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Braiding for Categorical Algebras and Crossed Modules of Algebras II: Leibniz Algebras

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Abstract. In this paper, we study the category of braided categorical Leibniz algebras and braided crossed modules of Leibniz algebras, and we relate these structures with the categories of braided categorical Lie algebras and braided crossed modules of Lie algebras using the Loday-Pirashvili category.

Introduction

This manuscript is the second part of the article [6]. The first part deals with study braiding for crossed modules and internal categories of associative and Lie algebras. In this work, we will consider braidings for the corresponding structures in the Leibniz algebras case.

Crossed modules for associative algebras [2], Lie algebras [9] and Leibniz algebras [10] act in an analogous way to the crossed modules of groups [13]. Moreover, it is known that the categories of these crossed modules are equivalent to their respective internal categories, and the notion of braiding for categorical groups provides an equivalent category to the category of braided crossed modules of groups (see [1, 8]).

Leibniz algebras appear in mathematics as a "non-antisymmetric" case of Lie algebras. Bearing this in mind, in this paper, we will show how to extend the idea of braiding for crossed modules and internal categories of Lie algebras to the Leibniz setting. After introducing these notions, we will prove the equivalence between braided crossed modules of Leibniz algebras and braided categorical Leibniz algebras, and we will show the parallelism between its examples and the ones given for groups, associative algebras and Lie algebras (see [6]).

For extending the notion of braiding, we will use the Loday-Pirashvili category [11], allowing us to see Leibniz algebras as a special case of Lie algebras in the category of linear maps with a certain structure of the braided monoidal category.

This paper is organized as follows. In the preliminaries, we will recall some ideas from [6] and a few basic definitions and properties about crossed modules of Leibniz algebras, including their relationship with crossed modules of Lie algebras. In Section 2, we show the internalization of the notion of a crossed module with a left Lie action of Lie objects in an arbitrary category. We will also define braidings for crossed

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modules of Lie objects and categorical Lie objects. We will apply this definition to the Loday-Pirashvili category $\mathcal{LM}_K$, and we will obtain the concepts of braiding for crossed modules of Leibniz algebras and categorical Leibniz algebras. In Section 3, we will prove the equivalence between braided categories in the Leibniz algebras case. Finally, in Section 4, we will show that the non-abelian tensor product of Leibniz algebras gives an example of a braided crossed module of Leibniz algebras.

1. Preliminaries

1.1. Crossed Modules of Lie algebras

We will recall the definitions of braiding for categorical Lie K-algebras and crossed modules of Lie K-algebras and some results on them (see [6]).

Definition 1.1. *Let $\mathcal{C} = (C_1, C_0, s, t, e, k)$ be a categorical Lie K-algebra.*
A braiding on $\mathcal{C}$ is a K-bilinear map $\tau\colon C_0 \times C_0 \to C_1$, $(a, b) \mapsto \tau_{a,b}$, satisfying the following properties:

$$\tau_{a,b}\colon [a, b] \to [b, a], \tag{LieT1}$$

$$\begin{array}{ccc} [s(x), s(y)] & \xrightarrow{[x,y]} & [t(x), t(y)] \\ \Big\downarrow{\scriptstyle \tau_{s(x),s(y)}} & & \Big\downarrow{\scriptstyle \tau_{t(x),t(y)}} \\ [s(y), s(x)] & \xrightarrow{[y,x]} & [t(y), t(x)], \end{array} \tag{LieT2}$$

$$\tau_{[a,b],c} = \tau_{a,[b,c]} - \tau_{b,[a,c]}, \tag{LieT3}$$

$$\tau_{a,[b,c]} = \tau_{[a,b],c} - \tau_{[a,c],b}, \qquad a, b, c \in C_0,\ x, y \in C_1. \tag{LieT4}$$

Definition 1.2. *Let $\mathcal{X} = (M, N, \cdot, \partial)$ be a crossed module of Lie K-algebras.*
A braiding (or Peiffer lifting) on the crossed module $\mathcal{X}$ is a K-bilinear map $\{-, -\}\colon N \times N \to M$, satisfying:

$$\partial\{n, n'\} = [n, n'], \tag{BLie1}$$
$$\{\partial m, \partial m'\} = [m, m'], \tag{BLie2}$$
$$\{\partial m, n\} = -n \cdot m, \tag{BLie3}$$
$$\{n, \partial m\} = n \cdot m, \tag{BLie4}$$
$$\{n, [n', n'']\} = \{[n, n'], n''\} - \{[n, n''], n'\}, \tag{BLie5}$$
$$\{[n, n'], n''\} = \{n, [n', n'']\} - \{n', [n, n'']\}, \qquad m, m' \in M,\ n, n', n'' \in N. \tag{BLie6}$$

If $\{-, -\}$ is a braiding on $\mathcal{X}$, we will say that $(M, N, \cdot, \partial, \{-, -\})$ is a braided crossed module of Lie K-algebras.

Definition 1.3. *A K-algebra $(M, [-, -])$ is called a Leibniz K-algebra if the Leibniz identity is satisfied, i.e.*

$$[x, [y, z]] = [[x, y], z] - [[x, z], y], \qquad x, y, z \in M.$$

Remember, as we saw in [6], that we are working in internal categories where all the internal morphisms are internal isomorphisms. We have the following properties.

Lemma 1.4. *Let (C_1, C_0, s, t, e, k) be an internal (associative, Lie, Leibniz) K-algebra or a categorical group whose operation is denoted by "+". Then the following rule for the composition is true*

$$k((x, y)) = x - e(t(x)) + y = x - e(s(y)) + y, \qquad (x, y) \in C_1 \times_{C_0} C_1.$$

Proposition 1.5 ([6]). *Let K be a field of* $\operatorname{char}(K) \neq 2$ *and (C_1, C_0, s, t, e, k) a categorical Lie K-algebra.*
If $\tau\colon C_0 \times C_0 \to C_1$ is a K-bilinear map satisfying (LieT1) *and* (LieT2)*, then*

$$\tau_{a,[b,c]} = [e(a), \tau_{b,c}] \qquad and \qquad \tau_{[b,c],a} = [\tau_{b,c}, e(a)].$$

In particular, by the anticommutativity, we have that $\tau_{a,[b,c]} = -\tau_{[b,c],a}$.

1.2. Crossed Modules of Leibniz algebras

The definition of crossed modules of Leibniz K-algebras, "non-antisymmetric" case of Lie K-algebras, was introduced by Loday and Pirashvili in [10].

Definition 1.6. *Let N and M be two Leibniz K-algebras. A Leibniz action of N on M is a pair $\cdot = (\cdot_1, \cdot_2)$ where $\cdot_1 : N \times M \to M$ and $\cdot_2 : M \times N \to M$ are K-bilinear maps and the following properties are satisfied*

$$n \cdot_1 [m, m'] = [n \cdot_1 m, m'] - [n \cdot_1 m', m], \tag{ALeib1}$$
$$[m, n \cdot_1 m'] = [m \cdot_2 n, m'] - [m, m'] \cdot_2 n, \tag{ALeib2}$$
$$[m, m' \cdot_2 n] = [m, m'] \cdot_2 n - [m \cdot_2 n, m'], \tag{ALeib3}$$
$$m \cdot_2 [n, n'] = (m \cdot_2 n) \cdot_2 n' - (m \cdot_2 n') \cdot_2 n, \tag{ALeib4}$$
$$n \cdot_1 (m \cdot_2 n') = (n \cdot_1 m) \cdot_2 n' - [n, n'] \cdot_1 m, \tag{ALeib5}$$
$$n \cdot_1 (n' \cdot_1 m) = [n, n'] \cdot_1 m - (n \cdot_1 m) \cdot_2 n', \qquad m, m' \in M,\ n, n' \in N. \tag{ALeib6}$$

Remark 1.7. *If we change the notation of $\cdot_1$ and $\cdot_2$ by $[-,-]$ in both cases, the axioms of the Leibniz actions are all possible rewritings of the Leibniz identity when we choose two elements in M and one in N (the first three) or one in M and two N (the last three).*

In particular, we have that the pair $([-,-],[-,-])$ where $[-,-]$ is the Leibniz bracket of the Leibniz K-algebra M is a Leibniz action of M on itself.

Definition 1.8. *A crossed module of Leibniz K-algebras is a 4-tuple $(M, N, \cdot, \partial)$ where M and N are Leibniz K-algebras, $\cdot = (\cdot_1, \cdot_2)$ is a Leibniz action of N on M, $\partial : M \to N$ is a Leibniz K-homomorphism, and the following properties are satisfied:*

- *∂ is an N-equivariant Leibniz K-homomorphism (we suppose that the bracket gives the action in N), i.e. $\partial(n \cdot_1 m) = [n, \partial(m)]$ and $\partial(m \cdot_2 n) = [\partial(m), n]$,*
- *$\partial(m) \cdot_1 m' = [m, m'] = m \cdot_2 \partial(m') \quad m, m' \in M, n \in N \qquad$ (Peiffer identity).*

Example 1.9. *If M is a Leibniz K-algebra then $(M, M, ([-,-],[-,-]), \mathrm{Id}_M)$ is a crossed module of Leibniz K-algebras (see [6] for the Lie case).*

The next propositions give a relation between crossed modules of Lie and Leibniz K-algebras.

Proposition 1.10. *Let M and N be two Lie K-algebras. Then, $\cdot$ is a Lie action of N on M if and only if $(\cdot, \cdot^-)$ is a Leibniz action of N on M, where $\cdot^- : M \times N \to M$ is defined by $m \cdot^- n := -n \cdot m$.*

That is, the Lie action is a particular case of a Leibniz action when the action is "anticommutative".

Proposition 1.11. *Let M and N be Lie K-algebras. Then, $(M, N, \cdot, \partial)$ is a crossed module of Lie K-algebras if and only if $(M, N, (\cdot, \cdot^-), \partial)$ is a crossed module of Leibniz K-algebras.*

Definition 1.12. *Let $(M, N, \cdot, \partial)$ and $(M', N', *, \partial')$ be crossed modules of Leibniz K-algebras. A homomorphism is a pair of Leibniz K-homomorphisms, $f_1 : M \to M'$ and $f_2 : N \to N'$ such that*

$$f_1(n \cdot_1 m) = f_2(n) *_1 f_1(m), \quad f_1(m \cdot_2 n) = f_1(m) *_2 f_2(n),\ n \in N, m \in M, \quad \text{and} \quad \partial' \circ f_1 = f_2 \circ \partial.$$

We will denote by $\boldsymbol{X}(\boldsymbol{LeibAlg}_K)$ the category of crossed modules of Leibniz K-algebras and its homomorphisms.

Remark 1.13. *As in the case of groups and Lie K-algebras, we have an equivalence between the categories $\boldsymbol{X}(\boldsymbol{LeibAlg}_K)$ and $\boldsymbol{ICat}(\boldsymbol{LeibAlg}_K)$. A proof of this can be found in [3].*

$\boldsymbol{X}(\boldsymbol{LieAlg}_K)$ can be seen as a full subcategory of the category $\boldsymbol{X}(\boldsymbol{LeibAlg}_K)$ using Proposition 1.11 (we actually have a functorial isomorphism between a full subcategory of $\boldsymbol{X}(\boldsymbol{LeibAlg}_K)$ and $\boldsymbol{X}(\boldsymbol{LieAlg}_K)$).

Since the pullbacks in $\mathbf{LieAlg}_K$ *and* $\mathbf{LeibAlg}_K$ *are the same, it is immediate to show that* $\mathbf{ICat}(\mathbf{LieAlg}_K)$ *is a full subcategory of* $\mathbf{ICat}(\mathbf{LeibAlg}_K)$*. We know that the equivalence in the Leibniz case generalizes the equivalence in the Lie case. The bracket gives the action in the functors (which was presented in [3]), and then, it is anticommutative when we have Lie K-algebras. We only have to check that the Leibniz semidirect product generalizes the Lie semidirect product, but this is immediate from definition (since* $m \cdot_2 n' = -n' \cdot_1 m$ *is the Lie case).*

Definition 1.14. *Let M and N be two Leibniz K-algebras and* $\cdot$ *a Leibniz action of N on M. The semidirect product, denoted by* $M \rtimes N$*, is the K-vector space* $M \times N$ *with the bracket*

$$[(m,n),(m',n')] := ([m,m'] + n \cdot_1 m' + m \cdot_2 n', [n,n']), \quad m,m' \in M,\ n,n' \in N.$$

2. Braiding for categorical Leibniz algebras and crossed modules of Leibniz algebras

In this section, we will use the idea of Loday and Pirashvili ([11]) to see the Leibniz K-algebras as a particular case of a Lie algebra in the monoidal category of linear maps $\mathcal{LM}_K$, also known as the Loday-Pirashvili category ([4, 12]). Using this, we will try to define the concept of braiding in the case of Leibniz algebras taking advantage of the fact that they will be a particular case of braidings for the corresponding ideas over Lie objects in that category.

First, we will introduce some notation.

Let $\boldsymbol{C}$ be a category with coproducts $\oplus$. If we have $A \xrightarrow{f} C \xleftarrow{g} B$ we denote the unique morphism given by the universal property of the coproduct as $f \boxplus g \colon A \oplus B \to C$.

We let the notation $\oplus$ in morphisms for the coproduct bifunctor and $+$ for the addition in an additive category.

The definition of the category $\mathcal{LM}_K$ can be seen in [11].

Definition 2.1. *The category* $\mathcal{LM}_K$ *is a monoidal category with the following data:*

- *As objects we take the K-linear maps.*
- *If* $\begin{array}{c} M \\ \downarrow{\scriptstyle f} \\ N \end{array}$ *and* $\begin{array}{c} L \\ \downarrow{\scriptstyle g} \\ H \end{array}$ *are two linear maps, a morphism between them is a pair* $\alpha = (\alpha_1, \alpha_2)$, $\alpha_1 \colon M \to L$ *and* $\alpha_2 \colon N \to H$ *of K-linear maps such that the following diagram is commutative:*

$$\begin{array}{ccc} M & \xrightarrow{\alpha_1} & L \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\ N & \xrightarrow{\alpha_2} & H. \end{array}$$

- *The tensor product* $\begin{array}{c} M \\ \downarrow{\scriptstyle f} \\ N \end{array} \otimes \begin{array}{c} L \\ \downarrow{\scriptstyle g} \\ H \end{array} := \begin{array}{c} (M \otimes H) \oplus (N \otimes L) \\ \downarrow{\scriptstyle (f \otimes \mathrm{Id}_H) \boxplus (\mathrm{Id}_N \otimes g)} \\ N \otimes H \end{array}$*, where* $\otimes$ *between K-vector spaces is the usual tensor product and* $\oplus$ *is the direct sum of vector spaces. In morphisms the tensor product is given by* $(f_1, f_2) \otimes (g_1, g_2) = ((f_1 \otimes g_2) \oplus (f_2 \otimes g_1), g_1 \otimes g_2)$.

This is a braided monoidal category with the braiding given by the isomorphism $\mathcal{T}_{f,g} = (\mathcal{T}^1_{f,g}, \mathcal{T}^2_{f,g}) \colon \begin{array}{c} M \\ \downarrow{\scriptstyle f} \\ N \end{array} \otimes \begin{array}{c} L \\ \downarrow{\scriptstyle g} \\ H \end{array} \to \begin{array}{c} L \\ \downarrow{\scriptstyle g} \\ H \end{array} \otimes \begin{array}{c} M \\ \downarrow{\scriptstyle f} \\ N \end{array}$*, with the K-linear isomorphism* $\mathcal{T}^2_{f,g} = \mathcal{T}_{N,H} \colon N \otimes H \to H \otimes N$*, the usual braiding for the K-vector tensor product, and* $\mathcal{T}^1_{f,g} \colon (M \otimes H) \oplus (N \otimes L) \to (L \otimes N) \oplus (H \otimes M)$ *given by* $\mathcal{T}^1_{f,g}((m \otimes h) + (n \otimes l)) = (l \otimes n) + (h \otimes m)$.

Remark 2.2. *$\mathcal{LM}_K$ is an additive category with* $\begin{array}{c}\{0\}\\ \downarrow{\scriptstyle 0}\\ \{0\}\end{array}$ *as zero object, where we have* $\begin{array}{c}M\\ \downarrow{\scriptstyle f}\\ N\end{array} \times \begin{array}{c}L\\ \downarrow{\scriptstyle g}\\ H\end{array} := \begin{array}{c}M\times L\\ \downarrow{\scriptstyle f\times g}\\ N\times H\end{array}$ *as product and the usual abelian group structure on morphisms.*

Analogously to [11], we can categorify the idea of Lie K-algebra and define it in the $\mathcal{LM}_K$ category. We generalize this scheme to a semigroupal category (see [6]).

Definition 2.3. *Let $C = (\boldsymbol{C}, \otimes, a, \mathcal{T})$ be a braided semigroupal category where $\boldsymbol{C}$ is an additive category.*
We say that a pair (A, μ) with $A \in \mathrm{Ob}(\boldsymbol{C})$ and $\mu\colon A \otimes A \to A$ is a Lie object in C if and only if we have that:

$$0 = \mu \circ \mathcal{T}_{A,A} + \mu,$$
$$0 = \mu \circ (\mathrm{Id}_A \otimes \mu) \circ a_{A,A,A} + \mu \circ (\mu \otimes \mathrm{Id}_A) \circ a^{-1}_{A,A,A} \circ (\mathrm{Id}_A \otimes \mathcal{T}_{A,A}) \circ a_{A,A,A} - \mu \circ (\mu \otimes \mathrm{Id}_A).$$

We will say that a morphism $f\colon (A, \mu_A) \to (B, \mu_B)$ is a Lie morphism if it satisfies the following diagram:

$$\begin{array}{ccc} A \otimes A & \xrightarrow{\mu_A} & A \\ \big\downarrow{\scriptstyle f\otimes f} & & \big\downarrow{\scriptstyle f} \\ B \otimes B & \xrightarrow{\mu_B} & B. \end{array}$$

*So, we have the category **Lie**(C).*

Example 2.4. *Let (V, μ) be a Lie object. If we take in $\mathbf{Vect}_K$ the usual tensor product, then $\mu\colon V \otimes V \to V$ can be seen as a K-bilinear map, $\mu(a, b) =: [a, b]$, which satisfies*

$$[a, b] = -[b, a] \qquad \text{and} \qquad [a, [b, c]] + [[a, c], b] - [[a, b], c] = 0.$$

The first is the anticommutativity and the second one is the Leibniz identity.
It is clear that, if $\mathrm{char}(K) \neq 2$, $\boldsymbol{Lie}(\mathbf{Vect}_K)$ is isomorphic to $\mathbf{LieAlg}_K$.

Since the generalization is only true for $\mathrm{char}(K) \neq 2$, we will assume it for the rest of the paper.
We want to explain what are an object and a morphism in $\boldsymbol{Lie}(\mathcal{LM}_K)$.

Definition 2.5. *Let M and N be Lie K-algebras and $\alpha\colon M \to N$ a Lie K-homomorphism.*

Let $(V, \cdot)$ be a right (resp. left) M-module and $(W, *)$ a right (resp. left) N-module. A K-linear map $V \xrightarrow{f} W$ is $(\alpha\colon M \to N, \cdot, *)$-equivariant if we have that

$$f(v \cdot m) = f(v) * \alpha(m) \quad (\text{resp. } f(m \cdot v) = \alpha(m) * f(v)), \qquad \text{for } v \in V,\ m \in M.$$

When $N = M$ and $\alpha = \mathrm{Id}_M$ we said that f is $(M, \cdot, *)$-equivariant.

Let $(V, \cdot)$ be a left M-module and $(W, *)$ a right N-module. A K-linear map $V \xrightarrow{f} W$ is $(\alpha\colon M \to N, \cdot, *)$-equivariant if we have that

$$f(m \cdot v) = -f(v) * \alpha(m), \quad \text{for } v \in V,\ m \in M.$$

When $N = M$ and $\alpha = \mathrm{Id}_M$ we said that f is $(M, \cdot, *)$-equivariant.

Remark 2.6. *It is easy to check that if $(M, [-, -])$ is a Lie K-algebra, then $(M, [-, -])$ is a right and left $(M, [-, -])$-module.*
Moreover, if $\cdot$ is a Lie action of N in M, we have that $(M, \cdot)$ is a left N-module.

Using this, we can see in [11] that a Lie object in $\mathcal{LM}_K$ is the following data:

Definition 2.7. *A Lie object in $\mathcal{LM}_K$ is a triple $\left(\begin{array}{c} M \\ \downarrow f \\ N \end{array}, *_N^M, [-,-]_N\right)$ where*

- $(N, [-,-]_N)$ *is a Lie K-algebra.*
- $*_N^M \colon M \times N \to M$ *is such that* $(M, *_N^M)$ *is an* $(N, [-,-]_N)$*-module.*
- f *is* $((N, [-,-]_N), *_N^M, [-,-]_N)$*-equivariant.*

As in the case of Lie K-algebras, we will denote a Lie object in $\mathcal{LM}_K$ using the K-linear map on which it is defined when there is no confusion.

Remark 2.8. *The "anticommutative" property of Lie object for $\mathcal{LM}_K$ allows to recover the Lie product $\mu = (\mu_1, \mu_2)$ for $\begin{array}{c} M \\ \downarrow f \\ N \end{array}$ with the maps $\mu_2 = [-,-]_N$ and $\mu_1 \colon (M \otimes N) \oplus (N \otimes M) \to M$, with $\mu_1((m \otimes n) + (n' \otimes m')) = m *_N^M n - m' *_N^M n'$.*

Definition 2.9. *Let $\begin{array}{c} M \\ \downarrow f \\ N \end{array}$ and $\begin{array}{c} L \\ \downarrow g \\ H \end{array}$ be Lie objects. A Lie morphism in $\mathcal{LM}_K$ between them is an $\mathcal{LM}_K$ morphism (α_1, α_2) such that:*

- $\alpha_2 \colon N \to H$ *is a Lie K-homomorphism.*
- $\alpha_1 \colon M \to L$ *is an* $(\alpha_2 \colon N \to H, *_N^M, *_H^L)$*-equivariant map.*

In [11] is shown a way to see the Leibniz K-algebras as a particular case of Lie objects in $\mathcal{LM}_K$. We show it in the next example.

Example 2.10. *Let M be a Leibniz K-algebra.*

We denote for I_M the ideal generated by elements of the form $[x, x]$ with $x \in M$. It is evident that the quotient Leibniz K-algebra is a Lie K-algebra. We will denote its Lie bracket as $\overline{[-,-]}$, and the elements of the quotient as $\overline{m}$ with $m \in M$.

$\mathrm{Lie}(M) := \frac{M}{I_M}$ *is known as Lieization (note that if M is a Lie K-algebra, then* $\mathrm{Lie}(M)$ *is trivially naturally isomorphic to M), and it is functorial.*

We consider the following Lie object in $\mathcal{LM}_K$:

We take $\begin{array}{c} M \\ \downarrow \pi_M \\ \mathrm{Lie}(M) \end{array}$ where $\pi(m) = \overline{m}$ is the natural map. It is a Lie object in $\mathcal{LM}_K$ with the following data:

- $m *_{\mathrm{Lie}(M)}^M \overline{m'} = [m, m'],$
- $[\overline{m}, \overline{m'}]_{\mathrm{Lie}(M)} = \overline{[\overline{m}, \overline{m'}]} := [m, m'].$

It is evident that π is $(\mathrm{Lie}(M), *_{\mathrm{Lie}(M)}^M, [-,-]_{\mathrm{Lie}(M)})$-equivariant.

So, we have a functor $\Phi \colon \boldsymbol{LeibAlg}_K \to \boldsymbol{Lie}(\mathcal{LM}_K)$, that is trivially full. This functor is also injective on objects and morphisms, because there is a functor $\Psi \colon \boldsymbol{Lie}(\mathcal{LM}_K) \to \boldsymbol{LeibAlg}_K$ such that $\Psi \circ \Phi = \mathrm{Id}_{\boldsymbol{LeibAlg}_K}$ (see [11]). The functor Ψ on objects is described in the following proposition.

Proposition 2.11 ([11]). *Let $\begin{array}{c} M \\ \downarrow f \\ N \end{array}$ be a Lie object in $\mathcal{LM}_K$. Then $(M, [-,-])$, where $[m, m'] := m *_N^M f(m')$, is a Leibniz K-algebra.*

In [4], we can see that the previous construction can be extended to crossed modules of Lie algebras in $\mathcal{LM}_K$. They did a crossed module with a right action. In this paper, we will define which is a crossed module with a left action, or simply a crossed module of Lie objects.

Definition 2.12. *Let $C = (\boldsymbol{C}, \otimes, a, \mathcal{T})$ be a braided semigroupal category where $\boldsymbol{C}$ is an additive category.*

If (A, μ_A) and (B, μ_B) are Lie objects, then a (left) Lie action of (B, μ_B) on (A, μ_A) is a morphism $p\colon B \otimes A \to A$ such that

$$p \circ (\mu_B \otimes \mathrm{Id}_A) = p \circ (\mathrm{Id}_B \otimes p) \circ a_{B,B,A} \circ (\mathrm{Id}_{(B\otimes B)\otimes A} - (\tau_{B,B} \otimes \mathrm{Id}_A)),$$
$$p \circ (\mathrm{Id}_B \otimes \mu_A) \circ a_{B,A,A} = \mu_A \circ (p \otimes \mathrm{Id}_A) \circ (\mathrm{Id}_{(B\otimes B)\otimes A} - (a^{-1}_{B,A,A} \circ (\mathrm{Id}_B \otimes \tau_{A,A}) \circ a_{B,A,A})).$$

We said that $((A, \mu_A), (B, \mu_B), p, \partial)$ is a crossed module of Lie objects if p is a Lie action of (B, μ_B) on (A, μ_A) and $\partial\colon (A, \mu_A) \to (B, \mu_B)$ is a Lie morphism such that

$$\partial \circ p = \mu_B \circ (\mathrm{Id}_B \otimes \partial),$$
$$\mu_A = p \circ (\partial \otimes \mathrm{Id}_A).$$

A morphism between two crossed modules of Lie objects $((A, \mu_A), (B, \mu_B), p, \partial)$ and $((C, \mu_C), (D, \mu_D), q, \delta)$ is a pair of Lie morphisms (α, β), $\alpha\colon (A, \mu_A) \to (C, \mu_C)$ and $\beta\colon (B, \mu_B) \to (D, \mu_D)$, which satisfies the following diagrams:

$$\begin{array}{ccc} B \otimes A & \xrightarrow{p} & A \\ \downarrow{\scriptstyle \beta\otimes\alpha} & & \downarrow{\scriptstyle \alpha} \\ D \otimes C & \xrightarrow{q} & C, \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{\partial} & B \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} \\ C & \xrightarrow{\delta} & D. \end{array}$$

We have the category $\boldsymbol{XLie}(C)$ with the usual composition in $\boldsymbol{C} \times \boldsymbol{C}$ for pairs of morphisms of Lie morphisms.

Example 2.13. *We have that $\boldsymbol{XLie}(\boldsymbol{Vect}_K)$ and $\boldsymbol{X}(\boldsymbol{LieAlg}_K)$ are isomorphic categories with the usual tensor product in $\boldsymbol{Vect}_K$ (we assume $\mathrm{char}(K) \neq 2$).*

Now, we describe the category $\boldsymbol{XLie}(\mathcal{LM}_K)$.

Definition 2.14. *Let $\begin{smallmatrix} M \\ \downarrow f \\ N \end{smallmatrix}$ and $\begin{smallmatrix} L \\ \downarrow g \\ H \end{smallmatrix}$ be Lie objects in $\mathcal{LM}_K$. A (left) Lie action of $\begin{smallmatrix} L \\ \downarrow g \\ H \end{smallmatrix}$ on $\begin{smallmatrix} M \\ \downarrow f \\ N \end{smallmatrix}$ in $\mathcal{LM}_K$ is a triple $\bar{\cdot} = (\cdot_1, \cdot_2, \xi_\cdot)$ where*

- *$\cdot_1\colon H \times M \to M$ is a K-bilinear map such that $(M, \cdot_1)$ is a left H-module;*
- *$\cdot_2\colon H \times N \to N$ is a Lie action of H on N;*
- *$\xi_\cdot\colon L \times N \to M$ is a K-bilinear map;*

such that the following properties are satisfied:

- *$\cdot_1$ and $\cdot_2$ are compatible actions with $*^M_N$. That is, for $h \in H$, $n \in N$, $m \in M$, we have*

$$h \cdot_1 (m *^M_N n) = (h \cdot_1 m) *^M_N n + m *^M_N (h \cdot_2 n);$$

- *f is an $(H, \cdot_1, \cdot_2)$-equivariant map;*
- *$\xi_\cdot$ satisfies, for $l \in L$, $n, n' \in N$, $h \in H$, the following equalities*

$$f(\xi_\cdot(l, n)) = g(l) \cdot_2 n,$$
$$\xi_\cdot(l *^L_H h, n) = \xi_\cdot(l, h \cdot_2 n) - h \cdot_1 \xi_\cdot(l, n),$$
$$\xi_\cdot(l, [n, n']_N) = \xi_\cdot(l, n) *^M_N n' - \xi_\cdot(l, n') *^M_N n.$$

Remark 2.15. *An action is, in fact, a pair* $\bar{\cdot} = (\bar{\cdot}_1, \bar{\cdot}_2)$*, with the two maps*

$$\bar{\cdot}_1 : (L \otimes N) \oplus (H \otimes N) \to M \qquad \text{and} \qquad \bar{\cdot}_2 : H \otimes N \to N$$

satisfying the general properties, but we can easily obtain the previous definition taking $\cdot_2 := \bar{\cdot}_2$ *and recovering* $\bar{\cdot}_1((l \otimes n) + (h \otimes m)) =: \xi_{\bar{\cdot}}(l, n) + h \cdot_1 m$.

Definition 2.16. *A crossed module of Lie objects in* $\mathcal{LM}_K$ *is a 4-tuple* $\left(\begin{smallmatrix} M \\ \downarrow f \\ N \end{smallmatrix}, \begin{smallmatrix} L \\ \downarrow g \\ H \end{smallmatrix}, \bar{\cdot}, \partial\right)$ *where* $\begin{smallmatrix} M \\ \downarrow f \\ N \end{smallmatrix}$ *and* $\begin{smallmatrix} L \\ \downarrow g \\ H \end{smallmatrix}$ *are Lie objects in* $\mathcal{LM}_K$*,* $\bar{\cdot}$ *is a Lie action of* $\begin{smallmatrix} L \\ \downarrow g \\ H \end{smallmatrix}$ *on* $\begin{smallmatrix} M \\ \downarrow f \\ N \end{smallmatrix}$*, and* $\partial = (\partial_1, \partial_2) \colon \begin{smallmatrix} M \\ \downarrow f \\ N \end{smallmatrix} \to \begin{smallmatrix} L \\ \downarrow g \\ H \end{smallmatrix}$ *is a Lie morphism in* $\mathcal{LM}_K$ *such that*

- $(N, H, \cdot_2, \partial_2)$ *is a crossed module of Lie K-algebras;*
- ∂_1 *is an* $(H, \cdot_1, *^L_H)$*-equivariant map;*
- $\partial_1(\xi_{\bar{\cdot}}(l, n)) = l *^L_N \partial_2(h)$ *and* $\xi_{\bar{\cdot}}(\partial_1(m), n) = m *^M_N n = -\partial_2(n) \cdot_1 m$, $\quad h \in H, l \in L, m \in M, n \in N$.

Definition 2.17. *Let* $\left(\begin{smallmatrix} M \\ \downarrow f \\ N \end{smallmatrix}, \begin{smallmatrix} L \\ \downarrow g \\ H \end{smallmatrix}, \bar{\cdot}, \partial\right)$ *and* $\left(\begin{smallmatrix} X \\ \downarrow k \\ Y \end{smallmatrix}, \begin{smallmatrix} V \\ \downarrow h \\ W \end{smallmatrix}, \bar{\star}, \delta\right)$ *be crossed modules of Lie objects in* $\mathcal{LM}_K$*. A morphism of crossed modules of Lie objects in* $\mathcal{LM}_K$ *is a pair* (α, β) *of Lie morphisms* $\alpha = (\alpha_1, \alpha_2) \colon \begin{smallmatrix} M \\ \downarrow f \\ N \end{smallmatrix} \to \begin{smallmatrix} X \\ \downarrow k \\ Y \end{smallmatrix}$ *and* $\beta = (\beta_1, \beta_2) \colon \begin{smallmatrix} L \\ \downarrow g \\ H \end{smallmatrix} \to \begin{smallmatrix} V \\ \downarrow h \\ W \end{smallmatrix}$ *such that*

- $(\alpha_2, \beta_2) \colon (N, H, \cdot_2, \partial_2) \to (Y, W, \star_2, \delta_2)$ *is an homomorphism of crossed modules of Lie K-algebras;*
- $\alpha_1(\xi_{\bar{\cdot}}(l, n)) = \xi_{\bar{\star}}(\beta_1(l), \alpha_2(n))$*, for* $l \in L$*,* $n \in N$*;*
- α_1 *is an* $(H \xrightarrow{\beta_2} W, \cdot_1, \star_1)$*-equivariant map;*
- $\beta_1 \circ \partial_1 = \delta_1 \circ \alpha_1$.

As in the case of Leibniz K-algebras we want to have a pair of functors between the categories $\boldsymbol{XLie}(\mathcal{LM}_K)$ and $\boldsymbol{XLeibAlg}_K$. For this purpose, we give the following propositions of which we omit their proofs because they are immediate. The first is symmetrical to the construction we can see in [4] for crossed modules with right actions.

Proposition 2.18. *Let* $(M, N, (\cdot_1, \cdot_2), \partial)$ *be a crossed module of Leibniz K-algebras.*

Then $\left(\begin{smallmatrix} M \\ \downarrow \pi_M \\ \frac{M}{[M,N]_x} \end{smallmatrix}, \begin{smallmatrix} N \\ \downarrow \pi_N \\ \mathrm{Lie}(N) \end{smallmatrix}, \bar{\bar{\cdot}}, \bar{\partial}\right)$ *is a crossed module of Lie objects in* $\mathcal{LM}_K$*, where*

- $\frac{M}{[M,N]_x}$ *is the Lie K-algebra quotient of M by the ideal* $[M, N]_x$ *whose generators are* $[m, m]$ *for* $m \in M$ *and* $n \cdot_1 m + m \cdot_2 n$ *for* $n \in N$*,* $m \in M$*; we denote the natural map by* $\pi_M \colon M \to \frac{M}{[M,N]_x}$*, and the elements of* $\frac{M}{[M,N]_x}$ *by* $\overline{m}$*,*
- $\bar{\cdot}_1 \colon \mathrm{Lie}(N) \times M \to M$, $(\overline{n}, m) \mapsto -m \cdot_2 n$,
- $\bar{\cdot}_2 \colon \mathrm{Lie}(N) \times \frac{M}{[M,N]_x} \to \frac{M}{[M,N]_x}$, $(\overline{n}, \overline{m}) \mapsto \overline{n \cdot_1 m} = \overline{-m \cdot_2 n}$,
- $\xi_{\bar{\cdot}} \colon N \times \frac{M}{[M,N]_x} \to M$, $(n, \overline{m}) \to n \cdot_1 m$,

- $\overline{\partial}_1 \colon M \to N,\ m \mapsto \partial(m)$,
- $\overline{\partial}_2 \colon \frac{M}{[M,N]_x} \to \mathrm{Lie}(N),\ \overline{m} \mapsto \overline{\partial m}$.

Remark 2.19. *We will say that the bottom part* $(\frac{M}{[M,N]_x}, \mathrm{Lie}(N), \bar{\cdot}_2, \overline{\partial}_2)$ *is the Lieization of the crossed module of Leibniz K-algebras. In this way we found a similar relation with the Leibniz and Lie object case.*

This Lieization satisfies again that applied on a crossed module of Lie K-algebras, thought as a crossed module of Leibniz K-algebras with the action $(\cdot, \cdot^-)$, *is naturally isomorphic to itself. That occurs because, in the quotient, the second generators are null too:*

$$n \cdot_1 m + m \cdot_2 n = n \cdot m + m \cdot^- n = n \cdot m - n \cdot m = 0.$$

Proposition 2.20. *Let* $(\begin{smallmatrix} M \\ \downarrow_f \\ N \end{smallmatrix}, \begin{smallmatrix} L \\ \downarrow_g \\ H \end{smallmatrix}, \bar{\cdot}, \partial)$ *be a crossed module of Lie objects in* $\mathcal{LM}_K$, *then* $(M, L, (\tilde{\cdot}_1, \tilde{\cdot}_2), \partial_1)$ *is a crossed module of Leibniz K-algebras, where*

- *The Leibniz brackets are:* $[m, m'] = m *^M_N f(m')$ *for* $m, m' \in M$ *and* $[l, l'] = l *^L_H g(l')$ *for* $l, l' \in M$;
- $\tilde{\cdot}_1 \colon L \times N \to M$ *is defined by* $l \tilde{\cdot}_1 m = \xi.(l, f(m))$ *for* $l \in L$, $m \in M$;
- $\tilde{\cdot}_2 \colon M \times L \to M$ *is defined by* $m \tilde{\cdot}_2 l = -g(l) \cdot_1 m$ *for* $l \in L$, $m \in M$.

We have the functors $\boldsymbol{X(LeibAlg_K)} \underset{X\Psi}{\overset{X\Phi}{\rightleftarrows}} \boldsymbol{XLie}(\mathcal{LM}_K)$ satisfying $X\Psi \circ X\Phi = \mathrm{Id}_{\boldsymbol{X(LeibAlg_K)}}$, and so, the functor $X\Phi$ is a full inclusion functor.

2.1. Braiding for Crossed modules of Lie objects in $\mathcal{LM}_K$ and crossed modules of Leibniz algebras

We want to define the notion of braiding for crossed modules of Leibniz algebras. We will use the idea that the braiding for crossed module of Leibniz K-algebras must be a particular case of braiding for Lie objects in $\mathcal{LM}_K$, satisfying symmetrical properties to the previous ones.

Definition 2.21. *Let* $\mathcal{C} = (\boldsymbol{C}, \otimes, a, \mathcal{T})$ *be a braided semigroupal category where* $\boldsymbol{C}$ *is an additive category. Let* $\mathcal{X} = ((A, \mu_A), (B, \mu_B), p, \partial)$ *be a crossed module of Lie objects in* $\boldsymbol{C}$.

A braiding (or Peiffer lifting) on $\mathcal{X}$ *is a morphism* $\mathfrak{T} \colon B \otimes B \to A$ *satisfying:*

$$\begin{aligned}
\partial \circ \mathfrak{T} &= \mu_B, \\
\mathfrak{T} \circ (\partial \otimes \partial) &= \mu_A, \\
-\mathfrak{T} \circ (\partial \otimes \mathrm{Id}_B) &= p \circ \mathcal{T}_{A,B}, \\
\mathfrak{T} \circ (\mathrm{Id}_B \otimes \partial) &= p, \\
\mathfrak{T} \circ (\mathrm{Id}_B \otimes \mu_B) \otimes a_{B,B,B} &= \mathfrak{T} \circ (\mu_B \otimes \mathrm{Id}_B) \circ (\mathrm{Id}_{(B \otimes B) \otimes B} - (a^{-1}_{B,B,B} \circ (\mathrm{Id}_B \otimes \mathcal{T}_{B,B}) \circ a_{B,B,B})), \\
\mathfrak{T} \circ (\mu_B \otimes \mathrm{Id}_B) &= \mathfrak{T} \circ (\mathrm{Id}_B \otimes \mu_B) \circ a_{B,B,B} \circ (\mathrm{Id}_{(B \otimes B) \otimes B} - (\mathcal{T}_{B,B} \otimes \mathrm{Id}_B)).
\end{aligned}$$

$((A, \mu_A), (B, \mu_B), p, \partial, \mathfrak{T})$ *will be called a braided crossed module of Lie objects in* $\boldsymbol{C}$.

A morphism $(\alpha, \beta) \colon ((A, \mu_A), (B, \mu_B), p, \partial, \mathfrak{T}) \to ((C, \mu_C), (D, \mu_D), q, \delta, \mathfrak{Y})$ *of braided crossed modules of Lie objects is a morphism of crossed modules of Lie objects in the category* $\boldsymbol{C}$ *satisfying the following commutative diagram*

$$\begin{array}{ccc}
B \otimes B & \xrightarrow{\ \mathfrak{T}\ } & A \\
\big\downarrow{\scriptstyle \beta \otimes \beta} & & \big\downarrow{\scriptstyle \alpha} \\
D \otimes D & \xrightarrow{\ \mathfrak{Y}\ } & B.
\end{array}$$

We denote this new category as $\boldsymbol{BXLie}(\boldsymbol{C})$.

Example 2.22. *As in the previous cases, we have that* $\boldsymbol{BXLie}(\boldsymbol{Vect}_K)$ *and* $\boldsymbol{BX}(\boldsymbol{LieAlg}_K)$ *are isomorphic, taking in* $\boldsymbol{Vect}_K$ *the usual tensor product.*

$\boldsymbol{BXLie}(\mathcal{LM}_K)$ is described in the following definitions.

Definition 2.23. *Let* $\mathcal{X} = (\begin{smallmatrix} M \\ \downarrow_f \\ N \end{smallmatrix}, \begin{smallmatrix} L \\ \downarrow_g \\ H \end{smallmatrix}, \bar{\cdot}, \partial)$ *be a crossed module of Lie objects in* $\mathcal{LM}_K$.

A braiding (or Peiffer lifting) for $\mathcal{X}$ *is given by a triple of maps* $T_{\{-,-\}} = (\{-,-\}_{LH}, \{-,-\}_{HL}, \{-,-\}_2)$ *where*

- $\{-,-\}_2 \colon H \times H \to N$ *is a K-bilinear map such that* $(N, H, \cdot_2, \partial_2, \{-,-\}_2)$ *is a braided crossed module of Lie K-algebras.*
- $\{-,-\}_{LH} \colon L \times H \to M$ *and* $\{-,-\}_{HL} \colon H \times L \to M$ *are K-bilinear maps, which with* $\{-,-\}_2$ *satisfy the following properties for* $l \in L$, $h, h' \in H$, $m \in M$, $n \in N$:

$$
\begin{aligned}
f(\{l,h\}_{LH}) &= \{g(l),h\}_2, & f(\{h,l\}_{HL}) &= \{h,g(l)\}_2,\\
\partial_1\{l,h\}_{LH} &= l *^L_H h, & \partial_1\{h,l\}_{HL} &= -l *^L_H h,\\
\{\partial_1(m),\partial_2(n)\}_{LH} &= m *^M_N n, & \{\partial_2(n),\partial_1(m)\}_{HL} &= -m *^M_N n,\\
\{\partial_1(m),h\}_{LH} &= -h \cdot_1 m, & \{\partial_2(n),l\}_{HL} &= -\xi_{\cdot}(l,n),\\
\{l,\partial_2(n)\} &= \xi_{\cdot}(l,n), & \{h,\partial_1(m)\} &= h \cdot_1 m,\\
\{l,[h,h']_H\}_{LH} &= \{l *^L_H h, h'\}_{LH} - \{l *^L_H h', h\}_{LH}, &&\\
\{[h,h']_H,l\}_{HL} &= -\{h, l *^L_H h'\}_{HL} - \{l *^L_H h, h'\}_{LH}, &&\\
\{l,[h,h']_H\}_{LH} &= \{l *^L_H h, h'\}_{LH} + \{h, l *^L_H h'\}_{HL}, &&\\
\{[h,h']_H,l\}_{HL} &= -\{h, l *^L_H h'\}_{HL} + \{h', l *^L_H h\}_{HL}. &&
\end{aligned}
$$

We will say that $(\begin{smallmatrix} M \\ \downarrow_f \\ N \end{smallmatrix}, \begin{smallmatrix} L \\ \downarrow_g \\ H \end{smallmatrix}, \bar{\cdot}, \partial, T_{\{-,-\}})$ *is a braided crossed module of Lie objects in* $\mathcal{LM}_K$.

Remark 2.24. *A braiding is a pair* $T_{\{-,-\}} = (T^1_{\{-,-\}}, T^2_{\{-,-\}})$, *but for simplicity we denote* $T^1_{\{-,-\}} \colon (L \otimes H) \oplus (H \otimes L) \to M$ *with* $T^1_{\{-,-\}}((l \otimes h) + (h' \otimes l')) = \{l,h\}_{LH} + \{h',l'\}_{HL}$ *and* $T^2_{\{-,-\}}(h,h') = \{h,h'\}_2$.

Definition 2.25. *Let* $(\begin{smallmatrix} M \\ \downarrow_f \\ N \end{smallmatrix}, \begin{smallmatrix} L \\ \downarrow_g \\ H \end{smallmatrix}, \bar{\cdot}, \partial, T_{\{-,-\}})$ *and* $(\begin{smallmatrix} X \\ \downarrow_k \\ Y \end{smallmatrix}, \begin{smallmatrix} V \\ \downarrow_h \\ W \end{smallmatrix}, \bar{\star}, \delta, T_{\{-,-\}'})$ *be braided crossed modules of Lie objects in* $\mathcal{LM}_K$. *A morphism of braided crossed modules of Lie objects in* $\mathcal{LM}_K$ *is a morphism* (α, β) *of crossed modules of Lie objects in* $\mathcal{LM}_K$ *satisfying:*

- $(\alpha_2, \beta_2) \colon (N, H, \cdot_2, \partial_2, \{-,-\}_2) \to (Y, W, \star_2, \delta_2, \{-,-\}'_2)$ *is an homomorphism of braided crossed modules of Lie K-algebras,*
- $\alpha_1(\{l,h\}_{LH}) = \{\beta_1(l), \beta_2(h)\}'_{VW}$, *for* $l \in L$, $h \in H$,
- $\alpha_1(\{h,l\}_{HL}) = \{\beta_2(h), \beta_1(l)\}'_{WV}$, *for* $l \in L$, $h \in H$.

We want to use the concept of braiding on crossed modules of Lie objects in $\mathcal{LM}_K$ to obtain a definition for crossed modules of Leibniz K-algebras. For that, we will take a braiding on $(\begin{smallmatrix} M \\ \downarrow_{\pi_M} \\ \frac{M}{[M,N]_X} \end{smallmatrix}, \begin{smallmatrix} N \\ \downarrow_{\pi_N} \\ \mathrm{Lie}(N) \end{smallmatrix}, \bar{\bar{\cdot}}, \bar{\partial})$. If we try to take one K-bilinear map $\{-,-\}$ we would find problems with the way of defining the corresponding maps because we have that the first properties add one more quotient that we would like to be trivial for

Lie K-algebras, or if we take it to be trivial, the rest of properties prevent it from being made for the general case of Leibniz K-algebras (if we take $\{n,\overline{n'}\}_{N\,\mathrm{Lie}(N)} = \{n,n'\} = \{\overline{n},n'\}_{\mathrm{Lie}(N)N}$ for example, the third and fourth property leads us to prove that M must be Lie K-algebra).

For this, as in the case of the two actions, we will take for braiding two K-bilinear maps $\{-,-\},\langle -,-\rangle\colon N\times N \to M$, and define $\{n,\overline{n'}\}_{N\,\mathrm{Lie}(N)} = \{n,n'\}$, $\{\overline{n},n'\}_{\mathrm{Lie}(N)N} = -\langle n',n\rangle$ and $\{\overline{n},\overline{n'}\}_2 = \overline{\{n,n'\}} = \overline{-\langle n',n\rangle}$, where we can see that we introduce a new quotient in M.

Definition 2.26. *Let $\mathcal{X} = (M,N,(\cdot_1,\cdot_2),\partial)$ be a crossed module of Leibniz K-algebras.*

A braiding (or Peiffer lifting) on $\mathcal{X}$ is a pair $(\{-,-\},\langle -,-\rangle)$ of K-bilinear maps $\{-,-\},\langle -,-\rangle\colon N\times N\to M$, $(n,n')\mapsto\{n,n'\}$ and $(n,n')\mapsto\langle n,n'\rangle$, satisfying:

$$\partial\{n,n'\} = [n,n'] = \partial\langle n,n'\rangle, \tag{BLeib1}$$
$$\{\partial m,\partial m'\} = [m,m'] = \langle\partial m,\partial m'\rangle, \tag{BLeib2}$$
$$\{\partial m,n\} = m\cdot_2 n = \langle\partial m,n\rangle, \tag{BLeib3}$$
$$\{n,\partial m\} = n\cdot_1 m = \langle n,\partial m\rangle, \tag{BLeib4}$$
$$\{n,[n',n'']\} = \{[n,n'],n''\}-\{[n,n''],n'\}, \tag{BLeib5}$$
$$\langle n,[n',n'']\rangle = \{[n,n'],n''\}-\langle[n,n''],n'\rangle, \tag{BLeib6}$$
$$\{n,[n',n'']\} = \{[n,n'],n''\}-\langle[n,n''],n'\rangle, \tag{BLeib7}$$
$$\langle n,[n',n'']\rangle = \langle[n,n'],n''\rangle-\langle[n,n''],n'\rangle, \qquad n,n',n''\in N,\ m,m'\in M. \tag{BLeib8}$$

In this case, we say that $(M,N,(\cdot_1,\cdot_2),\partial,(\{-,-\},\langle -,-\rangle))$ is a braided crossed module of Leibniz K-algebras.

Definition 2.27. *An homomorphism (f_1,f_2) of braided crossed modules of Leibniz K-algebras*

$$(M,N,(\cdot_1,\cdot_2),\partial,(\{-,-\},\langle -,-\rangle)) \xrightarrow{(f_1,f_2)} (M',N',(*_1,*_2),\partial',(\{-,-\}',\langle -,-\rangle'))$$

is an homomorphism between the corresponding crossed modules of Leibniz K-algebras satisfying:

$$f_1(\{n,n'\}) = \{f_2(n),f_2(n')\}', \tag{LeibHB1}$$
$$f_1(\langle n,n'\rangle) = \langle f_2(n),f_2(n')\rangle', \qquad n,n'\in N. \tag{LeibHB2}$$

We denote the category of braided crossed modules of Leibniz K-algebras and its homomorphisms by $\boldsymbol{BX}(\boldsymbol{LeibAlg}_K)$.

We want to know how to introduce the braided crossed modules of Lie K-algebras as a particular case. The next two properties answer this question:

Proposition 2.28. *Let $(M,N,(\cdot_1,\cdot_2),\partial,(\{-,-\},\langle -,-\rangle))$ be a braided crossed module of Leibniz K-algebras.*

If for all $n,n'\in N$ it is satisfied that $\{n,n'\} = -\langle n',n\rangle$, then we have the following properties:

- $m\cdot_2 n = -n\cdot_1 m$.
- $(M,N,\cdot_1,\partial,\{-,-\})$ *is a braided crossed module of Lie K-algebras.*

Proof. We will check first that M and N are Lie K-algebras.

By using (BLeib1), we have that for all $n,n'\in N$, $\partial\langle n,n'\rangle = [n,n']$. Then, if we use that $\langle n,n'\rangle = -\{n',n\}$ we obtain, again for (BLeib1):

$$[n,n] = \partial\langle n,n'\rangle = -\partial\{n',n\} = -[n,n'].$$

We conclude that N is a Lie K-algebra (we are working in a field of $\mathrm{char}(K)\neq 2$).

Now we take $m,m'\in M$. By (BLeib2) we have that $\langle\partial m,\partial m'\rangle = [m,m']$. Using again $\langle\partial m,\partial m'\rangle = -\{\partial m',\partial m\}$ and (BLeib2) we have

$$[m,m'] = \langle\partial m,\partial m'\rangle = -\{\partial m',\partial m\} = -[m',m].$$

We will check that $m \cdot_2 n = -n \cdot_1 m$, for $m \in M$, $n \in N$. We have

$$m \cdot_2 n = \langle \partial m, n \rangle = -\{n, \partial m\} = -n \cdot_1 m,$$

where we used (BLeib3) in the first equality and (BLeib4) in the third.

Now, we know that $(M, N, \cdot_1, \partial)$ is a crossed module of Lie K-algebras using Proposition 1.11.

We will prove the equivalences for the axioms of braiding.

The first equality of properties (BLeib1)–(BLeib4) coincides, respectively, with (BLie1)–(BLie4) (in the case of (BLeib3) remember that $m \cdot_2 n = -n \cdot_1 m$).

The second identity of (BLeib1) and (BLeib2) is immediate because of the anticommutativity of the bracket, while the second (BLeib3) is equivalent to (BLie4) and the second equality of (BLeib4) is to (BLie3) (again using that $n \cdot_1 m = -m \cdot_2 n$).

It is clear that (BLeib5) and (BLie5) are identical, and it is straightforward to prove that (BLeib8) is equivalent to (BLie6).

To see the last equivalences, we must prove an earlier property, which is satisfied for both braidings under our assumptions:

If $n, n', n'' \in N$, then $\{[n, n'], n''\} = -\{n'', [n, n']\}$.

We will start in the Lie case (we suppose we have an action $\cdot$).

$$\{[n, n'], n''\} = \{\partial\{n, n'\}, n''\} = -n'' \cdot \{n, n'\} = -\{n'', \partial\{n, n'\}\} = -\{n'', [n, n']\},$$

where we use (BLie1), (BLie3) and (BLie4).

In the Leibniz case, it is not true in general, because we need $m \cdot_2 n = -n \cdot_1 m$.

$$\{[n, n'], n''\} = \{\partial\{n, n'\}, n''\} = \{n, n'\} \cdot_2 n'' = -n'' \cdot_1 \{n, n'\} = -\{n'', \partial\{n, n'\}\} = -\{n'', [n, n']\},$$

where we use (BLeib1), (BLeib3) and (BLeib4).

With this property we can prove that (BLeib6) is equivalent to (BLie6), and (BLeib7) is equivalent to (BLie5). In particular $(M, N, \cdot_1, \partial\{-,-\})$ is a braided crossed module of Lie K-algebras. □

The next two propositions are immediate, and the second one gives the construction of the functor.

Proposition 2.29. *Let M and N be Lie K-algebras. Then, $(M, N, \cdot, \partial, \{-,-\})$ is a crossed module of Lie K-algebras if and only if $(M, N, (\cdot, \cdot^-), \partial, (\{-,-\}, \{-,-\}^-))$ is a crossed module of Leibniz K-algebras.*

$\cdot^- \colon M \times N \to N$ and $\{-,-\}^- \colon N \times N \to M$ are defined as $m \cdot^- n = -n \cdot m$ and $\{n, n'\}^- = -\{n', n\}$.

Proposition 2.30. *Let $(M, N, (\cdot_1, \cdot_2), \partial, (\{-,-\}, \langle -,- \rangle))$ be a braided crossed module of Leibniz K-algebras. Then $\left(\begin{array}{c} M \\ \downarrow^{\pi_M} \\ \frac{M}{\{M,N\}_x} \end{array}, \begin{array}{c} N \\ \downarrow^{\pi_N} \\ \mathrm{Lie}(N) \end{array}, \bar{\bar{\cdot}}, \bar{\partial}, (\{-,-\}_{N\,\mathrm{Lie}(N)}, \{-,-\}_{\mathrm{Lie}(N)N}, \{-,-\}_2) \right)$ is a braided crossed module of Lie objects in $\mathcal{LM}_K$, where*

- *$\frac{M}{\{M,N\}_x}$ is the Lie K-algebra quotient of M by the ideal $\{M,N\}_x$ whose generators are $[x,x]$ for $x \in M$, $n \cdot_1 m + m \cdot_2 n$ for $n \in N$, $m \in M$, and $\{n, n'\} + \langle n', n \rangle$ for $n, n' \in N$; we denote the natural map by $\pi_M \colon M \to \frac{M}{\{M,N\}_x}$, and the elements of $\frac{M}{\{M,N\}_x}$ by $\overline{m}$,*
- *$\bar{\cdot}_1 \colon \mathrm{Lie}(N) \times M \to M$, $(\overline{n}, m) \mapsto -m \cdot_2 n$,*
- *$\bar{\cdot}_2 \colon \mathrm{Lie}(N) \times \frac{M}{\{M,N\}_x} \to \frac{M}{\{M,N\}_x}$, $(\overline{n}, \overline{m}) \mapsto \overline{n \cdot_1 m} = \overline{-m \cdot_2 n}$,*
- *$\xi_{\bar{\cdot}} \colon N \times \frac{M}{\{M,N\}_x} \to M$, $(n, \overline{m}) \to n \cdot_1 m$,*
- *$\bar{\partial}_1 \colon M \to N$, $m \mapsto \partial(m)$,*
- *$\bar{\partial}_2 \colon \frac{M}{\{M,N\}_x} \to \mathrm{Lie}(N)$, $\overline{m} \mapsto \overline{\partial m}$,*

- $\{-,-\}_{N\,\mathrm{Lie}(N)}\colon N\times \mathrm{Lie}(N)\to M$, $(n,\overline{n'})\mapsto \{n,n'\}$,
- $\{-,-\}_{\mathrm{Lie}(N)N}\colon \mathrm{Lie}(N)\times N\to M$, $(\overline{n},n')\mapsto -\langle n',n\rangle$,
- $\{-,-\}_2\colon \mathrm{Lie}(n)\times \mathrm{Lie}(N)\to M$, $(\overline{n},\overline{n'})\mapsto \overline{\{n,n'\}}=\overline{-\langle n',n\rangle}$.

Remark 2.31. *As in the previous cases, $(\frac{M}{\{M,N\}_x},\mathrm{Lie}(N),\bar{\cdot}_2,\overline{\partial}_2,\{-,-\}_2)$ will be called Lieization, and it is functorial.*

If we apply this Lieization on a crossed module of Lie K-algebras, thought as a crossed module of Leibniz K-algebras with the action $(\cdot,\cdot^-)$ and the braiding $(\{-,-\},\{-,-\}^-)$, the third generators are null too:

$$\{n,n'\}+\langle n',n\rangle=\{n,n'\}+\{n',n\}^-=\{n,n'\}-\{n,n'\}=0.$$

In the Lie case, we obtain a natural isomorphism to itself after doing the Lieization.

Proposition 2.32. *Let $\left(\begin{smallmatrix}M\\ \downarrow f\\ N\end{smallmatrix},\begin{smallmatrix}L\\ \downarrow g\\ H\end{smallmatrix},\bar{\cdot},\partial,T_{\{-,-\}}\right)$ be a braided crossed module of Lie objects in $\mathcal{LM}_K$.*

Then $(M,L,(\bar{\cdot}_1,\bar{\cdot}_2),\partial_1,(\{-,-\}_{T_{\{-,-\}}},\langle-,-\rangle_{T_{\{-,-\}}}))$ is a braided crossed module of Leibniz K-algebras, where

- *The Leibniz brackets are: $[m,m']=m*_N^M f(m')$ for $m,m'\in M$ and $[l,l']=l*_H^L g(l')$ for $l,l'\in M$;*
- *$\bar{\cdot}_1\colon L\times N\to M$ is defined by $l\,\bar{\cdot}_1 m=\xi.(l,f(m))$ for $l\in L$, $m\in M$;*
- *$\bar{\cdot}_2\colon M\times L\to M$ is defined as $m\bar{\cdot}_2 l=-g(l)\cdot_1 m$ for $l\in L$, $m\in M$;*
- *$\{-,-\}_{T_{\{-,-\}}}\colon L\times L\to M$ is defined as $\{l,l'\}_{T_{\{-,-\}}}=\{l,g(l')\}_{LH}$ for $l,l'\in L$;*
- *$\langle-,-\rangle_{T_{\{-,-\}}}\colon L\times L\to M$ is defined as $\langle l,l'\rangle_{T_{\{-,-\}}}=-\{g(l'),l\}_{HL}$ for $l,l'\in L$.*

Thus, we have the functors $\boldsymbol{BX}(\boldsymbol{LeibAlg}_K)\underset{BX\Psi}{\overset{BX\Phi}{\rightleftarrows}}\boldsymbol{BXLie}(\mathcal{LM}_K)$ satisfying $BX\Psi\circ BX\Phi=\mathrm{Id}_{\boldsymbol{BX}(\boldsymbol{LeibAlg}_K)}$, and so, the functor $BX\Phi$ is a full inclusion functor.

2.2. Braiding for categorical Lie objects in $\mathcal{LM}_K$ and categorical Leibniz algebras

We also want to define a braiding for categorical Leibniz K-algebras. As in the crossed module case, we will use the idea of the category $\mathcal{LM}_K$. We only need to show that $\mathcal{LM}_K$ is a category with pullbacks.

Remark 2.33. *$\mathcal{LM}_K$ is a category with pullbacks.*

If we have the morphisms $\begin{smallmatrix}A\\ \downarrow f\\ B\end{smallmatrix}\overset{\alpha}{\to}\begin{smallmatrix}X\\ \downarrow h\\ Y\end{smallmatrix}\overset{\beta}{\leftarrow}\begin{smallmatrix}C\\ \downarrow g\\ D\end{smallmatrix}$, then $\left(\begin{smallmatrix}A\times_X C\\ \downarrow f\times_h g\\ B\times_Y D\end{smallmatrix},(\pi_A,\pi_B),(\pi_C,\pi_D)\right)$ is their pullback.

It is easy to check that $\boldsymbol{Lie}(\mathcal{LM}_K)$ has the same pullback with the operations $[(b,d),(b',d')]_{B\times_Y D}:=([b,b']_B,[d,d']_D)$ and $(a,c)*_{B\times_Y D}^{A\times_X C}(b,d):=(a*_B^A b,c*_D^C d)$. So, we can speak about categorical Lie objects in $\mathcal{LM}_K$.

As in the crossed module case, we have the following result.

Proposition 2.34. *Let (C_1,C_0,s,t,e,k) be a categorical Leibniz K-algebra.*

Then $\left(\begin{smallmatrix}C_1\\ \downarrow \pi_{C_1}\\ \mathrm{Lie}(C_1)\end{smallmatrix},\begin{smallmatrix}C_0\\ \downarrow \pi_{C_0}\\ \mathrm{Lie}(C_0)\end{smallmatrix},(s,\mathrm{Lie}(s)),(t,\mathrm{Lie}(t)),(e,\mathrm{Lie}(e)),(k,\overline{k})\right)$ is a categorical Lie object in $\mathcal{LM}_K$, where we denote the Lieization functor as $\mathrm{Lie}\colon \boldsymbol{LeibAlg}_K\to\boldsymbol{LieAlg}_K$, the composition morphism $\overline{k}\colon \mathrm{Lie}(C_1)\times_{\mathrm{Lie}(C_0)}\mathrm{Lie}(C_1)\to\mathrm{Lie}(C_1)$ is defined as $\overline{k}(\overline{x},\overline{y})=\overline{\mathring{k}(x,y)}$ and $\mathring{k}\colon C_1\times C_1\to C_1$ is the extension of k to the Leibniz product, defined as $\mathring{k}(x,y)=x+y-e(s(y))$.

Remark 2.35. *$\mathring{k}$ is an extension of k, since the same formula is satisfied for composition, as can be seen in Lemma 1.4.*

One can ask why not to extend k as $\mathring{k}'(x, y) = x + y - e(t(x))$, which is not identical to $\mathring{k}$ in the general case. But, in this case we have that the result is the same

$$\overline{\mathring{k}(x, y)} = \overline{x} + \overline{y} - \mathrm{Lie}(e)(\mathrm{Lie}(s)(\overline{y})) = \overline{x} + \overline{y} - \mathrm{Lie}(e)(\mathrm{Lie}(t)(\overline{x})) = \overline{\mathring{k}'(x, y)},$$

since $(\overline{x}, \overline{y}) \in \mathrm{Lie}(C_1) \times_{\mathrm{Lie}(C_0)} \mathrm{Lie}(C_1)$ *implies* $\mathrm{Lie}(s)(\overline{y}) = \mathrm{Lie}(t)(\overline{x})$.

Remark 2.36. *We again have in the bottom part the Lieization, and in the case of Lie K-algebras thought as Leibniz K-algebras, we obtain the identity.*

Proposition 2.37. *If $\left(\begin{smallmatrix} C_1 \\ \downarrow f_1 \\ D_1 \end{smallmatrix}, \begin{smallmatrix} C_0 \\ \downarrow f_0 \\ D_0 \end{smallmatrix}, s, t, e, k \right)$ is a categorical Lie object in $\mathcal{LM}_K$, then $(C_1, C_0, s_1, t_1, e_1, k_1)$ is a categorical Leibniz K-algebra, where $[x, y]_{C_1} = x *_{D_1}^{C_1} y$ and $[a, b]_{C_0} = a *_{D_0}^{C_0} b$, for $x, y \in C_1, a, b \in C_0$.*

Analogously to the crossed modules case, we have once again for internal categories the pair of functors $\boldsymbol{ICat}(\boldsymbol{LeibAlg}_K) \underset{I\Psi}{\overset{I\Phi}{\rightleftarrows}} \boldsymbol{ICat}(\boldsymbol{Lie}(\mathcal{LM}_K))$ satisfying $I\Psi \circ I\Phi = \mathrm{Id}_{\boldsymbol{ICat}(\boldsymbol{LeibAlg}_K)}$, and so, the functor $I\Phi$ is a full inclusion functor.

This new inclusion functor allows us to define a braiding on categorical Leibniz K-algebras using the idea of braiding of Lie objects in $\mathcal{LM}_K$.

Proposition 2.38. *Let $\mathcal{C} = (\mathbf{C}, \otimes, a, \mathcal{T})$ be a braided semigroupal category where $\mathbf{C}$ is an additive category with pullbacks. Then $\boldsymbol{Lie}(\mathcal{C})$ has pullbacks.*

Proof. If we have two Lie morphisms

$$\begin{array}{ccc} & & (A, \mu_A) \\ & & \downarrow f \\ (B, \mu_B) & \xrightarrow{g} & (C, \mu_C), \end{array}$$

the pullback is given by $((A \times_C B, \mu_{A \times_C B}), \pi_A, \pi_B)$, where $A \times_C B$ is the pullback in $\mathbf{C}$

$$\begin{array}{ccc} A \times_C B & \xrightarrow{\pi_A} & A \\ \downarrow \pi_B & & \downarrow f \\ B & \xrightarrow{g} & C, \end{array}$$

and $\mu_{A \times_C B}$ is the unique morphism such that $\pi_X \circ \mu_{A \times_C B} = \mu_X \circ (\pi_X \otimes \pi_X)$ for $X \in \{A, B\}$ constructed by the universal property of pullbacks in $\mathbf{C}$ in the following diagram:

$$\begin{array}{ccccc} (A \times_C B) \otimes (A \times_C B) & & \xrightarrow{\pi_A \otimes \pi_A} & & A \otimes A \\ & \searrow^{\mu_{A \times_C B}} & & & \downarrow \mu_A \\ \downarrow \pi_B \otimes \pi_B & & A \times_C B & \xrightarrow{\pi_A} & A \\ & & \downarrow \pi_B & & \downarrow f \\ B \otimes B & \xrightarrow{\mu_B} & B & \xrightarrow{g} & C. \end{array}$$

It is straightforward to see that $\mu_{A\times_C B}$ is well defined. Now, we will prove that $(A \times_C B, \mu_{A\times_C B})$ is a Lie object checking the first axiom. For simplicity of notation, we will denote $D := A \times_C B$. Let $X \in \{A, B\}$. Using universal properties we have:

$$\pi_X \circ (-\mu_D \circ \mathcal{T}_{D,D}) = -\pi_X \circ \mu_D \circ \mathcal{T}_{D,D} = -\mu_X \circ (\pi_X \otimes \pi_X) \circ \mathcal{T}_{D,D}.$$

Since $\mathcal{T}$ is a natural isomorphism and that (X, μ_X) is a Lie object, we get

$$\pi_X \circ (-\mu_D \circ \mathcal{T}_{D,D}) = -\mu_X \circ \mathcal{T}_{X,X} \circ (\pi_X \otimes \pi_X) = \mu_X \circ (\pi_X \otimes \pi_X).$$

We conclude that $\mu_D = -\mu_D \circ \mathcal{T}_{D,D}$ because μ_D is the unique morphism that satisfies the previous equality for $X \in \{A, B\}$.

Now, we will check the second axiom of Lie object.

For the first summand, we have:

$$\begin{aligned}\pi_X \circ \mu_D \circ (\mathrm{Id}_D \otimes \mu_D) \circ a_{D,D,D} &= \mu_X \circ (\pi_X \otimes \pi_X) \circ (\mathrm{Id}_D \otimes \mu_D) \circ a_{D,D,D}\\ &= \mu_X \circ (\pi_X \otimes (\pi_X \circ \mu_D)) \circ a_{D,D,D} = \mu_X \circ (\pi_X \otimes (\mu_X \circ (\pi_X \otimes \pi_X))) \circ a_{D,D,D}\\ &= \mu_X \circ (\mathrm{Id}_X \otimes \mu_X) \circ (\pi_X \otimes (\pi_X \otimes \pi_X)) \circ a_{D,D,D}.\end{aligned}$$

Using that a is a natural isomorphism, we have

$$\pi_X \circ \mu_D \circ (\mathrm{Id}_D \otimes \mu_D) \circ a_{D,D,D} = \mu_X \circ (\mathrm{Id}_X \otimes \mu_X) \circ a_{X,X,X} \circ ((\pi_X \otimes \pi_X) \otimes \pi_X).$$

Doing the same for the second and third summands (the naturalness of a gives the same naturalness to a^{-1}), we have that:

$$\begin{aligned}&\pi_X \circ \mu_D \circ (\mu_D \otimes \mathrm{Id}_D) \circ a^{-1}_{D,D,D} \circ (\mathrm{Id}_D \otimes \mathcal{T}_{D,D}) \circ a_{D,D,D}\\ &= \mu_X \circ (\mu_X \otimes \mathrm{Id}_X) \circ a^{-1}_{X,X,X} \circ (\mathrm{Id}_X \otimes \mathcal{T}_{X,X}) \circ a_{X,X,X} \circ ((\pi_X \otimes \pi_X) \otimes \pi_X),\end{aligned}$$

$$\pi_X \circ (-\mu_D \circ (\mu_D \otimes \mathrm{Id}_D)) = -\mu_X \circ (\mu_X \otimes \mathrm{Id}_X) \circ ((\pi_X \otimes \pi_X) \otimes \pi_X).$$

Adding the three last equalities and using the distributivity of the composition, we have $\pi_X \circ \mathcal{L}_D = \mathcal{L}_X \circ ((\pi_X \otimes \pi_X) \otimes \pi_X)$, where denote by $\mathcal{L}_Y$ the morphism that is in the first term of the equality of the second axiom for a Lie object (Y, μ_Y). Since (X, μ_X) is a Lie object, we get $\mathcal{L}_X = {}_{(X\otimes X)\otimes X}0_X$, and so $\pi_X \circ \mathcal{L}_D = {}_{(D\otimes D)\otimes D}0_X$.

Now, by the universal property, we have that $\mathcal{L}_D = {}_{(D\otimes D)\otimes D}0_D$ and therefore (D, μ_D) is a Lie object.

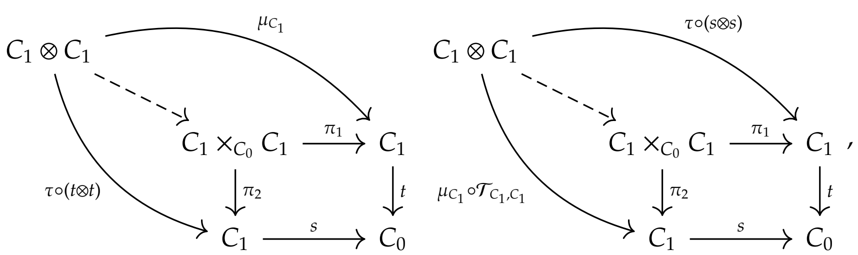
To conclude the proof it is enough to check that the morphism given by the pullback in $\boldsymbol{C}$ is a Lie morphism, but this is a routine verification. □

Definition 2.39. *Let $C = (\boldsymbol{C}, \otimes, a, \mathcal{T})$ be a braided semigroupal category where $\boldsymbol{C}$ is an additive category with pullbacks.*

Let $\mathfrak{C} = ((C_1, \mu_{C_1}), (C_0, \mu_{C_0}), s, t, e, k)$ be a categorical Lie object in $\boldsymbol{Lie}(C)$.

A braiding on $\mathfrak{C}$ is a morphism $\tau\colon C_0 \otimes C_0 \to C_1$ satisfying:

- $s \circ \tau = \mu_{C_0}$ *and* $t \circ \tau = \mu_{C_0} \circ \mathcal{T}_{C_0,C_0}$,
- *We define* $C_0 \otimes C_0 \xrightarrow{\mu_{C_1}\times_{C_0}(\tau\circ(t\otimes t)),(\tau\circ(s\otimes s))\times_{C_0}(\mu_{C_1}\circ\mathcal{T})} C_1 \times_{C_0} C_1$ *as the two unique morphisms which satisfy the universal property, respectively, in the following diagrams:*



and the equality

$$k \circ (\mu_{C_1} \times_{C_0} (\tau \circ (t \otimes t))) = k \circ ((\tau \circ (s \otimes s)) \times_{C_0} (\mu_{C_1} \circ \mathcal{T})).$$

- *It must satisfy*

$$\tau \circ (\mathrm{Id}_{C_0} \otimes \mu_{C_0}) \otimes a_{C_0,C_0,C_0} = \tau \circ (\mu_{C_0} \otimes \mathrm{Id}_{C_0}) \circ (\mathrm{Id}_{(C_0 \otimes C_0) \otimes C_0} - (a^{-1}_{C_0,C_0,C_0} \circ (\mathrm{Id}_{C_0} \otimes \mathcal{T}_{C_0,C_0}) \circ a_{C_0,C_0,C_0})),$$
$$\tau \circ (\mu_{C_0} \otimes \mathrm{Id}_{C_0}) = \tau \circ (\mathrm{Id}_{C_0} \otimes \mu_{C_0}) \circ a_{C_0,C_0,C_0} \circ (\mathrm{Id}_{(C_0 \otimes C_0) \otimes C_0} - (\mathcal{T}_{C_0,C_0} \otimes \mathrm{Id}_{C_0})).$$

We will say that $((C_1, \mu_{C_1}), (C_0, \mu_{C_0}), s, t, e, k, \tau)$ *is a braided categorical Lie object in* $\mathcal{C}$.

An internal functor $((C_1, \mu_{C_1}), (C_0, \mu_{C_0}), s, t, e, k, \tau) \xrightarrow{(F_1,F_0)} ((C'_1, \mu_{C'_1}), (C'_0, \mu_{C'_0}), s', t', e', k', \tau')$ *is said to be a braided internal functor of braided categorical Lie objects in* $\mathcal{C}$ *if it satisfies the following diagram:*

$$\begin{array}{ccc} C_0 \otimes C_0 & \xrightarrow{\tau} & C_1 \\ \downarrow{\scriptstyle F_0 \otimes F_0} & & \downarrow{\scriptstyle F_1} \\ C'_0 \otimes C'_0 & \xrightarrow{\tau'} & C_1. \end{array}$$

We denote this new category as $\boldsymbol{BICat}(\boldsymbol{Lie}(\mathcal{C}))$.

Example 2.40. *We have that the categories* $\boldsymbol{BICat}(\boldsymbol{Lie}(\boldsymbol{Vect}_K))$ *and* $\boldsymbol{BICat}(\boldsymbol{LieAlg}_K)$ *are isomorphic, taking in* $\boldsymbol{Vect}_K$ *the usual tensor product (we assume* $\mathrm{char}(K) \neq 2$*).*

Definition 2.41. *Let* $\mathcal{C} = \left(\begin{array}{c} C_1 \\ \downarrow{\scriptstyle f_1} \\ D_1 \end{array}, \begin{array}{c} C_0 \\ \downarrow{\scriptstyle f_0} \\ D_0 \end{array}, s, t, e, k \right)$ *be a categorical Lie object in* $\mathcal{LM}_K$.

A braiding on $\mathcal{C}$ *is a triple* $\tau = (\tau^{C_0,D_0}, \tau^{D_0,C_0}, \tau^2)$ *where*

- $\tau^2 \colon D_0 \times D_0 \to D_1$ *is a K-bilinear map such that* $(D_1, D_0, s, t, e, k, \tau_2)$ *is a braided crossed module of Lie K-algebras,*
- $\tau^{D_0,C_0} \colon D_0 \times C_0 \to C_1$ *and* $\tau^{C_0,D_0} \colon C_0 \times D_0 \to C_1$ *are K-bilinear maps which, with* τ^2*, satisfy the following properties for* $c \in C_0$, $d, d' \in D_0$, $x \in C_1$, $y \in D_1$*:*

$$f_1(\tau^{C_0,D_0}_{c,d}) = \tau^2_{f_0(c),d} \qquad \text{and} \qquad f_1(\tau^{D_0,C_0}_{d,c}) = \tau^2_{d,f_0(c)},$$
$$\tau^{C_0,D_0}_{c,d} \colon c *^{C_0}_{D_0} d \to -c *^{C_0}_{D_0} d \qquad \text{and} \qquad \tau^{D_0,C_0}_{d,c} \colon -c *^{C_0}_{D_0} d \to c *^{C_0}_{D_0} d.$$

The following diagrams are satisfied in the internal category:

$$\begin{array}{ccc} s_1(x) *^{C_0}_{D_0} s_2(y) & \xrightarrow{x *^{C_1}_{C_0} y} & t_1(x) *^{C_0}_{D_0} t_2(y) \\ \downarrow{\scriptstyle \tau^{C_0,D_0}_{s_1(x),s_2(y)}} & & \downarrow{\scriptstyle \tau^{C_0,D_0}_{t_1(x),t_2(y)}} \\ -s_1(x) *^{C_0}_{D_0} s_2(y) & \xrightarrow{-x *^{C_1}_{D_1} y} & -t_1(x) *^{C_0}_{D_0} t_2(y) \end{array}, \qquad \begin{array}{ccc} -s_1(x) *^{C_0}_{D_0} s_2(y) & \xrightarrow{-x *^{C_1}_{C_0} y} & -t_1(x) *^{C_0}_{D_0} t_2(y) \\ \downarrow{\scriptstyle \tau^{D_0,C_0}_{s_2(y),s_1(x)}} & & \downarrow{\scriptstyle \tau^{D_0,C_0}_{t_2(y),t_1(x)}} \\ s_1(x) *^{C_0}_{D_0} s_2(y) & \xrightarrow{x *^{C_1}_{D_1} y} & t_1(x) *^{C_0}_{D_0} t_2(y) \end{array}.$$

Moreover, we have the following properties:

$$\tau^{C_0,D_0}_{c,[d,d']_{D_0}} = \tau^{C_0,D_0}_{c *^{C_0}_{D_0} d, d'} - \tau^{C_0,D_0}_{c *^{C_0}_{D_0} d', d},$$
$$\tau^{D_0,C_0}_{[d,d']_{D_0},c} = -\tau^{D_0,C_0}_{d, c *^{C_0}_{D_0} d'} - \tau^{C_0,D_0}_{c *^{C_0}_{D_0} d, d'},$$
$$\tau^{C_0,D_0}_{c,[d,d']_{D_0}} = \tau^{C_0,D_0}_{c *^{C_0}_{D_0} d, d'} + \tau^{D_0,C_0}_{d, c *^{C_0}_{D_0} d'},$$
$$\tau^{D_0,C_0}_{[d,d']_{D_0},c} = -\tau^{D_0,C_0}_{d, c *^{C_0}_{D_0} d'} + \tau^{D_0,C_0}_{d', c *^{C_0}_{D_0} d}.$$

We will say that $\left(\begin{smallmatrix} C_1 \\ \downarrow f_1 \\ D_1 \end{smallmatrix}, \begin{smallmatrix} C_0 \\ \downarrow f_0 \\ D_0 \end{smallmatrix}, s, t, e, k, \tau\right)$ *is a braided categorical Lie object in* $\mathcal{LM}_K$.

Remark 2.42. *A braiding is a pair* $\tau = (\tau_1, \tau_2)$ *but, for simplicity, we denote* $\tau_2(d, d') = \tau^2_{d,d'}$ *and* $\tau_1 \colon (C_0 \otimes D_0) \oplus (D_0 \otimes C_0) \to C_1$ *by the expression* $\tau_1((c \otimes d) + (d' \otimes c')) = \tau^{C_0,D_0}_{c,d} + \tau^{D_0,C_0}_{d',c'}$.

Definition 2.43. *Let* $\left(\begin{smallmatrix} C_1 \\ \downarrow f_1 \\ D_1 \end{smallmatrix}, \begin{smallmatrix} C_0 \\ \downarrow f_0 \\ D_0 \end{smallmatrix}, s, t, e, k, \tau\right)$ *and* $\left(\begin{smallmatrix} C'_1 \\ \downarrow g_1 \\ D'_1 \end{smallmatrix}, \begin{smallmatrix} C'_0 \\ \downarrow g_0 \\ D'_0 \end{smallmatrix}, s', t', e', k', \psi\right)$ *be braided categorical Lie objects in* $\mathcal{LM}_K$. *A braided internal functor between categorical Lie objects in* $\mathcal{LM}_K$ *is an internal functor* $((F^1_1, F^0_1), (F^1_0, F^0_0))$ *between the respective categorical Lie objects which satisfies:*

- $(F^0_1, F^0_0) \colon (D_1, D_0, s_2, t_2, e_2, k_2, \tau_2) \to (D'_1, D'_0, s'_2, t'_2, e'_2, k'_2, \psi_2)$ *is a braided internal functor between categorical Lie K-algebras.*
- $F^1_1(\tau^{C_0,D_0}_{c,d}) = \psi^{C'_0,D'_0}_{F^1_0(c),F^0_0(d)}$ *for* $c \in C_0$, $d \in D_0$.
- $F^1_1(\tau^{D_0,C_0}_{d,c}) = \psi^{D'_0,C'_0}_{F^0_0(d),F^1_0(c)}$ *for* $c \in C_0$, $d \in D_0$.

To introduce a braiding for the categorical Leibniz K-algebras with the previous scheme, we will use two K-bilinear maps $\tau, \psi \colon C_0 \times C_0 \to C_1$, as in the case of a braiding of crossed modules of Leibniz K-algebras. Consider for the inclusion Lie object in $\mathcal{LM}_K$ the braiding $\bar{\tau}$ defined by $\bar{\tau}^{C_0,\mathrm{Lie}(C_0)}_{a,\bar{b}} = \tau_{a,b}$, $\bar{\tau}^{\mathrm{Lie}(C_0),C_0}_{\bar{a},b} = -\psi_{b,a}$ and $\bar{\tau}^2_{\bar{a},\bar{b}} = \overline{\tau_{a,b}} = \overline{-\psi_{b,a}}$, where we introduce a quotient in C_1 whose elements we will denote as $\bar{x}$.

Definition 2.44. *A braiding for the categorical Leibniz K-algebra* (C_1, C_0, s, t, e, k) *is a pair* (τ, ψ) *of K-bilinear maps* $\tau, \psi \colon C_0 \times C_0 \to C_1$, $(a, b) \mapsto \tau_{a,b}$ *and* $(a, b) \mapsto \psi_{a,b}$, *satisfying:*

$$\tau_{a,b} \colon [a, b] \to -[a, b] \quad \text{and} \quad \psi_{a,b} \colon [a, b] \to -[a, b], \tag{LeibT1}$$

$$\begin{array}{ccc} [s(x), s(y)] & \xrightarrow{[x,y]} & [t(x), t(y)] \\ \downarrow{\scriptstyle \tau_{s(x),s(y)}} & & \downarrow{\scriptstyle \tau_{t(x),t(y)}} \\ -[s(x), s(y)] & \xrightarrow{-[x,y]} & -[t(x), t(y)], \end{array} \qquad \begin{array}{ccc} [s(x), s(y)] & \xrightarrow{[x,y]} & [t(x), t(y)] \\ \downarrow{\scriptstyle \psi_{s(x),s(y)}} & & \downarrow{\scriptstyle \psi_{t(x),t(y)}} \\ -[s(x), s(y)] & \xrightarrow{-[x,y]} & -[t(x), t(y)], \end{array} \tag{LeibT2}$$

$$\tau_{a,[b,c]} = \tau_{[a,b],c} - \tau_{[a,c],b}, \tag{LeibT3}$$
$$\psi_{a,[b,c]} = \tau_{[a,b],c} - \psi_{[a,c],b}, \tag{LeibT4}$$
$$\tau_{a,[b,c]} = \tau_{[a,b],c} - \psi_{[a,c],b}, \tag{LeibT5}$$
$$\psi_{a,[b,c]} = \psi_{[a,b],c} - \psi_{[a,c],b}, \qquad a, b, c \in C_0,\ x, y \in C_1. \tag{LeibT6}$$

We will say that $(C_1, C_0, s, t, e, k, (\tau, \psi))$ *is a braided categorical Leibniz K-algebra.*

Definition 2.45. *Let* $(C_1, C_0, s, t, e, k, (\tau, \psi))$ *and* $(C'_1, C'_0, s', t', e', k', (\tau', \psi'))$ *be two braided categorical Leibniz K-algebras.*

An internal functor $(C_1, C_0, s, t, e, k) \xrightarrow{(F_1,F_0)} (C'_1, C'_0, s', t', e', k')$ *is said to be a braided internal functor between two braided categorical Leibniz K-algebras if it satisfies:*

$$F_1(\tau_{a,b}) = \tau'_{F_0(a),F_0(b)}, \tag{LeibHT1}$$
$$F_1(\psi_{a,b}) = \psi'_{F_0(a),F_0(b)}, \qquad a, b \in C_0. \tag{LeibHT2}$$

We denote the category of braided categorical Leibniz K-algebras and braided internal functors between them as ***BICat*(*LeibAlg*$_K$)**.

We want to see the braided categorical Lie K-algebras as a particular case of braided categorical Leibniz K-algebras.

Proposition 2.46. *Let C_1 and C_0 be Lie K-algebras. Then, $(C_1, C_0, s, t, e, k, \tau)$ is a braided categorical Lie K-algebra if and only if $(C_1, C_0, s, t, e, k, (\tau, \tau^-))$ is a braided categorical Leibniz K-algebra. $\tau^- \colon C_0 \times C_0 \to C_1$ is defined as $\tau^-_{a,b} = -\tau_{b,a}$.*

Proof. Immediately, (LeibT1) and (LeibT2) can be rewritten as (LieT1) and (LieT2), respectively, using the anticommutativity. Moreover, it is clear that (LeibT3) and (LieT4) are identical, and that (LeibT6) is equivalent to (LieT3).

To see the last equivalences, (LeibT4) with (LieT3), and (LeibT5) with (BLie4), we must prove $\tau_{[a,b],c} = -\tau_{c,[a,b]}$, for $a, b, c \in C_0$.

- In the Lie case, it is true using Proposition 1.5.
- In the Leibniz case it is not true in general, because we need $\tau_{a,b} = -\psi_{b,a}$; but using (LeibT4) and (LeibT5) we can observe that $\tau_{a,[b,c]} = \psi_{a,[b,c]} = \tau^-_{a,[b,c]} = -\tau_{[b,c],a}$.

□

Proposition 2.47. *Let $(C_1, C_0, s, t, e, k, (\tau, \psi))$ be a braided categorical Leibniz K-algebra. Then $\left(\begin{array}{c} C_1 \\ \downarrow{\scriptstyle \pi_{C_1}} \\ \frac{C_1}{[\tau_{C_0,C_0}]} \end{array}, \begin{array}{c} C_0 \\ \downarrow{\scriptstyle \pi_{C_0}} \\ \mathrm{Lie}(C_0) \end{array}, (s,\bar{s}), (t,\bar{t}), (e,\bar{e}), (k,\widetilde{k}), \bar{\tau} \right)$ is a braided categorical Lie object in $\mathcal{LM}_K$, where $\frac{C_1}{[\tau_{C_0,C_0}]}$ is the Lie K-algebra which is a Leibniz quotient of C_1 by the ideal generated by elements of the form $[x, x]$ and $\tau_{a,b} + \psi_{b,a}$, $x \in C_1$, $a, b \in C_0$; and the maps are the following ones:*

- $\bar{s} \colon \frac{C_1}{[\tau_{C_0,C_0}]} \to \mathrm{Lie}(C_0)$ *defined as* $\bar{s}(\bar{x}) = \overline{s(x)}$ *for* $\bar{x} \in \frac{C_1}{[\tau_{C_0,C_0}]}$;
- $\bar{t} \colon \frac{C_1}{[\tau_{C_0,C_0}]} \to \mathrm{Lie}(C_0)$ *defined as* $\bar{t}(\bar{x}) = \overline{t(x)}$ *for* $\bar{x} \in \frac{C_1}{[\tau_{C_0,C_0}]}$;
- $\bar{e} \colon \mathrm{Lie}(C_0) \to \frac{C_1}{[\tau_{C_0,C_0}]}$ *defined as* $\bar{e}(\bar{a}) = \overline{e(a)}$ *for* $\bar{a} \in \mathrm{Lie}(C_0)$;
- $\widetilde{k} \colon \frac{C_1}{[\tau_{C_0,C_0}]} \times_{\mathrm{Lie}(C_0)} \frac{C_1}{[\tau_{C_0,C_0}]} \to \frac{C_1}{[\tau_{C_0,C_0}]}$ *defined as* $\bar{k}((\bar{x}, \bar{y})) = \overline{\mathring{k}(x, y)}$ *for* $(\bar{x}, \bar{y}) \in \frac{C_1}{[\tau_{C_0,C_0}]} \times_{\mathrm{Lie}(C_0)} \frac{C_1}{[\tau_{C_0,C_0}]}$, *where* $\mathring{k}$ *is again the extension to the product* $\mathring{k}(x, y) = x + y - e(s(y))$ *(we can take* $\mathring{k}'(x, y) = x + y - e(t(x))$ *too, because in the quotient it will not change anything);*
- $\bar{\tau}^{C_0,\mathrm{Lie}(C_0)} \colon C_0 \times \mathrm{Lie}(C_0) \to C_1$ *defined as* $\bar{\tau}^{C_0,\mathrm{Lie}(C_0)}_{a,\bar{b}} = \tau_{a,b}$ *for* $a \in C_0$, $\bar{b} \in \mathrm{Lie}(C_0)$;
- $\bar{\tau}^{\mathrm{Lie}(C_0),C_0} \colon \mathrm{Lie}(C_0) \times C_0 \to C_1$ *defined as* $\bar{\tau}^{\mathrm{Lie}(C_0),C_0}_{\bar{a},b} = -\psi_{b,a}$ *for* $\bar{a} \in \mathrm{Lie}(C_0)$, $b \in C_0$;
- $\bar{\tau}^2 \colon \mathrm{Lie}(C_0) \times \mathrm{Lie}(C_0) \to \frac{C_1}{[\tau_{C_0,C_0}]}$ *defined as* $\tau^2_{\bar{a},\bar{b}} = \overline{\tau_{a,b}} = \overline{-\psi_{b,a}}$ *for* $\bar{a}, \bar{b} \in \mathrm{Lie}(C_0)$.

Remark 2.48. *The bottom part $(\frac{C_1}{[\tau_{C_0,C_0}]}, \mathrm{Lie}(C_0), \bar{s}, \bar{t}, \bar{e}, \widetilde{k}, \bar{\tau}_2)$ will be called Lieization, and it is again functorial.*

If we apply this Lieization on a braided categorical Lie K-algebra, thought as a crossed module of Leibniz K-algebras with the action with the braiding (τ, τ^-), the new generator is null

$$\tau_{a,b} + \psi_{b,a} = \tau_{a,b} + \tau^-_{b,a} = \tau_{a,b} - \tau_{a,b} = 0.$$

Proposition 2.49. *Let $\left(\begin{array}{c} C_1 \\ \downarrow{\scriptstyle f_1} \\ D_1 \end{array}, \begin{array}{c} C_0 \\ \downarrow{\scriptstyle f_0} \\ D_0 \end{array}, s, t, e, k, \tau \right)$ be a braided categorical Lie object in $\mathcal{LM}_K$.*

Then $(C_1, C_0, s_1, t_1, e_1, k_1, (\bar{\tau}^\tau, \bar{\psi}^\tau))$ is a braided categorical Leibniz K-algebra, where $[x, y]_{C_1} = x *^{C_1}_{D_1} y$ and $[a, b]_{C_0} = a *^{C_0}_{D_0} b$ for $x, y \in C_1, a, b \in C_0$, and $\bar{\tau}^\tau_{a,b} = \tau^{C_0,D_0}_{a,f_0(b)}$, $\bar{\psi}^\tau_{a,b} = -\tau^{D_0,C_0}_{f(b),a}$ for $a, b \in C_0$.

We have again the pair of functors $\boldsymbol{BICat}(\boldsymbol{LeibAlg}_K) \underset{BI\Psi}{\overset{BI\Phi}{\rightleftarrows}} \boldsymbol{BICat}(\boldsymbol{Lie}(\mathcal{LM}_K))$ satisfying $BI\Psi \circ BI\Phi = \mathrm{Id}_{\boldsymbol{BICat}(\boldsymbol{LeibAlg}_K)}$, and so, the functor $BI\Phi$ is a full inclusion functor.

3. The equivalence between the categories of braided crossed modules and braided internal categories in the case of Leibniz algebras

First, we will prove that the categories $\boldsymbol{BICat}(\boldsymbol{LeibAlg}_K)$ and $\boldsymbol{BX}(\boldsymbol{LeibAlg}_K)$ are equivalent, as in the case of groups and Lie K-algebras. Moreover, the equivalence must generalize the Lie K-algebras case (i.e. the braidings of the Leibniz K-algebras must satisfy $\{n, n'\} = -\langle n', n\rangle$ and $\tau_{a,b} = -\psi_{b,a}$ and the functors for the Lie case would be recovered) and must be an extension of the one given to the non-braiding case.

Proposition 3.1. *Let $\mathcal{X} = (M, N, (\cdot_1, \cdot_2), \partial, (\{-,-\}, \langle -,-\rangle))$ be a braided crossed module of Leibniz K-algebras. Then $C_{\mathcal{X}} := (M \rtimes N, N, \bar{s}, \bar{t}, \bar{e}, \bar{k}, (\bar{\tau}, \bar{\psi}))$ is a braided categorical Leibniz K-algebra where*

- $\bar{s}\colon M \rtimes N \to N,\ \bar{s}((m, n)) = n,$
- $\bar{t}\colon M \rtimes N \to N,\ \bar{t}((m, n)) = \partial m + n,$
- $\bar{e}\colon N \to M \rtimes N,\ \bar{e}(n) = (0, n),$
- $\bar{k}\colon (M \rtimes N) \times_N (M \rtimes N) \to M \rtimes N$, *where the source is the pullback of $\bar{t}$ with $\bar{s}$, defined as* $k(((m, n), (m', \partial m + n))) = (m + m', n),$
- $\bar{\tau}\colon N \times N \to M \rtimes N,\ \bar{\tau}_{n,n'} = (-2\{n, n'\}, [n, n']),$
- $\bar{\psi}\colon N \times N \to M \rtimes N,\ \bar{\psi}_{n,n'} = (-2\langle n, n'\rangle, [n, n']).$

Proof. We only need to check the braiding axioms, since $(M \rtimes N, N, \bar{s}, \bar{t}, \bar{e}, \bar{k})$ is a categorical Leibniz K-algebra (see [3]).

We will start with (LeibT1). Let $n, n' \in N$.

$$\bar{s}(\bar{\tau}_{n,n'}) = \bar{s}((-2\{n, n'\}, [n, n'])) = [n, n'],$$
$$\bar{t}(\bar{\tau}_{n,n'}) = \bar{t}((-2\{n, n'\}, [n, n'])) = -2\partial\{n, n'\} + [n, n'] = -2[n, n'] + [n, n'] = -[n, n'],$$

where we use (BLeib1). In the same way we can prove this property of $\bar{\psi}$ by the symmetry of the construction.

We will prove now (LeibT2). Again, we will only check this for $\bar{\tau}$. Let $x = (m, n), y = (m', n') \in M \rtimes N$.

We need to show that $\tau_{t(x),t(y)} \circ [x, y] = -[x, y] \circ \tau_{s(x),s(y)}$. Now, we will write the equalities in function of the data given by the braided crossed module.

$$\begin{aligned}
&\tau_{t(x),t(y)} \circ [x, y]\\
&= \bar{k}(([(m, n), (m', n')], (-2\{\bar{t}((m, n)), \bar{t}((m', n'))\}, [\bar{t}((m, n)), \bar{t}((m', n'))])))\\
&= \bar{k}(([(m, n), (m', n')], (-2\{\partial m + n, \partial m' + n'\}, [\partial m + n, \partial m' + n'])))\\
&= \bar{k}((([m, m'] + n \cdot_1 m' + m \cdot_2 n', [n, n']), (-2\{\partial m + n, \partial m' + n'\}, [\partial m + n, \partial m' + n'])))\\
&= ([m, m'] + n \cdot_1 m' + m \cdot_2 n' - 2\{\partial m + n, \partial m' + n'\}, [n, n'])\\
&= ([m, m'] + n \cdot_1 m' + m \cdot_2 n' - 2\{\partial m, \partial m'\} - 2\{\partial m, n'\} - 2\{n, \partial m'\} - 2\{n, n'\}, [n, n'])\\
&= ([m, m'] + n \cdot_1 m' + m \cdot_2 n' - 2[m, m'] - 2(m \cdot_2 n') - 2(n \cdot_1 m') - 2\{n, n'\}, [n, n'])\\
&= (-[m, m'] - n \cdot_1 m' - m \cdot_2 n' - 2\{n, n'\}, [n, n']),
\end{aligned}$$

where we use (BLeib2), (BLeib3) and (BLeib4) in the sixth equality. In the other way,

$$\begin{aligned}
&-[x,y]\circ\tau_{s(x),s(y)}\\
&=\bar{k}(((-2\{\bar{s}((m,n)),\bar{s}((m,n'))\},[\bar{s}((m,n)),\bar{s}((m',n'))]),-[(m,n),(m',n')]))\\
&=\bar{k}(((-2\{n,n'\},[n,n']),-[(m,n),(m',n')]))\\
&=\bar{k}(((-2\{n,n'\},[n,n']),(-[m,m']-n\cdot_1 m'-m\cdot_2 n',-[n,n'])))\\
&=(-2\{n,n'\}-[m,m']-n\cdot_1 m'-m\cdot_2 n',[n,n']).
\end{aligned}$$

We will satisfy (LeibT3) below. Let $n,n',n''\in N$. Then

$$\begin{aligned}
\bar{\tau}_{n,[n',n'']}&=(-2\{n,[n',n'']\},[n,[n',n'']])\\
&=(-2(\{[n,n'],n''\}-\{[n,n''],n'\}),[[n,n'],n'']-[[n,n''],n'])\\
&=(-2\{[n,n'],n''\},[[n,n'],n''])-(-2\{[n,n''],n'\},[[n,n''],n'])\\
&=\bar{\tau}_{[n,n'],n''}-\bar{\tau}_{[n,n''],n'},
\end{aligned}$$

where we use (BLeib5) and the Leibniz identity in the second equality.

The same argument is valid for (LeibT6), using (BLeib8) and by the symmetry of the properties.

Finally, we will show that (LeibT4) and (LeibT5) are satisfied.

$$\begin{aligned}
\bar{\psi}_{n,[n',n'']}&=(-2\langle n,[n',n'']\rangle,[n,[n',n'']])\\
&=(-2(\{[n,n'],n''\}-\langle[n,n''],n'\rangle),[[n,n'],n'']-[[n,n''],n'])\\
&=(-2\{[n,n'],n''\},[[n,n'],n''])-(-2\langle[n,n''],n'\rangle,[[n,n''],n'])\\
&=\bar{\tau}_{[n,n'],n''}-\bar{\psi}_{[n,n''],n'}\\
&=(-2(\{[n,n'],n''\}-\langle[n,n''],n'\rangle),[[n,n'],n'']-[[n,n''],n'])\\
&=(-2\{n,[n',n'']\},[n,[n',n'']])=\bar{\tau}_{n,[n',n'']},
\end{aligned}$$

where we use (BLeib6) along with the Leibniz identity in the second equality; and (BLeib7) with the Leibniz identity in the penultimate equality. □

Remark 3.2. *Note that if $\mathcal{X}$ is a braided crossed module of Lie K-algebras, then*

$$\bar{\tau}_{n,n'}=(-2\{n,n'\},[n,n'])=-(-2\langle n',n\rangle,[n',n])=-\bar{\psi}_{n',n}$$

and we recover the construction for the Lie case (see [5]).

Proposition 3.3. *We have a functor $C\colon \boldsymbol{BX}(\boldsymbol{LeibAlg}_K)\to\boldsymbol{BICat}(\boldsymbol{LeibAlg}_K)$ defined as*

$$C(\mathcal{X}\xrightarrow{(f_1,f_2)}\mathcal{X}'):=C_{\mathcal{X}}\xrightarrow{(f_1\times f_2,f_2)}C_{\mathcal{X}'},\ \textit{where } C_{\mathcal{X}} \textit{ is constructed in the previous proposition}.$$

Proof. It is enough to see that $(f_1\times f_2,f_2)$ is a braided internal functor of braided categorical Leibniz K-algebras, since $(f_1\times f_2,f_2)$ is an internal functor between the respective internal categories (see [3]).

We will satisfy (LeibHT1). Let $n,n'\in N$.

$$\begin{aligned}
(f_1\times f_2)(\bar{\tau}_{n,n'})&=(f_1\times f_2)((-2\{n,n'\},[n,n']))=(-2f_1(\{n,n'\}),f_2([n,n']))\\
&=(-2\{f_2(n),f_2(n')\}',[f_2(n),f_2(n')])=\bar{\tau}'_{f_2(n),f_2(n')},
\end{aligned}$$

where we use (LeibHB1) in the penultimate equality.

Again, the same argument is valid to prove (LeibHT2), using (LeibHB2) and because of the symmetry of the braiding's properties and the construction. □

Proposition 3.4. *Let $C=(C_1,C_0,s,t,e,k,(\tau,\psi))$ be a braided categorical Leibniz K-algebra.*

Then $\mathcal{X}_C:=(\ker(s),C_0,({}^{e}\cdot,\cdot^{e}),\partial_t,(\{-,-\}_\tau,\langle-,-\rangle_\psi))$ is a braided crossed module of Leibniz K-algebras where

- ${}^{e}\cdot : C_0 \times \ker(s) \to \ker(s),\ a\, {}^{e}\cdot\, x := [e(a), x]$,
- $\cdot^{e} : \ker(s) \times C_0 \to \ker(s),\ x \cdot^{e} a := [x, e(a)]$,
- $\partial_t := t|_{\ker(s)}$,
- $\{-,-\}_\tau : C_0 \times C_0 \to \ker(s),\ \{a,b\}_\tau := \frac{e([a,b])-\tau_{a,b}}{2}$,
- $\langle -,-\rangle_\psi : C_0 \times C_0 \to \ker(s),\ \langle a,b\rangle_\psi := \frac{e([a,b])-\psi_{a,b}}{2}$.

Proof. It is enough to show that $(\{-,-\}_\tau, \langle -,-\rangle_\psi)$ is a braiding on the crossed module of Leibniz K-algebras $(\ker(s), C_0, (\cdot^{e}, {}^{e}\cdot), \partial_t)$ (see [3]).

First let us see that it is well defined because the image falls in C_1 which is not $\ker(s)$. We will check it only for $\{-,-\}_\tau$, since for $\langle -,-\rangle_\psi$ we will have a completely symmetric argument. Let $a, b \in C_0$, and using (LeibT1), we have

$$s(\{a,b\}_\tau) = s\Big(\frac{e([a,b]) - \tau_{a,b}}{2}\Big) = \frac{[a,b]-[a,b]}{2} = 0.$$

To check (BLeib1)–(BLeib4) we will only prove it for $\{-,-\}_\tau$.

First, we will check (BLeib1). Let $a, b \in C_0$, and using (LeibT1), we get

$$\partial_t\{a,b\}_\tau = t\Big(\frac{e([a,b]) - \tau_{a,b}}{2}\Big) = \frac{[a,b] - (-[a,b])}{2} = \frac{2[a,b]}{2} = [a,b].$$

We will see now (BLeib2). Let $x, y \in \ker(s)$. Then

$$\{\partial_t x, \partial_t y\}_\tau = \frac{e([\partial_t x, \partial_t y]) - \tau_{\partial_t x, \partial_t y}}{2} = \frac{e([t(x), t(y)]) - \tau_{t(x),t(y)}}{2}.$$

Let us see that $\frac{e([t(x),t(y)])-\tau_{t(x),t(y)}}{2} = [x,y]$.

By axiom (LeibT2), we have $k(([x,y], \tau_{t(x),t(y)})) = k((\tau_{s(x),s(y)}, -[x,y]))$. As $x \in \ker(s)$, we have that $s(x) = 0$ (in the same way y), and $\tau_{s(x),s(y)} = 0$ by K-bilinearity. So, we have $k((\tau_{s(x),s(y)}, -[x,y])) = k((0, -[x,y]))$, and therefore $k(([x,y], \tau_{t(x),t(y)})) = k((0, -[x,y]))$. Using now the K-linearity of k in the previous expression, we obtain $0 = k(([x,y], \tau_{t(x),t(y)} + [x,y]))$.

Since $t(\tau_{t(x),t(y)} + [x,y]) = -[t(x),t(y)] + [t(x),t(y)] = 0 = s(e(0))$ we can talk about $k((\tau_{t(x),t(y)} + [x,y], e(0)))$. Further $k((\tau_{t(x),t(y)} + [x,y], e(0))) = \tau_{t(x),t(y)} + [x,y]$ by the internal category axioms.

Adding both equalities and by using the K-linearity of k we get

$$k(([x,y] + \tau_{t(x),t(y)} + [x,y], \tau_{t(x),t(y)} + [x,y])) = \tau_{t(x),t(y)} + [x,y].$$

Therefore, by grouping, we have

$$k((2[x,y] + \tau_{t(x),t(y)}, \tau_{t(x),t(y)} + [x,y])) = \tau_{t(x),t(y)} + [x,y].$$

By using that $\ker(s)$ is an ideal and the fact that x or y are in $\ker(s)$, we have $s(\tau_{t(x),t(y)} + [x,y]) = [t(x),t(y)] - 0 = [t(x),t(y)]$, and so it makes sense to speak about the composition $k((e([t(x),t(y)]), \tau_{t(x),t(y)} + [x,y]))$, which is equal to $\tau_{t(x),t(y)} + [x,y]$.

Subtracting both equalities and using the K-linearity of k, we obtain

$$k((2[x,y] + \tau_{t(x),t(y)} - e([t(x),t(y)]), 0)) = 0.$$

Again, using the properties for internal categories, we have

$$\begin{aligned} 0 &= k((2[x,y] + \tau_{t(x),t(y)} - e([t(x),t(y)]), 0)) \\ &= k((2[x,y] + \tau_{t(x),t(y)} - e([t(x),t(y)]), e(0))) \\ &= 2[x,y] + \tau_{t(x),t(y)} - e([t(x),t(y)]), \end{aligned}$$

which gives us the required equality, since $\mathrm{char}(K) \neq 2$.

As an observation to the above, in the part of the proof where we use that $x, y \in \ker(s)$, it is sufficient that one of the two is in that kernel. Therefore, by repeating the argument, we have the following equalities for $x \in \ker(s)$ and $y \in C_1$:

$$\frac{e([t(x), t(y)]) - \tau_{t(x),t(y)}}{2} = [x, y], \qquad \frac{e([t(y), t(x)]) - \tau_{t(y),t(x)}}{2} = [y, x].$$

With these equalities, we will check (BLeib3) and (BLeib4).

Let $a \in C_0$ and $x \in \ker(s)$. Then

$$\begin{aligned}\{\partial_t x, a\}_\tau &= \frac{e([t(x), t(e(a))]) - \tau_{t(x),t(e(a))}}{2} = [x, e(a)] = x \cdot^{e} a,\\ \{a, \partial_t x\}_\tau &= \frac{e([t(e(a)), t(x)]) - \tau_{t(e(a)),t(x)}}{2} = [e(a), x] = a \,{}^{e}\!\cdot x.\end{aligned}$$

We will see now the last conditions, starting with (BLeib5). Let $a, b, c \in C_0$.

$$\begin{aligned}\{a, [b, c]\}_\tau &= \frac{e([a, [b, c]]) - \tau_{a,[b,c]}}{2} = \frac{e([[a, b], c]) - e([[a, c], b]) - \tau_{[a,b],c} + \tau_{[a,c],b}}{2}\\ &= \frac{e([[a, b], c]) - \tau_{[a,b],c}}{2} - \frac{e([[a, c], b]) - \tau_{[a,c],b}}{2} = \{[a, b], c\}_\tau - \{[a, c], b\}_\tau,\end{aligned}$$

where we use (LeibT3) and the Leibniz identity in the second equality. By symmetry we can prove (BLeib8), using (LeibT6).

To conclude we will check (BLeib6) and (BLeib7).

$$\begin{aligned}\langle a, [b, c]\rangle_\psi &= \frac{e([a, [b, c]]) - \psi_{a,[b,c]}}{2} = \frac{e([[a, b], c]) - e([[a, c], b]) - \tau_{[a,b],c} + \psi_{[a,c],b}}{2}\\ &= \frac{e([[a, b], c]) - \tau_{[a,b],c}}{2} - \frac{e([[a, c], b]) - \psi_{[a,c],b}}{2} = \{[a, b], c\}_\tau - \langle [a, c], b\rangle_\psi\\ &= \frac{e([[a, b], c]) - e([[a, c], b]) - \tau_{[a,b],c} + \psi_{[a,c],b}}{2} = \frac{e([a, [b, c]]) - \tau_{a,[b,c]}}{2}\\ &= \{a, [b, c]\}_\tau,\end{aligned}$$

where we use (LeibT4) in the second equality together with Leibniz identity and (LeibT5) in the penultimate equality with the Leibniz identity. □

Remark 3.5. *Note that if* C *is a braided categorical Lie K-algebra, then*

$$\{a, b\}_\tau = \frac{e([a, b]) - \tau_{a,b}}{2} = -\frac{e([b, a]) - \psi_{b,a}}{2} = -\langle b, a\rangle_\psi$$

and we recover the construction for the Lie case.

Proposition 3.6. *We have a functor* $\mathcal{X}\colon \boldsymbol{ICat}(\boldsymbol{LeibAlg}_K) \to \boldsymbol{BX}(\boldsymbol{LeibAlg}_K)$ *defined as*

$$\mathcal{X}(C \xrightarrow{(F_1,F_0)} C') = \mathcal{X}_C \xrightarrow{(F_1^s,F_0)} \mathcal{X}_{C'},$$

where $\mathcal{X}_C$ *is constructed in the previous proposition and* $F_1^s\colon \ker(s) \to \ker(s')$ *is defined as* $F_1^s(x) = F_1(x)$ *for* $x \in \ker(s)$.

Proof. $\mathcal{X}$ is a functor between the categories without braiding (see [3]). So, we only must check the axioms of the homomorphisms of braided crossed of Leibniz K-algebras.

We will start with (LeibHB1). Let $a, b \in C_0$.

$$\begin{aligned} F_1^s(\{a,b\}_\tau) &= F_1\Big(\frac{e([a,b]) - \tau_{a,b}}{2}\Big) = \frac{F_1(e([a,b])) - F_1(\tau_{a,b})}{2} \\ &= \frac{e'(F_0([a,b])) - \tau'_{F_0(a),F_0(b)}}{2} = \frac{e'([F_0(a),F_0(b)]) - \tau'_{F_0(a),F_0(b)}}{2} \\ &= \{F_0(a), F_0(b)\}_{\tau'}, \end{aligned}$$

where we use (LeibHT1) in the third equality.

Again, we can prove (LeibHB2) using the same argument, (LeibHT2) and the symmetry. □

Remark 3.7. *Note that, if* $(M, N, (\cdot_1, \cdot_2), \partial, (\{-,-\}, \langle -,-\rangle))$ *then* $\ker(\bar{s}) = \{(m, 0) \in M \rtimes N \mid m \in M\}$, *where* $\bar{s}$ *is defined for the functor* C.

Proposition 3.8. *The categories* $\textbf{BX}(\textbf{LeibAlg}_K)$ *and* $\textbf{ICat}(\textbf{LeibAlg}_K)$ *are equivalent categories.*

Further, the functors C *and* $\mathcal{X}$ *are inverse equivalences, where the natural isomorphisms* $\mathrm{Id}_{\textbf{BX}(\textbf{LeibAlg}_K)} \overset{\alpha}{\cong} \mathcal{X} \circ C$ *and* $\mathrm{Id}_{\textbf{ICat}(\textbf{LeibAlg}_K)} \overset{\beta}{\cong} C \circ \mathcal{X}$ *are given by:*

- *If* $\mathcal{Z} = (M, N, (\cdot_1, \cdot_2), \partial, (\{-,-\}, \langle -,-\rangle))$ *is a braided crossed module of Leibniz K-algebras, then* $\alpha_{\mathcal{Z}} = (\alpha_M, \mathrm{Id}_N)$, *where* $\alpha_M \colon M \to (M, 0)$ *is defined by* $\alpha_M(m) = (m, 0)$;
- *If* $\mathcal{D} = (C_1, C_0, s, t, e, k, (\tau, \psi))$ *is a braided categorical Leibniz K-algebra, then* $\beta_{\mathcal{D}} = (\beta_s, \mathrm{Id}_{C_0})$, *where* $\beta_{C_1} \colon C_1 \to \ker(s) \rtimes C_0$ *is defined by* $\beta_{C_1}(x) = (x - e(s(x)), s(x))$.

Proof. C and $\mathcal{X}$ are natural isomorphisms in the categories without braiding (see [3]). So, it is enough to show that they are morphisms between braided objects.

Let $\mathcal{Z} = (M, N, (\cdot_1, \cdot_2), \partial, (\{-,-\}, \langle -,-\rangle))$ a braided crossed module of Leibniz K-algebras. Let us see that $\alpha_{\mathcal{Z}} = (\alpha_M, \mathrm{Id}_N)$ satisfies (LeibHB1).

$$\begin{aligned} \mathrm{Id}_N(\{n,n'\}_{\bar{\tau}}) = \{n,n'\}_{\bar{\tau}} &= \frac{\bar{e}([n,n']) - \bar{\tau}_{n,n'}}{2} = \frac{(0,[n,n']) - (-2\{n,n'\},[n,n'])}{2} \\ &= \frac{(2\{n,n'\},0)}{2} = (\{n,n'\},0) = \alpha_M(\{n,n'\}), \qquad n,n' \in N. \end{aligned}$$

Analogously (LeibHB2) is proven, by the similarity of definitions.

Let $\mathcal{D} = (C_1, C_0, s, t, e, k, (\tau, \psi))$ be a braided categorical Leibniz K-algebra. We will check that $\beta_{\mathcal{D}} = (\beta_s, \mathrm{Id}_{C_0})$ satisfies (LeibHT1) and (LeibHT2). We only show the proof for (LeibHT1), since the one for (LeibHT2) is similar.

Let us consider $a, b \in C_0$. We have:

$$\begin{aligned} \mathrm{Id}_{C_0}(\bar{\tau}_{a,b}) = \bar{\tau}_{a,b} &= (-2\{a,b\}_\tau, [a,b]) = (-2\frac{e([a,b]) - \tau_{a,b}}{2}, [a,b]) \\ &= (\tau_{a,b} - e([a,b]), [a,b]) = (\tau_{a,b} - e(s(\tau_{a,b})), s(\tau_{a,b})) = \beta_{C_1}(\tau_{a,b}). \end{aligned}$$

□

4. The non-abelian tensor product as example of braiding

If $(M, [-,-])$ is a Leibniz K-algebra, then $([-,-],[-,-])$ is a braiding on $(M, M, ([-,-],[-,-]), \mathrm{Id}_M)$. This example is analogous for the case of Leibniz K-algebras of the models $(G, G, \mathrm{Conj}, \mathrm{Id}_G, [-,-])$ for groups and $(M, M, [-,-], \mathrm{Id}_M, [-,-])$ for Lie K-algebras. Further, this example generalizes the Lie example, since $[y, x] = -[x, y]$ in this case.

We will give another symmetric instance in the three frameworks: the non-abelian tensor product. The non-abelian tensor product of Leibniz K-algebras was introduced by Gnedbaye in [7], where the tensor product is denoted as $M \star N$, and its generators as $m * n$ and $n * m$. In the general case it does not give rise to confusion, but in the case $M = N$ these generators would be denoted in the same way, giving rise to confusion. To avoid this, we change the nomenclature, meaning $m * n$ as $m \otimes n$ and $n * m$ as $n \circledast m$.

Definition 4.1. *Let M and N two Leibniz K-algebras together with two Leibniz actions $\cdot = (\cdot_1, \cdot_2)$ of M on N and $* = (*_1, *_2)$ of N on M.*

The non-abelian tensor product of M and N, denoted by $M \star N$, is the Leibniz K-algebra generated by the symbols $m \otimes n$ and $n \circledast m$ with $m \in M$, $n \in N$, together with the relations:

$$\begin{aligned} &\lambda(m \otimes n) = \lambda m \otimes n = m \otimes \lambda n, \\ &\lambda(n \circledast m) = \lambda n \circledast m = n \circledast \lambda m, \end{aligned} \tag{RTLeib1}$$

$$\begin{aligned} &(m + m') \otimes n = m \otimes n + m' \otimes n, \\ &m \otimes (n + n') = m \otimes n + m \otimes n', \\ &(n + n') \circledast m = n \circledast m + n' \circledast m, \\ &n \circledast (m + m') = n \circledast m + n \circledast m', \end{aligned} \tag{RTLeib2}$$

$$\begin{aligned} &m \otimes [n, n'] = (m *_2 n) \otimes n' - (m *_2 n') \otimes n, \\ &n \circledast [m, m'] = (n \cdot_2 m) \circledast m' - (n \cdot_2 m') \circledast m, \\ &[m, m'] \otimes n = (m \cdot_1 n) \circledast m' - m \otimes (n \cdot_2 m'), \\ &[n, n'] \circledast m = (n *_1 m) \otimes n' - n \circledast (m *_2 n'), \end{aligned} \tag{RTLeib3}$$

$$\begin{aligned} &m \otimes (m' \cdot_1 n) = -m \otimes (n \cdot_2 m'), \\ &n \circledast (n' *_1 m) = -n \circledast (m *_2 n'), \end{aligned} \tag{RTLeib4}$$

$$\begin{aligned} &(m *_2 n) \otimes (m' \cdot_1 n') = [m \otimes n, m' \otimes n'] = (m \cdot_1 n) \circledast (m' *_2 n'), \\ &(m *_2 n) \otimes (n' \cdot_2 m') = [m \otimes n, n' \circledast m'] = (m \cdot_1 n) \circledast (n' *_1 m'), \\ &(n *_1 m) \otimes (n' \cdot_2 m') = [n \circledast m, n' \circledast m'] = (n \cdot_2 m) \circledast (n' *_1 m'), \\ &(n *_1 m) \otimes (m' \cdot_1 n') = [n \circledast m, m' \otimes n'] = (n \cdot_2 m) \circledast (m' *_2 n'), \qquad m, m' \in M,\ n, n' \in N. \end{aligned} \tag{RTLeib5}$$

Proposition 4.2 ([7]). *Let M be a Leibniz K-algebra.*

Then $(M \star M, M, (\cdot_1, \cdot_2), \partial)$ is a crossed module of Leibniz K-algebras, where $M \star M$ is the non-abelian tensor product of M with itself using the actions given by the Leibniz bracket, where

- *the left action on generators is given by $m \cdot_1 (m_1 \otimes m_2) = [m, m_1] \otimes m_2 - [m, m_2] \circledast m_1$, $m \cdot_1 (m_1 \circledast m_2) = [m, m_1] \circledast m_2 - [m, m_2] \otimes m_1$;*
- *the right action on generators is given by $(m_1 \otimes m_2) \cdot_2 m = [m_1, m] \otimes m_2 + m_1 \otimes [m_2, m]$, $(m_1 \circledast m_2) \cdot_2 m = [m_1, m] \circledast m_2 + m_1 \circledast [m_2, m]$;*
- *the map ∂ is defined on generators as $\partial(m_1 \otimes m_2) = [m_1, m_2] = \partial(m_1 \circledast m_2)$.*

Remark 4.3. *We will show how are the relations* (RTLeib3)–(RTLeib5) *for the non-abelian tensor product $M \star M$*

with the action $([-,-],[-,-])$ *on itself:*

$$m_1 \otimes [m_2, m_3] = [m_1, m_2] \otimes m_3 - [m_1, m_3] \otimes m_2, \quad \text{(RTLeib3)}$$
$$m_1 \circledast [m_2, m_3] = [m_1, m_2] \circledast m_3 - [m_1, m_3] \circledast m_2,$$
$$[m_1, m_2] \otimes m_3 = [m_1, m_3] \circledast m_2 - m_1 \otimes [m_3, m_2],$$
$$[m_1, m_2] \circledast m_3 = [m_1, m_3] \otimes m_2 - m_1 \circledast [m_3, m_2],$$
$$m_1 \otimes [m_2, m_3] = -m_1 \otimes [m_3, m_2], \quad \text{(RTLeib4)}$$
$$m_1 \circledast [m_2, m_3] = -m_1 \circledast [m_3, m_2],$$
$$[m_1, m_2] \otimes [m_3, m_4] = [m_1 \otimes m_2, m_3 \otimes m_4] = [m_1, m_2] \circledast [m_3, m_4], \quad \text{(RTLeib5)}$$
$$[m_1, m_2] \otimes [m_3, m_4] = [m_1 \otimes m_2, m_3 \circledast m_4] = [m_1, m_2] \circledast [m_3, m_4],$$
$$[m_1, m_2] \otimes [m_3, m_4] = [m_1 \circledast m_2, m_3 \circledast m_4] = [m_1, m_2] \circledast [m_3, m_4],$$
$$[m_1, m_2] \otimes [m_3, m_4] = [m_1 \circledast m_2, m_3 \otimes m_4] = [m_1, m_2] \circledast [m_3, m_4], \qquad m_1, m_2, m_3, m_4 \in M.$$

The following example shows the necessity of a pair of braidings for the Leibniz K-algebras case since they will be different.

Example 4.4. *Let M be a Leibniz K-algebra.*

The pair of K-bilinear maps $\{-,-\}, \langle -,- \rangle \colon M \times M \to M \otimes M$ *defined as* $\{m_1, m_2\} = m_1 \otimes m_2$ *and* $\langle m_1, m_2 \rangle = m_1 \circledast m_2$ *is a braiding on the crossed module of Leibniz K-algebras* $(M \star M, M, (\cdot_1, \cdot_2), \partial)$.

First, will check (BLeib1).

$$\partial\{m_1, m_2\} = \partial(m_1 \otimes m_2) = [m_1, m_2] = \partial(m_1 \circledast m_2) = \partial\langle m_1, m_2 \rangle, \qquad m_1, m_2 \in M.$$

Now, we will prove (BLeib2).

$$\{\partial(m_1 \otimes m_2), \partial(m_3 \otimes m_4)\} = [m_1, m_2] \otimes [m_3, m_4] = [m_1 \otimes m_2, m_3 \otimes m_4],$$
$$\{\partial(m_1 \otimes m_2), \partial(m_3 \circledast m_4)\} = [m_1, m_2] \otimes [m_3, m_4] = [m_1 \otimes m_2, m_3 \circledast m_4],$$
$$\{\partial(m_1 \circledast m_2), \partial(m_3 \otimes m_4)\} = [m_1, m_2] \otimes [m_3, m_4] = [m_1 \circledast m_2, m_3 \otimes m_4],$$
$$\{\partial(m_1 \circledast m_2), \partial(m_3 \circledast m_4)\} = [m_1, m_2] \otimes [m_3, m_4] = [m_1 \circledast m_2, m_3 \circledast m_4],$$
$$\langle \partial(m_1 \otimes m_2), \partial(m_3 \otimes m_4) \rangle = [m_1, m_2] \circledast [m_3, m_4] = [m_1 \otimes m_2, m_3 \otimes m_4],$$
$$\langle \partial(m_1 \otimes m_2), \partial(m_3 \circledast m_4) \rangle = [m_1, m_2] \circledast [m_3, m_4] = [m_1 \otimes m_2, m_3 \circledast m_4],$$
$$\langle \partial(m_1 \circledast m_2), \partial(m_3 \otimes m_4) \rangle = [m_1, m_2] \circledast [m_3, m_4] = [m_1 \circledast m_2, m_3 \otimes m_4],$$
$$\langle \partial(m_1 \circledast m_2), \partial(m_3 \circledast m_4) \rangle = [m_1, m_2] \circledast [m_3, m_4] = [m_1 \circledast m_2, m_3 \circledast m_4],$$

where in all the cases we used (RTLeib5).

Before to check the following axioms, we need to check a property that can be proven using (RTLeib3) *and* (RTLeib4).

Using (RTLeib4) *in the last equality of relation* (RTLeib3) *and rewriting that equality and the second one, we get*

$$m_1 \circledast [m_2, m_3] = [m_1, m_2] \circledast m_3 - [m_1, m_3] \circledast m_2,$$
$$m_1 \circledast [m_2, m_3] = [m_1, m_2] \circledast m_3 - [m_1, m_3] \otimes m_2.$$

Subtracting, we obtain the equality $[m_1, m_3] \otimes m_2 = [m_1, m_3] \circledast m_2$*. Using this last equality and the first and second equality of* (RTLeib3)*, we obtain*

$$m_1 \otimes [m_2, m_3] = [m_1, m_2] \otimes m_3 - [m_1, m_3] \otimes m_2 = [m_1, m_2] \circledast m_3 - [m_1, m_3] \circledast m_2 = m_1 \circledast [m_2, m_3].$$

Let us satisfy now the first equality of (BLeib3) *with* $m, m_1, m_2 \in M$,

$$\begin{aligned}\{\partial(m_1 \otimes m_2), m\} &= [m_1, m_2] \otimes m = [m_1, m] \circledast m_2 - m_1 \otimes [m, m_2] \\ &= [m_1, m] \circledast m_2 + m_1 \otimes [m_2, m] = [m_1, m] \otimes m_2 + m_1 \otimes [m_2, m] = (m_1 \otimes m_2) \cdot_2 m,\end{aligned}$$

where we use (RTLeib3) *and* (RTLeib4).

The second equality is analogous:

$$\begin{aligned}\{\partial(m_1 \circledast m_2), m\} &= [m_1, m_2] \otimes m = [m_1, m] \circledast m_2 + m_1 \otimes [m_2, m] \\ &= [m_1, m] \circledast m_2 + m_1 \circledast [m_2, m] = (m_1 \circledast m_2) \cdot_2 m.\end{aligned}$$

Using the exchange properties between $\otimes$ *and* $\circledast$ *again, we will see the remaining equalities:*

$$\begin{aligned}\langle \partial(m_1 \otimes m_2), m\rangle &= [m_1, m_2] \circledast m = [m_1, m_2] \otimes m = (m_1 \otimes m_2) \cdot_2 m, \\ \langle \partial(m_1 \circledast m_2), m\rangle &= [m_1, m_2] \circledast m = [m_1, m_2] \otimes m = (m_1 \circledast m_2) \cdot_2 m.\end{aligned}$$

Now we will check the next axiom, (BLeib4)*, where we will use again that we can exchange the symbols if in one side is the bracket. Starting with the first equality, we have*

$$\begin{aligned}\{m, \partial(m_1 \otimes m_2)\} &= m \otimes [m_1, m_2] = [m, m_1] \otimes m_2 - [m, m_2] \otimes m_1 \\ &= [m, m_1] \otimes m_2 - [m, m_2] \circledast m_1 = m \cdot_1 (m_1 \otimes m_2),\end{aligned}$$

where we use (RTLeib3)*. Analogously we obtain the second equality:*

$$\begin{aligned}\{m, \partial(m_1 \circledast m_2)\} &= m \otimes [m_1, m_2] = [m, m_1] \otimes m_2 - [m, m_2] \otimes m_1 \\ &= [m, m_1] \circledast m_2 - [m, m_2] \otimes m_1 = m \cdot_1 (m_1 \circledast m_2).\end{aligned}$$

So, the following properties are immediate:

$$\begin{aligned}\langle m, \partial(m_1 \otimes m_2)\rangle &= m \circledast [m_1, m_2] = m \otimes [m_1, m_2] = m \cdot_1 (m_1 \otimes m_2), \\ \langle m, \partial(m_1 \circledast m_2)\rangle &= m \circledast [m_1, m_2] = m \otimes [m_1, m_2] = m \cdot_1 (m_1 \circledast m_2).\end{aligned}$$

To finalize, we will prove (BLeib5)*, because, if it is satisfied;* (BLeib6)–(BLeib8) *will be fulfilled using the following properties:*

$$\begin{aligned}\{m, [m', m'']\} &= m \otimes [m', m''] = m \circledast [m', m''] = \langle m, [m', m'']\rangle, \\ \{[m, m'], m''\} &= [m, m'] \otimes m'' = [m, m'] \circledast m'' = \langle [m, m'], m''\rangle.\end{aligned}$$

By using (RTLeib3)*, we have* (BLeib5)*:*

$$\{m, [m', m'']\} = m \otimes [m', m''] = [m, m'] \otimes m'' - [m, m''] \otimes m' = \{[m, m'], m''\} - \{[m, m''], m'\},$$

Remark 4.5. *Note that the actions can be written with a simpler notation, given by*

$$\begin{aligned}&m \cdot_1 (m_1 \otimes m_2) = m \cdot_1 (m_1 \circledast m_2) = m \otimes [m_1, m_2] = m \circledast [m_1, m_2], \\ &(m_1 \otimes m_2) \cdot_2 m = (m_1 \circledast m_2) \cdot_2 m = [m_1, m_2] \otimes m = [m_1, m_2] \circledast m.\end{aligned}$$

Remark 4.6. *Example 4.4 generalizes the Lie example, since if we have that* $m_1 \circledast m_2 = -m_2 \otimes m_1$ *as a new relation, we obtain the Lie non-abelian tensor product of M with itself using the adjoint action.*

References

[1] D. Conduché, Modules croisés généralisés de longueur 2, J. Pure Appl. Algebra 34(2-3) (1984) 155–178.

[2] P. Dedecker, A. S.-T. Lue, A nonabelian two-dimensional cohomology for associative algebras, Bull. Amer. Math. Soc. 72 (1966) 1044–1050.

[3] R. Fernández-Casado, Relations between crossed modules of different algebras, Ph.D. thesis, Universidad de Santiago de Compostela, `http://hdl.handle.net/10347/13839`, 2015.

[4] R. Fernández-Casado, X. García-Martínez, M. Ladra, A natural extension of the universal enveloping algebra functor to crossed modules of Leibniz algebras, Appl. Categ. Structures 25 (6) (2017) 1059–1076.

[5] A. Fernández-Fariña, Trenzas y algebras de Lie categóricas, Master's thesis, Universidad de Santiago de Compostela, 2016.

[6] A. Fernández-Fariña, M. Ladra, Braiding for categorical algebras and crossed modules of algebras I: Associative algebras and Lie algebras, J. Algebra Appl. 19 (9) (2020) 2050176, 30 pp.
[7] A. V. Gnedbaye, A non-abelian tensor product of Leibniz algebras, Ann. Inst. Fourier (Grenoble) 49 (4) (1999) 1149–1177.
[8] A. Joyal, R. Street, Braided monoidal categories, Macquarie Math. Reports No. 860081 (1986).
[9] C. Kassel, J.-L. Loday, Extensions centrales d'algèbres de Lie, Ann. Inst. Fourier (Grenoble) 32 (1982) 119–142.
[10] J.-L. Loday, T. Pirashvili, Universal enveloping algebras of Leibniz algebras and (co)homology, Math. Ann. 296 (1993) 139–158.
[11] J.-L. Loday, T. Pirashvili, The tensor category of linear maps and Leibniz algebras, Georgian Math. J. 5 (3) (1998) 263–276.
[12] A. Rovi, Lie algebroids in the Loday-Pirashvili category, SIGMA Symmetry Integrability Geom. Methods Appl. 11 (2015) Paper 079, 16 pp.
[13] J. H. C. Whitehead, Combinatorial homotopy II, Bull. Amer. Math. Soc. 55 (1949) 453–496.