtained from Lemma 5.1. Therefore, we have:

$$\Psi_{j+1}^{\frac{1}{2}} \le \Psi_0^{\frac{1}{2}} - \frac{j\kappa}{2p\left(1+(\frac{\log\Psi_j}{5p})^2\right)}.$$

Now, letting $j = L-1$, we obtain that:

$$0 \le \Psi_L^{\frac{1}{2}} \le \Psi_0^{\frac{1}{2}} - \frac{(L-1)\kappa}{2p\left(1+(\frac{\log\Psi_0}{5p})^2\right)},$$

which implies that:

$$L \le 1 + \frac{2p\left(1+(\frac{\log\Psi_0}{5p})^2\right)}{\kappa}\Psi_0^{\frac{1}{2}}.$$

This completes the proof of the theorem. □

As our interest is to compute the worst case iteration complexity for the large-update IPMs in the large neighborhood of the central path, we set $\tau = O(n)$ and $\theta = \Theta(1)$. Again, we note that (31) implies that $\Psi_0 = O(n)$. Therefore, using Lemma 5.2, the following upper bound is obtained for the total number of inner iterations in an outer iteration:

$$L \le \left\lfloor 1 + \frac{2p\left(1+(\frac{\log\Psi_0}{5p})^2\right)}{\kappa}\Psi_0^{\frac{1}{2}}\right\rfloor = \left\lfloor O\left(\sqrt{n}p\left(1+(\frac{\log n}{5p})^2\right)\right)\right\rfloor. \tag{36}$$

As it has been stated in Lemma I.36 of [30], the total number of outer iterations in the large update methods for reaching $n\mu \le \varepsilon$ is bounded above by $O\left(\frac{1}{\theta}\log\frac{n}{\varepsilon}\right)$. Therefore, the total number of iterations in Algorithm 1 is obtained by multiplying the total number of inner and outer iterations. Hence, the total number of iterations to get an ε solution for the problems (P) and (D), i.e., a solution that satisfies $x^T s = n\mu \le \varepsilon$, is obtained as follows:

$$O\left(\sqrt{n}p\left(1+(\frac{\log n}{5p})^2\right)\log\frac{n}{\varepsilon}\right). \tag{37}$$

Note that, for $p = O(\log n)$, the relation (37) is simplified as follows:

$$O\left(\sqrt{n}\log n\log\frac{n}{\varepsilon}\right).$$

Now, due to [1], this bound yields the so far best known iteration bound for large update methods in terms of trigonometric kernel functions.

The net goal in this section is to compute the iteration complexity bound for small update method. To this end, we set $\Psi_0 = O(1)$ and $\theta = \Theta(\frac{1}{\sqrt{n}})$. Therefore, using Theorem 5.2, we have the following theorem:

Theorem 5.3. *Let $\Psi_0 = O(1)$ and $\theta = \Theta(\frac{1}{\sqrt{n}})$. Then, the complexity bound for small-update for IPM based on the new kernel function is denoted by:*

$$O\Big(p\sqrt{n}\log\frac{n}{\epsilon}\Big). \tag{38}$$

Letting $p = O(1)$ in (38). Then the worst case iteration complexity for small-update IPMs is denoted by $O\Big(\sqrt{n}\log\frac{n}{\epsilon}\Big)$, which matches to the currently best known iteration bound for small-update methods.

6. Numerical results

In this section, we provide numerical results of performing Algorithm 1 on a test problem given in [5]. We have implemented Algorithm 1 with the kernel function given by (11) along with six existing kernel functions in the literature. These kernel function are listed in Table 1. All the considered algorithms are coded in MATLAB 8.2.0.701 (R2013b) and run on a PC with Intel Core i5–7200U CPU and 12GB of RAM memory by double precision format.

Moreover, we have chosen the step size in the inner iterations of all approaches as an approximate value of the default step size in the related references.

For the considered test problem [5], we have $n = 2m$, and for all $1 \le i \le m$, the parameters of the problem are:

$$A = [I_m, I_m], \quad b = 2e_m \quad c = [-e_m; 0_m],$$

where I_m denotes the identity matrix of size $m \times m$, 0_m and e_m are the zero vector and the all-one vector of length m, respectively.

The strictly feasible initial point is given by:

$$x^0 = [e_m; e_m], \quad s^0 = [e_m; 2e_m], \quad y^0 = -2e_m.$$

We set the parameters of Algorithm 1 as below:

$$\mu^0 \;=\; 1, \quad \varepsilon = 10^{-8}, \quad \tau = 3, \quad \theta \in \{0.95, 0.99\}.$$

Moreover, we select $m \in \{375, 750, 1500, 3000, 7500\}$ and $p \in \{1, 2, 3, 4, 4.5\}$ in the setting of the new proposed kernel function. Note that the values of p are considered according to the fact that the algorithms obtain the best iteration complexity when $p = O(\log n)$.

The total number of inner iterations of performing Algorithm 1 based on the kernel functions defined in Table 1, are given in Tables 2 and 3. In these tables, "Iter", "Time" and "gap" stand for the number of iterations, CPU time (in second) and the value of $c^T x - b^T y$, respectively. Furthermore, $\psi_{7,p}$ stands for the new proposed kernel function with different values p.

Based on the results in Tables 2 and 3, one can see that the new proposed kernel function outperforms the other considered kernel functions.

Table 1: Considered kernel functions.

i	Kernel functions ψ_i	References
1	$\frac{t^2-1}{2} - \int_1^t \exp\left(\frac{1}{t}-1\right)dx$	[1]
2	$\frac{t^2-1}{2} - \log(t)$	[1]
3	$\frac{t^2-1}{2} - \int_1^t \exp\Big(3(\tan(\frac{\pi}{2+2x})-1)\Big)dx$	[28]
4	$\frac{t^2-1}{2} + \frac{4}{\pi}\cot(\frac{\pi t}{1+t})$	[15]
5	$\frac{t^2-1}{2} + \frac{6}{\pi}\tan(\frac{1-t}{2+4t}\pi)$	[7]
6	$\frac{t^2-1-\log(t)}{2} + \frac{t^{1-q}-1}{2(q-1)},\ q=2$	[4]
7	$\frac{t^2-1}{2} - \int_1^t \exp^{5p\tan(\frac{1-x}{2+4x}\pi)}dx, \quad p \geq 1$	New kernel function

7. Conclusion

In this paper, we propose a new kernel function with trigonometric barrier term and analyze the worst case iteration complexity of large-update primal-dual interior point method based on this kernel function in the large neighborhood of the central path for linear optimization problems. Using some mild and easy to check conditions, worst case iteration complexity analysis for the large update primal dual IPMs based on the new kernel function is provided. As usual, the e-convexity property of the kernel function plays an important role in deriving a default value for the step size. Our analysis shows that, with the specific choice of the function's parameter, the so far best known worst case iteration complexity of Algorithm 1, i.e. $O\left(\sqrt{n}\log n \log \frac{n}{\varepsilon}\right)$, is achieved. Numerical results shows that the new proposed kernel function is well promising and outperforms some existing kernel functions in the literature.

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Table 2: Iterations numbers of performing Algorithm 1 with $\theta = 0.95$.

ψ_i		m				
		375	750	1500	3000	7500
ψ_1	Iter	200	266	315	339	423
	Time	24.38	89.42	368.64	2168.76	14537.45
	gap	8.42E-09	1.62E-08	3.12E-09	5.62E-09	2.91E-09
ψ_2	Iter	189	257	304	326	436
	Time	24.16	85.80	325.76	1984.98	14872.87
	gap	8.25E-09	2.98E-08	3.09E-09	6.41E-09	7.81E-09
ψ_3	Iter	235	298	312	324	418
	Time	37.14	108.54	341.39	1973.87	13985.65
	gap	8.19E-09	1.65E-08	1.67E-08	3.62E-09	8.48E-09
ψ_4	Iter	176	223	283	319	397
	Time	19.81	72.82	327.71	1832.61	13565.71
	gap	8.34E-09	1.96E-08	1.02E-08	2.05E-09	5.12E-09
ψ_5	Iter	194	261	308	345	461
	Time	20.47	79.81	359.62	2171.01	14891.38
	gap	8.37E-09	1.61E-08	2.67E-09	4.35E-09	8.19E-09
ψ_6	Iter	189	272	319	342	439
	Time	22.94	81.37	381.06	2153.87	14624.43
	gap	2.15E-08	1.51E-08	2.74E-09	4.29E-09	8.01E-09
$\psi_{7,1}$	Iter	155	189	221	287	347
	Time	17.88	65.48	241.58	1589.37	10674.87
	gap	8.32E-09	1.13E-08	3.12E-09	4.43E-09	7.61E-09
$\psi_{7,2}$	Iter	159	194	224	297	349
	Time	18.87	70.15	251.35	1638.54	10834.98
	gap	8.47E-09	1.61E-08	3.13E-09	4.95E-09	8.34E-09
$\psi_{7,3}$	Iter	167	198	254	303	352
	Time	19.11	71.43	291.60	1668.32	10943.23
	gap	1.34E-08	1.48E-08	3.21E-09	5.01E-09	6.11E-09
$\psi_{7,4}$	Iter	168	206	248	301	351
	Time	20.76	75.32	279.00	1645.29	11035.63
	gap	1.12E-08	7.35E-09	2.12E-08	4.12E-09	5.97E-09
$\psi_{7,4.5}$	Iter	171	216	254	312	349
	Time	19.21	72.62	287.02	1673.07	10759.67
	gap	1.43E-08	1.49E-08	1.67E-08	5.27E-09	6.43E-09

Table 3: Iterations numbers of performing Algorithm 1 with $\theta = 0.99$.

ψ_i		m				
		375	750	1500	3000	7500
	Iter	182	237	263	282	359
ψ_1	Time	24.09	88.08	324.04	1763.23	11452.41
	gap	1.64E-09	3.19E-09	6.19E-09	1.22E-08	2.17E-09
	Iter	173	245	281	302	381
ψ_2	Time	23.25	80.35	302.14	1691.41	11191.40
	gap	1.51E-09	3.14E-09	6.01E-09	1.26E-08	2.43E-09
	Iter	221	251	288	307	373
ψ_3	Time	32.14	103.07	338.97	1837.98	12342.28
	gap	1.45E-09	2.99E-09	6.71E-09	1.67E-08	2.01E-09
	Iter	151	209	246	271	325
ψ_4	Time	19.34	79.31	302.17	1692.68	11109.14
	gap	1.44E-09	3.07E-09	6.43E-09	1.81E-08	2.71E-09
	Iter	171	231	266	315	383
ψ_5	Time	19.01	81.12	312.72	1701.01	11601.54
	gap	1.39E-09	2.97E-09	6.22E-09	1.27E-08	4.91E-010
	Iter	156	191	250	292	347
ψ_6	Time	22.07	85.12	319.45	1757.16	11910.15
	gap	1.44E-09	3.09E-09	6.43E-09	1.89E-08	4.32E-010
	Iter	137	173	201	243	281
$\psi_{7,1}$	Time	18.01	61.09	264.09	1431.01	8912.09
	gap	1.41E-09	3.08E-09	6.52E-09	2.01E-08	1.49E-09
	Iter	141	173	212	239	279
$\psi_{7,2}$	Time	18.34	63.31	271.49	1451.98	9001.54
	gap	1.32E-09	3.02E-09	7.03E-09	1.41E-08	1.79E-09
	Iter	142	183	228	251	281
$\psi_{7,3}$	Time	18.91	67.34	282.43	1482.00	8946.13
	gap	1.34E-09	3.00E-09	6.41E-09	1.50E-08	3.01E-09
	Iter	142	182	231	262	282
$\psi_{7,4}$	Time	19.01	69.70	280.14	1401.08	9067.21
	gap	8.31E-010	2.82E-09	6.53E-09	7.83E-09	6.98E-010
	Iter	144	189	241	261	284
$\psi_{7,4.5}$	Time	19.35	71.02	289.09	1410.43	9101.32
	gap	7.78E-010	2.01E-09	3.05E-09	6.12E-09	8.01E-010

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