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Volterra-Type Operators from $F(p,q,s)$ Space to Bloch-Orlicz and Zygmund-Orlicz Spaces

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Abstract. In this paper, we discussed the equivalent conditions for the boundedness and compactness of several Volterra-type operators acting from general $F(p,q,s)$ space to Bloch-Orlicz and Zygmund-Orlicz spaces.

1. Introduction

Let $\mathbb{D}$ be the unit disk in the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ (or $S(\mathbb{D})$) the collection of all analytic functions (or all analytic self-maps) on $\mathbb{D}$. Given an analytic self-map $\phi : \mathbb{D} \to \mathbb{D}$, the composition operator $C_\phi : H(\mathbb{D}) \to H(\mathbb{D})$ is defined by

$$C_\phi f = f \circ \phi, \ f \in H(\mathbb{D}).$$

The systematic study of composition operators acting on various spaces of analytic functions has been very popular in recent years. In particular, the problems of relating operator-theoretic properties of C_ϕ to function-theoretic properties of ϕ are interesting and have been widely investigated. We refer the readers to consult [1, 5, 7, 12, 19, 20] and so on.

In this paper, we fix our attention on the boundedness and compactness of some Volterra-type operators defined below. Similarly, the mentioned questions and other operator theoretic properties of Volterra-type operators expressed in terms of function theoretic conditions on symbols have been a subject of high interest, which can be found in [6, 10, 11, 13–16] and their reference therein. Now we formulate four integral-type operators.

(a) Given $h \in H(\mathbb{D})$, the operator T^h is defined by

$$T^h f(z) = \int_0^z f(t)h(t)dt, \ f \in H(\mathbb{D}), \ z \in \mathbb{D}.$$

(b) Given $h \in H(\mathbb{D})$, the operator T_h is defined by

$$T_h f(z) = \int_0^z f(t)h'(t)dt, \ f \in H(\mathbb{D}), \ z \in \mathbb{D}.$$

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(c) Let $\phi \in S(\mathbb{D})$ and $h \in H(\mathbb{D})$, the operator P_ϕ^h is defined by

$$P_\phi^h f(z) = \int_0^z f(\phi(t))h(t)dt, \ f \in H(\mathbb{D}), \ z \in \mathbb{D}.$$

(d) Let $\phi \in S(\mathbb{D})$ and $h \in H(\mathbb{D})$, the operator $T_h C_\phi$ is defined by

$$T_h C_\phi f(z) = \int_0^z f(\phi(t))h'(t)dt, \ f \in H(\mathbb{D}), \ z \in \mathbb{D}.$$

Indeed, these Volterra-type operators have close connections. On the one hand, when $\phi = id$ the identity map, then

$$P_{id}^h = T^h \text{ and } T_h C_{id} = T_h.$$

That means the operators T^h and T_h are special cases of P_ϕ^h and $T_h C_\phi$, respectively. On the other hand, if we let $h = k' \in H(\mathbb{D})$ in P_ϕ^h, then $P_\phi^{k'} = T_k C_\phi$. Inspired by the above observations, we mainly provide the investigations concerning P_ϕ^h, then the analogous results for other Volterra-type operators follow immediately. Like composition operators, it is known that these type of operators are also appeared in the study of operator theory on holomorphic function spaces. However, it seems that most of papers do not include the estimate for these Volterra-type operators acting from general $F(p,q,s)$ into Bloch (or Zygmund)-Orlicz spaces even on the unit disk $\mathbb{D}$. Motivated by the works in [2, 4, 10, 16], we continue this line of research and extend a number of results on Volterra-type operators.

For $0 < p, s < \infty, -2 < q < \infty$, a function $f \in H(\mathbb{D})$ is said to belong to the general function space $F(p,q,s) = F(p,q,s)(\mathbb{D})$ if

$$\|f\|_{F(p,q,s)}^p = |f(0)|^p + \sup_{u \in \mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^q (1-|\phi_u(z)|^2)^s dA(z) < \infty,$$

where $\phi_u(z) = (u-z)/(1-\overline{u}z), \ u \in \mathbb{D}$. It is known that

$$1 - |\varphi_u(z)|^2 = \frac{(1-|u|^2)(1-|z|^2)}{|1-\langle z,u\rangle|^2}.$$

The family of spaces $F(p,q,s)$ was first introduced by Zhao [18]. It is called general function space, which contains, as special cases, many classical holomorphic function spaces, such as *BMOA* space, Q_p space, Bergman space, Hardy space, Bloch space, if we take special parameters of p, q, s. Notice that $F(p,q,s)$ is the space of constant functions if $q + s \le -1$. For the definition of these spaces described above, we recommend the readers to [21].

Let μ be a weight, which is a positive continuous function on $\mathbb{D}$. The μ-Bloch space $\mathcal{B}_\mu = \mathcal{B}_\mu(\mathbb{D})$ consists of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{\mathcal{B}_\mu} = |f(0)| + \sup_{z \in \mathbb{D}} \mu(z)|f'(z)| < \infty,$$

and $\mathcal{B}_\mu$ is a Banach space under the norm $\|f\|_{\mathcal{B}_\mu}$. In particular, if $\mu(z) = (1-|z|^2)^\alpha$, it leads to

$$\mathcal{B}^\alpha = \{f \in H(\mathbb{D}), \ \|f\|_{\mathcal{B}^\alpha} = |f(0)| + \sup_{z \in \mathbb{D}} (1-|z|^2)^\alpha |f'(z)| < \infty\},$$

which degenerates the classical Bloch space $\mathcal{B}$ for $\alpha = 1$. In the usual sense, the μ-Zygmund space $\mathcal{Z}_\mu = \mathcal{Z}_\mu(\mathbb{D})$ includes all $f \in H(\mathbb{D})$ verifying

$$\|f\|_{\mathcal{Z}_\mu} = |f(0)| + |f'(0)| + \sup_{z \in \mathbb{D}} \mu(z)|f'(z)| < \infty,$$

which is a complete norm on $\mathcal{Z}_\mu$.

Recently, Ramos Fernández used Young's functions to define the Bloch-Orlicz space in [8], which is a generalization of the Bloch space (cf. [3, 16]). More precisely, let $\varphi : [0,+\infty) \to [0,+\infty)$ be an $\mathcal{N}$-function, that is, φ is a strictly increasing convex function with $\varphi(0) = 0$, which implies that $\lim_{t\to\infty} \varphi(t) = +\infty$. The Bloch-Orlicz space related with the function φ, denoted by $\mathcal{B}^{\varphi}$, is the calss of all $f \in H(\mathbb{D})$ such that

$$\sup_{z\in\mathbb{D}}(1-|z|^2)\varphi(\lambda|f'(z)|) < \infty$$

for some $\lambda > 0$ depending on f. Suppose that φ^{-1} is further continuously differentiable. If φ^{-1} is not differentiable everywhere, we set the function

$$\psi(t) = \int_0^t \frac{\varphi(x)}{x}dx, \quad t \geq 0,$$

then ψ is differentiable, whence ψ^{-1} is differentiable everywhere on $[0,\infty)$. Since φ is a strictly increasing, convex function satisfying $\varphi(0) = 0$, hence the function $\varphi(t)/t,\ t > 0$, is increasing and

$$\varphi(t) \geq \psi(t) \geq \int_{t/2}^t \frac{\varphi(x)}{x}dx \geq \varphi\left(\frac{t}{2}\right) \text{ for all } t \geq 0.$$

Hence $\mathcal{B}^{\varphi} = \mathcal{B}^{\psi}$. Due to the convexity of φ, the Minkowski's functional

$$\|f\|_{\varphi} = \inf\left\{k > 0 : S_{\varphi}\left(\frac{f'}{k}\right) \leq 1\right\},$$

defines a seminorm for $\mathcal{B}^{\varphi}$, which in this case is well-known as *Luxemburg's seminorm*, where

$$S_{\varphi}(f) = \sup_{z\in\mathbb{D}}(1-|z|^2)\varphi(|f(z)|).$$

It has been proved $\mathcal{B}^{\varphi}$ is a Banach space under the norm

$$\|f\|_{\mathcal{B}^{\varphi}} = |f(0)| + \|f\|_{\varphi}.$$

Observing from the fact

$$S_{\varphi}\left(\frac{f'}{\|f\|_{\mathcal{B}^{\varphi}}}\right) \leq 1,$$

it leads to the following Lemma.

Lemma 1.1. *[8, Corollary 4] The Bloch-Orlicz space is isometrically equal to μ_1-Bloch space, where*

$$\mu_1(z) = \frac{1}{\varphi^{-1}\left(\frac{1}{1-|z|^2}\right)}, \; z \in \mathbb{D}.$$

Whence for any $f \in \mathcal{B}^{\varphi}$,

$$\|f\|_{\mathcal{B}^{\varphi}} = |f(0)| + \sup_{z\in\mathbb{D}} \mu_1(z)|f'(z)|.$$

As an apparent generalization, we recall the α-Bloch-Orlicz space $\mathcal{B}^{\varphi}_{\alpha} = \mathcal{B}^{\varphi}_{\alpha}(\mathbb{D})$ (cf. [4]) for $\alpha > 0$, which is the class of all $f \in H(\mathbb{D})$ satisfying

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^{\alpha}\varphi(\lambda|f'(z)|) < \infty$$

for some $\lambda > 0$ depending on f. And then $\mathcal{B}^{\varphi}_{\alpha}$ is also a Banach space endowed with the norm

$$\|f\|_{\mathcal{B}^{\varphi}_{\alpha}} = |f(0)| + \|f\|_{\varphi,\alpha},$$

where

$$\|f\|_{\varphi,\alpha} = \inf\left\{k > 0 : \ S_{\varphi,\alpha}\left(\frac{f'}{k}\right) \le 1\right\}$$

and

$$S_{\varphi,\alpha}(f) = \sup_{z\in\mathbb{D}}(1-|z|^2)^{\alpha}\varphi(|f(z)|). \tag{1.1}$$

From a standard result

$$S_{\varphi,\alpha}\left(\frac{f'}{\|f\|_{\mathcal{B}^{\varphi}_{\alpha}}}\right) \le 1, \tag{1.2}$$

the lemma below follows analogously.

Lemma 1.2. *The α-Bloch-Orlicz space is isometrically equal to μ_α-Bloch space, where*

$$\mu_\alpha(z) = \frac{1}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^{\alpha}}\right)}, \ z \in \mathbb{D}.$$

Hence $\mathcal{B}^{\varphi}_{\alpha}$ is also a Banach space under the norm

$$\|f\|_{\mathcal{B}^{\varphi}_{\alpha}} = |f(0)| + \sup_{z\in\mathbb{D}} \mu_\alpha(z)|f'(z)|.$$

The Luxemburg seminorm together with (1.2) imply

$$S_{\varphi,\alpha}(f') \le 1 \Leftrightarrow \|f\|_{\mathcal{B}^{\alpha}_{\varphi}} \le 1, \tag{1.3}$$

for any $f \in \mathcal{B}^{\varphi}_{\alpha}$. Using that we can define the β-Zygmund-Orlicz space $\mathcal{Z}^{\varphi}_{\beta} = \mathcal{Z}^{\varphi}_{\beta}(\mathbb{D})$ for $\beta > 0$, which contains all $f \in H(\mathbb{D})$ satisfying

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^{\beta}\varphi(\lambda|f''(z)|) < \infty,$$

for some $\lambda > 0$ depending on f. Same as the α-Bloch-Orlicz space, since φ is convex, the Minkowski functional

$$\|f\|_{\mathcal{Z}^{\varphi}_{\beta}} = \inf\left\{k > 0 : \ S_{\varphi,\beta}\left(\frac{f''}{k}\right) \le 1\right\}$$

defines a seminorm for $\mathcal{Z}^{\varphi}_{\beta}$, and $S_{\varphi,\beta}$ is given in (1.1). Moreover, $\mathcal{Z}^{\varphi}_{\beta}$ is a Banach space endowed with the norm

$$\|f\|_{\mathcal{Z}^{\varphi}_{\beta}} = |f(0)| + |f'(0)| + \|f\|_{\mathcal{Z}^{\varphi}_{\beta}}.$$

Lemma 1.3. *For any $f \in \mathcal{Z}^{\varphi}_{\beta} \setminus \{0\}$, the following relations hold*

$$S_{\varphi,\beta}\left(\frac{f''}{\|f\|_{\mathcal{Z}^{\varphi}_{\beta}}}\right) \le 1,$$
$$S_{\varphi,\beta}(f'') \le 1 \Leftrightarrow \|f\|_{\mathcal{Z}^{\varphi}_{\beta}} \le 1. \tag{1.4}$$

As a consequence of Lemma 1.3, the β-Zygmund-Orlicz space is isometrically equal to μ_β-Zygmund space with

$$\mu_\beta(z) = \frac{1}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^{\beta}}\right)}, \ z \in \mathbb{D}.$$

Furthermore, the equivalent norm

$$\|f\|_{\mathcal{Z}_\beta^\varphi} = |f(0)| + |f'(0)| + \sup_{z\in\mathbb{D}} \mu_\beta(z)|f''(z)|$$

makes $\mathcal{Z}_\beta^\varphi$ a Banach space.

There have been many significant developments in the study of the bounded and compact Volterra-type operators acting on various spaces of analytic functions. However, there is no treatment considering these operators acting on α-Bloch-Orlicz spaces and β-Zygmund-Orlicz spaces even on the unit disk. At present we mainly deal with the boundedness and compactness of several Volterra-type operators defined from the general space $F(p,q,s)$ to the α-Bloch-Orlicz space or β-Zygmund-Orlicz space. The organization of this paper is as follows, we collect some lemmas in Section 2 for later use. After that we provide the necessary and sufficient conditions for the boundedness and compactness of P_ϕ^h acting from $F(p,q,s)$ to $\mathcal{B}_\varphi^\alpha$ or $\mathcal{Z}_\varphi^\beta$ in Section 3 and Section 4, respectively. Finally we deduce some corollaries for remaining Volterra-type operators.

Besides, note the notation $A \lesssim B$ will be used for two nonnegative quantities A and B if $A \leq CB$ for an unimportant constant $C > 0$. For simplicity, we always suppose $0 < p,\ s < \infty,\ -2 < q < \infty,\ q+s > -1$ and $\alpha,\ \beta > 0$.

2. some Lemmas

Lemma 2.1. *If $f \in F(p,q,s)$, then $f \in \mathcal{B}^{(2+q)/p}$ and*

$$\|f\|_{\mathcal{B}^{(2+q)/p}} \lesssim \|f\|_{F(p,q,s)}. \tag{2.1}$$

Lemma 2.2. *[11, Lemma 2.2] For $0 < \alpha < \infty$, if $f \in \mathcal{B}^\alpha$, then for every $z \in \mathbb{D}$, there exists a constant $C_1 > 0$ fulfilling*

$$|f(z)| \leq \begin{cases} C_1\|f\|_{\mathcal{B}^\alpha}, & 0<\alpha<1; \\ C_1\|f\|_{\mathcal{B}^\alpha}\log\frac{2}{1-|z|^2}, & \alpha = 1; \\ \frac{C_1\|f\|_{\mathcal{B}^\alpha}}{(1-|z|^2)^{\alpha-1}}, & \alpha > 1. \end{cases} \tag{2.2}$$

The lemma below can be deduced by the standard arguments in [1, Proposition 3.11], consequently we omit the details.

Lemma 2.3. *Let $\varphi : [0,\infty) \to [0,\infty)$ be an $\mathcal{N}$-function and Y stand for the α-Bloch-Orlicz space $\mathcal{B}_\alpha^\varphi$ (or β-Zygmund-Orlicz space $\mathcal{Z}_\beta^\varphi$). Then $P_\phi^h : F(p,q,s) \to Y$ is compact if and only if $P_\phi^h : F(p,q,s) \to Y$ is bounded and, for any bounded sequence $\{f_n\}_{n\in\mathbb{N}}$ in $F(p,q,s)$ which converges to zero uniformly on $\mathbb{D}$ as $n\to\infty$, one has $\|P_\phi^h f_n\|_Y \to 0$ as $n \to \infty$.*

3. P_ϕ^h from $F(p,q,s)$ to α-Bloch-Orlicz space

In this section, we exhibit the sufficient and necessary conditions ensuring the boundedness and compactness of the operator $P_\phi^h : F(p,q,s) \to \mathcal{B}_\alpha^\varphi$.

Theorem 3.1. *Let $\varphi : [0,\infty) \to [0,\infty)$ be an $\mathcal{N}$-function, $\phi \in S(\mathbb{D})$ and $h \in H(\mathbb{D})$. Then $P_\phi^h : F(p,q,s) \to \mathcal{B}_\alpha^\varphi$ is bounded if and only if*

$$M_1 := \sup_{z\in\mathbb{D}} \frac{|h(z)|}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^\alpha}\right)} < \infty, \quad for \ \ 0 < \frac{2+q}{p} < 1; \tag{3.1}$$

$$M_2 := \sup_{z\in\mathbb{D}} \frac{|h(z)|\log\frac{2}{1-|\phi(z)|^2}}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^\alpha}\right)} < \infty, \quad for \ \ \frac{2+q}{p} = 1 \ \ and \ \ s > 2; \tag{3.2}$$

$$M_3 := \sup_{z\in\mathbb{D}} \frac{|h(z)|}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^\alpha}\right)(1-|\phi(z)|^2)^{\frac{2+q}{p}-1}} < \infty, \quad for \ \ \frac{2+q}{p} > 1. \tag{3.3}$$

Proof. Sufficiency. (*i*) For the case $0 < \frac{2+q}{p} < 1$, we suppose (3.1) is true. For any $f \in F(p,q,s)$, by Lemma 2.2 and (2.1) we conclude

$$\begin{aligned}
&S_{\varphi,\alpha}\left(\frac{(P_\phi^h f)'(z)}{M_1C_1\|f\|_{F(p,q,s)}}\right)\\
&= \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha\varphi\left(\frac{|f(\phi(z))h(z)|}{M_1C_1\|f\|_{F(p,q,s)}}\right)\\
&\le \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha\varphi\left(\varphi^{-1}\left(\frac{1}{(1-|z|^2)^\alpha}\right)\frac{|f(\phi(z))|}{C_1\|f\|_{F(p,q,s)}}\right)\\
&\le \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha\varphi\left(\varphi^{-1}\left(\frac{1}{(1-|z|^2)^\alpha}\right)\frac{\|f\|_{\mathcal{B}^{(2+q)/p}}}{\|f\|_{F(p,q,s)}}\right)\\
&\le \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha\varphi\left(\varphi^{-1}\left(\frac{1}{(1-|z|^2)^\alpha}\right)\right) \le 1,
\end{aligned}$$

which implies that

$$\left\|\frac{P_\phi^h f}{M_1C_1\|f\|_{F(p,q,s)}}\right\|_{\mathcal{B}_\varphi^\alpha} \le 1.$$

That means $\|P_\phi^h f\|_{\mathcal{B}_\varphi^\alpha} \le M_1C_1\|f\|_{F(p,q,s)}$ for any $f \in F(p,q,s)$, which yields the boundedness of $P_\phi^h : F(p,q,s) \to \mathcal{B}_\alpha^\varphi$ in this case.

(*ii*) For the case $\frac{2+q}{p} = 1$, we suppose (3.2) holds. For any $f \in F(p,q,s)$, by Lemma 2.2 and (2.1) we deduce that

$$\begin{aligned}
&S_{\varphi,\alpha}\left(\frac{(P_\phi^h f)'(z)}{M_2C_1\|f\|_{F(p,q,s)}}\right)\\
&= \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha\varphi\left(\frac{|f(\phi(z))h(z)|}{M_2C_1\|f\|_{F(p,q,s)}}\right)\\
&\le \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha\varphi\left(\varphi^{-1}\left(\frac{1}{(1-|z|^2)^\alpha}\right)\frac{|f(\phi(z))|}{C_1\log\frac{2}{1-|\phi(z)|^2}\|f\|_{F(p,q,s)}}\right)\\
&\le \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha\varphi\left(\varphi^{-1}\left(\frac{1}{(1-|z|^2)^\alpha}\right)\right) \le 1,
\end{aligned}$$

which accounts for

$$\left\|\frac{P_\phi^h f}{M_2C_1\|f\|_{F(p,q,s)}}\right\|_{\mathcal{B}_\varphi^\alpha} \le 1.$$

By similar arguments, we verify that $P^h_\phi : F(p,q,s) \to \mathcal{B}^\varphi_\alpha$ is bounded for $\frac{2+q}{p} = 1$.

(*iii*) For the case $\frac{2+q}{p} > 1$, suppose (3.3) is true. For any $f \in F(p,q,s)$, by Lemma 2.2 and (2.1) we derive

$$\begin{aligned}
& S_{\varphi,\alpha}\left(\frac{(P^h_\phi f)'(z)}{M_3C_1\|f\|_{F(p,q,s)}}\right) \\
&= \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha \varphi\left(\frac{|f(\phi(z))h(z)|}{M_3C_1\|f\|_{F(p,q,s)}}\right) \\
&\le \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha \varphi\left(\varphi^{-1}\left(\frac{1}{(1-|z|^2)^\alpha}\right)(1-|\phi(z)|^2)^{\frac{2+q}{p}-1}\frac{|f(\phi(z))|}{C_1\|f\|_{F(p,q,s)}}\right) \\
&\le \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha \varphi\left(\varphi^{-1}\left(\frac{1}{(1-|z|^2)^\alpha}\right)\right) \le 1,
\end{aligned}$$

which means that

$$\left\|\frac{P^h_\phi f}{M_3C_1\|f\|_{F(p,q,s)}}\right\|_{\mathcal{B}^\alpha_\varphi} \le 1.$$

Hence the operator $P^h_\phi : F(p,q,s) \to \mathcal{B}^\varphi_\alpha$ is bounded for $\frac{2+q}{p} > 1$.

Necessity. Assume the operator $P^h_\phi : F(p,q,s) \to \mathcal{B}^\varphi_\alpha$ is bounded, consequently, there exists $C > 0$ such that

$$\|P^h_\phi f\|_{\mathcal{B}^\varphi_\alpha} \le C\|f\|_{F(p,q,s)}, \quad \text{for any } f \in F(p,q,s).$$

That is to say

$$\left\|\frac{P^h_\phi f}{C\|f\|_{F(p,q,s)}}\right\|_{\mathcal{B}^\varphi_\alpha} \le 1, \quad \text{for any } f \in F(p,q,s).$$

In light of (1.3), the above inequality reveals that

$$S_{\varphi,\alpha}\left(\left(\frac{P^h_\phi f}{C\|f\|_{F(p,q,s)}}\right)'\right) = S_{\varphi,\alpha}\left(\frac{f\circ\phi\, h}{C\|f\|_{F(p,q,s)}}\right) \le 1, \quad \text{for any } f \in F(p,q,s). \tag{3.4}$$

Replacing f by the test function $f_0(z) = 1 \in F(p,q,s)$ in (3.4) shows that

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha\varphi\left(\frac{|h(z)|}{C}\right) \le 1 \Rightarrow \sup_{z\in\mathbb{D}} \frac{|h(z)|}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^\alpha}\right)} < \infty. \tag{3.5}$$

(*i*) For the case $0 < \frac{2+q}{p} < 1$, the desired formula (3.1) can be deduced from the boundedness of $P^h_\phi : F(p,q,s) \to \mathcal{B}^\varphi_\alpha$ and (3.5).

(*ii*) For the case $\frac{2+q}{p} = 1$ and $s > 2$, letting $a \in \mathbb{D}$, define the function

$$f_a(z) = \log\frac{2}{1-z\overline{\phi(a)}}, \quad z \in \mathbb{D}. \tag{3.6}$$

Applying [9, Proposition 1.4.10] with $s > 2$, we verify that

$$f_a \in F(p,q,s) \text{ satisfying } \sup_{a\in\mathbb{D}}\|f_a\|_{F(p,q,s)} \le 1.$$

Indeed, by a direct calculations, it holds

$$\begin{aligned}\|f_a\|^p_{F(p,q,s)} &\le \sup_{u\in\mathbb{D}}\int_{\mathbb{D}}\left|\frac{\overline{\phi(a)}}{1-z\overline{\phi(a)}}\right|^p\frac{(1-|z|^2)^{q+s}(1-|u|^2)^s}{|1-z\overline{u}|^{2s}}dA(z)\\ &\le \sup_{u\in\mathbb{D}}(1-|u|^2)^s\int_{\mathbb{D}}\frac{(1-|z|^2)^{s-2}}{|1-z\overline{u}|^{2s}}dA(z)\\ &\le \sup_{u\in\mathbb{D}}\frac{(1-|u|^2)^s}{(1-|u|^2)^s}\le C,\end{aligned}$$

with $p=q+2$ and $s>2$. In view of (3.4), it follows that

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha\varphi\left(\frac{|f_a(\phi(z))h(z)|}{C\|f_a\|_{F(p,q,s)}}\right)\le 1.$$

Then

$$(1-|a|^2)^\alpha\varphi\left(\frac{|f_a(\phi(a))h(a)|}{C\|f_a\|_{F(p,q,s)}}\right)\le 1\Rightarrow\frac{|f_a(\phi(a))h(a)|}{\varphi^{-1}\left(\frac{1}{(1-|a|^2)^\alpha}\right)}<\infty. \tag{3.7}$$

Since $a\in\mathbb{D}$ is arbitrary, thus we say that

$$\sup_{a\in\mathbb{D}}\frac{\log\frac{2}{1-|\phi(a)|^2}|h(a)|}{\varphi^{-1}\left(\frac{1}{(1-|a|^2)^\alpha}\right)}<\infty.$$

That is the formula (3.2) holds for $s>2$.

(iii) For the case $\frac{2+q}{p}>1$, letting $a\in\mathbb{D}$, define the function

$$f_a(z)=\frac{1-|\phi(a)|^2}{(1-z\overline{\phi(a)})^{\frac{2+q}{p}}},\ z\in\mathbb{D}.$$

By a direct calculation and [9, Proposition 1.4.10], it holds $f_a\in F(p,q,s)$ satisfying $\sup_{a\in\mathbb{D}}\|f_a\|_{F(p,q,s)}\le 1$. At this time, the formula (3.7) implies that

$$\sup_{a\in\mathbb{D}}\frac{|h(a)|}{\varphi^{-1}\left(\frac{1}{(1-|a|^2)^\alpha}\right)(1-|\phi(a)|^2)^{\frac{2+q}{p}-1}}<\infty,$$

and then the desire result (3.3) is valid. This concludes the proof. □

Remark 3.2. *The above results could also be directly deduced from Lemma 1.2.*

Theorem 3.3. *Let $\varphi:[0,\infty)\to[0,\infty)$ be an $\mathcal{N}$-function, $\phi\in S(\mathbb{D})$ and $h\in H(\mathbb{D})$. Then $P^h_\phi:F(p,q,s)\to\mathcal{B}^\varphi_\alpha$ is compact if and only if $P^h_\phi:F(p,q,s)\to\mathcal{B}^\varphi_\alpha$ is bounded and*

$$\lim_{|\phi(z)|\to1}\frac{|h(z)|}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^\alpha}\right)}=0,\ \ for\ \ 0<\frac{2+q}{p}<1; \tag{3.8}$$

$$\lim_{|\phi(z)|\to1}\frac{|h(z)|\log\frac{2}{1-|\phi(z)|^2}}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^\alpha}\right)}=0,\ \ for\ \ \frac{2+q}{p}=1\ \ and\ \ s>2; \tag{3.9}$$

$$\lim_{|\phi(z)|\to1}\frac{|h(z)|}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^\alpha}\right)(1-|\phi(z)|^2)^{\frac{2+q}{p}-1}}=0,\ \ for\ \ \frac{2+q}{p}>1. \tag{3.10}$$

Proof. Sufficiency. Suppose $P_\phi^h : F(p,q,s) \to \mathcal{B}_\alpha^\varphi$ is bounded and (3.8)–(3.10) hold. By Lemma 1.2, choosing $f_0(z) = 1 \in F(p,q,s)$ and applying the boundedness of $P_\phi^h : F(p,q,s) \to \mathcal{B}_\alpha^\varphi$, we verify (3.5), that is, $M_1 < \infty$. Let $\{f_n\}$ be a sequence in $F(p,q,s)$ with $\sup_{n\in\mathbb{N}} \|f_n\|_{F(p,q,s)} \leq K$ and f_n converging to zero uniformly on compact subsets of $\mathbb{D}$ as $n \to \infty$. By Lemma 2.3, it suffices to show that $\|P_\phi^h f_n\|_{\mathcal{B}_\alpha^\varphi} \to 0$ as $n \to \infty$. For any $0 < r < 1$, we claim that

$$\begin{aligned}
\|P_\phi^h f_n\|_{\mathcal{B}_\alpha^\varphi} &= |P_\phi^h f_n(0)| + \sup_{z\in\mathbb{D}} \frac{1}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^\alpha}\right)} |(P_\phi^h f_n)'(z)| \\
&= \sup_{z\in\mathbb{D}} \frac{1}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^\alpha}\right)} |f_n(\phi(z))h(z)| \\
&= \sup_{\{z\in\mathbb{D}:\, |\phi(z)|\leq r\}} \frac{1}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^\alpha}\right)} |f_n(\phi(z))h(z)| \\
&\quad + \sup_{\{z\in\mathbb{D}:\, |\phi(z)|> r\}} \frac{1}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^\alpha}\right)} |f_n(\phi(z))h(z)| \\
&\leq M_1 \sup_{\{w\in\mathbb{D}:\, |w|\leq r\}} |f_n(w)| + \sup_{\{z\in\mathbb{D}:\, |\phi(z)|> r\}} \frac{1}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^\alpha}\right)} |f_n(\phi(z))h(z)|.
\end{aligned}$$

(i) For the case $0 < \frac{2+q}{p} < 1$, by (3.8), for every $\epsilon > 0$, there exists $0 < r_1 < 1$ such that

$$\frac{|h(z)|}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^\alpha}\right)} < \epsilon, \text{ whenever } |\phi(z)| > r_1.$$

Due to Lemma 2.2 we obtain that

$$\begin{aligned}
\|P_\phi^h f_n\|_{\mathcal{B}_\alpha^\varphi} &\leq M_1 \sup_{\{w\in\mathbb{D}:\, |w|\leq r_1\}} |f_n(w)| + KC_1 \sup_{\{z\in\mathbb{D}:\, |\phi(z)|> r_1\}} \frac{|h(z)|}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^\alpha}\right)} \\
&< M_1 \sup_{\{w\in\mathbb{D}:\, |w|\leq r_1\}} |f_n(w)| + KC_1\epsilon.
\end{aligned}$$

Since f_n converges to zero uniformly on compact subsets of $\mathbb{D}$ as $n \to \infty$, we conclude that

$$\lim_{n\to\infty} \|P_\phi^h f_n\|_{\mathcal{B}_\alpha^\varphi} \leq KC_1\epsilon.$$

(ii) For the case $\frac{2+q}{p} = 1$, by (3.9), for every $\epsilon > 0$, there is $0 < r_2 < 1$ satisfying

$$\frac{|h(z)| \log \frac{2}{1-|\phi(z)|^2}}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^\alpha}\right)} < \epsilon, \text{ whenever } |\phi(z)| > r_2.$$

Analogously, we show that

$$\begin{aligned}
\|P_\phi^h f_n\|_{\mathcal{B}_\alpha^\varphi} &\leq M_1 \sup_{\{w\in\mathbb{D}:\, |w|\leq r_2\}} |f_n(w)| + KC_1 \sup_{\{z\in\mathbb{D}:\, |\phi(z)|> r_2\}} \frac{|h(z)| \log \frac{2}{1-|\phi(z)|^2}}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^\alpha}\right)} \\
&< M_1 \sup_{\{w\in\mathbb{D}:\, |w|\leq r_2\}} |f_n(w)| + KC_1\epsilon.
\end{aligned}$$

Furthermore, we arrive at $\lim_{n\to\infty} \|P_\phi^h f_n\|_{\mathcal{B}_\alpha^\varphi} \leq KC_1\epsilon$.

(*iii*) For $\frac{2+q}{p} > 1$, by (3.10), for every $\epsilon > 0$, there exists $0 < r_3 < 1$ such that

$$\frac{|h(z)|}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^\alpha}\right)(1-|\phi(z)|^2)^{\frac{2+q}{p}-1}} < \epsilon, \text{ for } |\phi(z)| > r_3.$$

By the similar arguments, it yields that

$$\begin{aligned} \|P_\phi^h f_n\|_{\mathcal{B}_\alpha^\varphi} &\le M_1 \sup_{\{w\in\mathbb{D}:\,|w|\le r_3\}} |f_n(w)| \\ &+ KC_1 \sup_{\{z\in\mathbb{D}:\,|\phi(z)|>r_3\}} \frac{|h(z)|}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^\alpha}\right)(1-|\phi(z)|^2)^{\frac{2+q}{p}-1}} \\ &< M_1 \sup_{\{w\in\mathbb{D}:\,|w|\le r_3\}} |f_n(w)| + KC_1\epsilon. \end{aligned}$$

We also derive that $\lim_{n\to\infty} \|P_\phi^h f_n\|_{\mathcal{B}_\alpha^\varphi} \le KC_1\epsilon$.

Considering ϵ is arbitrary, $\|P_\phi^h f_n\|_{\mathcal{B}_\alpha^\varphi} \to 0$ as $n \to \infty$ holds for the cases $0 < \frac{2+q}{p} < 1$, $\frac{2+q}{p} = 1$ and $\frac{2+q}{p} > 1$, respectively. Combining with Lemma 2.3, it follows $P_\phi^h : F(p,q,s) \to \mathcal{B}_\alpha^\varphi$ is compact.

Necessity. Assume $P_\phi^h : F(p,q,s) \to \mathcal{B}_\alpha^\varphi$ is compact. The boundedness clearly follows. Let $\{z_k\}_{k\in\mathbb{N}}$ be a sequence in $\mathbb{D}$ such that $\lim_{k\to\infty} |\phi(z_k)| = 1$. Set

$$\begin{aligned} f_{1,k}(z) &= \frac{(1-|\phi(z_k)|^2)^{\frac{2+q}{p}}}{(1-z\overline{\phi(z_k)})^{\frac{2+q}{p}}}, \quad \text{for } 0 < \frac{2+q}{p} < 1; \\ f_{2,k}(z) &= \left(\log\frac{2}{1-z\overline{\phi(z_k)}}\right)^2 \left(\log\frac{2}{1-|\phi(z_k)|^2}\right)^{-1}, \quad \text{for } \frac{2+q}{p} = 1 \text{ and } s > 2; \\ f_{3,k}(z) &= \frac{1-|\phi(z_k)|^2}{(1-z\overline{\phi(z_k)})^{\frac{2+q}{p}}}, \quad \text{for } \frac{2+q}{p} > 1. \end{aligned}$$

By [9, Proposition 1.4.10], it is trivial to verify that $f_{1,k}, f_{2,k}$ (with $s > 2$) and $f_{3,k} \in F(p,q,s)$ for $k \in \mathbb{N}$ and $f_{i,k} \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $k \to \infty$ for $i = 1,2,3$. By Lemma 2.3, it yields $\lim_{k\to\infty} \|P_\phi^h f_{i,k}\|_{\mathcal{B}_\alpha^\varphi} = 0$ for $i = 1,2,3$, which offers that

$$\begin{aligned} \|P_\phi^h f_{i,k}\|_{\mathcal{B}_\alpha^\varphi} &= |P_\phi^h f_{i,k}(0)| + \sup_{z\in\mathbb{D}} \frac{1}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^\alpha}\right)} |f_{i,k}(\phi(z))h(z)|] \\ &\ge \frac{1}{\varphi^{-1}\left(\frac{1}{(1-|z_k|^2)^\alpha}\right)} |f_{i,k}(\phi(z_k))h(z_k)|. \end{aligned} \tag{3.11}$$

Putting $f_{1,k}$, $f_{2,k}$ and $f_{3,k}$ into (3.11), we check that

$$\|P_\phi^h f_{i,k}\|_{\mathcal{B}_\alpha^\varphi} \ge \begin{cases} |h(z_k)|/\varphi^{-1}\left(\frac{1}{(1-|z_k|^2)^\alpha}\right), & 0 < \frac{2+q}{p} < 1; \\ |h(z_k)| \log\frac{2}{1-|\phi(z_k)|^2} / \varphi^{-1}\left(\frac{1}{(1-|z_k|^2)^\alpha}\right), & \frac{2+q}{p} = 1 \text{ and } s > 2; \\ |h(z_k)| / \left[\varphi^{-1}\left(\frac{1}{(1-|z_k|^2)^\alpha}\right)(1-|\phi(z_k)|^2)^{\frac{2+q}{p}-1}\right], & \frac{2+q}{p} > 1. \end{cases}$$

Letting $k \to \infty$ in the above inequalities, we conclude (3.8)–(3.10). The proof is finished. □

4. P_ϕ^h from $F(p,q,s)$ to β-Zygmund-Orlicz space

In this section, the properties of $P_\phi^h : F(p,q,s) \to \mathcal{Z}_\beta^\varphi$ are discussed in details.

Theorem 4.1. *Let $\varphi : [0,\infty) \to [0,\infty)$ be an $\mathcal{N}$-function, $\phi \in S(\mathbb{D})$ and $h \in H(\mathbb{D})$. Then the operator $P_\phi^h : F(p,q,s) \to \mathcal{Z}_\beta^\varphi$ is bounded if and only if*

$$L := \sup_{z\in\mathbb{D}} \frac{|\phi'(z)h(z)|}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^\beta}\right)(1-|\phi(z)|^2)^{\frac{2+q}{p}}} < \infty, \tag{4.1}$$

and

$$L_1 := \sup_{z\in\mathbb{D}} \frac{|h'(z)|}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^\beta}\right)} < \infty, \quad \text{for } 0 < \frac{2+q}{p} < 1; \tag{4.2}$$

$$L_2 := \sup_{z\in\mathbb{D}} \frac{|h'(z)|\log\frac{2}{1-|\phi(z)|^2}}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^\beta}\right)} < \infty, \quad \text{for } \frac{2+q}{p} = 1 \text{ and } s > 2; \tag{4.3}$$

$$L_3 := \sup_{z\in\mathbb{D}} \frac{|h'(z)|}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^\beta}\right)(1-|\phi(z)|^2)^{\frac{2+q}{p}-1}} < \infty, \quad \text{for } \frac{2+q}{p} > 1. \tag{4.4}$$

Proof. Sufficiency. Suppose (4.1)–(4.4) hold. For any $f \in F(p,q,s)$, observe that

$$\begin{aligned}
&S_{\varphi,\beta}\left(\frac{(P_\phi^h f)''(z)}{C\|f\|_{F(p,q,s)}}\right)\\
&= \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta \varphi\left(\frac{|f'(\phi(z))\phi'(z)h(z) + f(\phi(z))h'(z)|}{C\|f\|_{F(p,q,s)}}\right)\\
&\le \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta \varphi\left(\frac{|f'(\phi(z))\phi'(z)h(z)| + |f(\phi(z))h'(z)|}{C\|f\|_{F(p,q,s)}}\right),
\end{aligned} \tag{4.5}$$

where the constant C will be determined later.

(i) For $0 < \frac{2+q}{p} < 1$, by Lemma 2.2, we reformulate (4.5) into

$$\begin{aligned}
&S_{\varphi,\beta}\left(\frac{(P_\phi^h f)''(z)}{C\|f\|_{F(p,q,s)}}\right)\\
&\le \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta \varphi\left(\frac{(1-|\phi(z)|^2)^{\frac{2+q}{p}}|f'(\phi(z))\phi'(z)h(z)|}{(1-|\phi(z)|^2)^{\frac{2+q}{p}}C\|f\|_{F(p,q,s)}} + \frac{C_1|h'(z)|}{C}\right)\\
&\le \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta \varphi\left(\frac{|\phi'(z)h(z)|}{C(1-|\phi(z)|^2)^{\frac{2+q}{p}}} + \frac{C_1|h'(z)|}{C}\right)\\
&= \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta \varphi\left(\frac{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^\beta}\right)}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^\beta}\right)}\left[\frac{|\phi'(z)h(z)|}{C(1-|\phi(z)|^2)^{\frac{2+q}{p}}} + \frac{C_1|h'(z)|}{C}\right]\right)\\
&\le \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta \varphi\left(\varphi^{-1}\left(\frac{1}{(1-|z|^2)^\beta}\right)\frac{L+C_1L_1}{C}\right).
\end{aligned} \tag{4.6}$$

(ii) For $\frac{2+q}{p}=1$, by Lemma 2.2, (4.5) becomes into

$$\begin{aligned}
&S_{\varphi,\beta}\left(\frac{(P_\phi^h f)''(z)}{C\|f\|_{F(p,q,s)}}\right)\\
&\leq \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta\varphi\left(\frac{|\phi'(z)h(z)|}{C(1-|\phi(z)|^2)^{\frac{2+q}{p}}}+\frac{C_1|h'(z)|\log\frac{2}{1-|\phi(z)|^2}}{C}\right)\\
&=\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta\varphi\left(\frac{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^\beta}\right)}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^\beta}\right)}\left[\frac{|\phi'(z)h(z)|}{C(1-|\phi(z)|^2)^{\frac{2+q}{p}}}+\frac{C_1}{C}|h'(z)|\log\frac{2}{1-|\phi(z)|^2}\right]\right)\\
&\leq \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta\varphi\left(\varphi^{-1}\left(\frac{1}{(1-|z|^2)^\beta}\right)\frac{L+C_1L_2}{C}\right). \qquad (4.7)
\end{aligned}$$

(iii) For $\frac{2+q}{p}>1$, by Lemma 2.2, we rewrite (4.5) into

$$\begin{aligned}
&S_{\varphi,\beta}\left(\frac{(P_\phi^h f)''(z)}{C\|f\|_{F(p,q,s)}}\right)\\
&\leq \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta\varphi\left(\frac{|\phi'(z)h(z)|}{C(1-|\phi(z)|^2)^{\frac{2+q}{p}}}+\frac{C_1|h'(z)|}{C(1-|\phi(z)|^2)^{\frac{2+q}{p}-1}}\right)\\
&\leq \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta\varphi\left(\varphi^{-1}\left(\frac{1}{(1-|z|^2)^\beta}\right)\frac{L+C_1L_3}{C}\right). \qquad (4.8)
\end{aligned}$$

In (4.6)–(4.8), C_1 is given in Lemma 2.2. We choose the constant C large enough satisfying $L+C_1L_i\leq C$ for $i=1,2,3$. Hence (4.6)–(4.8) were transformed into

$$S_{\varphi,\beta}\left(\frac{(P_\phi^h f)''(z)}{C\|f\|_{F(p,q,s)}}\right)\leq \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta\varphi\left(\varphi^{-1}\left(\frac{1}{(1-|z|^2)^\beta}\right)\right)\leq 1.$$

The above formula and (1.4) imply that $\|P_\phi^h f\|_{\mathcal{Z}_\beta^\varphi}\leq C\|f\|_{F(p,q,s)}$. Apparently, $P_\phi^h: F(p,q,s)\to\mathcal{Z}_\beta^\varphi$ is bounded.

Necessity. Suppose that $P_\phi^h: F(p,q,s)\to\mathcal{Z}_\beta^\varphi$ is bounded. Hence there is a constant $C>0$ such that

$$\|P_\phi^h f\|_{\mathcal{Z}_\beta^\varphi}\leq C\|f\|_{F(p,q,s)} \quad \text{for all } f\in F(p,q,s).$$

By (1.4), we conclude that

$$S_{\varphi,\beta}\left(\frac{(P_\phi^h f)''(z)}{C\|f\|_{F(p,q,s)}}\right)\leq 1.$$

More precisely, it is

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta\varphi\left(\frac{|f'(\phi(z))\phi'(z)h(z)+f(\phi(z))h'(z)|}{C\|f\|_{F(p,q,s)}}\right)\leq 1. \qquad (4.9)$$

Put $f_0(z)=1$ or $f_0(z)=z$ into (4.9), which yields (4.2) and

$$\sup_{z\in\mathbb{D}}\frac{|\phi'(z)h(z)+\phi(z)h'(z)|}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^\beta}\right)}<\infty. \qquad (4.10)$$

Hence $L_1 < \infty$ for $0 < \frac{2+q}{p} < 1$, then (4.2) together with (4.10) indicate

$$\hat{L}_1 := \sup_{z\in\mathbb{D}} \frac{|\phi'(z)h(z)|}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^\beta}\right)} < \infty. \tag{4.11}$$

For $a \in \mathbb{D}$, define

$$f_a(z) = \frac{(1-|\phi(a)|^2)^{1+\frac{2+q}{p}}}{(1-z\overline{\phi(a)})^{2\frac{2+q}{p}}} - \frac{1-|\phi(a)|^2}{(1-z\overline{\phi(a)})^{\frac{2+q}{p}}},\ z \in \mathbb{D},$$

belonging to $F(p,q,s)$ with $\sup_{a\in\mathbb{D}} \|f\|_{F(p,q,s)} \le 1$ by [9, Proposition 1.4.10] . Furthermore,

$$f_a(\phi(a)) = 0 \text{ and } f_a'(\phi(a)) = \frac{2+q}{p} \frac{\overline{\phi(a)}}{(1-|\phi(a)|^2)^{\frac{2+q}{p}}}.$$

Putting f_a into (4.9), we show that

$$\begin{aligned}
&(1-|a|^2)^\beta \varphi\left(\frac{|f_a'(\phi(a))\phi'(a)h(a)|}{C\|f_a\|_{F(p,q,s)}}\right)\\
&= (1-|a|^2)^\beta \varphi\left(\frac{|f_a'(\phi(a))\phi'(a)h(a) + f_a(\phi(a))h'(a)|}{C\|f_a\|_{F(p,q,s)}}\right)\\
&\le \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta \varphi\left(\frac{|f_a'(\phi(z))\phi'(z)h(z) + f_a(\phi(z))h'(z)|}{C\|f_a\|_{F(p,q,s)}}\right) \le 1.
\end{aligned}$$

Hence

$$\frac{|f_a'(\phi(a))\phi'(a)h(a)|}{C\|f_a\|_{F(p,q,s)}} \le \varphi^{-1}\left(\frac{1}{(1-|a|^2)^\beta}\right),$$

which is equivalent to saying that

$$\frac{2+q}{p} \frac{|\overline{\phi(a)}||\phi'(a)h(a)|}{C\|f_a\|_{F(p,q,s)}(1-|\phi(a)|^2)^{\frac{2+q}{p}}} \le \varphi^{-1}\left(\frac{1}{(1-|a|^2)^\beta}\right).$$

Then

$$\frac{|\overline{\phi(a)}||\phi'(a)h(a)|}{\varphi^{-1}\left(\frac{1}{(1-|a|^2)^\beta}\right)(1-|\phi(a)|^2)^{\frac{2+q}{p}}} \le \|f_a\|_{F(p,q,s)} < \infty.$$

In general

$$\sup_{z\in\mathbb{D}} \frac{|\phi(z)||\phi'(z)h(z)|}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^\beta}\right)(1-|\phi(z)|^2)^{\frac{2+q}{p}}} < \infty. \tag{4.12}$$

Now we split into two cases to show (4.1).

($Case-1$) If $|\phi(z)| \le 1/2$, by (4.11) we get that

$$\sup_{\{z\in\mathbb{D}:\,|\phi(z)|\le 1/2\}} \frac{|\phi'(z)h(z)|}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^\beta}\right)(1-|\phi(z)|^2)^{\frac{2+q}{p}}} \le \sup_{z\in\mathbb{D}} \frac{|\phi'(z)h(z)|}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^\beta}\right)} < \infty.$$

$(Case-2)$ If $|\phi(z)|>1/2$, by (4.12) we obtain that

$$\sup_{\{z\in\mathbb{D}:\,|\phi(z)|>1/2\}}\frac{|\phi'(z)h(z)|}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^\beta}\right)(1-|\phi(z)|^2)^{\frac{2+q}{p}}}$$
$$\le\sup_{z\in\mathbb{D}}\frac{|\phi(z)||\phi'(z)h(z)|}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^\beta}\right)(1-|\phi(z)|^2)^{\frac{2+q}{p}}}<\infty.$$

Combining the above two cases, we prove (4.1) is valid.

(i) For $0<\frac{2+q}{p}<1$, take $a\in\mathbb{D}$ and define

$$f_a(z)=\frac{(1-|\phi(a)|^2)^{2\frac{2+q}{p}}}{(1-z\overline{\phi(z)})^{2\frac{2+q}{p}}}-2\frac{(1-|\phi(a)|^2)^{\frac{2+q}{p}}}{(1-z\overline{\phi(a)})^{\frac{2+q}{p}}},\ z\in\mathbb{D},$$

which is in $F(p,q,s)$ with $\sup_{a\in\mathbb{D}}\|f_a\|_{F(p,q,s)}\le 1$ from [9, Proposition 1.4.10]. And it holds

$$f_a(\phi(a))=-1\ \text{ and }\ f_a'(\phi(a))=0.$$

By (4.9), it yields that

$$(1-|a|^2)^\beta\varphi\left(\frac{|h'(a)|}{C\|f\|_{F(p,q,s)}}\right)\le 1,$$

which implies

$$\frac{|h'(a)|}{C\|f\|_{F(p,q,s)}}\le\varphi^{-1}\left(\frac{1}{(1-|a|^2)^\beta}\right).$$

Hence the desired formula (4.2) follows.

(ii) For $\frac{2+q}{p}=1$ and $s>2$, given $a\in\mathbb{D}$, set the function

$$f_a(z)=\log\frac{2}{1-z\overline{\phi(a)}},\ z\in\mathbb{D},$$

belonging to $F(p,q,s)$ with $\sup_{a\in\mathbb{D}}\|f_a\|_{F(p,q,s)}\le 1$ from [9, Proposition 1.4.10]. By a direct calculation,

$$f_a(\phi(a))=\log\frac{2}{1-|\phi(a)|^2}\ \text{ and }\ f_a'(\phi(a))=\frac{\overline{\phi(a)}}{1-|\phi(a)|^2}.$$

Hence replacing f by f_a in (4.9), we arrive at

$$(1-|a|^2)^\beta\varphi\left(\frac{|\overline{\phi(a)}\phi'(a)h(a)/(1-|\phi(a)|^2)+\log\frac{2}{1-|\phi(a)|^2}h'(a)|}{C\|f_a\|_{F(p,q,s)}}\right)$$
$$=(1-|a|^2)^\beta\varphi\left(\frac{|f_a'(\phi(a))\phi'(a)h(a)+f_a(\phi(a))h'(a)|}{C\|f_a\|_{F(p,q,s)}}\right)$$
$$\le\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta\varphi\left(\frac{|f_a'(\phi(z))\phi'(z)h(z)+f_a(\phi(z))h'(z)|}{C\|f_a\|_{F(p,q,s)}}\right)\le 1.$$

Therefore,

$$\frac{|\overline{\phi(a)}\phi'(a)h(a)/(1-|\phi(a)|^2)+\log\frac{2}{1-|\phi(a)|^2}h'(a)|}{C\|f_a\|_{F(p,q,s)}}\le\varphi^{-1}\left(\frac{1}{(1-|a|^2)^\beta}\right).$$

Furthermore,

$$\frac{|h'(a)|\log\frac{2}{1-|\phi(a)|^2}}{\varphi^{-1}\left(\frac{1}{(1-|a|^2)^\beta}\right)} \leq C\|f_a\|_{F(p,q,s)} + \frac{|\phi(a)\phi'(a)h(a)|}{\varphi^{-1}\left(\frac{1}{(1-|a|^2)^\beta}\right)(1-|\phi(a)|^2)}.$$

Employing (4.1), we obtain that

$$\sup_{z\in\mathbb{D}}\frac{|h'(z)|\log\frac{2}{1-|\phi(z)|^2}}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^\beta}\right)} \leq \sup_{z\in\mathbb{D}}\frac{|\phi(z)\phi'(z)h(z)|}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^\beta}\right)(1-|\phi(z)|^2)} < \infty.$$

As a consequence, (4.3) holds for the case $\frac{2+q}{p} = 1$ and $s > 2$.

(*iii*) For $\frac{2+q}{p} > 1$, considering $a \in \mathbb{D}$, define

$$f_a(z) = \frac{(1-|\phi(a)|^2)^{1+\frac{2+q}{p}}}{(1-z\overline{\phi(a)})^{2\frac{2+q}{p}}} - 2\frac{1-|\phi(a)|^2}{(1-z\overline{\phi(a)})^{\frac{2+q}{p}}},\ z\in\mathbb{D},$$

which is in $F(p,q,s)$ with $\sup_{a\in\mathbb{D}}\|f_a\|_{F(p,q,s)} \leq 1$. By a direct calculation,

$$f_a(\phi(a)) = -\frac{1}{(1-|\phi(a)|^2)^{\frac{2+q}{p}-1}} \quad\text{and}\quad f_a'(\phi(a)) = 0.$$

Putting f_a into (4.9), we verify that

$$\begin{aligned}
&(1-|a|^2)^\beta\varphi\left(\frac{|h'(a)|}{(1-|\phi(a)|^2)^{\frac{2+q}{p}-1}C\|f_a\|_{F(p,q,s)}}\right)\\
&= (1-|a|^2)^\beta\varphi\left(\frac{|f_a(\phi(a))h'(a)|}{C\|f_a\|_{F(p,q,s)}}\right)\\
&= (1-|a|^2)^\beta\varphi\left(\frac{|f_a'(\phi(a))\phi'(a)h(a) + f_a(\phi(a))h'(a)|}{C\|f_a\|_{F(p,q,s)}}\right)\\
&\leq \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta\varphi\left(\frac{|f_a'(\phi(z))\phi'(z)h(z) + f_a(\phi(z))h'(z)|}{C\|f_a\|_{F(p,q,s)}}\right) \leq 1,
\end{aligned}$$

which implies that

$$\sup_{a\in\mathbb{D}}\frac{|h'(a)|}{\varphi^{-1}\left(\frac{1}{(1-|a|^2)^\beta}\right)(1-|\phi(a)|^2)^{\frac{2+q}{p}-1}} < \infty.$$

Then (4.4) is true for the case $\frac{2+q}{p} > 1$. The proof is complete. □

Theorem 4.2. *Let $\varphi : [0,\infty) \to [0,\infty)$ be an $\mathcal{N}$-function, $\phi \in S(\mathbb{D})$ and $h \in H(\mathbb{D})$. Then the operator* P_ϕ^h :

$F(p,q,s) \to \mathcal{Z}_\beta^\varphi$ is compact if and only if $P_\phi^h : F(p,q,s) \to \mathcal{Z}_\beta^\varphi$ is bounded and

$$\lim_{|\phi(z)|\to 1} \frac{|\phi'(z)h(z)|}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^\beta}\right)(1-|\phi(z)|^2)^{\frac{2+q}{p}}} = 0, \tag{4.13}$$

and

$$\lim_{|\phi(z)|\to 1} \frac{|h'(z)|}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^\beta}\right)} = 0, \ \ \textit{for} \ \ 0 < \frac{2+q}{p} < 1; \tag{4.14}$$

$$\lim_{|\phi(z)|\to 1} \frac{|h'(z)| \log \frac{2}{1-|\phi(z)|^2}}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^\beta}\right)} = 0, \ \ \textit{for} \ \ \frac{2+q}{p} = 1 \ \ \textit{and} \ \ s > 2; \tag{4.15}$$

$$\lim_{|\phi(z)|\to 1} \frac{|h'(z)|}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^\beta}\right)(1-|\phi(z)|^2)^{\frac{2+q}{p}-1}} = 0, \ \ \textit{for} \ \ \frac{2+q}{p} > 1. \tag{4.16}$$

Proof. Sufficiency. Assume the operator $P_\phi^h : F(p,q,s) \to \mathcal{Z}_\beta^\varphi$ is bounded and (4.13)–(4.16) hold. By the boundedness of $P_\phi^h : F(p,q,s) \to \mathcal{Z}_\beta^\varphi$ and let $f_0(z) = 1$ or $f_0(z) = z$, we prove that $L_1 < \infty$ and $\hat{L}_1 < \infty$ in (4.2) and (4.11), respectively.

Let $\{f_n\}$ be a sequence in $F(p,q,s)$ with $\sup_{n\in\mathbb{N}} \|f_n\|_{F(p,q,s)} \le K$ and f_n converging to zero uniformly on compact subsets of $\mathbb{D}$ as $n \to \infty$. By Lemma 2.3, we will show that $\|P_\phi^h f_n\|_{\mathcal{Z}_\beta^\varphi} \to 0$ as $n \to \infty$. Indeed, for $r \in (0,1)$ we express the norm into

$$\begin{aligned}
\|P_\phi^h f_n\|_{\mathcal{Z}_\beta^\varphi} &= |P_\phi^h f_n(0)| + |(P_\phi^h f_n)'(0)| + \sup_{z\in\mathbb{D}} \frac{1}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^\beta}\right)} |(P_\phi^h f_n)''(z)| \\
&= |f_n(\phi(0))h(0)| + \sup_{z\in\mathbb{D}} \frac{|f_n'(\phi(z))\phi'(z)h(z) + f_n(\phi(z))h'(z)|}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^\beta}\right)} \\
&\le |f_n(\phi(0))h(0)| + \sup_{\{z\in\mathbb{D}:\ |\phi(z)|\le r\}} \frac{|f_n'(\phi(z))\phi'(z)h(z)| + |f_n(\phi(z))h'(z)|}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^\beta}\right)} \\
&\quad + \sup_{\{z\in\mathbb{D}:\ |\phi(z)|> r\}} \frac{|f_n'(\phi(z))\phi'(z)h(z)| + |f_n(\phi(z))h'(z)|}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^\beta}\right)}.
\end{aligned}$$

(i) For $0 < \frac{2+q}{p} < 1$, in view of (4.13) and (4.14), we claim that for every $\epsilon > 0$, there is $0 < r_1 < 1$ satisfying

$$\frac{|\phi'(z)h(z)|}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^\beta}\right)(1-|\phi(z)|^2)^{\frac{2+q}{p}}} < \frac{\epsilon}{2}, \tag{4.17}$$

$$\frac{|h'(z)|}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^\beta}\right)} < \frac{\epsilon}{2}, \tag{4.18}$$

for $|\phi(z)| > r_1$. Based on (4.17), (4.18) and the described norm above, we give that

$$\begin{aligned}
\|P_\phi^h f_n\|_{\mathcal{Z}_\beta^\varphi} &\leq |f_n(\phi(0))h(0)| + \sup_{\{z\in\mathbb{D}:\, |\phi(z)|\leq r_1\}} \frac{|f_n'(\phi(z))\phi'(z)h(z)| + |f_n(\phi(z))h'(z)|}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^\beta}\right)} \\
&+ \sup_{\{z\in\mathbb{D}:\, |\phi(z)|>r_1\}} \frac{|f_n'(\phi(z))\phi'(z)h(z)| + |f_n(\phi(z))h'(z)|}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^\beta}\right)} \\
&\leq |f_n(\phi(0))h(0)| + \hat{L}_1 \sup_{\{w\in\mathbb{D}:\, |w|\leq r_1\}} |f_n'(w)| + L_1 \sup_{\{w\in\mathbb{D}:\, |w|\leq r_1\}} |f_n(w)| \\
&+ K \sup_{\{z\in\mathbb{D}:\, |\phi(z)|>r_1\}} \frac{|\phi'(z)h(z)|}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^\beta}\right)(1-|\phi(z)|^2)^{\frac{2+q}{p}}} \\
&+ K \sup_{\{z\in\mathbb{D}:\, |\phi(z)|>r_1\}} \frac{|h'(z)|}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^\beta}\right)} \\
&< |f_n(\phi(0))h(0)| + \hat{L}_1 \sup_{\{w\in\mathbb{D}:\, |w|\leq r_1\}} |f_n'(w)| \\
&\quad + L_1 \sup_{\{w\in\mathbb{D}:\, |w|\leq r_1\}} |f_n(w)| + K\epsilon. \qquad (4.19)
\end{aligned}$$

(ii) For $\frac{2+q}{p} = 1$, in light of (4.13) and (4.15), then for every $\epsilon > 0$, there is $0 < r_2 < 1$ such that

$$\begin{aligned}
&\frac{|\phi'(z)h(z)|}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^\beta}\right)(1-|\phi(z)|^2)^{\frac{2+q}{p}}} < \frac{\epsilon}{2}, \\
&\frac{|h'(z)|\log\frac{2}{1-|\phi(z)|^2}}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^\beta}\right)} < \frac{\epsilon}{2},
\end{aligned}$$

for $|\phi(z)| > r_2$. Similarly, it follows that

$$\begin{aligned}
\|P_\phi^h f_n\|_{\mathcal{Z}_\beta^\varphi} &\leq |f_n(\phi(0))h(0)| + \hat{L}_1 \sup_{\{w\in\mathbb{D}:\, |w|\leq r_2\}} |f_n'(w)| + L_1 \sup_{\{w\in\mathbb{D}:\, |w|\leq r_2\}} |f_n(w)| \\
&+ K \sup_{\{z\in\mathbb{D}:\, |\phi(z)|>r_2\}} \frac{|\phi'(z)h(z)|}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^\beta}\right)(1-|\phi(z)|^2)^{\frac{2+q}{p}}} \\
&+ K \sup_{\{z\in\mathbb{D}:\, |\phi(z)|>r_2\}} \frac{|h'(z)|\log\frac{2}{1-|\phi(z)|^2}}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^\beta}\right)} \\
&< |f_n(\phi(0))h(0)| + \hat{L}_1 \sup_{\{w\in\mathbb{D}:\, |w|\leq r_2\}} |f_n'(w)| \\
&\quad + L_1 \sup_{\{w\in\mathbb{D}:\, |w|\leq r_2\}} |f_n(w)| + K\epsilon. \qquad (4.20)
\end{aligned}$$

(iii) For $\frac{2+q}{p} > 1$, in view of (4.13) and (4.16), it yields that for every $\epsilon > 0$, there exists $0 < r_3 < 1$ fulfilling

$$\begin{aligned}
&\frac{|\phi'(z)h(z)|}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^\beta}\right)(1-|\phi(z)|^2)^{\frac{2+q}{p}}} < \frac{\epsilon}{2}, \\
&\frac{|h'(z)|}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^\beta}\right)(1-|\phi(z)|^2)^{\frac{2+q}{p}-1}} < \frac{\epsilon}{2},
\end{aligned}$$

for $|\phi(z)| > r_3$. It can analogously be shown that

$$\begin{aligned}
\|P^h_\phi f_n\|_{\mathcal{Z}^\varphi_\beta} &\leq |f_n(\phi(0))h(0)| + \hat{L}_1 \sup_{\{w\in\mathbb{D}:\,|w|\leq r_3\}} |f'_n(w)| + L_1 \sup_{\{w\in\mathbb{D}:\,|w|\leq r_3\}} |f_n(w)| \\
&+ K \sup_{\{z\in\mathbb{D}:\,|\phi(z)|>r_3\}} \frac{|\phi'(z)h(z)|}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^\beta}\right)(1-|\phi(z)|^2)^{\frac{2+q}{p}}} \\
&+ K \sup_{\{z\in\mathbb{D}:\,|\phi(z)|>r_3\}} \frac{|h'(z)|}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^\beta}\right)(1-|\phi(z)|^2)^{\frac{2+q}{p}-1}} \\
&< |f_n(\phi(0))h(0)| + \hat{L}_1 \sup_{\{w\in\mathbb{D}:\,|w|\leq r_3\}} |f'_n(w)| \\
&\quad + L_1 \sup_{\{w\in\mathbb{D}:\,|w|\leq r_3\}} |f_n(w)| + K\epsilon. \qquad (4.21)
\end{aligned}$$

Summarizing (4.19)–(4.21), by Cauchy estimate, we conclude that

$$\lim_{n\to\infty} \|P^h_\phi f_n\|_{\mathcal{Z}^\varphi_\beta} \leq K\epsilon. \qquad (4.22)$$

Since ϵ is arbitrary, by (4.22) and Lemma 2.3, the operator $P^h_\phi : F(p,q,s) \to \mathcal{Z}^\varphi_\beta$ is compact.

Necessity. Assume $P^h_\phi : F(p,q,s) \to \mathcal{Z}^\varphi_\beta$ is compact. The boundedness clearly follows. Let $\{z_k\}_{k\in\mathbb{N}}$ be a sequence in $\mathbb{D}$ such that $\lim_{k\to\infty} |\phi(z_k)| = 1$. Then set

$$\begin{aligned}
\widehat{f_{0,k}}(z) &= \frac{(1-|\phi(z_k)|^2)^{1+\frac{2+q}{p}}}{(1-z\overline{\phi(z_k)})^{2\frac{2+q}{p}}} - \frac{1-|\phi(z_k)|^2}{(1-z\overline{\phi(z_k)})^{\frac{2+q}{p}}}; \\
\widehat{f_{1,k}}(z) &= \frac{(1-|\phi(z_k)|^2)^{2\frac{2+q}{p}}}{(1-z\overline{\phi(z_k)})^{2\frac{2+q}{p}}} - 2\frac{(1-|\phi(z_k)|^2)^{\frac{2+q}{p}}}{(1-z\overline{\phi(z_k)})^{\frac{2+q}{p}}}, \quad \text{for } 0 < \frac{2+q}{p} < 1; \\
\widehat{f_{2,k}}(z) &= \left(\log \frac{2}{1-z\overline{\phi(z_k)}}\right)^2 \left(\log \frac{2}{1-|\phi(z_k)|^2}\right)^{-1}, \quad \text{for } \frac{2+q}{p} = 1 \text{ and } s > 2; \\
\widehat{f_{3,k}}(z) &= \frac{(1-|\phi(z_k)|^2)^{1+\frac{2+q}{p}}}{(1-z\overline{\phi(z_k)})^{2\frac{2+q}{p}}} - 2\frac{1-|\phi(z_k)|^2}{(1-z\overline{\phi(z_k)})^{\frac{2+q}{p}}}, \quad \text{for } \frac{2+q}{p} > 1.
\end{aligned}$$

Similar to the proof in Theorem 3.3, the desired equations (4.13)–(4.16) follow. This ends the proof. □

5. Some corollaries

In this section, we present some corollaries without proof, which can seen as special cases in the above two sections.

(1) Let $\phi = id$ the identity map in P^h_ϕ, then $P^h_{id} = T^h$, combining with Theorems 3.1, 3.3, 4.1 and 4.2, four corollaries about the boundedness and compactness of $T^h : F(p,q,s) \to \mathcal{B}^\varphi_\alpha$ (or $\mathcal{Z}^\varphi_\beta$) follow.

Corollary 5.1. *Let $\varphi : [0,\infty) \to [0,\infty)$ be an $\mathcal{N}$-function and $h \in H(\mathbb{D})$. Then the operator $T^h : F(p,q,s) \to \mathcal{B}^\varphi_\alpha$ is* *bounded if and only if*

$$\sup_{z\in\mathbb{D}} \frac{|h(z)|}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^{\alpha}}\right)} < \infty, \ \textit{for} \ \ 0 < \frac{2+q}{p} < 1;$$
$$\sup_{z\in\mathbb{D}} \frac{|h(z)| \log \frac{2}{1-|z|^2}}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^{\alpha}}\right)} < \infty, \ \ \textit{for} \ \ \frac{2+q}{p} = 1 \ \ \textit{and} \ \ s > 2;$$
$$\sup_{z\in\mathbb{D}} \frac{|h(z)|}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^{\alpha}}\right)(1-|z|^2)^{\frac{2+q}{p}-1}} < \infty, \ \ \textit{for} \ \ \frac{2+q}{p} > 1.$$

Corollary 5.2. *Let $\varphi : [0,\infty) \to [0,\infty)$ be an $\mathcal{N}$-function and $h \in H(\mathbb{D})$. Then the operator $T^h : F(p,q,s) \to \mathcal{B}^{\varphi}_{\alpha}$ is compact if and only if $T^h : F(p,q,s) \to \mathcal{B}^{\varphi}_{\alpha}$ is bounded and*

$$\lim_{|z|\to 1} \frac{|h(z)|}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^{\alpha}}\right)} = 0, \ \textit{for} \ \ 0 < \frac{2+q}{p} < 1;$$
$$\lim_{|z|\to 1} \frac{|h(z)| \log \frac{2}{1-|z|^2}}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^{\alpha}}\right)} = 0, \ \ \textit{for} \ \ \frac{2+q}{p} = 1 \ \ \textit{and} \ \ s > 2;$$
$$\lim_{|z|\to 1} \frac{|h(z)|}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^{\alpha}}\right)(1-|z|^2)^{\frac{2+q}{p}-1}} = 0, \ \ \textit{for} \ \ \frac{2+q}{p} > 1.$$

Corollary 5.3. *Let $\varphi : [0,\infty) \to [0,\infty)$ be an $\mathcal{N}$-function and $h \in H(\mathbb{D})$. Then the operator $T^h : F(p,q,s) \to \mathcal{Z}^{\varphi}_{\beta}$ is bounded if and only if*

$$\sup_{z\in\mathbb{D}} \frac{|h(z)|}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^{\beta}}\right)(1-|z|^2)^{\frac{2+q}{p}}} < \infty,$$

and

$$\sup_{z\in\mathbb{D}} \frac{|h'(z)|}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^{\beta}}\right)} < \infty, \ \ \textit{for} \ 0 < \frac{2+q}{p} < 1;$$
$$\sup_{z\in\mathbb{D}} \frac{|h'(z)| \log \frac{2}{1-|z|^2}}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^{\beta}}\right)} < \infty, \ \ \textit{for} \ \frac{2+q}{p} = 1 \ \ \textit{and} \ \ s > 2;$$
$$\sup_{z\in\mathbb{D}} \frac{|h'(z)|}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^{\beta}}\right)(1-|z|^2)^{\frac{2+q}{p}-1}} < \infty, \ \ \textit{for} \ \frac{2+q}{p} > 1.$$

Corollary 5.4. *Let $\varphi : [0,\infty) \to [0,\infty)$ be an $\mathcal{N}$-function and $h \in H(\mathbb{D})$. Then the operator $T^h : F(p,q,s) \to \mathcal{Z}^{\varphi}_{\beta}$ is* *compact if and only if* $T^h : F(p,q,s) \to \mathcal{Z}^{\varphi}_{\beta}$ *is bounded and*

$$\lim_{|z|\to 1} \frac{|h(z)|}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^{\beta}}\right)(1-|z|^2)^{\frac{2+q}{p}}} = 0,$$

and

$$\lim_{|z|\to 1} \frac{|h'(z)|}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^{\beta}}\right)} = 0, \ \ \textit{for } 0 < \frac{2+q}{p} < 1;$$

$$\lim_{|z|\to 1} \frac{|h'(z)| \log \frac{2}{1-|z|^2}}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^{\beta}}\right)} = 0, \ \ \textit{for } \frac{2+q}{p} = 1 \ \ \textit{and} \ \ s > 2;$$

$$\lim_{|z|\to 1} \frac{|h'(z)|}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^{\beta}}\right)(1-|z|^2)^{\frac{2+q}{p}-1}} = 0, \ \ \textit{for } \frac{2+q}{p} > 1.$$

(2) Let $h = k' \in H(\mathbb{D})$ in P^h_{ϕ}, then $P^{k'}_{\phi} = T_k C_{\phi}$, which together with Theorems 3.1, 3.3, 4.1 and 4.2 imply some corollaries for the boundedness and compactness of $T_k C_{\phi} : F(p,q,s) \to \mathcal{B}^{\varphi}_{\alpha}$ (or $\mathcal{Z}^{\varphi}_{\beta}$).

Corollary 5.5. *Let* $\varphi : [0,\infty) \to [0,\infty)$ *be an* $\mathcal{N}$*-function,* $\phi \in S(\mathbb{D})$ *and* $k \in H(\mathbb{D})$*. Then the operator* $T_k C_{\phi} : F(p,q,s) \to \mathcal{B}^{\varphi}_{\alpha}$ *is bounded if and only if*

$$\sup_{z\in\mathbb{D}} \frac{|k'(z)|}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^{\alpha}}\right)} < \infty, \ \ \textit{for } \ 0 < \frac{2+q}{p} < 1;$$

$$\sup_{z\in\mathbb{D}} \frac{|k'(z)| \log \frac{2}{1-|\phi(z)|^2}}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^{\alpha}}\right)} < \infty, \ \ \textit{for } \ \frac{2+q}{p} = 1 \ \ \textit{and} \ \ s > 2;$$

$$\sup_{z\in\mathbb{D}} \frac{|k'(z)|}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^{\alpha}}\right)(1-|\phi(z)|^2)^{\frac{2+q}{p}-1}} < \infty, \ \ \textit{for } \ \frac{2+q}{p} > 1.$$

Corollary 5.6. *Let* $\varphi : [0,\infty) \to [0,\infty)$ *be an* $\mathcal{N}$*-function,* $\phi \in S(\mathbb{D})$ *and* $k \in H(\mathbb{D})$*. Then the operator* $T_k C_{\phi} : F(p,q,s) \to \mathcal{B}^{\varphi}_{\alpha}$ *is compact if and only if* $T_k C_{\phi} : F(p,q,s) \to \mathcal{B}^{\varphi}_{\alpha}$ *is bounded and*

$$\lim_{|\phi(z)|\to 1} \frac{|k'(z)|}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^{\alpha}}\right)} = 0, \ \ \textit{for } \ 0 < \frac{2+q}{p} < 1;$$

$$\lim_{|\phi(z)|\to 1} \frac{|k'(z)| \log \frac{2}{1-|\phi(z)|^2}}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^{\alpha}}\right)} = 0, \ \ \textit{for } \ \frac{2+q}{p} = 1 \ \ \textit{and} \ \ s > 2;$$

$$\lim_{|\phi(z)|\to 1} \frac{|k'(z)|}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^{\alpha}}\right)(1-|\phi(z)|^2)^{\frac{2+q}{p}-1}} = 0, \ \ \textit{for } \ \frac{2+q}{p} > 1.$$

Corollary 5.7. *Let* $\varphi : [0,\infty) \to [0,\infty)$ *be an* $\mathcal{N}$*-function,* $\phi \in S(\mathbb{D})$ *and* $k \in H(\mathbb{D})$*. Then the operator* $T_k C_{\phi}$:

$F(p,q,s) \to \mathcal{Z}_\beta^\varphi$ is bounded if and only if

$$\sup_{z\in\mathbb{D}} \frac{|\phi'(z)k'(z)|}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^\beta}\right)(1-|\phi(z)|^2)^{\frac{2+q}{p}}} < \infty,$$

and

$$\sup_{z\in\mathbb{D}} \frac{|k''(z)|}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^\beta}\right)} < \infty, \ \ \text{for } 0 < \frac{2+q}{p} < 1;$$

$$\sup_{z\in\mathbb{D}} \frac{|k''(z)|\log\frac{2}{1-|\phi(z)|^2}}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^\beta}\right)} < \infty, \ \ \text{for } \frac{2+q}{p} = 1 \ \text{ and } \ s > 2;$$

$$\sup_{z\in\mathbb{D}} \frac{|k''(z)|}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^\beta}\right)(1-|\phi(z)|^2)^{\frac{2+q}{p}-1}} < \infty, \ \ \text{for } \frac{2+q}{p} > 1.$$

Corollary 5.8. *Let $\varphi : [0,\infty) \to [0,\infty)$ be an $\mathcal{N}$-function, $\phi \in S(\mathbb{D})$ and $k \in H(\mathbb{D})$. Then the operator T_kC_ϕ : $F(p,q,s) \to \mathcal{Z}_\beta^\varphi$ is compact if and only if $T_kC_\phi : F(p,q,s) \to \mathcal{Z}_\beta^\varphi$ is bounded and*

$$\lim_{|\phi(z)|\to 1} \frac{|\phi'(z)k'(z)|}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^\beta}\right)(1-|\phi(z)|^2)^{\frac{2+q}{p}}} = 0,$$

and

$$\lim_{|\phi(z)|\to 1} \frac{|k''(z)|}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^\beta}\right)} = 0, \ \ \text{for } 0 < \frac{2+q}{p} < 1;$$

$$\lim_{|\phi(z)|\to 1} \frac{|k''(z)|\log\frac{2}{1-|\phi(z)|^2}}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^\beta}\right)} = 0, \ \ \text{for } \frac{2+q}{p} = 1 \ \text{ and } \ s > 2;$$

$$\lim_{|\phi(z)|\to 1} \frac{|k''(z)|}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^\beta}\right)(1-|\phi(z)|^2)^{\frac{2+q}{p}-1}} = 0, \ \ \text{for } \frac{2+q}{p} > 1.$$

(3) Let $\phi = id$ the identity map in T_kC_ϕ, then $T_kC_{id} = T_k$, which together with Corollaries 5.5-5.8 imply some corollaries for the operator $T_k : F(p,q,s) \to \mathcal{B}_\alpha^\varphi$ (or $\mathcal{Z}_\beta^\varphi$).

Corollary 5.9. *Let $\varphi : [0,\infty) \to [0,\infty)$ be an $\mathcal{N}$-function and $k \in H(\mathbb{D})$. Then the operator $T_k : F(p,q,s) \to \mathcal{B}_\alpha^\varphi$ is bounded if and only if*

$$\sup_{z\in\mathbb{D}} \frac{|k'(z)|}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^\alpha}\right)} < \infty, \ \ \text{for } \ 0 < \frac{2+q}{p} < 1;$$

$$\sup_{z\in\mathbb{D}} \frac{|k'(z)|\log\frac{2}{1-|z|^2}}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^\alpha}\right)} < \infty, \ \ \text{for } \ \frac{2+q}{p} = 1 \ \text{ and } \ s > 2;$$

$$\sup_{z\in\mathbb{D}} \frac{|k'(z)|}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^\alpha}\right)(1-|z|^2)^{\frac{2+q}{p}-1}} < \infty, \ \ \text{for } \ \frac{2+q}{p} > 1.$$

Corollary 5.10. *Let $\varphi : [0,\infty) \to [0,\infty)$ be an $\mathcal{N}$-function and $k \in H(\mathbb{D})$. Then the operator $T_k : F(p,q,s) \to \mathcal{B}_\alpha^\varphi$ is* *compact if and only if $T_k : F(p,q,s) \to \mathcal{B}_\alpha^\varphi$ is bounded and*

$$\lim_{|z|\to 1} \frac{|k'(z)|}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^\alpha}\right)} = 0, \text{ for } 0 < \frac{2+q}{p} < 1;$$

$$\lim_{|z|\to 1} \frac{|k'(z)| \log \frac{2}{1-|z|^2}}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^\alpha}\right)} = 0, \text{ for } \frac{2+q}{p} = 1 \text{ and } s > 2;$$

$$\lim_{|z|\to 1} \frac{|k'(z)|}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^\alpha}\right)(1-|z|^2)^{\frac{2+q}{p}-1}} = 0, \text{ for } \frac{2+q}{p} > 1.$$

Corollary 5.11. *Let $\varphi : [0,\infty) \to [0,\infty)$ be an $\mathcal{N}$-function and $k \in H(\mathbb{D})$. Then the operator $T_k : F(p,q,s) \to \mathcal{Z}_\beta^\varphi$ is bounded if and only if*

$$\sup_{z\in\mathbb{D}} \frac{|k'(z)|}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^\beta}\right)(1-|z|^2)^{\frac{2+q}{p}}} < \infty,$$

and

$$\sup_{z\in\mathbb{D}} \frac{|k''(z)|}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^\beta}\right)} < \infty, \text{ for } 0 < \frac{2+q}{p} < 1;$$

$$\sup_{z\in\mathbb{D}} \frac{|k''(z)| \log \frac{2}{1-|z|^2}}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^\beta}\right)} < \infty, \text{ for } \frac{2+q}{p} = 1 \text{ and } s > 2;$$

$$\sup_{z\in\mathbb{D}} \frac{|k''(z)|}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^\beta}\right)(1-|z|^2)^{\frac{2+q}{p}-1}} < \infty, \text{ for } \frac{2+q}{p} > 1.$$

Corollary 5.12. *Let $\varphi : [0,\infty) \to [0,\infty)$ be an $\mathcal{N}$-function and $k \in H(\mathbb{D})$. Then the operator $T_k : F(p,q,s) \to \mathcal{Z}_\beta^\varphi$ is compact if and only if $T_k : F(p,q,s) \to \mathcal{Z}_\beta^\varphi$ is bounded and*

$$\lim_{|z|\to 1} \frac{|k'(z)|}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^\beta}\right)(1-|z|^2)^{\frac{2+q}{p}}} = 0,$$

and

$$\lim_{|z|\to 1} \frac{|k''(z)|}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^\beta}\right)} = 0, \text{ for } 0 < \frac{2+q}{p} < 1;$$

$$\lim_{|z|\to 1} \frac{|k''(z)| \log \frac{2}{1-|z|^2}}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^\beta}\right)} = 0, \text{ for } \frac{2+q}{p} = 1 \text{ and } s > 2;$$

$$\lim_{|z|\to 1} \frac{|k''(z)|}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^\beta}\right)(1-|z|^2)^{\frac{2+q}{p}-1}} = 0, \text{ for } \frac{2+q}{p} > 1.$$

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