



On Special Subgroups of Fundamental Group

Fatemah Ayatollah Zadeh Shirazi^a, Fatemeh Ebrahimifar^a, Mohammad Ali Mahmoodi^a

^aFaculty of Mathematics, Statistics and Computer Science, College of Science, University of Tehran, Enghelab Ave., Tehran, Iran

Abstract. Suppose α is a nonzero cardinal number, \mathcal{I} is an ideal on arc connected topological space X , and $\mathfrak{F}_{\mathcal{I}}^{\alpha}(X)$ is the subgroup of $\pi_1(X)$ (the first fundamental group of X) generated by homotopy classes of $\alpha^{\mathcal{I}}$ -loops. The main aim of this text is to study $\mathfrak{F}_{\mathcal{I}}^{\alpha}(X)$ s and compare them. Most interest is in $\alpha \in \{\omega, c\}$ and $\mathcal{I} \in \{\mathcal{P}_{fin}(X), \{\emptyset\}\}$, where $\mathcal{P}_{fin}(X)$ denotes the collection of all finite subsets of X . We denote $\mathfrak{F}_{\{\emptyset\}}^{\alpha}(X)$ with $\mathfrak{F}^{\alpha}(X)$. We prove the following statements:

- for arc connected topological spaces X and Y if $\mathfrak{F}^{\alpha}(X)$ is isomorphic to $\mathfrak{F}^{\alpha}(Y)$ for all infinite cardinal number α , then $\pi_1(X)$ is isomorphic to $\pi_1(Y)$;
- there are arc connected topological spaces X and Y such that $\pi_1(X)$ is isomorphic to $\pi_1(Y)$ but $\mathfrak{F}^{\omega}(X)$ is not isomorphic to $\mathfrak{F}^{\omega}(Y)$;
- for arc connected topological space X we have $\mathfrak{F}^{\omega}(X) \subseteq \mathfrak{F}^c(X) \subseteq \pi_1(X)$;
- for Hawaiian earring \mathcal{X} , the sets $\mathfrak{F}^{\omega}(\mathcal{X})$, $\mathfrak{F}^c(\mathcal{X})$, and $\pi_1(\mathcal{X})$ are pairwise distinct.

So $\mathfrak{F}^{\alpha}(X)$ s and $\mathfrak{F}_{\mathcal{I}}^{\alpha}(X)$ s will help us to classify the class of all arc connected topological spaces with isomorphic fundamental groups.

1. Introduction

The main aim of algebraic topology is “classifying the topological spaces”. One of the first concepts introduced in algebraic topology is “fundamental group”. As it has been mentioned in [4, page1], fundamental groups are introduced by Poincaré. In this text we consider special subgroups of fundamental group. Explicitly we pay attention to path homotopy classes induced by loops which are “enough one to one”. We have the following sections:

1. Introduction
2. What is an $\alpha^{\mathcal{I}}$ -arc?
3. New subgroups
4. A useful remark
5. Primary properties of $\mathfrak{F}_{\mathcal{I}}^{\alpha}(X)$ s
6. Some preliminaries on Hawaiian earring
7. $\mathfrak{F}^c(\mathcal{X})$ is a proper subset of $\pi_1(\mathcal{X})$
8. $\mathfrak{F}_{\mathcal{P}_{fin}(\mathcal{Y})}^c(\mathcal{Y})$ is a proper subset of $\pi_1(\mathcal{Y})$

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Email addresses: fatemah@khayam.ut.ac.ir (Fatemah Ayatollah Zadeh Shirazi), ebrahimi.far64@ut.ac.ir (Fateme Ebrahimifar), m.a.mahmoodi@ut.ac.ir (Mohammad Ali Mahmoodi)

9. Main examples and counterexamples
10. Main Table
11. Two spaces having fundamental groups isomorphic to Hawaiian earring’s fundamental group
12. A distinguished counterexample
13. A diagram and a hint
14. A strategy for future and conjecture
15. Conclusion

Our main conventions located in section 2, although there are conventions in other sections too. Briefly, we introduce our new subgroups in Section 3 and obtain their primary properties in Section 5. Sections 6, 7 and 8 contain basic lemmas for our counterexamples in Section 9. Regarding these three sections 7, 8, and 9 we see $\mathfrak{F}^\omega(\mathcal{X}) \subset \mathfrak{F}^c(\mathcal{X}) \subset \pi_1(\mathcal{X})$ where \mathcal{X} is Infinite or Hawaiian earring and “ \subset ” means strict inclusion; also we see $\mathfrak{F}_{\mathcal{P}_{fin}(\mathcal{Y})}^\omega(\mathcal{Y}) \subset \mathfrak{F}_{\mathcal{P}_{fin}(\mathcal{Y})}^c(\mathcal{Y}) \subset \pi_1(\mathcal{Y})$ (\mathcal{Y} is introduced in Section 2). However Counterexamples of Section 9 are essential for Main Table in Section 10, which shows probable inclusion relations between different $\mathfrak{F}_I^\alpha(X)$ for a fix X (arc connected locally compact Hausdorff topological space), $\alpha \in \{\omega, c\}$ and $I \in \{\{\emptyset\}, \mathcal{P}_{fin}(X), \mathcal{P}(X)\}$ where $\mathcal{P}(X)$ is the power set of X and $\mathfrak{F}_{\mathcal{P}(X)}^\alpha(X)$ is just $\pi_1(X)$ (the fundamental group of X) by Section 5. We continue to discover the properties of “our new subgroups” in Sections 12 and 13, as a matter of fact in Sections 11 and 12 we see $\pi_1(\mathcal{X}) \cong \pi_1(\mathcal{W})$ and $\mathfrak{F}^\omega(\mathcal{X}) \not\cong \mathfrak{F}^\omega(\mathcal{W})$ (\mathcal{W} is introduced in Section 2), consequently we have a diagram and two problems in Section 13. As a matter of fact using the diagram of Section 13 and “Distinguished Example” in Section 12, we try to show “these new subgroups” can make meaningful subclasses of a class of arc connected locally compact Hausdorff topological spaces with the isomorphic fundamental groups.

Remembering all the conventions during reading the text is highly recommended.

Convention 1.1. A topological space X is an arc connected space, if for all $a, b \in X$ with $a \neq b$ there exists a continuous one to one map $f : [0, 1] \rightarrow X$ with $f(0) = a$ and $f(1) = b$. In this text all spaces assumed to be Hausdorff, locally compact, and arc connected with at least two elements.

Remark 1.2. Let X be an arbitrary set. We call $I \subseteq \mathcal{P}(X)$, an ideal on X , if:

- $I \neq \emptyset$,
- If $A, B \in I$, then $A \cup B \in I$,
- If $B \subseteq A$ and $A \in I$, then $B \in I$.

The collection of all finite subsets of X , $\mathcal{P}_{fin}(X)$, is one of the most famous ideals on X .

In this text ZFC+GCH (we recall that GCH or Generalized Continuum Hypothesis indicates that for transfinite cardinal number β , there is not any cardinal number γ with $\beta < \gamma < 2^\beta$, i.e. $2^\beta = \beta^+$ [3]) is assumed and by “ \subset ” we mean strict inclusion. Whenever G is a group isomorphic to group H , we write $G \cong H$. Also $G \not\cong H$ means that G is not isomorphic to H . Whenever $g \in G$ and $A \subseteq G$, then $\langle A \rangle$ denotes the subgroup of G generated by A , denote $\langle \{g\} \rangle$ simply by $\langle g \rangle$. We recall that ω is the cardinality of \mathbb{N} (the set of all natural numbers $\{1, 2, \dots\}$) and c is the cardinality of \mathbb{R} (the set of all real numbers). We denote the cardinality of A by $|A|$. For cardinal numbers (real numbers) α, β we denote the maximum of $\{\alpha, \beta\}$ by $\max(\alpha, \beta)$ also we denote the minimum of $\{\alpha, \beta\}$ by $\min(\alpha, \beta)$.

In addition for $n \in \mathbb{N}$, consider \mathbb{R}^n under Euclidean norm. Also consider $\mathbb{S}^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ as a subspace of \mathbb{R}^2 (or $\{e^{i\theta} : \theta \in [0, 2\pi]\}$ as a subspace of \mathbb{C} , the set of all complex numbers).

2. What is an α^I -arc?

The concept of α^I -arc is a generalization of α -arc which is originated from [1] and then in [2]. However a 1-arc or briefly arc is a one to one map $f : [0, 1] \rightarrow X$.

Definition 2.1. For nonzero cardinal number α , and ideal \mathcal{I} on X , the continuous map $f : Y \rightarrow X$ is called an $\alpha^{\mathcal{I}}$ -map if there exists $A \in \mathcal{I}$ such that for all $x \in X \setminus A$, $|f^{-1}(x)| < \alpha + 1$. In particular for infinite cardinal number α , the continuous map $f : Y \rightarrow X$ is an $\alpha^{\mathcal{I}}$ -map if there exists $A \in \mathcal{I}$ such that for all $x \in X \setminus A$, $|f^{-1}(x)| < \alpha$.

We call $\alpha^{\mathcal{I}}$ -map $f : [0, 1] \rightarrow X$, $\alpha^{\mathcal{I}}$ -arc. We call $\alpha^{\mathcal{I}}$ -map $f : [0, 1] \rightarrow X$ with $f(0) = f(1) = a$, an $\alpha^{\mathcal{I}}$ -loop with base point a .

We use briefly terms α -map, α -arc, and α -loop respectively instead of $\alpha^{\{\emptyset\}}$ -map, $\alpha^{\{\emptyset\}}$ -arc, and $\alpha^{\{\emptyset\}}$ -loop.

We want to study subgroups of $\pi_1(X)$ generated by path homotopy equivalence classes of α -loops and $\alpha^{\mathcal{I}}$ -loops for nonzero cardinal number α and ideal \mathcal{I} on X . We pay special attention to $\alpha^{\mathcal{I}}$ -loops for $\alpha \in \{\omega, c\}$ and $\mathcal{I} \in \{\mathcal{P}_{fin}(X), \{\emptyset\}\}$. We use the following spaces and loops in most counterexamples in this text.

Convention 2.2. Suppose $p \in \mathbb{N}$, let

$$\begin{aligned} \mathcal{X} &:= \left\{ \frac{1}{n}e^{2\pi i\theta} + \frac{i}{n} : n \in \mathbb{N}, \theta \in [0, 1] \right\} \\ & (= \bigcup_{n \in \mathbb{N}} \left\{ (x, y) \in \mathbb{R}^2 : x^2 + (y - \frac{1}{n})^2 = \frac{1}{n^2} \right\}) \quad (\text{Hawaiian earring}) \\ \mathcal{Y} &:= \bigcup \left\{ \frac{1}{2^{n+1}}\mathcal{X} + \frac{1}{n} : n \in \mathbb{N} \right\} \cup [0, 1] \\ \mathcal{Z} &:= \left\{ \frac{1}{k}e^{2\pi i(x-k-\frac{1}{4})} + \frac{i}{k} : k \in \{1, \dots, p\}, x \in [0, 1] \right\} \\ \mathcal{W} &:= \left\{ \frac{1}{2^{n+1}}e^{2\pi i\theta} + \frac{1}{n} + \frac{i}{2^{n+1}} : n \in \mathbb{N}, \theta \in [0, 1] \right\} \cup [0, 1] \\ \mathcal{C}_n &:= \left\{ \frac{1}{n}e^{2\pi it} + \frac{i}{n} : t \in [0, 1] \right\} \\ & (= \left\{ (x, y) \in \mathbb{R}^2 : x^2 + (y - \frac{1}{n})^2 = \frac{1}{n^2} \right\}) \\ & \quad (\text{circle with radius } \frac{1}{n} \text{ and center } \frac{i}{n} (n \in \mathbb{N})) \\ \mathcal{V} &:= \bigcup_{n \in \mathbb{N}} \left\{ (x, y, z) \in \mathbb{R}^3 : y^2 + (z - \frac{1}{n})^2 = \frac{1}{n^2} \wedge 0 \leq x \leq \frac{1}{n} \right\} \end{aligned}$$

moreover define $f_{\mathcal{X}} : [0, 1] \rightarrow \mathcal{X}$, $f_{\mathcal{Y}} : [0, 1] \rightarrow \mathcal{Y}$ and $f_{\mathcal{Z}} : [0, 1] \rightarrow \mathcal{Z}$ with:

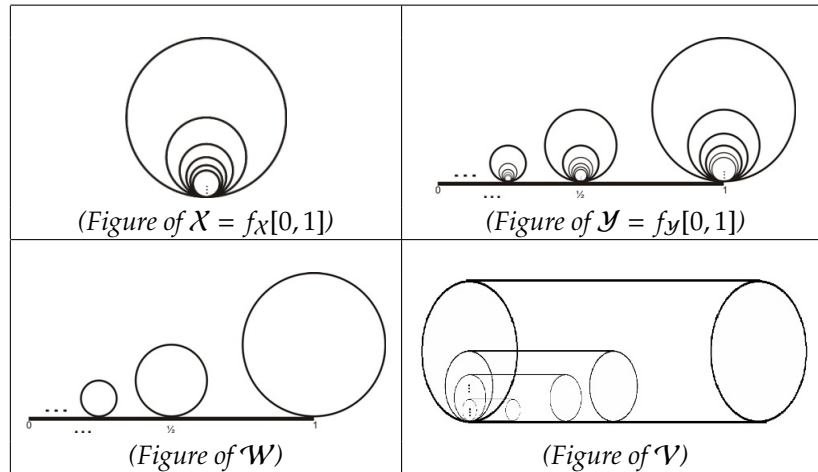
$$f_{\mathcal{X}}(x) = \begin{cases} \frac{1}{n}e^{2\pi i(n(n+1)x-n-\frac{1}{4})} + \frac{i}{n} & \frac{1}{n+1} \leq x \leq \frac{1}{n}, n \in \mathbb{N}, \\ 0 & x = 0, \end{cases}$$

$$f_{\mathcal{Y}}(x) = \begin{cases} \frac{f_{\mathcal{X}}(4xn(n+1) - (2n+1))}{2^{n+1}} + \frac{1}{n} & \frac{2n+1}{4n(n+1)} \leq x \leq \frac{1}{2n}, n \in \mathbb{N}, \\ 2(n+1)(2n-1)x + (2-2n) & \frac{1}{2(n+1)} \leq x \leq \frac{2n+1}{4n(n+1)}, n \in \mathbb{N}, \\ 2-2x & \frac{1}{2} \leq x \leq 1, \\ 0 & x = 0, \end{cases}$$

and

$$f_{\mathcal{Z}}(x) = \frac{1}{k}e^{2\pi i(px-k-\frac{1}{4})} + \frac{i}{k} \quad \left(\frac{k-1}{p} \leq x \leq \frac{k}{p}, k \in \{1, \dots, p\} \right).$$

Note: Consider 0 as the base point of all spaces in this convention



Example 2.3. 1) The map $f_X : [0, 1] \rightarrow X$ is an α -loop if and only if $\alpha > \omega$, since:

$$|f_X^{-1}(x)| = \begin{cases} 1 & x \neq 0, \\ \omega & x = 0. \end{cases}$$

In addition for each nonzero cardinal number α and ideal \mathcal{I} on X with $\{0\} \in \mathcal{I}$, $f_X : [0, 1] \rightarrow X$ is an $\alpha^{\mathcal{I}}$ -loop.

2) The map $f_Y : [0, 1] \rightarrow Y$ is an $\alpha^{\mathcal{I}}$ -loop if and only if " $\alpha > \omega$ " or " $\alpha \geq 2$ and $\{\frac{1}{n} : n \in \mathbb{N}\} \in \mathcal{I}$ ", since:

$$|f_Y^{-1}(x)| = \begin{cases} \omega & x \in \{\frac{1}{n} : n \in \mathbb{N}\}, \\ 2 & \text{otherwise.} \end{cases}$$

In particular $f_Y : [0, 1] \rightarrow Y$ is an $\alpha^{\mathcal{P}_{fin}(Y)}$ -loop if and only if $\alpha \geq c$.

3) The map $f_Z : [0, 1] \rightarrow Z$ is an α -loop if and only if $\alpha > p$. In addition for all nonzero cardinal number α and ideal \mathcal{I} on X with $\{0\} \in \mathcal{I}$, $f_X : [0, 1] \rightarrow X$ is an $\alpha^{\mathcal{I}}$ -loop.

3. New subgroups

In this section we introduce $\mathfrak{B}_I^\alpha(X)$ as a subgroup of $\pi_1(X)$.

We recall that for continuous maps $f, g : [0, 1] \rightarrow X$ with $f(1) = g(0)$, we have $f * g : [0, 1] \rightarrow X$ with $f * g(t) = f(2t)$ for $t \in [0, \frac{1}{2}]$ and $f * g(t) = g(2t - 1)$ for $t \in [\frac{1}{2}, 1]$. If $f : [0, 1] \rightarrow X$ is a continuous map, $[f]$ denotes its path homotopy equivalence class, where loops $f, g : [0, 1] \rightarrow X$ with same base point a are path homotopic (or $[f] = [g]$) if there exists continuous map $F : [0, 1] \times [0, 1] \rightarrow X$ with $F(s, 0) = f(s)$, $F(s, 1) = g(s)$ and $F(0, s) = F(1, s) = a$ for all $s \in [0, 1]$.

In the rest of this paper simply we use term "homotopy" or "homotopic" respectively instead of "path homotopy" or "path homotopic".

In addition for two loops $f, g : [0, 1] \rightarrow X$ with same base point a , we define $[f] * [g]$ as $[f * g]$. The class of all homotopy equivalence classes of loops with base point a under operation $*$ is a group which is denoted by $\pi_1(X, a)$. Whenever X is arc connected and $a, b \in X$ we have $\pi_1(X, a) \cong \pi_1(X, b)$ so $\pi_1(X, a)$ is denoted simply by $\pi_1(X)$.

Definition 3.1. For nonzero cardinal number α and ideal \mathcal{I} by $\mathfrak{P}_{\mathcal{I}}^{\alpha}(X, a)$ we mean subgroup of $\pi_1(X, a)$ generated by homotopy classes of $\alpha^{\mathcal{I}}$ -loops with base point a .

Theorem 3.2. For infinite cardinal number α and ideal \mathcal{I} on X , if $f, g : [0, 1] \rightarrow X$ are $\alpha^{\mathcal{I}}$ -arcs with $f(1) = g(0)$, then $f * g : [0, 1] \rightarrow X$ is an $\alpha^{\mathcal{I}}$ -arc. Moreover $\bar{f} : [0, 1] \rightarrow X$ with $\bar{f}(t) = f(1 - t)$ is an $\alpha^{\mathcal{I}}$ -arc too.

Proof. Use the fact that for all $x \in X$, $(f * g)^{-1}(x) = (\frac{1}{2}f^{-1}(x)) \cup (\frac{1}{2}g^{-1}(x) + \frac{1}{2})$, thus $|(f * g)^{-1}(x)| \leq |f^{-1}(x)| + |g^{-1}(x)|$. Also note to the fact that for all $x \in X$ we have $\bar{f}^{-1}(x) = \{1 - t : t \in f^{-1}(x)\}$, hence $|\bar{f}^{-1}(x)| = |f^{-1}(x)|$. \square

Theorem 3.3. For infinite cardinal number α , $a \in X$ and ideal \mathcal{I} on X , we have:

$$\mathfrak{P}_{\mathcal{I}}^{\alpha}(X, a) = \{[f] : f : [0, 1] \rightarrow X \text{ is an } \alpha^{\mathcal{I}}\text{-loop with base point } a\}.$$

Proof. Choose $b \in X \setminus \{a\}$. There exists a continuous one to one map $g : [0, 1] \rightarrow X$ with $g(0) = a$ and $g(1) = b$. Using Theorem 3.2, $g * \bar{g} : [0, 1] \rightarrow X$ is an $\alpha^{\mathcal{I}}$ -arc. Thus $[g * \bar{g}] \in \{[f] : f : [0, 1] \rightarrow X \text{ is an } \alpha^{\mathcal{I}}\text{-loop with base point } a\}$, and $\{[f] : f : [0, 1] \rightarrow X \text{ is an } \alpha^{\mathcal{I}}\text{-loop with base point } a\} \neq \emptyset$. Using Theorem 3.2, $\{[f] : f : [0, 1] \rightarrow X \text{ is an } \alpha^{\mathcal{I}}\text{-loop with base point } a\}$ is a subgroup of $\pi_1(X, a)$ which completes the proof. \square

Note 3.4. Using Theorem 3.3 for $a \in X$ and infinite cardinal number α , for the loop $g : [0, 1] \rightarrow X$ with base point a , $[g] \in \mathfrak{P}_{\mathcal{I}}^{\alpha}(X, a)$ if and only if there exists an $\alpha^{\mathcal{I}}$ -loop $f : [0, 1] \rightarrow X$ with base point a homotopic to $g : [0, 1] \rightarrow X$.

Theorem 3.5. For all $a, b \in X$, ideal \mathcal{I} on X and infinite α , $\mathfrak{P}_{\mathcal{I}}^{\alpha}(X, a)$ and $\mathfrak{P}_{\mathcal{I}}^{\alpha}(X, b)$ are isomorphic groups.

Proof. For $a \neq b$, suppose $f : [0, 1] \rightarrow X$ is a continuous one to one map (1-arc) such that $f(0) = a$ and $f(1) = b$, and $\bar{f} : [0, 1] \rightarrow X$ is $\bar{f}(t) = f(1 - t)$ for all $t \in [0, 1]$. Using Theorem 3.2, $g : [0, 1] \rightarrow X$ is an $\alpha^{\mathcal{I}}$ -arc if and only if $\bar{f} * g * f : [0, 1] \rightarrow X$ is an $\alpha^{\mathcal{I}}$ -arc too, which leads to the desired result (note: $\varphi : \mathfrak{P}_{\mathcal{I}}^{\alpha}(X, a) \rightarrow \mathfrak{P}_{\mathcal{I}}^{\alpha}(X, b)$, with $\varphi([g]) = [\bar{f} * g * f]$ is an isomorphism). \square

By the following counterexample the infiniteness of α in Theorem 3.5 is essential.

Counterexample 3.6. Consider $X = \mathbb{S}^1 \cup [1, 2]$ as a subspace of \mathbb{R}^2 (X and $\mathbf{9}$ are homeomorph). If $a \in \mathbb{S}^1$ and $b \in (1, 2]$, then:

1. $\mathfrak{P}_{\mathcal{P}_{fin}(X)}^1(X, a) = \pi_1(X, a) \cong \mathbb{Z}$,
2. $\mathfrak{P}_{\mathcal{P}_{fin}(X)}^1(X, b) = \{e\}$ (where e is the identity of $\pi_1(X, b)$).

In particular $\mathfrak{P}_{\mathcal{P}_{fin}(X)}^1(X, a)$ and $\mathfrak{P}_{\mathcal{P}_{fin}(X)}^1(X, b)$ are nonisomorphic (although X is linear connected).

Proof. (1) By definition $\mathfrak{P}_{\mathcal{P}_{fin}(X)}^1(X, a) \subseteq \pi_1(X, a) (= \mathbb{Z})$. On the other hand $f : [0, 1] \rightarrow X$ is a $1 \frac{\mathcal{P}_{fin}(X)}{t \mapsto e^{2\pi it}}$ -arc and $\pi_1(X, a) = \langle [f] \rangle \supseteq \mathfrak{P}_{\mathcal{P}_{fin}(X)}^1(X, a)$, which completes the proof.

(2) Suppose $f : [0, 1] \rightarrow X$ with $f(0) = f(1) = b$ is a continuous map. If $f \neq b$, then there exists $c \in [1, 2] \setminus \{b\}$ with $c = \inf[0, 1]$. Let $s := \min(c, b)$ and $t := \max(c, b)$. For all $y \in (s, t)$ we have $|f^{-1}(y)| \geq 2$, and $(s, t) \notin \mathcal{P}_{fin}(X)$ (since (s, t) is infinite). Therefore f is not a $1 \frac{\mathcal{P}_{fin}(X)}{t \mapsto e^{2\pi it}}$ -loop, and the constant loop b is the unique $1 \frac{\mathcal{P}_{fin}(X)}{t \mapsto e^{2\pi it}}$ -loop with base point b , thus $\mathfrak{P}_{\mathcal{P}_{fin}(X)}^1(X, b) = \{[b]\} = \{e\}$. \square

Definition 3.7. Regarding Theorem 3.5 for infinite cardinal number α and ideal \mathcal{I} on X , we denote $\mathfrak{P}_{\mathcal{I}}^{\alpha}(X, a)$ simply by $\mathfrak{P}_{\mathcal{I}}^{\alpha}(X)$ (subgroup of $\pi_1(X)$ generated by homotopy classes of $\alpha^{\mathcal{I}}$ -loops). We denote $\mathfrak{P}_{\{\emptyset\}}^{\alpha}(X)$ by $\mathfrak{P}^{\alpha}(X)$ (subgroup of $\pi_1(X)$ generated by homotopy classes of α -loops).

So for infinite cardinal number α we have (use Note 3.4 and above discussion):

$$\mathfrak{P}_{\mathcal{I}}^{\alpha}(X) = \{[f] : f : [0, 1] \rightarrow X \text{ is an } \alpha^{\mathcal{I}}\text{-loop}\},$$

and

$$\mathfrak{P}^{\alpha}(X) = \{[f] : f : [0, 1] \rightarrow X \text{ is an } \alpha\text{-loop}\}.$$

4. A useful remark

For the remain of this text we use the following useful convention.

Convention 4.1. Suppose X and Y are closed subspaces of Z such that $X \cap Y = \{t\}$. For $f : [0, 1] \rightarrow X \cup Y$ define:

$$f^X(x) = \begin{cases} f(x) & f(x) \in X, \\ t & f(x) \in Y. \end{cases}$$

Remark 4.2. Suppose X and Y are closed (linear connected) subspaces of Z such that $X \cap Y = \{t\}$. For loops $g, h : [0, 1] \rightarrow X \cup Y$ with base point t we have:

- A. If $g, h : [0, 1] \rightarrow X \cup Y$ are homotopic loops, then $g^X, h^X : [0, 1] \rightarrow X$ are homotopic loops (therefore $g^X, h^X : [0, 1] \rightarrow X \cup Y$ are homotopic too).
- B. Let $g[0, 1] \subseteq X$ and $h[0, 1] \subseteq Y$. $g, h : [0, 1] \rightarrow X \cup Y$ are homotopic if and only if they are null-homotopic.
- C. Let $g[0, 1] \cup h[0, 1] \subseteq X$. $g, h : [0, 1] \rightarrow X \cup Y$ are homotopic if and only if $g, h : [0, 1] \rightarrow X$ are homotopic.
- D. $\pi_1(X, t)$ and $\pi_1(Y, t)$ are subgroups of $\pi_1(X \cup Y, t)$ and $\pi_1(X, t) \cap \pi_1(Y, t) = \{[t]\}$ where t denotes the constant arc with value t (as a matter of fact the map $\pi_1(X, t) \rightarrow \pi_1(X \cup Y, t)$ is a group embedding).
 $\quad \quad \quad [f] \mapsto [f]$

Proof. (A) Suppose $g, h : [0, 1] \rightarrow X \cup Y$ are homotopic loops, then there exists a continuous map $F : [0, 1] \times [0, 1] \rightarrow X \cup Y$ such that $F(s, 0) = g(s)$, $F(s, 1) = h(s)$ and $F(0, s) = F(1, s) = t$ for all $s \in [0, 1]$. Define continuous map $P : X \cup Y \rightarrow X$ with $P(z) = z$ for $z \in X$ and $P(z) = t$ for $z \in Y$. The map $P \circ F : [0, 1] \times [0, 1] \rightarrow X$ is continuous, moreover $P \circ F(s, 0) = g^X(s)$, $P \circ F(s, 1) = h^X(s)$ and $P \circ F(1, s) = P \circ F(0, s) = t$ for all $s \in [0, 1]$, thus $g^X, h^X : [0, 1] \rightarrow X \cup Y$ are homotopic.

(B) If $g, h : [0, 1] \rightarrow X \cup Y$ are homotopic, then by (A), $g^X, h^X : [0, 1] \rightarrow X \cup Y$ are homotopic. On the other hand $g^X = t$ (constant function t) and $h^X = h$, since $g[0, 1] \subseteq X$ and $h[0, 1] \subseteq Y$. Therefore $h : [0, 1] \rightarrow X \cup Y$ is null homotopic which leads to the desired result. \square

5. Primary properties of $\mathfrak{P}_I^\alpha(X)$ s

In this section we study primary properties of $\mathfrak{P}_I^\alpha(X)$ s. It is wellknown that $\Phi : \pi_1(X, x_0) \times \pi_1(Y, y_0) \rightarrow \pi_1(X \times Y, (x_0, y_0))$ with $\Phi([f], [g]) = [(f, g)]$ is an isomorphism (for example see [5, Theorem 60.1]) where for $f : [0, 1] \rightarrow X$ and $g : [0, 1] \rightarrow Y$ we have $(f, g) : [0, 1] \rightarrow X \times Y$ with $(f, g)(t) = (f(t), g(t))$ ($t \in [0, 1]$). For transfinite cardinal numbers α, β , ideal \mathcal{I} on X and ideal \mathcal{J} on Y we prove $\Phi(\mathfrak{P}_I^\alpha(X, x_0) \times \mathfrak{P}_J^\beta(Y, y_0)) \subseteq \mathfrak{P}_{I \times J}^{\max(\alpha, \beta)}(X \times Y, (x_0, y_0))$, hence $\mathfrak{P}_I^\alpha(X, x_0) \times \mathfrak{P}_J^\beta(Y, y_0)$ is isomorphic to a subgroup of $\mathfrak{P}_{I \times J}^{\max(\alpha, \beta)}(X \times Y, (x_0, y_0))$.

Theorem 5.1. For topological spaces X and Y we have (we recall that X and Y are arc connected locally compact Hausdorff topological spaces with at least two elements, moreover consider $x_0 \in X$, and $y_0 \in Y$):

1. For all $\alpha > c$, nonzero β and ideal \mathcal{I} on X we have $\mathfrak{P}_I^\alpha(X) = \pi_1(X) = \mathfrak{P}_{\mathcal{P}(X)}^\beta(X)$.
2. For nonzero cardinal numbers α, β , $x_0 \in X$, and ideals \mathcal{I}, \mathcal{J} on X we have:

- If $\alpha \leq \beta$, then $\mathfrak{P}_I^\alpha(X, x_0) \subseteq \mathfrak{P}_I^\beta(X, x_0)$.
- If $\mathcal{I} \subseteq \mathcal{J}$, then $\mathfrak{P}_I^\alpha(X, x_0) \subseteq \mathfrak{P}_J^\alpha(X, x_0)$.

Therefore for infinite α we have (base point is x_0 , whenever it is necessary):

- If $\alpha \leq \beta$, then $\mathfrak{P}_I^\alpha(X) \subseteq \mathfrak{P}_I^\beta(X)$.
- If $\mathcal{I} \subseteq \mathcal{J}$, then $\mathfrak{P}_I^\alpha(X) \subseteq \mathfrak{P}_J^\alpha(X)$;

- $\mathfrak{P}_{\mathcal{I} \cap \mathcal{J}}^\alpha(X) \subseteq \mathfrak{P}_{\mathcal{I}}^\alpha(X) \cap \mathfrak{P}_{\mathcal{J}}^\alpha(X)$.

3. For infinite cardinal numbers α, β and ideals \mathcal{I} on X and \mathcal{J} on Y we have

$$\Phi(\mathfrak{P}_{\mathcal{I}}^\alpha(X, x_0) \times \mathfrak{P}_{\mathcal{J}}^\beta(Y, y_0)) \subseteq \mathfrak{P}_{\mathcal{I} \times \mathcal{J}}^{\max(\alpha, \beta)}(X \times Y, (x_0, y_0)),$$

where $\mathcal{I} \times \mathcal{J}$ is ideal on $X \times Y$ generated by $\{A \times B : A \in \mathcal{I}, B \in \mathcal{J}\}$ and $\Phi([f], [g]) = [(f, g)]$ for loops $f : [0, 1] \rightarrow X$ and $g : [0, 1] \rightarrow Y$. Hence $\mathfrak{P}_{\mathcal{I}}^\alpha(X) \times \mathfrak{P}_{\mathcal{J}}^\beta(Y)$ is isomorphic to a subgroup of $\mathfrak{P}_{\mathcal{I} \times \mathcal{J}}^{\max(\alpha, \beta)}(X \times Y)$.

4. For infinite cardinal numbers α, β , ideal \mathcal{I} on X , and isomorphism $\Phi : \pi_1(X) \times \pi_1(Y) \rightarrow \pi_1(X \times Y)$ with $\Phi([f], [g]) = [(f, g)]$, we have:

a. $\Phi(\mathfrak{P}_{\mathcal{I}}^\alpha(X, x_0) \times \pi_1(Y, y_0)) \subseteq \mathfrak{P}_{\mathcal{I} \times \mathcal{P}(Y)}^\alpha(X \times Y, (x_0, y_0))$,

b. $\Phi(\mathfrak{P}_{\mathcal{I}}^\alpha(X, x_0) \times \mathfrak{P}^\beta(Y, y_0)) \subseteq \mathfrak{P}^\beta(X \times Y, (x_0, y_0))$;

c. $\Phi(\mathfrak{P}^\alpha(X, x_0) \times \mathfrak{P}^\beta(Y, y_0)) \subseteq \mathfrak{P}^{\min(\alpha, \beta)}(X \times Y, (x_0, y_0))$.

5. For infinite cardinal numbers α, β , ideal \mathcal{I} on X , ideal \mathcal{J} on Y , $\mathcal{K} := \{A \cup B : A \in \mathcal{I}, B \in \mathcal{J}\}$, if $X \cap Y = \{t\}$ and X, Y are (linear connected) closed subspaces of Z , then we have (note that \mathcal{K} is an ideal on $X \cup Y$) (see Convention 4.1 (D)):

a. $\mathfrak{P}_{\mathcal{I}}^\alpha(X, t) \mathfrak{P}_{\mathcal{J}}^\beta(Y, t) \subseteq \mathfrak{P}_{\mathcal{K}}^{\max(\alpha, \beta)}(X \cup Y, t)$,

b. $\mathfrak{P}^\alpha(X, t) \mathfrak{P}^\beta(Y, t) \subseteq \mathfrak{P}^{\max(\alpha, \beta)}(X \cup Y, t)$.

Proof. (1) and (2) are clear by definition.

(3) If $f : [0, 1] \rightarrow X$ is an α -arc with base point x_0 and $g : [0, 1] \rightarrow Y$ is a β -arc with base point y_0 , then there exist $A \in \mathcal{I}$ and $B \in \mathcal{J}$ such that for all $x \in X \setminus A$ and $y \in Y \setminus B$ we have $|f^{-1}(x)| < \alpha$ and $|g^{-1}(y)| < \beta$. For $h = (f, g) : [0, 1] \rightarrow X \times Y$ with $h(t) = (f(t), g(t))$ and $(z, w) \in (X \times Y) \setminus (A \times B)$ we have:

$$\begin{aligned} (z, w) \in (X \times Y) \setminus (A \times B) &\Rightarrow z \in X \setminus A \vee w \in Y \setminus B \\ &\Rightarrow |f^{-1}(z)| < \alpha \vee |g^{-1}(w)| < \beta \\ &\Rightarrow |h^{-1}(z, w)| \leq \min(|f^{-1}(z)|, |g^{-1}(w)|) < \max(\alpha, \beta) \end{aligned}$$

therefore $(f, g) : [0, 1] \rightarrow X \times Y$ is a $\max(\alpha, \beta)$ -arc, and

$$\Phi([f], [g]) = [(f, g)] \in \mathfrak{P}_{\mathcal{I} \times \mathcal{J}}^{\max(\alpha, \beta)}(X \times Y, (x_0, y_0)).$$

(4) (a) is a special case of item (3), since $\pi_1(Y, y_0) = \mathfrak{P}_{\mathcal{P}(Y)}^\alpha(Y, y_0)$.

For rest note that for all $(x, y) \in X \times Y$, continuous maps $f : [0, 1] \rightarrow X$, and $g : [0, 1] \rightarrow Y$ we have $(f, g)^{-1}(x, y) = f^{-1}(x) \cap g^{-1}(y)$, thus $|h^{-1}(x, y)| \leq \min(|f^{-1}(x)|, |g^{-1}(y)|)$.

(b) If $f : [0, 1] \rightarrow X$ is an α -arc and $g : [0, 1] \rightarrow Y$ is a β -arc, then for all $(x, y) \in X \times Y$ we have $|(f, g)^{-1}(x, y)| \leq \min(|f^{-1}(x)|, |g^{-1}(y)|) \leq |g^{-1}(y)| < \beta$. Therefore $(f, g) : [0, 1] \rightarrow X \times Y$ is a β -arc.

(c) If $f : [0, 1] \rightarrow X$ is an α -arc and $g : [0, 1] \rightarrow Y$ is a β -arc, then for all $(x, y) \in X \times Y$ we have $|(f, g)^{-1}(x, y)| \leq \min(|f^{-1}(x)|, |g^{-1}(y)|) < \min(\alpha, \beta)$. Therefore $(f, g) : [0, 1] \rightarrow X \times Y$ is a $\min(\alpha, \beta)$ -arc.

(5) Since $\mathfrak{P}_{\mathcal{I}}^\alpha(X, t) \subseteq \mathfrak{P}_{\mathcal{K}}^\alpha(X, t) \subseteq \mathfrak{P}_{\mathcal{K}}^{\max(\alpha, \beta)}(X, t) \subseteq \mathfrak{P}_{\mathcal{K}}^{\max(\alpha, \beta)}(X \cup Y, t)$ and similarly $\mathfrak{P}_{\mathcal{J}}^\beta(Y, t) \subseteq \mathfrak{P}_{\mathcal{K}}^{\max(\alpha, \beta)}(X \cup Y, t)$. \square

Example 5.2 ($\mathfrak{P}_{\mathcal{I}}^\alpha(X)$ for some well-known spaces X). We may find the following easy examples:

1. It's evident that for any contractible space X , nonzero cardinal number α and ideal \mathcal{I} on X , we have $\mathfrak{P}_{\mathcal{I}}^\alpha(X) = \{e\}$.

2. Let $X = \{e^{2\pi i\theta} : \theta \in [0, 1]\} (= \mathbb{S}^1)$. Then $\mathfrak{P}_{\mathcal{I}}^\alpha(X) = \pi_1(X)$, for all $\alpha \geq 2$ and ideal \mathcal{I} on X (since for $f : [0, 1] \rightarrow \mathbb{S}^1$ with $f(t) = e^{2\pi it}$ we have $[f] \in \mathfrak{P}^\alpha(\mathbb{S}^1) \subseteq \mathfrak{P}_{\mathcal{I}}^\alpha(\mathbb{S}^1) \subseteq \pi_1(\mathbb{S}^1)$ and $[f]$ is a generator of $\pi_1(\mathbb{S}^1)$, thus $\mathfrak{P}_{\mathcal{I}}^\alpha(\mathbb{S}^1) = \pi_1(\mathbb{S}^1) \cong \mathbb{Z}$).

3. With a similar method described in item (2), for all $\alpha \geq 2$ and ideal \mathcal{I} on $\mathbb{R}^2 \setminus \{0\}$ (punctured space), we have $\mathfrak{P}_{\mathcal{I}}^{\alpha}(\mathbb{R}^2 \setminus \{0\}) = \pi_1(\mathbb{R}^2 \setminus \{0\}) \cong \mathbb{Z}$.
4. Using (2) and a similar method described in Theorem 5.1 for all $\alpha \geq 2$ and ideal \mathcal{I} on $\mathbb{T} = \mathbb{S}^1 \times \mathbb{S}^1$ (Torus) we have $\mathfrak{P}_{\mathcal{I}}^{\alpha}(\mathbb{T}) = \pi_1(\mathbb{T})$.

6. Some preliminaries on Hawaiian earring

In this section we bring some useful properties of Infinite earring (Hawaiian earring) (see [5, page 500, Exercise 5] too)

Lemma 6.1. *If loop $f : [0, 1] \rightarrow \mathbb{S}^1$ is not null-homotopic and $f(0) = f(1) = 1$, then there exist $a, b \in [0, 1]$ such that $f(a, b) = \mathbb{S}^1 \setminus \{1\}$ and $f(a) = f(b) = 1$.*

Proof. In the following proof for $g : [u, v] \rightarrow X$ with $g_{[u,v]} : [0, 1] \rightarrow X$ we mean $g_{[u,v]}(t) = g(t(v-u) + u)$. Since $f : [0, 1] \rightarrow \mathbb{S}^1$ is uniformly continuous, there exists $\varepsilon > 0$ such that for all $s, t \in [0, 1]$ with $|s - t| < \varepsilon$ we have $|f(s) - f(t)| < \frac{1}{2}$.

Let $T = \{t \in [0, 1] : f(0) = f(t) = 1 \text{ and } f_{[0,t]} : [0, 1] \rightarrow \mathbb{S}^1 \text{ is not null-homotopic}\}$. We have $T \neq \emptyset$, since $1 \in T$. Suppose $\tau = \inf(T)$. Since f is continuous and $T \subseteq f^{-1}(1)$, thus $f(\tau) = 1$. We claim that $\tau \in T$. There exists $t \in T$ such that $0 \leq t - \tau < \varepsilon$, if $\tau = t \in T$ we are done, otherwise since $f_{[\tau,t]}([0, 1]) = f_{[\tau,t]} \subseteq \{x \in \mathbb{S}^1 : |x - 1| = |x - f(\tau)| < \frac{1}{2}\} \subseteq \mathbb{S}^1 \setminus \{-1\}$, thus $f_{[\tau,t]}$ is null-homotopic. On the other hand $[f_{[0,t]}] = [f_{[0,\tau]}] * [f_{[\tau,t]}] = [f_{[0,\tau]}]$ and $f_{[0,\tau]}$ is not null-homotopic, which indicates $\tau \in T$.

Let $S = \{s \in [0, \tau] : f(s) = f(\tau) = 1 \text{ and } f_{[s,\tau]} : [0, 1] \rightarrow \mathbb{S}^1 \text{ is not null-homotopic}\}$, so $0 \in S$ and $S \neq \emptyset$. Let $\sigma = \sup(S)$. Similar to first part of proof, $\sigma \in S$. It is clear that $\sigma < \tau$. Moreover $[f_{[0,\tau]}] = [f_{[0,\sigma]}] * [f_{[\sigma,\tau]}]$ and using the way of choose of τ , $f_{[0,\sigma]} : [0, 1] \rightarrow \mathbb{S}^1$ is null-homotopic, thus $[f_{[0,\tau]}] = [f_{[\sigma,\tau]}]$ and $f_{[\sigma,\tau]} : [0, 1] \rightarrow \mathbb{S}^1$ is not null-homotopic.

Since $f_{[\sigma,\tau]} : [0, 1] \rightarrow \mathbb{S}^1$ is not null-homotopic, $f_{[\sigma,\tau]}([0, 1]) = \mathbb{S}^1$.

On the other hand if there exists $\zeta \in (\sigma, \tau)$ such that $f(\zeta) = 1$. Respectively using the way of choose of τ and σ , two maps $f_{[0,\zeta]} : [0, 1] \rightarrow \mathbb{S}^1$ and $f_{[\zeta,\tau]} : [0, 1] \rightarrow \mathbb{S}^1$ are null-homotopic. Using $[f_{[0,\tau]}] = [f_{[0,\zeta]}] * [f_{[\zeta,\tau]}]$, $f_{[0,\sigma]} : [0, 1] \rightarrow \mathbb{S}^1$ is null-homotopic, which is a contradiction. Therefore for all $\zeta \in (\sigma, \tau)$ we have $f(\zeta) \neq 1$, which shows $f(\sigma, \tau) = \mathbb{S}^1 \setminus \{1\}$. \square

Lemma 6.2. *If $X = (\mathbb{S}^1 - 1) \cup (\mathbb{S}^1 + 1)$ (X and Figure 8 are homeomorph), $\rho : [0, 1] \rightarrow X$ with $\rho(t) = e^{4\pi i t} - 1$ for $t \in [0, \frac{1}{2}]$ and $\rho(t) = -e^{4\pi i t} + 1$ for $t \in [\frac{1}{2}, 1]$, and loop $f : [0, 1] \rightarrow X$ with $f(0) = f(1) = 0$ is homotopic to $\rho : [0, 1] \rightarrow X$, then there exist $a, b, c, d \in [0, 1]$ such that $a < b \leq c < d$, $f(a) = f(b) = f(c) = f(d) = 0$, $f(a, b) = (\mathbb{S}^1 - 1) \setminus \{0\}$ and $f(c, d) = (\mathbb{S}^1 + 1) \setminus \{0\}$.*

Proof. Let:

$$f^{\mathbb{S}^1-1}(t) = \begin{cases} f(t) & f(t) \in \mathbb{S}^1 - 1 \\ 0 & \text{otherwise} \end{cases}, \quad \rho^{\mathbb{S}^1-1}(t) = \begin{cases} \rho(t) & \rho(t) \in \mathbb{S}^1 - 1 \\ 0 & \text{otherwise} \end{cases}$$

$$f^{\mathbb{S}^1+1}(t) = \begin{cases} f(t) & f(t) \in \mathbb{S}^1 + 1 \\ 0 & \text{otherwise} \end{cases}, \quad \rho^{\mathbb{S}^1+1}(t) = \begin{cases} \rho(t) & \rho(t) \in \mathbb{S}^1 + 1 \\ 0 & \text{otherwise} \end{cases}$$

Two maps $f^{\mathbb{S}^1-1}, \rho^{\mathbb{S}^1-1} : [0, 1] \rightarrow \mathbb{S}^1 - 1$ are homotopic, since $f, \rho : [0, 1] \rightarrow X$ are homotopic. Since $\rho^{\mathbb{S}^1-1} : [0, 1] \rightarrow \mathbb{S}^1 - 1$ is not null-homotopic, by Lemma 6.1 there exists $a, b \in [0, 1]$ with $f^{\mathbb{S}^1-1}(a, b) = (\mathbb{S}^1 - 1) \setminus \{0\}$ and $f^{\mathbb{S}^1-1}(a) = f^{\mathbb{S}^1-1}(b) = 0$. For all $t \in (a, b)$ we have $f^{\mathbb{S}^1-1}(t) \neq 0$, therefore $f(t) = f^{\mathbb{S}^1-1}(t)$. Thus $f(a, b) = f^{\mathbb{S}^1-1}(a, b) = \mathbb{S}^1 - 1 \setminus \{0\}$. Moreover $f^{\mathbb{S}^1-1}(a) = f^{\mathbb{S}^1-1}(b) = 0$, thus $f(a), f(b) \in \mathbb{S}^1 + 1$. Using the continuity of f we have $f(a), f(b) \in \overline{f(a, b)} = \mathbb{S}^1 - 1$, therefore $f(a), f(b) \in \mathbb{S}^1 - 1 \cap \mathbb{S}^1 + 1 = \{0\}$ and $f(a) = f(b) = 0$. Let:

$$\Gamma_1 := \{(a, b) \in [0, 1] \times [0, 1] : f(a, b) = (\mathbb{S}^1 - 1) \setminus \{0\}, f(a) = f(b) = 0\},$$

$$\Gamma_2 := \{(a, b) \in [0, 1] \times [0, 1] : f(a, b) = (\mathbb{S}^1 + 1) \setminus \{0\}, f(a) = f(b) = 0\}.$$

By the above discussion, $\Gamma_1 \neq \emptyset$. It is evident that for all distinct $(a, b), (a', b') \in \Gamma_1$ we have $(a, b) \cap (a', b') = \emptyset$. Since $f : [0, 1] \rightarrow X$ is uniformly continuous there exists $\delta > 0$ such that:

$$\forall u, v \in [0, 1] \quad (|u - v| < \delta \Rightarrow |f(u) - f(v)| < 1)$$

which leads to:

$$\forall u, v \in [0, 1] \quad (|u - v| < \delta \Rightarrow f(u, v) \neq \mathbb{S}^1 - 1)$$

so for all $(a, b) \in \Gamma_1$ we have $b - a \geq \delta$.

Thus Γ_1 is finite, since Γ_1 is a nonempty collection of disjoint subintervals of $[0, 1]$ with $b - a \geq \delta$ for all $(a, b) \in \Gamma_1$.

In a similar way Γ_2 is a nonempty finite collection of disjoint subintervals of $[0, 1]$.

It is evident that for all $(a, b) \in \Gamma_1$ and $(c, d) \in \Gamma_2$ we have $(a, b) \cap (c, d) \neq \emptyset$ (since $f(a, b) \cap f(c, d) = ((\mathbb{S}^1 - 1) \cap (\mathbb{S}^1 + 1)) \setminus \{0\} = \emptyset$), therefore $a < b \leq c < d$ or $c < d \leq a < b$.

If there exist $(a, b) \in \Gamma_1$ and $(c, d) \in \Gamma_2$ with $a < b \leq c < d$, we are done, otherwise suppose for all $(a, b) \in \Gamma_1$ and $(c, d) \in \Gamma_2$ we have $c < d \leq a < b$. Let

$$\Gamma_1 = \{(a_1, b_1), \dots, (a_n, b_n)\}, \Gamma_2 = \{(c_1, d_1), \dots, (c_m, d_m)\}.$$

and suppose

$$c_1 < d_1 \leq c_2 < d_2 \leq \dots \leq c_m < d_m \leq a_1 < b_1 \leq a_2 < b_2 \leq \dots \leq a_n < b_n$$

Using the same notations as in Lemma 6.1, if $d_1 < c_2$, then $f_{[d_1, c_2]} : [0, 1] \rightarrow X$ is null-homotopic (use Lemma 6.1 and consider Γ_1, Γ_2). if $p \in [0, 1]$ suppose $f_{[p, p]} : [0, 1] \rightarrow X$ is constant 0 function. So

$$f_{[0, c_1]}, f_{[d_1, c_2]}, f_{[d_2, c_3]}, \dots, f_{[d_{m-1}, c_m]}, f_{[d_m, a_1]}, f_{[b_1, a_2]}, f_{[b_{n-1}, a_n]}, f_{[a_n, 1]} : [0, 1] \rightarrow X$$

are null-homotopic. Thus

$$[f] = [f_{[c_1, d_1]}] * \dots * [f_{[c_m, d_m]}] * [f_{[a_1, b_1]}] * \dots * [f_{[a_n, b_n]}]$$

For all i, j we have $f_{[c_i, d_i]} \subseteq \mathbb{S}^1 + 1$ and $f_{[a_j, b_j]} \subseteq \mathbb{S}^1 - 1$. thus there exist $q_1, \dots, q_m, p_1, \dots, p_n \geq 0$ with

$$[f_{[c_i, d_i]}] = [\rho_{[\frac{1}{2}, 1]}]^{q_i} \quad (1 \leq i \leq m) \quad \text{and} \quad [f_{[a_j, b_j]}] = [\rho_{[0, \frac{1}{2}]}]^{p_j} \quad (1 \leq j \leq n)$$

(we recall that $\pi_1(X) = \pi_1(\mathbb{S}^1 - 1) * \pi_1(\mathbb{S}^1 + 1) = \langle [\rho_{[0, \frac{1}{2}]}] \rangle * \langle [\rho_{[\frac{1}{2}, 1]}] \rangle$, by van Kampen Theorem). Thus

$$[\rho_{[0, \frac{1}{2}]}] * [\rho_{[\frac{1}{2}, 1]}] = [\rho] = [f] = [\rho_{[\frac{1}{2}, 1]}]^{q_1 + \dots + q_m} * [\rho_{[0, \frac{1}{2}]}]^{p_1 + \dots + p_n}$$

which is a contradiction since $\pi_1(X)$ is nonabelian free group over two generators $[\rho_{[0, \frac{1}{2}]}]$ and $[\rho_{[\frac{1}{2}, 1]}]$. \square

Lemma 6.3. *If loop $f : [0, 1] \rightarrow \mathcal{Z}$ with $f(0) = f(1) = 0$ is homotopic to $f_{\mathcal{Z}} : [0, 1] \rightarrow \mathcal{Z}$, then there exist $s_1, t_1, s_2, t_2, \dots, s_p, t_p \in [0, 1]$ such that $s_1 < t_1 \leq s_2 < t_2 \leq \dots \leq s_p < t_p$, $f(s_j) = f(t_j) = 0$, $f(s_j, t_j) = \{\frac{1}{j}e^{2\pi i(x-j\frac{1}{4})} + \frac{i}{j} : x \in [0, 1]\} \setminus \{0\}$ for all $j \in \{1, \dots, p\}$.*

Proof. Use the same method described in Lemma 6.2 and note to the fact that $\pi_1(\mathcal{Z})$ is nonabelian free group over p generators $[h_{[\frac{k-1}{p}, \frac{k}{p}]}]$ for $k = 1, \dots, p$ where $h := f_{\mathcal{Z}}$ and using the notations of Lemma 6.1. \square

Note 6.4. *Consider loops $f, g : [0, 1] \rightarrow X$ such that $f(0) = f(1) = g(0) = g(1) = 0$. For nonempty subset Γ of \mathbb{N} and $h : [0, 1] \rightarrow X$ let:*

$$h^\Gamma(x) = \begin{cases} h(x) & h(x) \in \bigcup\{C_n : n \in \Gamma\}, \\ 0 & h(x) \in (X \setminus \bigcup\{C_n : n \in \Gamma\}) \cup \{0\}. \end{cases}$$

(As a matter of fact we denote $h^{\bigcup\{C_n : n \in \Gamma\}}$ (see Convention 4.1) briefly by h^Γ)

1. If $f, g : [0, 1] \rightarrow \mathcal{X}$ are homotopic, then $f^\Gamma, g^\Gamma : [0, 1] \rightarrow \bigcup\{C_n : n \in \Gamma\}$ are homotopic (equivalently $f^\Gamma, g^\Gamma : [0, 1] \rightarrow \mathcal{X}$ are homotopic).
2. For loop $h : [0, 1] \rightarrow \mathcal{X}$ with $h(0) = h(1) = 0$ define:

$$A(h) := \{n \in \mathbb{N} : h^{(n)} : [0, 1] \rightarrow C_n \text{ is not null - homotopic}\}.$$

Then $A(h)$ is a subset of:

$$\{n \in \mathbb{N} : \exists a, b \in [0, 1] (h(a, b) = C_n \setminus \{0\} \wedge h(a) = h(b) = 0)\}.$$

Moreover if $f, g : [0, 1] \rightarrow \mathcal{X}$ are homotopic, then $A(f) = A(g)$.

3. For loop $h : [0, 1] \rightarrow \mathcal{X}$ with $h(0) = h(1) = 0$, we have $|h^{-1}(0)| \geq |A(h)|$.
4. If $[f] \in \mathfrak{F}^\omega(\mathcal{X})$, then $|A(f)| < \omega$ and $A(f)$ is finite.

Proof.

1. Note to the fact that $A = \bigcup\{C_n : n \in \Gamma\}$ and $B = (\mathcal{X} \setminus A) \cup \{0\}$ are closed (linear connected) subsets of \mathcal{X} . Moreover $A \cap B = \{0\}$. Now use the same argument as in Convention 4.1.
2. If $n \in A(h)$, then $h^{(n)} : [0, 1] \rightarrow C_n$ is not null-homotopic. By Lemma 6.1 there exist $a, b \in [0, 1]$ with $h(a) = h(b) = 0$ and $h(a, b) = C_n \setminus \{0\}$. Use item (1) to complete the proof.
3. By (2) for all $n \in A(h)$ there exists $a_n, b_n \in [0, 1]$ with $h(a_n, b_n) = C_n \setminus \{0\}$ and $h(a_n) = h(b_n) = 0$. We claim that $A(h) \xrightarrow{n \rightarrow a_n} h^{-1}(0)$ is one to one. Suppose $n \neq m$ and $n, m \in A(h)$. By

$$h(a_n, b_n) \cap h(a_m, b_m) = (C_n \setminus \{0\}) \cap (C_m \setminus \{0\}) = \emptyset$$

we have $(a_n, b_n) \cap (a_m, b_m) = \emptyset$, thus $a_n \neq a_m$.

4. If $[f] \in \mathfrak{F}^\omega(\mathcal{X})$, then by Note 3.4 there exists ω -loop $k : [0, 1] \rightarrow \mathcal{X}$ with $k(0) = k(1) = 0$ homotopic to $f : [0, 1] \rightarrow \mathcal{X}$. By (3) we have $|A(k)| \leq |k^{-1}(0)| < \omega$. By item (2) we have $A(f) = A(k)$ which leads to $|A(f)| = |A(k)| \leq |k^{-1}(0)| < \omega$.

□

Note 6.5. For $(m, n) \in \mathbb{N} \times \mathbb{N}$ and loop $h : [0, 1] \rightarrow \mathcal{Y}$ with base point 0, define:

$$h^{(m,n)}(t) = \begin{cases} h(t) & h(t) \in \frac{1}{2^{m+1}}C_n + \frac{1}{m}, \\ \frac{1}{m} & h(t) \notin \frac{1}{2^{m+1}}C_n + \frac{1}{m}. \end{cases}$$

(As a matter of fact we denote $h^{\frac{1}{2^{m+1}}C_n + \frac{1}{m}}$ (see Convention 4.1) briefly by $h^{(m,n)}$)
Moreover for loop $h : [0, 1] \rightarrow \mathcal{Y}$ we define:

$$B(h) := \left\{ (m, n) \in \mathbb{N} \times \mathbb{N} : h^{(m,n)} : [0, 1] \rightarrow \frac{1}{2^{m+1}}C_n + \frac{1}{m} \text{ is not null - homotopic} \right\},$$

then $B(h)$ is a subset of:

$$\left\{ (m, n) \in \mathbb{N} \times \mathbb{N} : \exists a, b \in [0, 1] (h(a, b) = (\frac{1}{2^{m+1}}C_n + \frac{1}{m}) \setminus \{\frac{1}{m}\} \wedge h(a) = h(b) = \frac{1}{m}) \right\}$$

and for loops $f, g : [0, 1] \rightarrow \mathcal{Y}$ with $f(0) = f(1) = g(0) = g(1) = 0$, we have:

1. If $f, g : [0, 1] \rightarrow \mathcal{Y}$ are homotopic, then $B(f) = B(g)$.
2. For $m \in \mathbb{N}$, we have:

a. $|f^{-1}(\frac{1}{m})| \geq |\{n \in \mathbb{N} : (m, n) \in B(f)\}|$.

b. If $[f] \in \mathfrak{F}^\omega(\mathcal{Y})$, then $|\{n \in \mathbb{N} : (m, n) \in B(f)\}| < \omega$.

c. If \mathcal{I} is an ideal on \mathcal{Y} and $[f] \in \mathfrak{F}_{\mathcal{I}}^\omega(\mathcal{Y})$, then there exists $F \in \mathcal{I}$ such that for all $k \in \mathbb{N}$ with $\frac{1}{k} \in \mathcal{Y} \setminus F$, we have $|\{n \in \mathbb{N} : (k, n) \in B(f)\}| < \omega$.

Proof. If $(m, n) \in B(h)$, then $h^{(m,n)} : [0, 1] \rightarrow \frac{1}{2^{m+1}}C_n + \frac{1}{m}$ is not null-homotopic, thus by Lemma 6.1 there exist $a, b \in [0, 1]$ with $h(a, b) = (\frac{1}{2^{m+1}}C_n + \frac{1}{m}) \setminus \{\frac{1}{m}\}$ and $h(a) = h(b) = \frac{1}{m}$.

1) Suppose $f, g : [0, 1] \rightarrow \mathcal{Y}$ are homotopic. For $m, n \in \mathbb{N}$, two sets $\frac{1}{2^{m+1}}C_n + \frac{1}{m}$ and $(\mathcal{Y} \setminus (\frac{1}{2^{m+1}}C_n + \frac{1}{m})) \cup \{\frac{1}{m}\}$ are closed (linear) subsets of \mathcal{Y} with $(\frac{1}{2^{m+1}}C_n + \frac{1}{m}) \cap ((\mathcal{Y} \setminus (\frac{1}{2^{m+1}}C_n + \frac{1}{m})) \cup \{\frac{1}{m}\}) = \{\frac{1}{m}\}$. Thus using the same argument as in Convention 4.1 (note to the fact that base point in the proof of Convention 4.1 is not important) two maps $f^{(m,n)}, g^{(m,n)} : [0, 1] \rightarrow \frac{1}{2^{m+1}}C_n + \frac{1}{m}$ are homotopic, therefore $(m, n) \in B(f)$ if and only if $(m, n) \in B(g)$. So $B(f) = B(g)$.

2-a) For all $(m, n) \in B(f)$ there exists $a_{(m,n)}, b_{(m,n)} \in [0, 1]$ with $f(a_{(m,n)}, b_{(m,n)}) = (\frac{1}{2^{m+1}}C_n + \frac{1}{m}) \setminus \{\frac{1}{m}\}$ and $f(a_{(m,n)}) = f(b_{(m,n)}) = \frac{1}{m}$. We claim that

$$\{n \in \mathbb{N} : (m, n) \in B(f)\} \xrightarrow{n \mapsto a_{(m,n)}} f^{-1}(\frac{1}{m})$$

is one to one. Suppose $n \neq k$ and $(m, k), (m, n) \in B(f)$. By

$$\begin{aligned} & f(a_{(m,n)}, b_{(m,n)}) \cap f(a_{(m,k)}, b_{(m,k)}) \\ &= ((\frac{1}{2^{m+1}}C_n + \frac{1}{m}) \setminus \{\frac{1}{m}\}) \cap ((\frac{1}{2^{m+1}}C_k + \frac{1}{m}) \setminus \{\frac{1}{m}\}) = \emptyset \end{aligned}$$

we have $(a_{(m,n)}, b_{(m,n)}) \cap (a_{(m,k)}, b_{(m,k)}) = \emptyset$, thus $a_{(m,n)} \neq a_{(m,k)}$. Therefore $|f^{-1}(\frac{1}{m})| \geq |\{n \in \mathbb{N} : (m, n) \in B(f)\}|$

2-b) This item is a special case of (c) for $\mathcal{I} = \{\emptyset\}$.

2-c) If $[f] \in \mathfrak{F}_{\mathcal{I}}^\omega(\mathcal{Y})$, then by Note 3.4 there exists ω -loop $h : [0, 1] \rightarrow \mathcal{Y}$ homotopic to $f : [0, 1] \rightarrow \mathcal{Y}$ also we may suppose $h(0) = h(1) = 0$. There exists $F \in \mathcal{I}$ such that for all $z \in \mathcal{Y} \setminus F$ we have $|h^{-1}(z)| < \omega + 1$. In particular for all $k \in \mathbb{N}$ with $\frac{1}{k} \in \mathcal{Y} \setminus F$ we have $|h^{-1}(\frac{1}{k})| < \omega$, which leads to $|\{n \in \mathbb{N} : (k, n) \in B(h)\}| \leq |h^{-1}(\frac{1}{k})| < \omega$ by (a). Using (1) we have $B(f) = B(h)$, thus $|\{n \in \mathbb{N} : (k, n) \in B(h)\}| = |\{n \in \mathbb{N} : (k, n) \in B(f)\}| < \omega$. \square

7. $\mathfrak{F}^c(\mathcal{X})$ is a proper subset of $\pi_1(\mathcal{X})$

Here we want to prove $\pi_1(\mathcal{X}) \setminus \mathfrak{F}^c(\mathcal{X}) \neq \emptyset$ step by step.

Consider the following conventions in this section:

Usually in order to construct Cantor set, one may remove the following intervals step by step from $[0, 1]$:

$$\begin{aligned} (c_1^1, d_1^1) &= (\frac{1}{3}, \frac{2}{3}), \\ (c_2^1, d_2^1) &= (\frac{1}{9}, \frac{2}{9}), (c_2^2, d_2^2) = (\frac{7}{9}, \frac{8}{9}), \\ &\vdots \\ (c_n^1, d_n^1) &= (\frac{1}{3^n}, \frac{2}{3^n}), (c_n^2, d_n^2) = (\frac{2}{3} + \frac{1}{3^n}, \frac{2}{3} + \frac{2}{3^n}), \dots, (c_n^{2^{n-1}}, d_n^{2^{n-1}}) = (1 - \frac{2}{3^n}, 1 - \frac{1}{3^n}), \\ &\vdots \end{aligned}$$

So $M = [0, 1] \setminus \bigcup \{(c_n^i, d_n^i) : n \in \mathbb{N}, i \in \{1, \dots, 2^{n-1}\}\}$ is Cantor set. Now suppose:

$$\begin{aligned}
 (a_1, b_1) &= (c_1^1, d_1^1), \\
 (a_2, b_2) &= (c_2^1, d_2^1), (a_3, b_3) = (c_2^2, d_2^2), \\
 &\vdots \\
 (a_{2^{n-1}}, b_{2^{n-1}}) &= (c_n^1, d_n^1), (a_{2^{n-1}+1}, b_{2^{n-1}+1}) = (c_n^2, d_n^2), \dots, (a_{2^n-1}, b_{2^n-1}) = (c_n^{2^{n-1}}, d_n^{2^{n-1}}), \\
 &\vdots
 \end{aligned}$$

Define $g : [0, 1] \rightarrow \mathcal{X}$ with:

$$g(x) = \begin{cases} \frac{1}{n} e^{2\pi i \frac{x-a_n}{b_n-a_n}} + \frac{i}{n} & x \in (a_n, b_n), n \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$$

Suppose the loops $g, f : [0, 1] \rightarrow \mathcal{X}$ are homotopic with $f(0) = f(1) = 0$. Consider the above mentioned f and g in this section.

It is well-known that (see [6]):

$$M = \left\{ \sum_{n=1}^{\infty} \frac{x_n}{3^n} : \forall n \in \mathbb{N} \quad x_n \in \{0, 2\} \right\}.$$

For $x = \sum_{n=1}^{\infty} \frac{x_n}{3^n} \in M$ with $x_n \in \{0, 2\}$ ($n \in \mathbb{N}$). For $m \in \mathbb{N}$ choose $n_m^x \in \mathbb{N}$ such that:

$$a_{n_m^x} = \begin{cases} \min\{c_m^i : 1 \leq i \leq 2^{m-1}, x \leq c_m^i\} & x_m = 0 \\ \max\{c_m^i : 1 \leq i \leq 2^{m-1}, c_m^i \leq x\} & x_m = 2 \end{cases}$$

also let

$$E^x := \{n \in \mathbb{N} : x_n = 0\}, F^x := \{n \in \mathbb{N} : x_n = 2\}.$$

Finally consider:

$$K := \{x \in M : E^x \text{ and } F^x \text{ are infinite}\}.$$

We have the following sequel of lemmas and notes.

Lemma 7.1. For $x = \sum_{n=1}^{\infty} \frac{x_n}{3^n} \in M$ with $x_n \in \{0, 2\}$, we have:

$$a_{n_k^x} = \begin{cases} \sum_{n=1}^k \frac{x_n}{3^n} + \frac{1}{3^k} & x_k = 0, \\ \sum_{n=1}^k \frac{x_n}{3^n} - \frac{1}{3^k} & x_k = 2, \end{cases} \quad \text{and} \quad b_{n_k^x} = \begin{cases} \sum_{n=1}^k \frac{x_n}{3^n} + \frac{2}{3^k} & x_k = 0, \\ \sum_{n=1}^k \frac{x_n}{3^n} & x_k = 2. \end{cases} \tag{*}$$

And:

$$|a_{n_k^x} - x| \leq \frac{2}{3^k} \quad \text{and} \quad |b_{n_k^x} - x| \leq \frac{2}{3^k} \quad (\text{for all } k \in \mathbb{N}). \tag{**}$$

Proof. For each $k \in \mathbb{N}$ suppose

$$A_k = \left\{ \sum_{n=1}^k \frac{y_n}{3^n} : y_1, \dots, y_k \in \{0, 2\} \right\},$$

then we may suppose $A_k = \{w_k^1, \dots, w_k^{2^k}\}$ with $w_k^1 < w_k^2 < \dots < w_k^{2^k}$. It is easy to see that:

$$\begin{aligned} c_k^1 &= w_k^1 + \sum_{n \geq k+1} \frac{2}{3^n} = w_k^1 + \frac{1}{3^k} & , & \quad d_k^1 = w_k^2 \\ c_k^2 &= w_k^3 + \sum_{n \geq k+1} \frac{2}{3^n} = w_k^3 + \frac{1}{3^k} & , & \quad d_k^2 = w_k^4 \\ & \vdots & & \\ c_k^i &= w_k^{2^{i-1}} + \sum_{n \geq k+1} \frac{2}{3^n} = w_k^{2^{i-1}} + \frac{1}{3^k} & , & \quad d_k^i = w_k^{2^i} \\ & \vdots & & \\ c_k^{2^{k-1}} &= w_k^{2^{k-1}} + \sum_{n \geq k+1} \frac{2}{3^n} = w_k^{2^{k-1}} + \frac{1}{3^k} & , & \quad d_k^{2^{k-1}} = w_k^{2^k} \end{aligned}$$

so $(c_k^i, d_k^i) = (w_k^{2^{i-1}} + \frac{1}{3^k}, w_k^{2^i})$.

Now for $x = \sum_{n \in \mathbb{N}} \frac{x_n}{3^n} \in M$ with $x_1, x_2, \dots \in \{0, 2\}$ we have:

- For $p \in \mathbb{N}$ we have $x_p = 0$ and $x_{p+1} = x_{p+2} = \dots = 2$ if and only if there exists $i \in \{1, \dots, 2^{p-1}\}$ with $x = c_p^i$.
- For $p \in \mathbb{N}$ we have $x_p = 2$ and $x_{p+1} = x_{p+2} = \dots = 0$ if and only if there exists $i \in \{1, \dots, 2^{p-1}\}$ with $x = d_p^i$.
- $x \in K$ if and only if for all $p \in \mathbb{N}$ we have $p \notin \{c_p^i : 1 \leq i \leq 2^{p-1}\} \cup \{d_p^i : 1 \leq i \leq 2^{p-1}\}$ (and $x \in M$).

In particular if $x_k = 0$, then $a_{n_k^x} = \sum_{n=1}^k \frac{x_n}{3^n} + \frac{1}{3^k}$, in other words if $w_k^i = \sum_{n=1}^k \frac{x_n}{3^n}$, then $i = 2j - 1$ is odd and

$a_{n_k^x} = w_k^{2j-1} + \frac{1}{3^k} = \sum_{n=1}^k \frac{x_n}{3^n} + \frac{1}{3^k} = c_k^i$. Also if $x_k = 2$, then $\sum_{n=1}^k \frac{x_n}{3^n} \in A_k$ and there exists even $i = 2j$ such that

$\sum_{n=1}^k \frac{x_n}{3^n} = w_k^{2j}$, moreover $b_{n_k^x} = w_k^{2j}$. So we have (*), moreover considering the following inequalities will complete the proof:

$$|a_{n_k^x} - x| \leq \left| a_{n_k^x} - \sum_{n=1}^k \frac{x_n}{3^n} \right| + \sum_{n=k+1}^{\infty} \frac{x_n}{3^n} \leq \frac{1}{3^k} + \sum_{n=k+1}^{\infty} \frac{2}{3^n} = \frac{2}{3^k},$$

and:

$$\begin{aligned}
 |b_{n_k^x} - x| &= \left| \sum_{n=1}^{k-1} \frac{x_n}{3^n} + \frac{2}{3^k} - x \right| \\
 &= \left| \frac{2 - x_k}{3^k} - \sum_{n=k+1}^{\infty} \frac{x_n}{3^n} \right| \\
 &= \begin{cases} \sum_{n=k+1}^{\infty} \frac{x_n}{3^n} \leq \sum_{n=k+1}^{\infty} \frac{2}{3^n} = \frac{1}{3^k} & x_k = 2 \\ \frac{2}{3^k} - \sum_{n=k+1}^{\infty} \frac{x_n}{3^n} \leq \frac{2}{3^k} & x_k = 0 \end{cases}
 \end{aligned}$$

which shows (**). \square

Lemma 7.2. Let $x = \sum_{n \in \mathbb{N}} \frac{x_n}{3^n} \in M$ with $x_n \in \{0, 2\}$ ($n \in \mathbb{N}$), then we have:

1. $\lim_{k \rightarrow \infty} a_{n_k^x} = \lim_{k \rightarrow \infty} b_{n_k^x} = x$.
2. For $i < j$ if $x_i = x_j = 0$, then $x \leq a_{n_j^x} < b_{n_j^x} < a_{n_i^x} < b_{n_i^x}$.
3. For $i < j$ if $x_i = x_j = 2$, then $a_{n_i^x} < b_{n_i^x} < a_{n_j^x} < b_{n_j^x} \leq x$.

Proof. Use (**) in Lemma 7.1 in order to prove (1).

2) Suppose $i < j$ and $x_i = x_j = 0$, then by (*) in Lemma 7.1 we have:

$$\begin{aligned}
 b_{n_i^x} &> a_{n_i^x} = \sum_{n=1}^i \frac{x_n}{3^n} + \frac{1}{3^i} = \sum_{n=1}^i \frac{x_n}{3^n} + \sum_{n=i+1}^{\infty} \frac{2}{3^n} > \sum_{n=1}^i \frac{x_n}{3^n} + \sum_{n=i+1}^j \frac{2}{3^n} \\
 &> \sum_{n=1}^i \frac{x_n}{3^n} + \sum_{n=i+1}^{j-1} \frac{x_n}{3^n} + \frac{2}{3^j} = \sum_{n=1}^j \frac{x_n}{3^n} + \frac{2}{3^j} = b_{n_j^x} > a_{n_j^x} = \sum_{n=1}^j \frac{x_n}{3^n} + \frac{1}{3^j} \\
 &= \sum_{n=1}^j \frac{x_n}{3^n} + \sum_{n=j+1}^{\infty} \frac{2}{3^n} \geq \sum_{n=1}^j \frac{x_n}{3^n} + \sum_{n=j+1}^{\infty} \frac{x_n}{3^n} = x
 \end{aligned}$$

3) Use a similar method described in the proof of (2), to prove (3). \square

Lemma 7.3. There exists a sequence $((p_n, q_n) : n \in \mathbb{N})$ such that for all $n, m \in \mathbb{N}$ we have:

- $0 \leq p_n < q_n \leq 1$, $f(p_n, q_n) = C_n \setminus \{0\}$ and $f(p_n) = f(q_n) = 0$;
- if $a_n < b_n < a_m < b_m$, then $p_n < q_n < p_m < q_m$.

Proof. For all $n \in \mathbb{N}$, by Note 6.4 we have $f^{[n]}, g^{[n]} : [0, 1] \rightarrow C_n$ are homotopic loops, therefore $f^{[n]} : [0, 1] \rightarrow C_n$ is not null-homotopic. By Lemma 6.1 there exist $a, b \in [0, 1]$ with $f^{[n]}(a, b) = C_n \setminus \{0\}$ and $f^{[n]}(a) = f^{[n]}(b) = 0$, therefore $f(a, b) = C_n \setminus \{0\}$ and $f(a) = f(b) = 0$. On the other hand $f : [0, 1] \rightarrow X$ is uniformly continuous, thus

$$\Gamma_n := \{(a, b) \in [0, 1] \times [0, 1] : f(a, b) = C_n \setminus \{0\}, f(a) = f(b) = 0\}$$

is a finite nonempty set. For $k \in \mathbb{N}$, by considering $f^{\{1, \dots, k\}} : [0, 1] \rightarrow C_1 \cup \dots \cup C_k$, Note 6.4 and Lemma 6.3 there exist $(u_1, v_1) \in \Gamma_1, \dots, (u_k, v_k) \in \Gamma_k$ such that if $a_i < b_i < a_j < b_j$, then $u_i < v_i \leq u_j < v_j$ for all $i, j \in \{1, \dots, k\}$. Using the above mentioned note and finiteness of Γ_1 , there exists $(p_1, q_1) \in \Gamma_1$ such that $\sup\{k \in \mathbb{N} : \text{there}$

exist $u_2, v_2, u_3, v_3, \dots, u_k, v_k \in [0, 1]$ such that for $u_1 = p_1$ and $v_1 = q_1$ and all $i, j \in \{1, \dots, k\}$ we have $(u_i, v_i) \in \Gamma_i$ and if $a_i < b_i < a_j < b_j$, then $u_i < v_i \leq u_j < v_j = \infty$.

For $m \in \mathbb{N}$ if $(p_1, q_1) \in \Gamma_1, \dots, (p_m, q_m) \in \Gamma_m$ are such that $\sup\{k \in \mathbb{N} : \text{there exist } u_{m+1}, v_{m+1}, u_{m+2}, v_{m+2}, \dots, u_k, v_k \in [0, 1] \text{ such that for } u_1 = p_1, v_1 = q_1, u_2 = p_2, v_2 = q_2, \dots, u_m = p_m, v_m = q_m \text{ for all } i, j \in \{1, \dots, k\} \text{ we have } (u_i, v_i) \in \Gamma_i \text{ and if } a_i < b_i < a_j < b_j, \text{ then } u_i < v_i \leq u_j < v_j = \infty\} = \infty$. Since Γ_{m+1} is finite, there exists $(p_{m+1}, q_{m+1}) \in \Gamma_{m+1}$ such that $\sup\{k \in \mathbb{N} : \text{there exist } u_{m+2}, v_{m+2}, u_{m+3}, v_{m+3}, \dots, u_k, v_k \in [0, 1] \text{ such that for } u_1 = p_1, v_1 = q_1, u_2 = p_2, v_2 = q_2, \dots, u_{m+1} = p_{m+1}, v_{m+1} = q_{m+1} \text{ for all } i, j \in \{1, \dots, k\} \text{ we have } (u_i, v_i) \in \Gamma_i \text{ and if } a_i < b_i < a_j < b_j, \text{ then } u_i < v_i \leq u_j < v_j = \infty\} = \infty$.

The sequence $((p_n, q_n) : n \in \mathbb{N})$ is our desired sequence. \square

Lemma 7.4. Let $x = \sum_{n \in \mathbb{N}} \frac{x_n}{3^n} \in K(\subset M)$ with $x_n \in \{0, 2\}$ ($n \in \mathbb{N}$), and

$$E^x = \{n \in \mathbb{N} : x_n = 0\} = \{u_k : k \in \mathbb{N}\},$$

$$F^x = \{n \in \mathbb{N} : x_n = 2\} = \{v_k : k \in \mathbb{N}\}$$

such that $u_1 < u_2 < \dots$ and $v_1 < v_2 < \dots$, and consider the sequence $((p_n, q_n) : n \in \mathbb{N})$ as in Lemma 7.3, then we have:

1. The sequences $\{a_{n_{u_k}^x} : k \in \mathbb{N}\}$ and $\{b_{n_{u_k}^x} : k \in \mathbb{N}\}$ are strictly decreasing to x .
2. The sequences $\{a_{n_{v_k}^x} : k \in \mathbb{N}\}$ and $\{b_{n_{v_k}^x} : k \in \mathbb{N}\}$ are strictly increasing to x .
3. The sequences $\{p_{n_{u_k}^x} : k \in \mathbb{N}\}$ and $\{q_{n_{u_k}^x} : k \in \mathbb{N}\}$ are strictly decreasing.
4. The sequences $\{p_{n_{v_k}^x} : k \in \mathbb{N}\}$ and $\{q_{n_{v_k}^x} : k \in \mathbb{N}\}$ are strictly increasing.
5. $\lim_{k \rightarrow \infty} p_{n_{v_k}^x} = \lim_{k \rightarrow \infty} q_{n_{v_k}^x} \leq \lim_{k \rightarrow \infty} p_{n_{u_k}^x} = \lim_{k \rightarrow \infty} q_{n_{u_k}^x}$.

Proof.

Use Lemma 7.2 in order to prove (1) and (2).

3) By Lemma 7.2 (2), for all $k \in \mathbb{N}$ we have

$$a_{n_{u_{k+1}}^x} < b_{n_{u_{k+1}}^x} < a_{n_{u_k}^x} < b_{n_{u_k}^x},$$

which leads to $p_{n_{u_{k+1}}^x} < q_{n_{u_{k+1}}^x} < p_{n_{u_k}^x} < q_{n_{u_k}^x}$.

4) Lemma 7.2 (3), for all $k \in \mathbb{N}$ we have

$$a_{n_{v_k}^x} < b_{n_{v_k}^x} < a_{n_{v_{k+1}}^x} < b_{n_{v_{k+1}}^x},$$

which leads to $p_{n_{v_k}^x} < q_{n_{v_k}^x} < p_{n_{v_{k+1}}^x} < q_{n_{v_{k+1}}^x}$.

5) Using (3) and (4), we have $\lim_{k \rightarrow \infty} p_{n_{u_k}^x} = \lim_{k \rightarrow \infty} q_{n_{u_k}^x}$ and $\lim_{k \rightarrow \infty} p_{n_{v_k}^x} = \lim_{k \rightarrow \infty} q_{n_{v_k}^x}$. On the other hand for all $k \in \mathbb{N}$ we have $a_{n_{v_k}^x} < b_{n_{v_k}^x} < x < a_{n_{u_k}^x} < b_{n_{u_k}^x}$, thus

$$p_{n_{v_k}^x} < q_{n_{v_k}^x} < p_{n_{u_k}^x} < q_{n_{u_k}^x},$$

which leads to $\lim_{k \rightarrow \infty} p_{n_{v_k}^x} \leq \lim_{k \rightarrow \infty} p_{n_{u_k}^x}$. \square

Lemma 7.5. For $x = \sum_{n \in \mathbb{N}} \frac{x_n}{3^n} \in K$ with $x_n \in \{0, 2\}$ and $E^x = \{n \in \mathbb{N} : x_n = 0\} = \{u_k : k \in \mathbb{N}\}$ with $u_1 < u_2 < \dots$

under the same notations as in Lemma 7.3, by Lemma 7.4, $\{p_{n_{u_k}^x} : k \in \mathbb{N}\}$ is an strictly decreasing sequence (in $[0, 1]$). Let $\eta(x) = \lim_{k \rightarrow \infty} p_{n_{u_k}^x}$, then $\eta : K \rightarrow [0, 1]$ is strictly increasing, and for all $x \in K$ we have $f(\eta(x)) = 0$.

Proof. Consider $x, y \in K$ with $x < y$. Suppose $x = \sum_{n \in \mathbb{N}} \frac{x_n}{3^n}$ and $y = \sum_{n \in \mathbb{N}} \frac{y_n}{3^n}$ with $x_n, y_n \in \{0, 2\}$ for all $n \in \mathbb{N}$. Let

$$E^x = \{n \in \mathbb{N} : x_n = 0\} = \{u_k : k \in \mathbb{N}\},$$

$$E^y = \{n \in \mathbb{N} : y_n = 0\} = \{u'_k : k \in \mathbb{N}\},$$

with

$$u_1 < u_2 < \dots \quad \text{and} \quad u'_1 < u'_2 < \dots.$$

By Lemma 7.4 (1), $\{a_{n_{u_k}^x} : k \in \mathbb{N}\}$ is a strictly decreasing sequence to x , and $\{a_{n_{u'_k}^y} : k \in \mathbb{N}\}$ is a strictly decreasing sequence to y . Since $x < y$, there exists $m \in \mathbb{N}$ such that

$$x \leq \dots < a_{n_{u_{m+2}}^x} < a_{n_{u_{m+1}}^x} < a_{n_{u_m}^x} < y \leq \dots < a_{n_{u'_{m+2}}^y} < a_{n_{u'_{m+1}}^y} < a_{n_{u'_m}^y}.$$

Thus

$$\begin{aligned} x &\leq \dots < a_{n_{u_{m+2}}^x} < b_{n_{u_{m+2}}^x} < a_{n_{u_{m+1}}^x} < b_{n_{u_{m+1}}^x} < a_{n_{u_m}^x} < b_{n_{u_m}^x} \\ &< y \leq \dots < a_{n_{u'_{m+2}}^y} < b_{n_{u'_{m+2}}^y} < a_{n_{u'_{m+1}}^y} < b_{n_{u'_{m+1}}^y} < a_{n_{u'_m}^y} < b_{n_{u'_m}^y}. \end{aligned}$$

Using Lemma 7.3 we have:

$$\begin{aligned} &\dots < p_{n_{u_{m+2}}^x} < q_{n_{u_{m+2}}^x} < p_{n_{u_{m+1}}^x} < q_{n_{u_{m+1}}^x} < p_{n_{u_m}^x} < q_{n_{u_m}^x} \\ &< \dots < p_{n_{u'_{m+2}}^y} < q_{n_{u'_{m+2}}^y} < p_{n_{u'_{m+1}}^y} < q_{n_{u'_{m+1}}^y} < p_{n_{u'_m}^y} < q_{n_{u'_m}^y}. \end{aligned}$$

Therefore

$$\eta(x) = \lim_{k \rightarrow \infty} p_{n_{u_k}^x} \leq p_{n_{u_{m+1}}^x} < p_{n_{u_m}^x} \leq \lim_{k \rightarrow \infty} p_{n_{u'_k}^y} = \eta(y),$$

and $\eta : K \rightarrow [0, 1]$ is strictly increasing. Since $f(p_{n_{u_k}^x}) = 0$ for all $k \in \mathbb{N}$ and f is continuous, we have $f(\eta(x)) = 0$. \square

Lemma 7.6. $|f^{-1}(0)| \geq c$ and f is not a c -arc.

Proof. Consider $\eta : K \rightarrow [0, 1]$ as in Lemma 7.5. By Lemma 7.5 we have $|f^{-1}(0)| \geq |\eta(K)|$ and by Lemma 7.4 η is one to one, therefore $|\eta(K)| = |K| = c$. Thus $|f^{-1}(0)| \geq c$ and f is not a c -arc. \square

Theorem 7.7. We have

$$\mathfrak{F}^c(\mathcal{X}) \subset \pi_1(\mathcal{X}), \mathfrak{F}_{\mathcal{P}_{fin}(\mathcal{X})}^\omega(\mathcal{X}) \not\subset \mathfrak{F}^c(\mathcal{X}),$$

$$\mathfrak{F}^\omega(\mathcal{X}) \subset \mathfrak{F}_{\mathcal{P}_{fin}(\mathcal{X})}^\omega(\mathcal{X}) \text{ and } \mathfrak{F}_{\mathcal{P}_{fin}(\mathcal{X})}^\omega(\mathcal{X}) \not\subset \mathfrak{F}^c(\mathcal{X}).$$

Proof. Using Note 3.4, and Lemma 7.6, $[g] \notin \mathfrak{F}^c(\mathcal{X})$, thus $\mathfrak{F}^c(\mathcal{X}) \subset \pi_1(\mathcal{X})$. Using $[g] \in \mathfrak{F}_{\mathcal{P}_{fin}(\mathcal{X})}^\omega(\mathcal{X})$ shows $\mathfrak{F}_{\mathcal{P}_{fin}(\mathcal{X})}^\omega(\mathcal{X}) \not\subset \mathfrak{F}^c(\mathcal{X})$. Also $[g] \in \mathfrak{F}^\omega(\mathcal{X}) \setminus \mathfrak{F}_{\mathcal{P}_{fin}(\mathcal{X})}^\omega(\mathcal{X})$, thus $\mathfrak{F}^\omega(\mathcal{X})$ is a proper subgroup of $\mathfrak{F}_{\mathcal{P}_{fin}(\mathcal{X})}^\omega(\mathcal{X})$. Using $[g] \in \mathfrak{F}_{\mathcal{P}_{fin}(\mathcal{X})}^\omega(\mathcal{X}) \setminus \mathfrak{F}^c(\mathcal{X})$ will complete the proof. \square

8. $\mathfrak{F}_{\mathcal{P}_{fin}(\mathcal{Y})}^c(\mathcal{Y})$ is a proper subset of $\pi_1(\mathcal{Y})$

In this section we prove $\pi_1(\mathcal{Y}) \setminus \mathfrak{F}_{\mathcal{P}_{fin}(\mathcal{Y})}^c(\mathcal{Y}) \neq \emptyset$. We use the same notations as in Section 7.

Define $G : [0, 1] \rightarrow \mathcal{Y}$ with:

$$G(x) = \begin{cases} \frac{g(4xn(n+1) - (2n+1))}{2^{n+1}} + \frac{1}{n} & \frac{2n+1}{4n(n+1)} \leq x \leq \frac{1}{2n}, n \in \mathbb{N}, \\ 2(n+1)(2n-1)x + (2-2n) & \frac{1}{2(n+1)} \leq x \leq \frac{2n+1}{4n(n+1)}, n \in \mathbb{N}, \\ 2-2x & \frac{1}{2} \leq x \leq 1, \\ 0 & x = 0, \end{cases}$$

where $g : [0, 1] \rightarrow \mathcal{X}$ as in section 7 is:

$$g(x) = \begin{cases} \frac{1}{n} e^{2\pi i \frac{x-a_n}{b_n-a_n}} + \frac{i}{n} & x \in (a_n, b_n), n \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 8.1. Let $K, G : [0, 1] \rightarrow \mathcal{Y}$ are homotopic and $m \in \mathbb{N}$, then $|K^{-1}(\frac{1}{m})| = c$.

Proof. Choose $\theta \in (\frac{1}{m+1} + \frac{1}{2^{m+2}}, \frac{1}{m} - \frac{1}{2^{m+1}})$. Consider $h, \bar{h} : [0, 1] \rightarrow \mathcal{Y}$ with $h(x) = \theta x$ and $\bar{h}(x) = \theta(1 - x)$. Since $K, G : [0, 1] \rightarrow \mathcal{Y}$ are path homotopic with base point 0, $\bar{h} * K * h, \bar{h} * G * h : [0, 1] \rightarrow \mathcal{Y}$ are path homotopic with base point θ . Using Convention 4.1 two maps

$$(\bar{h} * K * h)^{(x,y) \in \mathcal{Y}:x \geq \theta}, (\bar{h} * G * h)^{(x,y) \in \mathcal{Y}:x \geq \theta} : [0, 1] \rightarrow \{(x, y) \in \mathcal{Y} : x \geq \theta\}$$

are path homotopic with base point θ . Let

$$K_1 = (\bar{h} * K * h)^{(x,y) \in \mathcal{Y}:x \geq \theta} \text{ and } G_1 = (\bar{h} * G * h)^{(x,y) \in \mathcal{Y}:x \geq \theta}.$$

If $m = 1$ let $K_2 = K_1$ and $G_2 = G_1$.

If $m > 1$, choose $\mu \in (\frac{1}{m} + \frac{1}{2^{m+1}}, \frac{1}{m-1} - \frac{1}{2^m})$. Consider $h_1, \bar{h}_1 : [0, 1] \rightarrow \mathcal{Y}$ with $h_1(x) = (\mu - \theta)x + \theta$ and $\bar{h}_1(x) = (\mu - \theta)(1 - x) + \theta$. Since $K_1, G_1 : [0, 1] \rightarrow \{(x, y) \in \mathcal{Y} : x \geq \theta\}$ are path homotopic with base point θ , $\bar{h}_1 * K_1 * h_1, \bar{h}_1 * G_1 * h_1 : [0, 1] \rightarrow \{(x, y) \in \mathcal{Y} : x \geq \theta\}$ are path homotopic with base point μ . Using Convention 4.1 two maps $(\bar{h}_1 * K_1 * h_1)^{(x,y) \in \mathcal{Y}:\theta \leq x \leq \mu}$ and $(\bar{h}_1 * G_1 * h_1)^{(x,y) \in \mathcal{Y}:\theta \leq x \leq \mu}$ from $[0, 1]$ to $\{(x, y) \in \mathcal{Y} : \theta \leq x \leq \mu\}$ are path homotopic with base point μ . Let $h_2(x) = (\frac{1}{m} - \mu)x + \mu$ and $\bar{h}_2(x) = (\frac{1}{m} - \mu)(1 - x) + \mu$ for $x \in [0, 1]$.

Now let:

$$K_2 = \begin{cases} \bar{h}_2 * (\bar{h}_1 * K_1 * h_1)^{(x,y) \in \mathcal{Y}:\theta \leq x \leq \mu} * h_2 & m > 1, \\ K_1 & m = 1, \end{cases}$$

and

$$G_2 = \begin{cases} \bar{h}_2 * (\bar{h}_1 * G_1 * h_1)^{(x,y) \in \mathcal{Y}:\theta \leq x \leq \mu} * h_2 & m > 1, \\ G_1 & m = 1, \end{cases}$$

also in order to be more convenient, whenever $m = 1$ let $\mu = 1$. Then $K_2, G_2 : [0, 1] \rightarrow \{(x, y) \in \mathcal{Y} : \theta \leq x \leq \mu\}$ ($\subseteq (\frac{1}{2^{m+1}}\mathcal{X} + \frac{1}{m}) \cup [\theta, \mu]$) are path homotopic with base point $\frac{1}{m}$. Hence there exists a continuous map $F : [0, 1] \times [0, 1] \rightarrow \{(x, y) \in \mathcal{Y} : \theta \leq x \leq \mu\}$ such that $F(0, s) = F(1, s) = \frac{1}{m}$, $F(s, 0) = K_2(s)$ and $F(s, 1) = G_2(s)$ for all $s \in [0, 1]$.

Define $\mathcal{K}, \mathcal{G} : [0, 1] \rightarrow \mathcal{X}$ and $\mathcal{F} : [0, 1] \times [0, 1] \rightarrow \mathcal{X}$ with:

$$\mathcal{K}(x) = \begin{cases} 2^{m+1}(K_2(x) - \frac{1}{m}) & K_2(x) \in \frac{1}{2^{m+1}}\mathcal{X} + \frac{1}{m}, \\ -ie^{\frac{i\pi(K_2(x) - \frac{1}{m})}{2}} + i & \theta \leq K_2(x) \leq \mu, \end{cases}$$

$$\mathcal{G}(x) = \begin{cases} 2^{m+1}(G_2(x) - \frac{1}{m}) & G_2(x) \in \frac{1}{2^{m+1}}\mathcal{X} + \frac{1}{m}, \\ -ie^{\frac{i\pi(G_2(x) - \frac{1}{m})}{2}} + i & \theta \leq G_2(x) \leq \mu, \end{cases}$$

$$\mathcal{F}(s, t) = \begin{cases} 2^{m+1}(F(s, t) - \frac{1}{m}) & F(s, t) \in \frac{1}{2^{m+1}}\mathcal{X} + \frac{1}{m}, \\ -ie^{\frac{i\pi(F(s,t) - \frac{1}{m})}{2}} + i & \theta \leq F(s, t) \leq \mu. \end{cases}$$

Using the gluing lemma, \mathcal{K}, \mathcal{G} and \mathcal{F} are continuous, moreover by the above definition, for all $s \in [0, 1]$ we have:

- the equality $F(0, s) = F(1, s) = \frac{1}{m}$, implies

$$\mathcal{F}(0, s) = \mathcal{F}(1, s) = -ie^{\frac{i\pi(\frac{1}{m}-\frac{1}{m})}{2}} + i = 0,$$

- two equalities $F(s, 0) = K_2(s)$ and $F(s, 1) = G_2(s)$, imply $\mathcal{F}(s, 0) = \mathcal{K}(s)$ and $\mathcal{F}(s, 1) = \mathcal{G}(s)$.

So $\mathcal{K}, \mathcal{G}; [0, 1] \rightarrow \mathcal{X}$ are path homotopic with base point 0. One could verify that $\mathcal{G}, g : [0, 1] \rightarrow \mathcal{X}$ are homotopic, thus $\mathcal{K}, g : [0, 1] \rightarrow \mathcal{X}$ are homotopic and by Lemma 7.6 in which we proved $|f^{-1}(0)| = c$ whenever $f, g : [0, 1] \rightarrow \mathcal{X}$ are homotopic, we have $|\mathcal{K}^{-1}(0)| \geq c$. Since $\mathcal{K}^{-1}(0)$ and $K^{-1}(\frac{1}{m})$ differs in a finite set, we have $|K^{-1}(\frac{1}{m})| = c$. \square

Theorem 8.2. We have $[G] \in \pi_1(\mathcal{Y}) \setminus \mathfrak{P}_{\mathcal{P}_{fm}(\mathcal{Y})}^c(\mathcal{Y})$.

Proof. If $[G] \in \mathfrak{P}_{\mathcal{P}_{fm}(\mathcal{Y})}^c(\mathcal{Y})$, then by Note 3.4, there exists $c \frac{\mathcal{P}_{fm}(\mathcal{Y})}{\text{loop}} K : [0, 1] \rightarrow \mathcal{Y}$ with $K(0) = K(1) = 0$ and $[F] = [G]$. Since $G : [0, 1] \rightarrow \mathcal{Y}$ is not null-homotopic, $K : [0, 1] \rightarrow \mathcal{Y}$ is not constant. Thus there exists $k \in \mathbb{N}$ such that for all $m \geq k$ we have $\frac{1}{m} \in K[0, 1]$. By Lemma 8.1, for all $m \geq k$ we have $|K^{-1}(\frac{1}{m})| = c$, thus $\{x \in \mathcal{Y} : |K^{-1}(x)| \neq c\}$ is infinite, which is a contradiction, since $K : [0, 1] \rightarrow \mathcal{Y}$ is a $c \frac{\mathcal{P}_{fm}(\mathcal{Y})}{\text{loop}}$. Therefore $[G] \notin \mathfrak{P}_{\mathcal{P}_{fm}(\mathcal{Y})}^c(\mathcal{Y})$. \square

9. Main examples and counterexamples

Now we are ready to present examples.

Example 9.1. Using Note 6.4 (4), since $A(f_X)(= \mathbb{N})$ is infinite, thus $[f_X] \notin \mathfrak{P}^\omega(\mathcal{X})$. On the other hand, using Example 2.3 (1), $f_X : [0, 1] \rightarrow \mathcal{X}$ is a c -loop, thus $[f_X] \in \mathfrak{P}^c(\mathcal{X}) \setminus \mathfrak{P}^\omega(\mathcal{X})$ and $\mathfrak{P}^\omega(\mathcal{X})$ is a proper subgroup of $\mathfrak{P}^c(\mathcal{X})$. Therefore by Theorem 2.3, we have:

$$\mathfrak{P}^\omega(\mathcal{X}) \subset \mathfrak{P}^c(\mathcal{X}) \subset \pi_1(\mathcal{X}).$$

Also using Theorem 2.3 again we have $\mathfrak{P}^\omega(\mathcal{X}) \subset \mathfrak{P}_{\mathcal{P}_{fm}(\mathcal{X})}^\omega(\mathcal{X})$, which leads to $\mathfrak{P}^\omega(\mathcal{X}) \subset \mathfrak{P}_{\mathcal{P}_{fm}(\mathcal{X})}^c(\mathcal{X})$, since $\mathfrak{P}_{\mathcal{P}_{fm}(\mathcal{X})}^\omega(\mathcal{X}) \subseteq \mathfrak{P}_{\mathcal{P}_{fm}(\mathcal{X})}^c(\mathcal{X})$. We recall that according to Theorem 2.3, $\mathfrak{P}_{\mathcal{P}_{fm}(\mathcal{X})}^\omega(\mathcal{X}) \not\subseteq \mathfrak{P}^c(\mathcal{X})$, which leads to $\mathfrak{P}_{\mathcal{P}_{fm}(\mathcal{X})}^c(\mathcal{X}) \not\subseteq \mathfrak{P}^c(\mathcal{X})$ since $\mathfrak{P}_{\mathcal{P}_{fm}(\mathcal{X})}^\omega(\mathcal{X}) \subseteq \mathfrak{P}_{\mathcal{P}_{fm}(\mathcal{X})}^c(\mathcal{X})$.

The following Example deal with Theorem 5.1. We again recall that $\Phi : \pi_1(X) \times \pi_1(Y) \rightarrow \pi_1(X \times Y)$, with $\Phi([f], [g]) = [(f, g)]$ (where $(f, g)(t) = (f(t), g(t))$ (for $t \in [0, 1]$ and loops $f : [0, 1] \rightarrow X, g : [0, 1] \rightarrow Y$) is a group isomorphism. Moreover as it was proved in Theorem 5.1 (4c), for infinite cardinal number α we have $\Phi(\mathfrak{P}^\alpha(X) \times \mathfrak{P}^\alpha(Y)) \subseteq \mathfrak{P}^\alpha(X \times Y)$. In the following we bring an example in which $\Phi(\mathfrak{P}^\alpha(X) \times \mathfrak{P}^\alpha(Y)) \neq \mathfrak{P}^\alpha(X \times Y)$, in particular we prove that $\Phi \upharpoonright_{\mathfrak{P}^\omega(\mathcal{X}) \times \mathfrak{P}^\omega(\mathcal{X})} : \mathfrak{P}^\omega(\mathcal{X}) \times \mathfrak{P}^\omega(\mathcal{X}) \rightarrow \mathfrak{P}^\omega(\mathcal{X} \times \mathcal{X})$ is a group monomorphism but it is not an isomorphism.

Example 9.2. Define $\bar{f}_X : [0, 1] \rightarrow \mathcal{X}$ with $\bar{f}_X(t) = f_X(1 - t)$. $(f_X, \bar{f}_X) : [0, 1] \rightarrow \mathcal{X} \times \mathcal{X}$ is an ω -arc since for all $(x, y) \in \mathcal{X} \times \mathcal{X}$, if $|(f_X, \bar{f}_X)^{-1}(x, y)| > 1$, then $x = y = 0$. Moreover $(f_X, \bar{f}_X)^{-1}(0, 0) \subseteq \{t \in [0, 1] : t, 1 - t \in \{\frac{1}{n} : n \in \mathbb{N}\}\} \cup \{0, 1\} = \{0, 1, \frac{1}{2}\}$. Therefore for all (x, y) we have $|(f_X, \bar{f}_X)^{-1}(x, y)| \leq 3 < \omega$ and $(f_X, \bar{f}_X) : [0, 1] \rightarrow \mathcal{X} \times \mathcal{X}$ is an ω -arc. Thus $\Phi([f_X], [\bar{f}_X]) = [(f_X, \bar{f}_X)] \in \mathfrak{P}^\omega(\mathcal{X} \times \mathcal{X})$. Since $\Phi : \pi_1(\mathcal{X}) \times \pi_1(\mathcal{X}) \rightarrow \pi_1(\mathcal{X} \times \mathcal{X})$ is a group isomorphism, there exist unique $([g], [h]) \in \pi_1(\mathcal{X}) \times \pi_1(\mathcal{X})$ with $\Phi([g], [h]) = [(f_X, \bar{f}_X)]$ therefore $[g] = [f_X]$ and $[h] = [\bar{f}_X]$. Using Example 9.1, $[f_X] \notin \mathfrak{P}^\omega(\mathcal{X})$, so $([g], [h]) = ([f_X], [\bar{f}_X]) \notin \mathfrak{P}^\omega(\mathcal{X}) \times \mathfrak{P}^\omega(\mathcal{X})$. So (note: Φ is one to one):

$$[(f_X, \bar{f}_X)] = \Phi([g], [h]) = \Phi([f_X], [\bar{f}_X]) \notin \Phi(\mathfrak{P}^\omega(\mathcal{X}) \times \mathfrak{P}^\omega(\mathcal{X}))$$

which shows $\Phi(\mathfrak{P}^\omega(\mathcal{X}) \times \mathfrak{P}^\omega(\mathcal{X})) \neq \mathfrak{P}^\omega(\mathcal{X} \times \mathcal{X})$.

Example 9.3. Using the same notations as in Note 6.5 we have $B(f_{\mathcal{Y}}) = \mathbb{N} \times \mathbb{N}$, therefore for all $m \in \mathbb{N}$, $\{n \in \mathbb{N} : (m, n) \in B(f_{\mathcal{Y}})\} (= \mathbb{N})$ is infinite. If $F \in \mathcal{P}_{fin}(\mathcal{Y})$, then F is finite and there exists $k \in \mathbb{N}$ with $\frac{1}{k} \in \mathcal{Y} \setminus F$. Using infiniteness of $\{n \in \mathbb{N} : (k, n) \in B(f_{\mathcal{Y}})\}$ and Note 6.5 (c) we have $[f_{\mathcal{Y}}] \notin \mathfrak{P}_{\mathcal{P}_{fin}(\mathcal{Y})}^{\omega}(\mathcal{Y})$. On the other hand using Example 2.3 (2), $f_{\mathcal{Y}} : [0, 1] \rightarrow \mathcal{Y}$ is a c -loop, thus $[f_{\mathcal{Y}}] \in \mathfrak{P}^c(\mathcal{Y})$. So $\mathfrak{P}_{\mathcal{P}_{fin}(\mathcal{Y})}^{\omega}(\mathcal{Y})$ is a proper subgroup of $\mathfrak{P}^c(\mathcal{Y})$ and $\pi_1(\mathcal{Y})$ (Hint: We can prove $\mathfrak{P}^{\omega}(\mathcal{Y})$ is a proper subgroup of $\mathfrak{P}_{\mathcal{P}_{fin}(\mathcal{Y})}^{\omega}(\mathcal{Y})$, thus $\mathfrak{P}^{\omega}(\mathcal{Y}) \subset \mathfrak{P}_{\mathcal{P}_{fin}(\mathcal{Y})}^{\omega}(\mathcal{Y}) \subset \pi_1(\mathcal{Y})$).

Example 9.4. Map $f_{\mathcal{Z}} : [0, 1] \rightarrow \mathcal{Z}$ is a $p + 1$ -arc and it is not homotopic with any k -arc $g : [0, 1] \rightarrow \mathcal{Z}$ for $k < p + 1$. However for all $\alpha \geq 2$ and ideal \mathcal{I} on \mathcal{Z} we have $\mathfrak{P}_{\mathcal{I}}^{\alpha}(\mathcal{Z}) = \pi_1(\mathcal{Z})$. For this aim, for all $k \in \{1, \dots, p\}$, define $f_k : [0, 1] \rightarrow \mathcal{Z}$ with $f_k(t) = \frac{1}{k}e^{2\pi i t(t-\frac{1}{k})} + \frac{i}{k}$. For all $\alpha \geq 2$ and ideal \mathcal{I} on \mathcal{Z} , we have $[f_k] \in \mathfrak{P}^2(\mathcal{Z}) \subseteq \mathfrak{P}_{\mathcal{I}}^{\alpha}(\mathcal{Z}) \subseteq \pi_1(\mathcal{Z})$. Since $\{[f_1], \dots, [f_p]\}$ generates $\pi_1(\mathcal{Z})$, thus $\mathfrak{P}_{\mathcal{I}}^{\alpha}(\mathcal{Z}) = \pi_1(\mathcal{Z})$.

Example 9.5. We recall that $\pi_1(\mathcal{Y}) \setminus \mathfrak{P}_{\mathcal{P}_{fin}(\mathcal{Y})}^c(\mathcal{Y}) \neq \emptyset$ by Theorem 8.2. However $[f_{\mathcal{Y}}] \in \mathfrak{P}_{\mathcal{P}_{fin}(\mathcal{Y})}^c(\mathcal{Y})$ (since $f_{\mathcal{Y}} : [0, 1] \rightarrow \mathcal{Y}$ is a c -loop, thus $[f_{\mathcal{Y}}] \in \mathfrak{P}_{\mathcal{P}_{fin}(\mathcal{Y})}^c(\mathcal{Y})$) One may show $[f_{\mathcal{Y}}] \notin \mathfrak{P}_{\mathcal{P}_{fin}(\mathcal{Y})}^{\omega}(\mathcal{Y})$, thus:

$$\mathfrak{P}_{\mathcal{P}_{fin}(\mathcal{Y})}^{\omega}(\mathcal{Y}) \subset \mathfrak{P}_{\mathcal{P}_{fin}(\mathcal{Y})}^c(\mathcal{Y}) \subset \pi_1(\mathcal{Y}).$$

10. Main Table

Table 10.1. We have the following Table:

$\underline{\hspace{2cm}}$ H	K	$\mathfrak{P}^{\omega}(X)$	$\mathfrak{P}_{\mathcal{P}_{fin}(X)}^{\omega}(X)$	$\mathfrak{P}^c(X)$	$\mathfrak{P}_{\mathcal{P}_{fin}(X)}^c(X)$	$\pi_1(X)$
$\mathfrak{P}^{\omega}(X)$	\subseteq	\subseteq	\subseteq	\subseteq	\subseteq	\subseteq
$\mathfrak{P}_{\mathcal{P}_{fin}(X)}^{\omega}(X)$	9.1	\subseteq	9.1	\subseteq	\subseteq	\subseteq
$\mathfrak{P}^c(X)$	9.1	9.3	\subseteq	\subseteq	\subseteq	\subseteq
$\mathfrak{P}_{\mathcal{P}_{fin}(X)}^c(X)$	9.1	9.3	9.1	\subseteq	\subseteq	\subseteq
$\pi_1(X)$	9.1	9.3	9.1	9.5	\subseteq	\subseteq

In the above table “ \subseteq ” means that in the corresponding case we have $H \subseteq K$.

In addition the number i,j means that in Example i,j there exists an example such that $H \not\subseteq K$ in the corresponding case.

11. Two spaces having fundamental groups isomorphic to Hawaiian earring’s fundamental group

In this section we prove in a sequel of Lemmas, that \mathcal{X} (Hawaiian earring) and \mathcal{W} are homeomorph with two deformation retracts of \mathcal{V} . Thus we have $\pi_1(\mathcal{X}) \cong \pi_1(\mathcal{V}) \cong \pi_1(\mathcal{W})$, which is important for our main counterexamples in next section.

We recall sign map $\text{sgn} : \mathbb{R} \rightarrow \{\pm 1, 0\}$ with $\text{sgn}(x) = \frac{x}{|x|}$ for $x \neq 0$ and $\text{sgn}(0) = 0$.

Note: In a connected topological space A , we call $x \in A$ a cut point of A if $A \setminus \{x\}$ is disconnected. It is evident that \mathcal{X} and \mathcal{W} are not homeomorphic since \mathcal{X} has just one cut point and \mathcal{W} has infinitely many cut points.

Lemma 11.1. For $x \in [0, 1]$, the map $\Phi_x : [0, \frac{1}{2}] \rightarrow \{w \in [-1, 1] : x + w \leq 0\} = [-1, 1] \cap (-\infty, -x] = [-1, -x]$ with:

$$\Phi_x(t) = \begin{cases} (1 - \sin(\pi t))(1 - \frac{x}{1-2t}) - 1 & t \in [0, \frac{1}{2}) \\ -1 & t = \frac{1}{2} \end{cases}$$

is a homeomorphism.

Proof. Suppose $z \in (-1, 1]$ and $z + x \leq 0$. The map $\varphi : [0, \frac{1}{2}) \rightarrow \mathbb{R}$ with $\varphi(t) = (1 - \sin(\pi t))(1 - \frac{x}{1-2t}) - 1$ is continuous, moreover $\varphi(0) = -x$ and $\lim_{t \rightarrow \frac{1}{2}^-} \varphi(t) = -1$. By $-1 < z \leq -x$ and the mean value theorem

there exists $t \in [0, \frac{1}{2})$ with $\varphi(t) = z$. In addition $\Phi_x \upharpoonright_{[0, \frac{1}{2})} = \varphi : [0, \frac{1}{2}) \rightarrow \mathbb{R}$ is strictly decreasing, therefore $\Phi_x : [0, \frac{1}{2}] \rightarrow [-1, -x]$ is a bijective continuous map which completes the proof. \square

Lemma 11.2. Using the same notations as in Lemma 11.1, $\widehat{\Phi} : \{(x, w) \in [0, 1] \times [-1, 0] : x + w \leq 0\} \rightarrow [0, \frac{1}{2}]$ with $\widehat{\Phi}(x, w) = \Phi_x^{-1}(w)$ is continuous.

Proof. Using Lemma 11.1, $\widehat{\Phi} : \{(x, w) \in [0, 1] \times [-1, 0] : x + w \leq 0\} \rightarrow [0, \frac{1}{2}]$ is well-defined. Let $A := \{(x, w) \in [0, 1] \times [-1, 0] : x + w \leq 0\}$. Consider $(x, w) \in A$, $s \in [0, \frac{1}{2}]$, and sequence $\{(x_n, w_n) : n \in \mathbb{N}\}$ such that $\lim_{n \rightarrow \infty} x_n = x$, $\lim_{n \rightarrow \infty} w_n = w$, $\lim_{n \rightarrow \infty} \widehat{\Phi}(x_n, w_n) = s$. Let $t = \widehat{\Phi}(x, w)$ and $t_n = \widehat{\Phi}(x_n, w_n)$ ($n \in \mathbb{N}$). We show $s = t$, i.e. $\lim_{n \rightarrow \infty} \widehat{\Phi}(x_n, w_n) = \widehat{\Phi}(x, w)$.

We have the following cases:

Case 1. $s \neq \frac{1}{2}$. In this case we may suppose for all $n \in \mathbb{N}$ we have $t_n \neq \frac{1}{2}$. For all $n \in \mathbb{N}$ we have $w_n = \Phi_{x_n}(t_n) = (1 - \sin(\pi t_n))(1 - \frac{x_n}{1 - 2t_n}) - 1$ moreover:

$$\begin{aligned} \Phi_x(t) &= w = \lim_{n \rightarrow \infty} w_n = \lim_{n \rightarrow \infty} \Phi_{x_n}(t_n) \\ &= \lim_{n \rightarrow \infty} (1 - \sin(\pi t_n))(1 - \frac{x_n}{1 - 2t_n}) - 1 \\ &= (1 - \sin(\pi s))(1 - \frac{x}{1 - 2s}) - 1 = \Phi_x(s) \end{aligned}$$

and $s = t$ since Φ_x is one to one according to Lemma 11.1.

Case 2. $s = \frac{1}{2}$ and for infinitely many of n s we have $t_n = \frac{1}{2}$. In this case we may suppose for all $n \in \mathbb{N}$ we have $t_n = \frac{1}{2}$. Thus we have:

$$\begin{aligned} \Phi_x(t) &= w = \lim_{n \rightarrow \infty} w_n = \lim_{n \rightarrow \infty} \Phi_{x_n}(t_n) \\ &= \lim_{n \rightarrow \infty} \Phi_{x_n}(\frac{1}{2}) = \lim_{n \rightarrow \infty} -1 = -1 = \Phi_x(\frac{1}{2}) \end{aligned}$$

and $s = \frac{1}{2} = t$ since Φ_x is one to one according to Lemma 11.1.

Case 3. $s = \frac{1}{2}$ and for infinitely many of n s we have $t_n \neq \frac{1}{2}$. In this case we may suppose for all $n \in \mathbb{N}$ we have $t_n \neq \frac{1}{2}$. Thus we have:

$$\begin{aligned} \Phi_x(t) &= w = \lim_{n \rightarrow \infty} w_n = \lim_{n \rightarrow \infty} \Phi_{x_n}(t_n) \\ &= \lim_{n \rightarrow \infty} (1 - \sin(\pi t_n))(1 - \frac{x_n}{1 - 2t_n}) - 1 \\ &= \lim_{n \rightarrow \infty} \frac{(1 - \sin(\pi t_n))}{1 - 2t_n} \lim_{n \rightarrow \infty} (1 - 2t_n - x_n) - 1 \\ &= 0 \times (1 - s - x) - 1 = -1 = \Phi_x(\frac{1}{2}) \end{aligned}$$

and $s = \frac{1}{2} = t$ since Φ_x is one to one according to Lemma 11.1.

Using the above cases $s = t$ and $\widehat{\Phi} : \{(x, w) \in [0, 1] \times [-1, 0] : x + w \leq 0\} \rightarrow [0, \frac{1}{2}]$ is continuous (otherwise since $[0, \frac{1}{2}]$ is compact, there exists $(x, w) \in A$ and sequence $\{(x_n, w_n) : n \in \mathbb{N}\}$ converging to (x, w) such that the sequence $\{\widehat{\Phi}(x_n, w_n) : n \in \mathbb{N}\}$ converges to a point $s \in [0, \frac{1}{2}] \setminus \{\widehat{\Phi}(x, w)\}$). \square

Lemma 11.3. Consider $X = \{(x, y, z) \in \mathbb{R}^3 : y^2 + z^2 = 1, 0 \leq x \leq 1\}$ and $\widehat{\Phi}$ as in Lemma 11.2. Let $M_1 = \{(x, y, z) \in X : x + z \leq 0\}$, the map $F_1 : [0, 1] \times M_1 \rightarrow X$ with $F_1(\mu, (x, y, z)) = (x', y', z')$ for:

$$\begin{cases} x' = x + (1 - 2(1 - x)\widehat{\Phi}(x, z) - x)\mu, \\ z' = (1 - \mu)z - \mu, \\ y' = \text{sgn}(y) \sqrt{1 - z'^2}, \end{cases}$$

is continuous.

Proof. Let $(\mu, (x, y, z)) \in [0, 1] \times M_1$, since $\widehat{\Phi}(x, z) \in [0, \frac{1}{2}]$, we have $1 - 2\widehat{\Phi}(x, z) \in [0, 1]$ which leads to (use $x, \mu \in [0, 1]$):

$$\begin{aligned} 0 \leq x(1 - \mu) &= x + (0 - x)\mu \leq x' = x + (1 - 2(1 - x)\widehat{\Phi}(x, z) - x)\mu \\ &\leq x + (1 - x)\mu \leq x + (1 - x) = 1 \end{aligned}$$

thus $x' \in [0, 1]$. Moreover using $\mu \in [0, 1]$ and $z \in [-1, 0]$ we have:

$$-1 = (1 - \mu)(-1) - \mu \leq (1 - \mu)z - \mu \leq (1 - \mu)0 - \mu = -\mu \leq 0,$$

thus $z' \in [-1, 0]$ using $y'^2 + z'^2 = 1$, $F_1 : [0, 1] \times M_1 \rightarrow X$ is well-defined.

Using Lemma 11.2, $F_1 : [0, 1] \times M_1 \rightarrow X$ is continuous. \square

Lemma 11.4. For $x \in [0, 1]$, the map $\Psi_x : [0, \frac{1}{2}] \rightarrow \{z \in [-1, 1] : x + z \geq 0\} = [-1, 1] \cap [-x, +\infty) = [-x, 1]$ with $\Psi_x(t) = \sin(\pi t) - (1 + \sin(\pi t) - 4t)x$ is a homeomorphism.

Proof. Suppose $z \in [-1, 1]$ and $z + x \geq 0$. Since $\Psi_x(0) = -x$ and $\Psi_x(\frac{1}{2}) = 1$ by the mean value theorem there exists $t \in [0, \frac{1}{2}]$ with $\Psi_x(t) = z$. Thus $\Psi_x : [0, \frac{1}{2}] \rightarrow [-x, 1]$ is a bijection continuous map which completes the proof. \square

Lemma 11.5. Using the same notations as in Lemma 11.4, $\widehat{\Psi} : \{(x, w) \in [0, 1] \times [-1, 1] : x + w \geq 0\} \rightarrow [0, \frac{1}{2}]$ with $\widehat{\Psi}(x, w) = \Psi_x^{-1}(w)$ is continuous.

Proof. Using Lemma 11.4, $\widehat{\Psi} : \{(x, w) \in [0, 1] \times [-1, 1] : x + w \geq 0\} \rightarrow [0, \frac{1}{2}]$ is well-defined. Let $B := \{(x, w) \in [0, 1] \times [-1, 1] : x + w \geq 0\}$. Consider $(x, w) \in B, s \in [0, \frac{1}{2}]$, and sequence $\{(x_n, w_n) : n \in \mathbb{N}\}$ such that $\lim_{n \rightarrow \infty} x_n = x, \lim_{n \rightarrow \infty} w_n = w, \lim_{n \rightarrow \infty} \widehat{\Psi}(x_n, w_n) = s$. Let $t = \widehat{\Psi}(x, w)$ and $t_n = \widehat{\Psi}(x_n, w_n)$ ($n \in \mathbb{N}$). We have:

$$\begin{aligned} \Psi_x(t) &= w = \lim_{n \rightarrow \infty} w_n = \lim_{n \rightarrow \infty} \Psi_{x_n}(t_n) \\ &= \lim_{n \rightarrow \infty} \sin(\pi t_n) - (1 + \sin(\pi t_n) - 4t_n)x_n \\ &= \sin(\pi s) - (1 + \sin(\pi s) - 4s)x = \Psi_x(s) \end{aligned}$$

and $s = t$ since Ψ_x is one to one according to Lemma 11.4. Using the above discussion and the compactness of $[0, \frac{1}{2}]$, $\widehat{\Psi} : \{(x, w) \in [0, 1] \times [-1, 1] : x + w \geq 0\} \rightarrow [0, \frac{1}{2}]$ is continuous. \square

Lemma 11.6. Consider $X = \{(x, y, z) \in \mathbb{R}^3 : y^2 + z^2 = 1, 0 \leq x \leq 1\}$ and $\widehat{\Psi}$ as in Lemma 11.5. Let $M_2 = \{(x, y, z) \in X : x + z \geq 0\}$, the map $F_2 : [0, 1] \times M_2 \rightarrow X$ with $F_2(\mu, (x, y, z)) = (x', y', z')$ for:

$$\begin{cases} x' = x + (1 - x)\mu, \\ z' = (1 - \mu)z + (4\widehat{\Psi}(x, z) - 1)\mu, \\ y' = \operatorname{sgn}(y)\sqrt{1 - z'^2}, \end{cases}$$

is continuous.

Proof. Let $(\mu, (x, y, z)) \in [0, 1] \times M_2$, since $x, \mu \in [0, 1]$ we have $0 \leq x \leq x + (1 - x)\mu \leq x + (1 - x) = 1$ and $x' \in [0, 1]$. Since $\widehat{\Psi}(x, z) \in [0, \frac{1}{2}]$ we have $1 - 4\widehat{\Psi}(x, z) \in [-1, 1]$. Now using $\mu \in [0, 1]$ and $1 - 4\widehat{\Psi}(x, z), z \in [-1, 1]$ we have

$$-1 = (1 - \mu)(-1) + (-1)\mu \leq (1 - \mu)z + (4\widehat{\Psi}(x, z) - 1)\mu \leq 1 - \mu + \mu = 1$$

therefore $z' \in [-1, 1]$ using $y'^2 + z'^2 = 1$, $F_2 : [0, 1] \times M_2 \rightarrow X$ is well-defined.

Using Lemma 11.5, $F_2 : [0, 1] \times M_2 \rightarrow X$ is continuous. \square

Construction 11.7. Consider $X = \{(x, y, z) \in \mathbb{R}^3 : y^2 + z^2 = 1, 0 \leq x \leq 1\}$, $Y = \{(x, y, z) \in X : x = 1 \vee z = -1\}$, $M_1 = \{(x, y, z) \in X : x + z \leq 0\}$ and $M_2 = \{(x, y, z) \in X : x + z \geq 0\}$. $F : [0, 1] \times X \rightarrow X$ with $F \upharpoonright_{M_1} = F_1$ as in Lemma 11.3 and $F \upharpoonright_{M_2} = F_2$ as in Lemma 11.6. Then we have:

1. $F : [0, 1] \times X \rightarrow X$ is continuous.
2. For all $(x, y, z) \in X$ we have $F(0, (x, y, z)) = (x, y, z)$ and $F(1, (x, y, z)) \in Y$
3. For all $(x, y, z) \in Y$ and $\mu \in [0, 1]$ we have $F(\mu, (x, y, z)) = (x, y, z)$.

Proof. (1) For all $x \in [0, 1]$ we have $\widehat{\Phi}(x, -x) = \widehat{\Psi}(x, -x) = 0$, so using Lemma 11.3, Lemma 11.6 and gluing lemma the map $F : [0, 1] \times X \rightarrow X$ is continuous.

(2) For $(x, y, z) \in X$, $F(0, (x, y, z)) = (x, y, z)$ is clear by definition of F_1 and F_2 . Suppose $F(1, (x, y, z)) = (x_1, y_1, z_1)$. If $(x, y, z) \in M_1$, then $z_1 = (1 - 1)z - 1 = -1$ and $F(1, (x, y, z)) = (x_1, y_1, z_1) \in Y$. If $(x, y, z) \in M_2$, then $x_1 = x + (1 - x)1 = 1$ and $F(1, (x, y, z)) = (x_1, y_1, z_1) \in Y$.

(3) Suppose $(x, y, z) \in Y$, $\mu \in [0, 1]$ and $F(\mu, (x, y, z)) = (x', y', z')$. We have the following cases:

Case 1. $z = -1$. In this case $y = 0$, $(x, y, z) \in M_1$ and $\widehat{\Phi}(x, z) = \widehat{\Phi}(x, -1) = \frac{1}{2}$. Thus $x' = x + (1 - 2(1 - x)^{\frac{1}{2}} - x)\mu = x$, $z' = (1 - \mu)(-1) - \mu = -1 = z$ and $y' = \text{sgn}(y) \sqrt{1 - z'^2} = \text{sgn}(y) \sqrt{1 - 1} = 0 = y$

Case 2. $x = 1$. In this case $y = 0$, $(x, y, z) \in M_2$ and $\widehat{\Psi}(x, z) = \widehat{\Psi}(1, z) = t$ implies $z = \Psi_1(t) = 4t - 1$, i.e. $\widehat{\Psi}(1, z) = \frac{z+1}{4}$. Thus $x' = 1 + (1 - 1)\mu = 1 = x$, $z' = (1 - \mu)z + (4 \times \frac{z+1}{4} - 1)\mu = z$, and $y' = \text{sgn}(y) \sqrt{1 - z'^2} = \text{sgn}(y) \sqrt{1 - z^2} = \text{sgn}(y)|y| = y$.

Considering the above cases we are done. \square

Construction 11.8. For $n \in \mathbb{N}$ let

$$X_n = \{(x, y, z) \in \mathbb{R}^3 : y^2 + (z - \frac{1}{n})^2 = \frac{1}{n^2}, 0 \leq x \leq \frac{1}{n}\},$$

and $X_0 = \bigcup\{X_n : n \in \mathbb{N}\}$, in this construction we want to define a map $F_0 : [0, 1] \times X_0 \rightarrow X_0$.

Considering the same notations as in Construction 11.7 for $m \in \mathbb{N}$ and $(x, y, z) \in X_m$ we have $(mx, my, mz - 1) \in X$. For $\mu \in [0, 1]$ if

$$F(\mu, (mx, my, mz - 1)) = (x'_m, y'_m, z'_m) \in X,$$

then $0 \leq x'_m \leq 1$ and $y'^2_m + z'^2_m = 1$, thus $0 \leq \frac{x'_m}{m} \leq \frac{1}{m}$ and

$$\left(\frac{y'_m}{m}\right)^2 + \left(\frac{z'_m + 1}{m} - \frac{1}{m}\right)^2 = \frac{1}{m^2},$$

therefore $(\frac{x'_m}{m}, \frac{y'_m}{m}, \frac{z'_m + 1}{m}) \in X_m$, let $F_m(\mu, (x, y, z)) = (\frac{x'_m}{m}, \frac{y'_m}{m}, \frac{z'_m + 1}{m})$, i.e.

$$F_m(\mu, (x, y, z)) = \frac{1}{m}F(\mu, (mx, my, mz - 1)) + (0, 0, \frac{1}{m}).$$

It is clear that $F_m : [0, 1] \times X_m \rightarrow X_m$ is continuous. Suppose $s, t \in \mathbb{N}$, $s < t$, $\mu \in [0, 1]$ and $(x, y, z) \in F_s \cap F_t$, then:

$$0 \leq x \leq \min(\frac{1}{t}, \frac{1}{s}) \wedge y^2 + (z - \frac{1}{s})^2 = \frac{1}{s^2} \wedge y^2 + (z - \frac{1}{t})^2 = \frac{1}{t^2}$$

which leads to $0 \leq x \leq \frac{1}{t} (< \frac{1}{s})$ and $y = z = 0$. Now, since $(sx, 0, -1), (tx, 0, -1) \in Y$ (in Construction 11.8), we have:

$$F(\mu, (sx, 0, -1)) = (sx, 0, -1), F(\mu, (tx, 0, -1)) = (sx, 0, -1),$$

and:

$$\begin{aligned} F_s(\mu, (x, y, z)) &= F_s(\mu, (x, 0, 0)) = \frac{1}{s}F(\mu, (sx, 0, -1)) + (0, 0, \frac{1}{s}) \\ &= \frac{1}{s}(sx, 0, -1) + (0, 0, \frac{1}{s}) = (x, 0, 0) = \frac{1}{t}(tx, 0, -1) + (0, 0, \frac{1}{t}) \\ &= \frac{1}{t}F(\mu, (tx, 0, -1)) + (0, 0, \frac{1}{t}) = F_t(\mu, (x, 0, 0)) = F_t(\mu, (x, y, z)) \end{aligned}$$

Therefore for $F_0 = \bigcup\{F_n : n \in \mathbb{N}\}$, $F_0 : [0, 1] \times X_0 \rightarrow X_0$ is well-defined.

Note: We recall that for $A \subseteq B$, we call A a deformation retract of B if there exists a continuous map $v : [0, 1] \times B \rightarrow A$ with $v(0, b) = b$, $v(1, b) \in A$, and $v(t, a) = a$ (for all $b \in B, a \in A, t \in [0, 1]$). It is well-known that if A is a deformation retract of B (and $a_0 \in A$), then $\Upsilon : \pi_1(A, a_0) \rightarrow \pi_1(B, a_0)$ is a group isomorphism, in particular $\pi_1(A) \cong \pi_1(B)$ [5, Theorem 58.3].

Lemma 11.9. For $n \in \mathbb{N}$ let

$$X_n = \{(x, y, z) \in \mathbb{R}^3 : y^2 + (z - \frac{1}{n})^2 = \frac{1}{n^2}, 0 \leq x \leq \frac{1}{n}\},$$

and

$$Y_n = \{(x, y, z) \in X_n : x = \frac{1}{n} \vee z = 0\},$$

then $Y_0 = \bigcup\{Y_n : n \in \mathbb{N}\}$ is a deformation retract of $X_0 = \bigcup\{X_n : n \in \mathbb{N}\}$.

Proof. Consider $F_0 : [0, 1] \times X_0 \rightarrow X_0$ as in Construction 11.8. We prove the following claims:

- Claim 1. $F_0 : [0, 1] \times X_0 \rightarrow X_0$ is continuous.
- Claim 2. $\forall (x, y, z) \in X_0 \quad (F_0(0, (x, y, z)) = (x, y, z) \wedge F_0(1, (x, y, z)) \in Y_0)$.
- Claim 3. $\forall (x, y, z) \in Y_0 \quad \forall \mu \in [0, 1] \quad F_0(\mu, (x, y, z)) = (x, y, z)$.

Proof of Claim 1. Since for all $n \in \mathbb{N}$, $F_n : [0, 1] \times X_n \rightarrow X_n$ is continuous, using the gluing lemma, $\bigcup\{F_i : 1 \leq i \leq n\} : [0, 1] \times \bigcup\{X_i : 1 \leq i \leq n\} \rightarrow \bigcup\{X_i : 1 \leq i \leq n\}$ is continuous.

If $(x, y, z) \in X_0 \setminus \{(0, 0, 0)\}$, then there exist $n \in \mathbb{N}$ and open neighborhood V of (x, y, z) in X_0 such that $V \subseteq \bigcup\{X_i : 1 \leq i \leq n\}$. Since $\bigcup\{F_i : 1 \leq i \leq n\} \upharpoonright_{[0,1] \times V} : [0, 1] \times \bigcup\{X_i : 1 \leq i \leq n\} \rightarrow \bigcup\{X_i : 1 \leq i \leq n\}$ is continuous, $\bigcup\{F_i : 1 \leq i \leq n\} : [0, 1] \times V \rightarrow X_0$ is continuous, i.e. $F_0 \upharpoonright_{[0,1] \times V} : [0, 1] \times V \rightarrow X_0$ is continuous, therefore F_0 is continuous in all points of $[0, 1] \times \{(x, y, z)\}$.

In order to show the continuity of $F_0 : [0, 1] \times X_0 \rightarrow X_0$, we should prove that it is continuous in all points $(\mu, (0, 0, 0))$ ($\mu \in [0, 1]$). Consider $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that $\frac{\sqrt{6}}{n} < \varepsilon$ for all $(x, y, z) \in X_0$ and $\mu, \lambda \in [0, 1]$ we have (consider $[0, 1] \times X_0$ and X_0 respectively under Euclidean norm of \mathbb{R}^4 and \mathbb{R}^3):

$$\begin{aligned} \|(\mu, (0, 0, 0)) - (\lambda, (x, y, z))\| &< \frac{1}{n} \\ \Rightarrow x &< \frac{1}{n} \\ \Rightarrow (x, y, z) &\in \bigcup\{X_i : i \geq n\} \\ \Rightarrow F_0(\lambda, (x, y, z)) &= \bigcup\{F_i : i \geq n\}(\lambda, (x, y, z)) \\ \Rightarrow F_0(\lambda, (x, y, z)) &\in \bigcup\{F_i : i \geq n\}([0, 1] \times \bigcup\{X_i : 1 \leq i \geq n\}) \\ \Rightarrow F_0(\lambda, (x, y, z)) &\in \bigcup\{F_i([0, 1] \times X_i) : i \geq n\} = \bigcup\{X_i : i \geq n\} \\ \Rightarrow \|F_0(\lambda, (x, y, z))\| &\leq \max\{\frac{\sqrt{6}}{i} : i \geq n\} = \frac{\sqrt{6}}{n} \\ \Rightarrow \|F_0(\lambda, (x, y, z)) - F_0(\mu, (0, 0, 0))\| &= \|F_0(\lambda, (x, y, z))\| \leq \frac{\sqrt{6}}{n} < \varepsilon \end{aligned}$$

(note to the fact that $X_n \subseteq [0, \frac{1}{n}] \times [-\frac{1}{n}, \frac{1}{n}] \times [0, \frac{2}{n}]$, thus for all $(u, v, w) \in X_n$ we have $\|(u, v, w)\| \leq \sqrt{\frac{1}{n^2} + \frac{1}{n^2} + \frac{4}{n^2}} = \frac{\sqrt{6}}{n}$) therefore $F_0 : [0, 1] \times X_0 \rightarrow X_0$ is continuous in $(\mu, (0, 0, 0))$ as well as it is continuous in other points of $[0, 1] \times X_0$.

Proof of Claim 2. Suppose $(x, y, z) \in X_0$, there exists $n \in \mathbb{N}$ such that $(x, y, z) \in X_n$, using Construction 11.7 (2), we have:

$$\begin{aligned} F_0(0, (x, y, z)) &= F_n(0, (x, y, z)) = \frac{1}{n}F(0, (nx, ny, nz - 1)) + (0, 0, \frac{1}{n}) \\ &= \frac{1}{n}(nx, ny, nz - 1) + (0, 0, \frac{1}{n}) = (x, y, z) \end{aligned}$$

and $F_0(1, (x, y, z)) = F_n(1, (x, y, z)) = \frac{1}{n}F(1, (nx, ny, nz - 1)) + (0, 0, \frac{1}{n})$, by Construction 11.7 (2) we have $F(1, (nx, ny, nz - 1)) \in Y$ which leads to $F_0(1, (x, y, z)) \in \frac{1}{n}Y + (0, 0, \frac{1}{n}) = Y_n \subseteq Y_0$.

Proof of Claim 3. Suppose $\mu \in [0, 1]$ and $(x, y, z) \in X_0$, there exists $n \in \mathbb{N}$ such that $(x, y, z) \in Y_n \subseteq X_n$, now we have (use Construction 11.7 (3)):

$$\begin{aligned} (x, y, z) \in Y_n &\Rightarrow ((x, y, z) \in X_n \wedge x = \frac{1}{n}) \vee ((x, y, z) \in X_n \wedge z = 0) \\ &\Rightarrow (y^2 + (z - \frac{1}{n})^2 = \frac{1}{n^2} \wedge x = \frac{1}{n}) \vee (0 \leq x \leq \frac{1}{n} \wedge y = z = 0) \\ &\Rightarrow (((ny)^2 + (nz - 1)^2 = 1 \wedge nx = 1) \\ &\quad \vee (0 \leq nx \leq 1 \wedge ny = 0 \wedge nz - 1 = -1)) \\ &\Rightarrow (nx, ny, nz - 1) \in Y \\ &\Rightarrow F(\mu, (nx, ny, nz - 1)) = (nx, ny, nz - 1) \end{aligned}$$

thus

$$\begin{aligned} F_0(\mu, (x, y, z)) &= F_n(\mu, (x, y, z)) = \frac{1}{n}F(\mu, (nx, ny, nz - 1)) + (0, 0, \frac{1}{n}) \\ &= \frac{1}{n}(nx, ny, nz - 1) + (0, 0, \frac{1}{n}) = (x, y, z) \end{aligned}$$

Which completes the proof of Claim 3.

Using Claims 1, 2, and 3, Y_0 is a deformation retract of X_0 . \square

Theorem 11.10. Under the same notations as in Construction 11.8 and Lemma 11.9, $Z_0 = \{(0, y, z) : \exists x(x, y, z) \in X_0\}$ is a deformation retract of X_0 . In particular $\pi_1(Y_0) \cong \pi_1(X_0) \cong \pi_1(Z_0)$.

Proof. The map $[0, 1] \times X_0 \rightarrow Z_0$ $(\mu, (x, y, z)) \mapsto ((1 - \mu)x, y, z)$ shows that Z_0 is a deformation retract of X_0 too. Now use [5, Theorem 58.3] to complete the proof. \square

Corollary 11.11. Two sets \mathcal{X} and \mathcal{W} are homeomorphic with deformation retracts of \mathcal{V} , therefore $\pi_1(\mathcal{X}) \cong \pi_1(\mathcal{V}) \cong \pi_1(\mathcal{W})$.

Proof. Under the same notations as in Theorem 11.10, \mathcal{X} and Z_0 are homeomorph, moreover Y_0 and \mathcal{W} are homeomorph too, also $X_0 = \mathcal{V}$. Now by Theorem 11.10 we have $\pi_1(\mathcal{X}) \cong \pi_1(\mathcal{V}) \cong \pi_1(\mathcal{W})$. \square

12. A distinguished counterexample

In Section 11 we have proved $\pi_1(\mathcal{X}) \cong \pi_1(\mathcal{W})$, in this section we prove $\mathfrak{F}^\omega(\mathcal{X}) \not\cong \mathfrak{F}^\omega(\mathcal{W})$.

Lemma 12.1. We have $|\mathfrak{F}^\omega(\mathcal{X})| = \omega$.

Proof. For $n \in \mathbb{N}$ consider $\rho_n : [0, 1] \rightarrow C_n$ with $\rho_n(t) = \frac{1}{n}e^{2\pi it - \frac{it}{2}} + \frac{i}{n}$, then ω -loops $\rho_n, \rho_m : [0, 1] \rightarrow \mathcal{X}$ are homotopic if and only if $n = m$. Therefore $\{[\rho_n] : n \in \mathbb{N}\}$ is an infinite subset of $\mathfrak{P}^\omega(\mathcal{X})$ which leads to $|\mathfrak{P}^\omega(\mathcal{X})| \geq \omega$. On the other hand as it has been mentioned in Note 6.4 (4), if $[f] \in \mathfrak{P}^\omega(\mathcal{X})$, then $|A(f)| < \omega$, which leads to $\mathfrak{P}^\omega(\mathcal{X}) \subseteq * \{\pi_1(C_n) : n \in \mathbb{N}\}$, thus

$$\begin{aligned} |\mathfrak{P}^\omega(\mathcal{X})| &\leq |* \{\pi_1(C_n) : n \in \mathbb{N}\}| \\ &= |\{\rho_{i_1}^{j_1} * \rho_{i_2}^{j_2} * \dots * \rho_{i_m}^{j_m} : m \in \mathbb{N}, i_1, j_1, i_2, j_2, \dots, i_m, j_m \in \mathbb{Z}\}| \\ &\leq |\bigcup_{m \in \mathbb{N}} \{(i_1, j_1, \dots, i_m, j_m) : i_1, j_1, \dots, i_m, j_m \in \mathbb{Z}\}| = \omega \end{aligned}$$

Hence $|\mathfrak{P}^\omega(\mathcal{X})| = \omega$. \square

Lemma 12.2. We have $|\mathfrak{P}^\omega(\mathcal{W})| = c$.

Proof. It is well-known that for all Hausdorff separable space A , $|C(A, \mathbb{R}^2)| \leq c$ where $C(A, B)$ denotes the collection of all continuous maps $\phi : A \rightarrow B$. Therefore

$$|\mathfrak{P}^\omega(\mathcal{W})| \leq |C([0, 1], \mathcal{W})| \leq |C([0, 1], \mathbb{R}^2)| = c.$$

Now for all $a = (a_n : n \in \mathbb{N}) \in \{0, 1\}^\mathbb{N}$ define $f_a : [0, 1] \rightarrow \mathcal{W}$ with:

$$f_a(x) = \begin{cases} \frac{1}{2^{n+1}}e^{2\pi i(4n(n+1)x - (2n+1)\frac{x}{2})} + \frac{i}{2^{n+1}} + \frac{1}{n} & \frac{2n+1}{4n(n+1)} \leq x \leq \frac{1}{2n}, a_n = 1, n \in \mathbb{N}, \\ 4x - \frac{1}{n+1} & \frac{1}{2(n+1)} \leq x \leq \frac{2n+1}{4n(n+1)}, a_n = 1, n \in \mathbb{N}, \\ 2x & \frac{1}{2(n+1)} \leq x \leq \frac{1}{2n}, a_n = 0, n \in \mathbb{N}, \\ 0 & x = 0, \\ 2 - 2x & \frac{1}{2} \leq x \leq 1, \end{cases}$$

then $f_a : [0, 1] \rightarrow \mathcal{W}$ is an ω -loop, thus $[f_a] \in \mathfrak{P}^\omega(\mathcal{W})$. We claim that $\psi : \{0, 1\}^\mathbb{N} \rightarrow \mathfrak{P}^\omega(\mathcal{X})$ with $\psi(a) = [f_a]$ ($a \in \{0, 1\}^\mathbb{N}$) is one to one. Let $a = (a_n : n \in \mathbb{N}), b = (b_n : n \in \mathbb{N}) \in \{0, 1\}^\mathbb{N}$ and $a \neq b$, then there exists $m \in \mathbb{N}$ such that $a_m \neq b_m$. Suppose $a_m = 0$ and $b_m = 1$. Let $W := \{\frac{1}{2^{m+1}}e^{2\pi i\theta} + \frac{1}{m} + \frac{i}{2^{m+1}} : \theta \in [0, 1]\}$. Since f_a^W is constant map $\frac{1}{m}$, $[f_a^W]$ is null-homotopic. However $[f_b^W]$ is not null-homotopic, thus $[f_a^W] \neq [f_b^W]$ which leads to $[f_a] \neq [f_b]$ according to Convention 4.1. Hence $\psi : \{0, 1\}^\mathbb{N} \rightarrow \mathfrak{P}^\omega(\mathcal{X})$ is one to one which leads to $|\mathfrak{P}^\omega(\mathcal{X})| \geq |\{0, 1\}^\mathbb{N}| = c$ and completes the proof. \square

Counterexample 12.3 (A Distinguished Counterexample). Two groups $\pi_1(\mathcal{X})$ and $\pi_1(\mathcal{W})$ are isomorphic and two groups $\mathfrak{P}^\omega(\mathcal{X})$ and $\mathfrak{P}^\omega(\mathcal{W})$ are non-isomorphic. Briefly $\pi_1(\mathcal{X}) \cong \pi_1(\mathcal{W})$ and $\mathfrak{P}^\omega(\mathcal{X}) \not\cong \mathfrak{P}^\omega(\mathcal{W})$ (use Lemma 12.1, Lemma 12.2, and Corollary 11.11).

13. A diagram and a hint

Consider the following diagram:

$$\begin{array}{ccccc} \forall \alpha \geq \omega \mathfrak{P}^\alpha(\mathcal{X}) \cong \mathfrak{P}^\alpha(\mathcal{Y}) & \xrightarrow{(i)} & \forall \alpha \geq c \mathfrak{P}^\alpha(\mathcal{X}) \cong \mathfrak{P}^\alpha(\mathcal{Y}) & \xrightarrow{(ii)} & \forall \alpha \geq 2^c \mathfrak{P}^\alpha(\mathcal{X}) \cong \mathfrak{P}^\alpha(\mathcal{Y}) \\ \uparrow (v) & & & & \uparrow (iii) \\ \pi_1(\mathcal{X}) \cong \pi_1(\mathcal{Y}) & \xleftarrow{(iv)} & & \xrightarrow{} & \exists \alpha \geq 2^c \mathfrak{P}^\alpha(\mathcal{X}) \cong \mathfrak{P}^\alpha(\mathcal{Y}) \end{array}$$

Arrows (III) and (IV) are valid regarding Theorem 5.1 (1). However by Counterexample 12.3, there exist X, Y such that $\pi_1(X) \cong \pi_1(Y)$ and $\mathfrak{P}^\omega(X) \cong \mathfrak{P}^\omega(Y)$, thus:

$$\pi_1(X) \cong \pi_1(Y) \wedge \neg(\forall \alpha \geq \omega \mathfrak{P}^\alpha(X) \cong \mathfrak{P}^\alpha(Y))$$

Hence the above diagram is valid. We have the following arising problems:

Problem 13.1. Find a counterexample for arrow (I), i.e. find X, Y such that $\pi_1(X) \cong \pi_1(Y)$, $\mathfrak{P}^c(X) \cong \mathfrak{P}^c(Y)$ and $\mathfrak{P}^\omega(X) \not\cong \mathfrak{P}^\omega(Y)$ (Hint: is it true that $\mathfrak{P}^c(X) \cong \mathfrak{P}^c(\mathcal{W})$).

Problem 13.2. Find a counterexample for arrow (II), i.e. find X, Y such that $\pi(X)_1 \cong \pi_1(Y)$ and $\mathfrak{P}^c(X) \not\cong \mathfrak{P}^c(Y)$.

14. A Strategy for Future and Conjecture

Let's extend of the idea of this text to homotopy group of order n . Let $b \in \mathbb{S}^n$ be a fixed point. For infinite cardinal number α and ideal \mathcal{I} on X which contains all finite subsets of X , if $f, g : \mathbb{S}^n \rightarrow X$ are $\alpha^{\mathcal{I}}$ -maps, with $f(b) = g(b)$, then it is easy to see that $f \vee g : \mathbb{S}^n \rightarrow X$ is $\alpha^{\mathcal{I}}$ -map too. So we may have the following definition.

Definition 14.1. For $a \in X$, by $\mathfrak{P}_{(n, \mathcal{I})}^\alpha(X, a)$ we mean subgroup of $\pi_n(X, a)$ generated by $\alpha^{\mathcal{I}}$ -maps with base point a .

It's evident by the definition that for ideals \mathcal{I}, \mathcal{J} on X containing finite subsets, transfinite cardinal number α , and $a \in X$, we have:

- If $\mathcal{I} \subseteq \mathcal{J}$, then $\mathfrak{P}_{(n, \mathcal{I})}^\alpha(X, a) \subseteq \mathfrak{P}_{(n, \mathcal{J})}^\alpha(X, a)$;
- $\mathfrak{P}_{(n, \mathcal{I} \cap \mathcal{J})}^\alpha(X, a) \subseteq \mathfrak{P}_{(n, \mathcal{I})}^\alpha(X, a) \cap \mathfrak{P}_{(n, \mathcal{J})}^\alpha(X, a)$.

Now we are ready to the following conjecture:

Conjecture. Arc connected spaces X and Y are homeomorph if and only if there exists a bijection $f : X \rightarrow Y$ such that for all nonzero cardinal number α and all ideal \mathcal{I} on X , $\mathfrak{P}_{\mathcal{I}}^\alpha(X) \cong \mathfrak{P}_{f(\mathcal{I})}^\alpha(Y)$.

One more idea for future study. Let's recall that in topological space Z and $a, b \in Z$ for nonzero cardinal number β , a collection Γ of maps $f : [0, 1] \rightarrow Z$ with $f(0) = a$ and $f(1) = b$, is called a β -separated family of maps between a and b if for all distinct $g, h \in \Gamma$ we have $|(g[0, 1] \cap h[0, 1]) \setminus \{a, b\}| < \beta$ [2, Definition 2.5].

Now for cardinal numbers $\alpha, \beta > 0$ and ideal \mathcal{I} on X we may consider the collection $S(\mathcal{I}, \alpha, \beta)$ consisting of all families Γ such that Γ is a collection of $\alpha^{\mathcal{I}}$ -loops with base point a and a β -separated family of maps between a and a . Suppose

$$\mathfrak{S}(\mathcal{I}, \alpha, \beta) := \{ \langle \{f\} : f \in \Gamma \rangle : \Gamma \in S(\mathcal{I}, \alpha, \beta) \},$$

then $\mathfrak{S}(\mathcal{I}, \alpha, \beta)$ is a "poset" under \subseteq and a collection of subgroups of $\pi_1(X)$. For arc connected spaces X and Y one may compare these "type" of collections of their fundamental groups to discover "differents" between X and Y .

15. Conclusion

In this paper, for arc connected locally compact Hausdorff topological space X (with at least two elements), $a \in X$, nonzero cardinal number α , and ideal \mathcal{I} on X we introduce $\mathfrak{P}_{\mathcal{I}}^\alpha(X, a)$ as a subgroup of $\pi_1(X, a)$. We prove that for transfinite α and $a, b \in X$ two groups $\mathfrak{P}_{\mathcal{I}}^\alpha(X, a)$ and $\mathfrak{P}_{\mathcal{I}}^\alpha(X, b)$ are isomorphic, therefore for transfinite α we denote $\mathfrak{P}_{\mathcal{I}}^\alpha(X, a)$ simply by $\mathfrak{P}_{\mathcal{I}}^\alpha(X)$ and $\mathfrak{P}_{\{\emptyset\}}^\alpha(X)$ simply by $\mathfrak{P}^\alpha(X)$. Moreover for $\alpha \geq 2^c$ we have $\mathfrak{P}_{\mathcal{I}}^\alpha(X) = \pi_1(X)$, hence the most interest is in $\omega \leq \alpha < 2^c$ using GCH we prefer to study $\alpha \in \{\omega, c\}$. We obtain that for Hawaiian earring (infinite earring) \mathcal{X} , three groups $\mathfrak{P}^\omega(\mathcal{X})$, $\mathfrak{P}^c(\mathcal{X})$, and $\mathfrak{P}^{2^c}(\mathcal{X}) (= \pi_1(\mathcal{X}))$ are pairwise distinct. Also we introduce \mathcal{Y} such that $\mathfrak{P}_{\mathcal{P}_{fin}(\mathcal{Y})}^\omega(\mathcal{Y})$, $\mathfrak{P}_{\mathcal{P}_{fin}(\mathcal{Y})}^c(\mathcal{Y})$, and $\mathfrak{P}_{\mathcal{P}_{fin}(\mathcal{Y})}^{2^c}(\mathcal{Y}) (= \pi_1(\mathcal{Y}))$ are pairwise distinct. We find \mathcal{W} such that $\pi_1(\mathcal{X}) \cong \pi_1(\mathcal{W})$ and $\mathfrak{P}^\omega(\mathcal{X}) \not\cong \mathfrak{P}^\omega(\mathcal{W})$, this example leads us to the fact that we can classify spaces with isomorphic first homotopy groups using the concept of $\mathfrak{P}^\alpha(-)$ s (first homotopy groups with respect to $\alpha \geq \omega$). However investigating the structure of our examples and specially Section 12, shows remarkable role of the number of (locally) cut points their and order in α -arcs, $\alpha^{\mathcal{I}}$ -arcs, and our constructed subgroups of first fundamental group.

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