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The Convex Properties and Norm Bounds for Operator Matrices Involving Contractions

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Abstract. In this note, the norm bounds and convex properties of special operator matrices $\widetilde{H}_n^{(m)}$ and $\widetilde{S}_n^{(m)}$ are investigated. When Hilbert space $\mathcal{K}$ is infinite dimensional, we firstly show that $\widetilde{H}_n^{(m)} = \widetilde{H}_{n+1}^{(m)}$ and $\widetilde{S}_n^{(m)} = \widetilde{S}_{n+1}^{(m)}$, for $m, n = 1, 2, \cdots$. Then we get that $\widetilde{H}_n^{(m)}$ is a convex and compact set in the ω^* topology. Moreover, some norm bounds for $\widetilde{H}_n^{(m)}$ and $\widetilde{S}_n^{(m)}$ are given.

1. Introduction

Let $\mathcal{H}$ be an infinite dimensional complex Hilbert space and $\mathcal{B}(\mathcal{H})$ be the set of all bounded linear operators on $\mathcal{H}$. For an operator $A \in \mathcal{B}(\mathcal{H})$, the adjoint of A is denoted by A^*. We write $A \in \mathcal{B}(\mathcal{H})^+$, if A is a positive operator, meaning $\langle Ax, x\rangle \geq 0$ for all $x \in \mathcal{H}$. As usual, denote by $\mathcal{R}(A)$, $\overline{\mathcal{R}(A)}$ and $|A| := (A^*A)^{\frac{1}{2}}$, the range of A, the closed linear span of $\mathcal{R}(A)$, and the absolute value of A, respectively. Also, $P(\mathcal{H})$ is the set of all orthogonal projections on $\mathcal{H}$ and $x \otimes y$ denotes the one rank linear operator $x \otimes y(z) := \langle z, y\rangle x$, $(z \in \mathcal{H})$, where $x \in \mathcal{H}$ and $y \in \mathcal{K}$. An operator A is called a contraction (strict contraction) if $\|A\| \leq 1$ ($\|A\| < 1$). Let $\mathcal{K}$ be Hilbert space (finite or infinite dimensional) and $\mathcal{K}^n := \underbrace{\mathcal{K} \oplus \mathcal{K} \oplus \cdots \oplus \mathcal{K}}_{n}$. For convenience, we write $A \in \mathcal{B}_1(\mathcal{H}, \mathcal{K})$ if and only if $\|A\| \leq 1$. For $A_{ij} \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, where $i, j = 1, 2, \cdots m$, we denote $m \times m$ operator matrices $(A_{ij})_{i,j=1}^m \in \mathcal{B}(\mathcal{H}^m, \mathcal{K}^m)$ by

$$(A_{ij})_{i,j=1}^m := \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1m} \\ \vdots & \vdots & \cdots & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mm} \end{pmatrix}.$$

For $m, n = 1, 2 \cdots$, we define the following operator matrices of $\widetilde{H}_n^{(m)}$ and $\widetilde{S}_n^{(m)}$ by

$$\widetilde{H}_n^{(m)} := \{(I - A_j^* A_i)_{i,j=1}^m : A_1, \cdots A_m \in \mathcal{B}(\mathcal{H}, \mathcal{K}^n),\ \|\sum_{i=1}^m A_i A_i^*\| \leq 1\}, \tag{1}$$

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and

$$\widetilde{S}_n^{(m)} := \{(I - A_j^*A_i)_{i,j=1}^m : A_1, \cdots A_m \in \mathcal{B}(\mathcal{H}, \mathcal{K}^n),\ (A_i^*A_j)_{i,j=1}^m \in P(\mathcal{H}^n)\}. \tag{2}$$

Clearly, for $n = 1, 2, \cdots$,

$$\widetilde{H}_n^{(m)} \subseteq \mathcal{B}(\mathcal{H}^m) \quad \text{and} \quad \widetilde{S}_n^{(m)} \subseteq \mathcal{B}(\mathcal{H}^m).$$

Also, set

$$\widetilde{H}^{(m)} = \bigcup_{n=1}^{\infty} \widetilde{H}_n^{(m)} \quad \text{and} \quad \widetilde{S}^{(m)} = \bigcup_{n=1}^{\infty} \widetilde{S}_n^{(m)}. \tag{3}$$

It is well known that contractions and their dilations are important and useful for operator and matrix theory. Many interesting results for contractions and their applications have been obtained in [2, 5, 10]. One speciality of our definition $\widetilde{H}_n^{(m)}$ and $\widetilde{S}_n^{(m)}$ is involved contractions. Another is that $\widetilde{H}_n^{(m)}$ is related to the Hua-type operator matrices ([4]). For strict contractions A_1 and A_2, 2×2 operator matrices $H(A_1, A_2) = ((I - A_j^*A_i)^{-1})_{i,j=1}^2$ and its cousin $G(A_1, A_2) = ((I - A_iA_j^*)^{-1})_{i,j=1}^2$ are well defined and they are called Hua-type operator matrices ([4]).

In more recent papers [7, 8, 11], the positivity and the norm estimation of Hua type operator matrices are studied. In particular, [8, Theorem 2.2 and Theorem 2.3] gives the equations

$$min\{\|H(A_1, A_2)\| : \|A_1\| < 1, \|A_2\| < 1\} = 2$$

and

$$min\{\|G(A_1, A_2)\| : \|A_1\| < 1, \|A_2\| < 1\} = 2.$$

The above two equations might hint naturally to two problems

$$sup\{\|\begin{pmatrix} I - A^*A & I - B^*A \\ I - A^*B & I - B^*B \end{pmatrix}\| : A, B \in \mathcal{B}_1(\mathcal{H})\} =? \tag{4}$$

and

$$sup\{\|\begin{pmatrix} I - A^*A & I - A^*B \\ I - B^*A & I - B^*B \end{pmatrix}\| : A, B \in \mathcal{B}_1(\mathcal{H})\} =? \tag{5}$$

Indeed, the second problem is easily characterized (Corollary 2.2). However, the first problem is difficult. The following Example 2.1 show that the upper bound is different between (4) and (5). More generally, for $A_1, \cdots A_m \in \mathcal{B}_1(\mathcal{H}, \mathcal{K})$, how to characterize the norm bound or other properties of $m \times m$ operator matrices $(I - A_j^*A_i)_{i,j=1}^m$ and $(I - A_i^*A_j)_{i,j=1}^m$?

In this note, we mainly investigate some convex properties and norm bound for operator matrices $\widetilde{H}_n^{(m)}$ and $\widetilde{S}_n^{(m)}$, which are based on $m \times m$ operator matrices $(I - A_j^*A_i)_{i,j=1}^m$. When $\mathcal{K}$ be an infinite dimensional Hilbert space, we firstly show that $\widetilde{H}_n^{(m)} = \widetilde{H}_{n+1}^{(m)} = \widetilde{H}^{(m)}$, for $n, m = 1, 2, \cdots$. Then we get that $\widetilde{H}^{(m)} \subseteq B(\mathcal{H}^m)$ is a convex and compact set in the ω^* topology. Moreover, some norm estimations for $\widetilde{H}^{(m)}$ and $\widetilde{S}^{(m)}$ are given.

2. Main results

In this note, we always assume that $\mathcal{H}$ is a separable complex Hilbert space. To show our main results, we need the following lemmas.

Lemma 2.1. For any operator matrix $\widetilde{A} = (A_{ij})_{i,j=1}^m$, where $A_{ij} \in B(\mathcal{H})$, for $i, j = 1, 2 \cdots m$, we have

$$\|\widetilde{A}\| \le \|(\|A_{ij}\|)_{i,j=1}^m\| \le \sum_{k=0}^{m-1} (\max_{1 \le i \le m} \{\|A_{i[i+k]}\|\}),$$

where $[i+k] = \begin{cases} m, & i+k \mid m; \\ i+k \ mod\ m, & i+k \nmid m. \end{cases}$

Proof. The first inequality is a direct calculation ([5, Theorem 1.1]).

Clearly,

$$\|(\|A_{ij}\|)_{i,j=1}^m\| = \|\begin{pmatrix} \|A_{11}\| & 0 & \cdots & 0 \\ 0 & \|A_{22}\| & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \|A_{mm}\| \end{pmatrix} + \begin{pmatrix} 0 & \|A_{12}\| & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \|A_{m-1m}\| \\ A_{m1} & 0 & \cdots & 0 \end{pmatrix} + \cdots + \begin{pmatrix} 0 & \cdots & 0 & \|A_{1m}\| \\ \|A_{21}\| & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \|A_{mm-1}\| & 0 \end{pmatrix}\|$$

and

$$\|\begin{pmatrix} \|A_{11}\| & 0 & \cdots & 0 \\ 0 & \|A_{22}\| & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \|A_{mm}\| \end{pmatrix}\| \leq \max_{1\leq i\leq m} \|A_{ii}\| = \max_{1\leq i\leq m}\{\|A_{i[i+k]}\|, \quad \text{where } k = 0.$$

Also,

$$\|\begin{pmatrix} 0 & \|A_{12}\| & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \|A_{m-1m}\| \\ \|A_{m1}\| & 0 & \cdots & 0 \end{pmatrix}\| = \max_{1\leq i\leq m}\{\|A_{i[i+k]}\|, \quad \text{for } k = 1,$$

and

$$\|\begin{pmatrix} 0 & \cdots & 0 & \|A_{1m}\| \\ \|A_{21}\| & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \|A_{mm-1}\| & 0 \end{pmatrix}\| = \max_{1\leq i\leq m}\{\|A_{i[i+k]}\|, \quad \text{for } k = m-1$$

So

$$\|\widetilde{A}\| \leq \|(\|A_{ij}\|)_{i,j=1}^m\| \leq \sum_{k=0}^{m-1}(\max_{1\leq i\leq m}\{\|A_{i[i+k]}\|\|\}).$$

□

For $n \times n$ operator matrices, some interesting results such as the estimation of operator norm and numerical radius have been obtained in [1, 5] and their references. The following lemma which gives a characterization of $n \times n$ positive operator matrices seems to be known. However, we can't find the references. As a corollary of this lemma, we might get a simpler proof of [5, Theorem 2.7](see Remark 2.1 (b)).

Lemma 2.2. Let $\mathcal{H}_i$ be Hilbert spaces for $i = 1, 2 \cdots m$ and $\widetilde{A} \in \mathcal{B}(\oplus_{i=1}^m \mathcal{H}_i)^+$. Then there exist Hilbert space $\widetilde{\mathcal{K}}$ and operators $A_i \in \mathcal{B}(\mathcal{H}_i, \widetilde{\mathcal{K}})$ such that

$$\widetilde{A} = \begin{pmatrix} A_1^*A_1 & A_1^*A_2 & \cdots & A_1^*A_m \\ \vdots & \vdots & \cdots & \vdots \\ A_m^*A_1 & A_m^*A_2 & \cdots & A_m^*A_m \end{pmatrix}.$$

Proof. As $\widetilde{A} \in \mathcal{B}(\oplus_{i=1}^m \mathcal{H}_i)^+$, then we conclude $\widetilde{A} = \widetilde{C}^*\widetilde{C}$ for some $\widetilde{C} \in \mathcal{B}(\oplus_{i=1}^m \mathcal{H}_i)$. Without loss of generality, we suppose that

$$\widetilde{C} = \begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1m} \\ \vdots & \ddots & \cdots & \vdots \\ C_{m1} & C_{m2} & \cdots & C_{mm} \end{pmatrix}.$$

Denote $\widetilde{\mathcal{K}} = \oplus_{i=1}^{m}\mathcal{H}_i$ and $A_i = \begin{pmatrix} C_{1i} \\ C_{2i} \\ \vdots \\ \vdots \\ C_{mi} \end{pmatrix} \in \mathcal{B}(\mathcal{H}_i, \widetilde{\mathcal{K}})$, for $i = 1, 2, \cdots m$. Thus

$$\widetilde{A} = \widetilde{C}^*\widetilde{C} = \begin{pmatrix} A_1^* \\ A_2^* \\ \vdots \\ A_m^* \end{pmatrix} \begin{pmatrix} A_1 & A_2 & \cdots & A_m \end{pmatrix} = \begin{pmatrix} A_1^*A_1 & A_1^*A_2 & \cdots & A_1^*A_m \\ \vdots & \vdots & \cdots & \vdots \\ A_m^*A_1 & A_m^*A_2 & \cdots & A_m^*A_m \end{pmatrix}.$$

□

Lemma 2.3. (*a*) Let $A_\tau \in \mathcal{B}_1(\mathcal{H})$ be a net of contractions, where τ is in a directed index set. If $\omega^* \lim_\tau A_\tau = A$, then $A \in \mathcal{B}_1(\mathcal{H})$.

(*b*) Let $B_\tau \in \mathcal{B}(\mathcal{H})$, $C, D \in \mathcal{B}(\mathcal{H})$. If $\omega^* \lim_\tau B_\tau = B$, then

$$\omega^* \lim_\tau B_\tau^* = B^* \quad \text{and} \quad \omega^* \lim_\tau CB_\tau D = CBD.$$

Proof. (a) As $\omega^* \lim_\tau A_\tau = A$, then for any $X \in T(\mathcal{H})$, we have $\lim_\tau tr(A_\tau X) = tr(AX)$. It is clear that for any index τ

$$|tr(A_\tau X)| \leq \|A_\tau\|\|X\|_1 \leq \|X\|_1.$$

So for all $X \in T(\mathcal{H})$, we get that $|tr(AX)| \leq \|X\|_1$, which implies $\|A\| \leq 1$.

(b) For any $X \in T(\mathcal{H})$, we have

$$|tr[(B_\tau^* - B^*)X]| = |tr[(B_\tau - B)X^*]| \longrightarrow 0$$

and

$$tr([CB_\tau D - CBD]X) = tr[C(B_\tau - B)DX] = tr[(B_\tau - B)DXC] \longrightarrow 0,$$

So $\omega^* \lim_\tau B_\tau^* = B^*$ and $\omega^* \lim_\tau CB_\tau D = CBD$. □

Lemma 2.4. ([3]) The set of all positive contractions ($\mathcal{B}(\mathcal{H})^+ \cap \mathcal{B}_1(\mathcal{H})$) is convex. And the extreme points of $\mathcal{B}(\mathcal{H})^+ \cap \mathcal{B}_1(\mathcal{H})$ is $P(\mathcal{H})$.

Lemma 2.5. If $M \in \mathcal{B}(\mathcal{H})^+ \cap \mathcal{B}_1(\mathcal{H})$, then $\|M - M^2\| \leq \frac{1}{4}$.

Proof. Using the spectral resolution of M ([6, Theorem 5.2.2]), we conclude that $M = \int_0^1 \lambda dE_\lambda$, where E_λ satisfy $\bigwedge_{\lambda \in [0,1]} E_\lambda = 0$ and $\bigvee_{\lambda \in [0,1]} E_\lambda = I$. Thus $M^2 = \int_0^1 \lambda^2 dE_\lambda$, which yields

$$M - M^2 = \int_0^1 (\lambda - \lambda^2) dE_\lambda.$$

For any unit vector $x \in \mathcal{H}$, we have

$$\langle (M - M^2)x, x \rangle = \int_0^1 (\lambda - \lambda^2) d(\langle E_\lambda x, x \rangle) \leq max_{\lambda \in [0,1]}(\lambda - \lambda^2) = \frac{1}{4},$$

which implies

$$\|M - M^2\| = \sup\{\langle (M - M^2)x, x \rangle : \|x\| = 1\} \leq \frac{1}{4}.$$

□

Lemma 2.6. $sup\{\|\widetilde{A}\| : \widetilde{A} \in \widetilde{S}^{(2)}\} \leq \frac{5}{2}$.

Proof. Let $\widetilde{A} \in \widetilde{S}^{(2)}$. Without loss of generality, we assume that

$$\widetilde{A} = \begin{pmatrix} I - A^*A & I - B^*A \\ I - A^*B & I - B^*B \end{pmatrix},$$

where $\begin{pmatrix} A^*A & A^*B \\ B^*A & B^*B \end{pmatrix} \in P(\mathcal{H} \oplus \mathcal{H})$ and $A, B \in \mathcal{B}(\mathcal{H}, \mathcal{K}^n)$. Then

$$AA^* + BB^* = \begin{pmatrix} A & B \end{pmatrix} \begin{pmatrix} A^* \\ B^* \end{pmatrix} \in P(\mathcal{K}^n).$$

Denoting $P := AA^* + BB^*$, then P is an orthogonal projection, so we have

$$AA^* = \begin{pmatrix} M & 0 \\ 0 & 0 \end{pmatrix} : \mathcal{R}(P) \oplus \mathcal{R}(P)^\perp \ \text{ and } \ BB^* = \begin{pmatrix} I - M & 0 \\ 0 & 0 \end{pmatrix} : \mathcal{R}(P) \oplus \mathcal{R}(P)^\perp,$$

where $M \in \mathcal{B}(\mathcal{R}(P))^+$ and $\|M\| \leq 1$.

Using the polar decomposition theorem, we conclude that $A = (AA^*)^{\frac{1}{2}} V$, where V is a partial isometry from initial space $\overline{\mathcal{R}(A^*)}$ onto final space $\overline{\mathcal{R}(A)}$. Thus

$$V = \begin{pmatrix} V_1 \\ 0 \end{pmatrix} : H \to \mathcal{R}(P) \oplus \mathcal{R}(P)^\perp,$$

since $\overline{\mathcal{R}(A)} = \overline{\mathcal{R}(AA^*)} \subseteq \mathcal{R}(P)$, which implies

$$A = (AA^*)^{\frac{1}{2}} V = \begin{pmatrix} M^{\frac{1}{2}} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} V_1 \\ 0 \end{pmatrix} = \begin{pmatrix} M^{\frac{1}{2}} V_1 \\ 0 \end{pmatrix}.$$

Similarly,

$$B = (BB^*)^{\frac{1}{2}} U = \begin{pmatrix} (I - M)^{\frac{1}{2}} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} U_1 \\ 0 \end{pmatrix} = \begin{pmatrix} (I - M)^{\frac{1}{2}} U_1 \\ 0 \end{pmatrix},$$

where U_1 is a partial isometry. Then

$$\widetilde{A} = \begin{pmatrix} I - A^*A & I - B^*A \\ I - A^*B & I - B^*B \end{pmatrix} = \begin{pmatrix} I - V_1^* M V_1 & I - U_1^* (I - M)^{\frac{1}{2}} M^{\frac{1}{2}} V_1 \\ I - V_1^* M^{\frac{1}{2}} (I - M)^{\frac{1}{2}} U_1 & I - U_1^* (I - M) U_1 \end{pmatrix}.$$

So Lemma 2.5 yields

$$\begin{aligned} \|\widetilde{A}\| &\leq \left\| \begin{pmatrix} I - V_1^* M V_1 & 0 \\ 0 & I - U_1^* (I - M) U_1 \end{pmatrix} \right\| \\ &\quad + \left\| \begin{pmatrix} 0 & I - U_1^* (I - M)^{\frac{1}{2}} M^{\frac{1}{2}} V_1 \\ I - V_1^* M^{\frac{1}{2}} (I - M)^{\frac{1}{2}} U_1 & 0 \end{pmatrix} \right\| \\ &\leq 1 + \|I - U_1^* (I - M)^{\frac{1}{2}} M^{\frac{1}{2}} V_1\| \\ &\leq 2 + \|(I - M)^{\frac{1}{2}} M^{\frac{1}{2}}\| \\ &= 2 + \|(I - M) M\|^{\frac{1}{2}} \\ &= \frac{5}{2}. \end{aligned}$$

□

The following Krein-Milman theorem is well known.

Lemma 2.7. ([9]) Let C be a non-empty convex compact set in a Hausdorff locally convex space $\mathcal{X}$. Then the set $\mathcal{E}$ of extreme points of C is non-empty and $C = \overline{co(\mathcal{E})}$.

Proposition 2.1. Let $\mathcal{K}$ be an infinite dimensional Hilbert space. Then $\widetilde{H}_n^{(m)} = \widetilde{H}_{n+1}^{(m)} = \widetilde{H}^{(m)}$, for $n, m = 1, 2, \cdots$.

Proof. Let $\widetilde{A} \in \widetilde{H}_n^{(m)}$. Then

$$\widetilde{A} = (I - A_j^* A_i)_{i,j=1}^m, \text{ where } A_1, A_2 \cdots A_m \in \mathcal{B}(\mathcal{H}, \mathcal{K}^n) \text{ and } \|\sum_{i=1}^m A_i A_i^*\| \leq 1.$$

Setting

$$B_i = \begin{pmatrix} A_i \\ 0 \end{pmatrix} : H \to \mathcal{K}^n \oplus \mathcal{K}, \text{ for } i = 1, 2 \cdots m,$$

then

$$B_j^* B_i = \begin{pmatrix} A_j^* & 0 \end{pmatrix} \begin{pmatrix} A_i \\ 0 \end{pmatrix} = A_j^* A_i.$$

Thus

$$\widetilde{A} = (I - A_j^* A_i)_{i,j=1}^m = (I - B_j^* B_i)_{i,j=1}^m.$$

Obviously,

$$B_i B_i^* = \begin{pmatrix} A_i \\ 0 \end{pmatrix} \begin{pmatrix} A_i^* & 0 \end{pmatrix} = \begin{pmatrix} A_i A_i^* & 0 \\ 0 & 0 \end{pmatrix} : \mathcal{K}^n \oplus \mathcal{K},$$

so

$$\|\sum_{i=1}^m B_i B_i^*\| = \|\sum_{i=1}^m A_i A_i^*\| \leq 1.$$

Hence $\widetilde{H}_n^{(m)} \subseteq \widetilde{H}_{n+1}^{(m)}$.

Suppose that $\widetilde{C} \in \widetilde{H}_{n+1}^{(m)}$ and

$$\widetilde{C} = (I - C_j^* C_i)_{i,j=1}^m, \text{ where } C_1, C_2 \cdots C_m \in \mathcal{B}(\mathcal{H}, \mathcal{K}^{n+1}).$$

As the Hilbert space $\mathcal{K}$ is infinite dimensional, there exists the unitary operator U from $\mathcal{K}^{n+1}$ onto $\mathcal{K}^n$. Let $D_i = UC_i \in \mathcal{B}(\mathcal{H}, \mathcal{K}^n)$, for $i = 1, 2 \cdots m$. Then $D_j^* D_i = C_j^* U^* U C_i = C_j^* C_i$, so

$$\widetilde{C} = (I - D_j^* D_i)_{i,j=1}^m,$$

where $D_1, D_2 \cdots D_m \in \mathcal{B}(\mathcal{H}, \mathcal{K}^n)$, and

$$\|\sum_{i=1}^m D_i D_i^*\| = \|\sum_{i=1}^m C_i C_i^*\| \leq 1.$$

which implies $\widetilde{H}_{n+1}^{(m)} \subseteq \widetilde{H}_n^{(m)}$. $\square$

By a similar proof, we have the following corollary.

Corollary 2.1. Let $\mathcal{K}$ be an infinite dimensional Hilbert space. Then $\widetilde{S}_n^{(m)} = \widetilde{S}_{n+1}^{(m)} = \widetilde{S}^{(m)}$, for $n, m = 1, 2, \cdots$.

Remark 2.1. (a) If Hilbert space $\mathcal{K}$ is finite dimensional, then $\widetilde{H}_n^{(m)} \subsetneq \widetilde{H}_{n+1}^{(m)} \subsetneq \widetilde{H}^{(m)}$ and $\widetilde{S}_n^{(m)} \subsetneq \widetilde{S}_{n+1}^{(m)} \subsetneq \widetilde{S}^{(m)}$. Also, the following proof of Theorem 2.1 (a) implies that $\widetilde{H}^{(m)}$ and $\widetilde{S}^{(m)}$ are convex sets.

(b) By Lemma 2.2, we can conclude the positivity $\widetilde{A} = (A_{ij})_{i,j=1}^m \in \mathcal{B}(\oplus_{i=1}^m \mathcal{H}_i)^+$ implies $\|\widetilde{A}\| \leq \sum_{i=1}^m \|A_{ii}\|$ which has been obtained in [5, Theorem 2.7].

Indeed, we conclude from Lemma 2.2 that $\widetilde{A} = (A_{ij})_{i,j=1}^m = (C_i^* C_j)_{i,j=1}^m$, for some $C_i \in \mathcal{B}(\mathcal{H}_i, \widetilde{\mathcal{K}})$, where $i = 1, 2, \cdots m$. Thus $A_{ii} = C_i^* C_i$, for $i = 1, 2, \cdots m$, so

$$\|\widetilde{A}\| = \|(C_i^* C_j)_{i,j=1}^m\| = \|\sum_{i=1}^m C_i C_i^*\| \leq \sum_{i=1}^m \|C_i C_i^*\| = \sum_{i=1}^m \|A_{ii}\|.$$

Proposition 2.2. Let $A_i (i = 1, 2, \ldots, m)$ be contractions. Then $\|(I - A_i^* A_j)_{i,j=1}^m\| \leq m$.

Proof. Obviously

$$(I - A_i^*A_j)_{i,j=1}^m \le \begin{pmatrix} I & I & \cdots & I \\ \vdots & \vdots & \ddots & \vdots \\ I & I & \cdots & I \end{pmatrix} \le diag(mI, mI, \cdots mI). \tag{6}$$

Since $\|A_i\| \le 1$ for $i = 1, 2, \cdots m$, we get that

$$\|(A_i^*A_j)_{i,j=1}^m\| = \|A_1A_1^* + A_2A_2^* + \cdots A_mA_m^*\| \le m.$$

So

$$\begin{pmatrix} I & I & \cdots & I \\ \vdots & \vdots & \vdots & \vdots \\ I & I & \cdots & I \end{pmatrix} - \begin{pmatrix} A_1^*A_1 & A_1^*A_2 & \cdots & A_1^*A_m \\ \vdots & \vdots & \cdots & \vdots \\ A_m^*A_1 & A_m^*A_2 & \cdots & A_m^*A_m \end{pmatrix} \ge diag(-mI, -mI, \cdots - mI). \tag{7}$$

Thus above two inequalities (6) and (7) imply the desired conclusion. □

Corollary 2.2. $\max\{\|(I - A_i^*A_j)_{i,j=1}^m\| : A_i \in \mathcal{B}_1(\mathcal{H}, \mathcal{K}), \text{ for } i = 1, 2 \cdots m\} = m.$

For another problem (4), we only have the following result.

Proposition 2.3. Let $A_i (i = 1, 2, \ldots, m)$ be contractions. Then $\|(I - A_j^*A_i)_{i,j=1}^m\| \le 2m - 1$.

Proof. By Lemma 2.1,

$$\|(I - A_j^*A_i)_{i,j=1}^m\| \le \sum_{k=0}^{m-1} (\max_{1 \le i \le m}\{\|I - A_{[i+k]}^*A_i\|\}).$$

Clearly,

$$\max_{1 \le i \le m}\{\|I - A_{[i+k]}^*A_i\|\} \le 1, \quad \text{for } k = 0$$

and

$$\max_{1 \le i \le m}\{\|I - A_{[i+k]}^*A_i\|\} \le 2, \quad \text{for } k = 1, 2, \cdots m - 1,$$

so

$$\|(I - A_j^*A_i)_{i,j=1}^m\| \le \sum_{k=0}^{m-1} (\max_{1 \le i \le m}\{\|I - A_{[i+k]}^*A_i\|\}) \le 2m - 1.$$

□

Remark 2.2. If Hilbert space $\mathcal{K}$ is finite dimensional, then using the fact that the essential norm is less than the norm for operators, we conclude that

$$min\{\|(I - A_j^*A_i)_{i,j=1}^m\| : A_1, A_2 \cdots A_m \in \mathcal{B}_1(\mathcal{H}, \mathcal{K})\} = m.$$

The following example shows that equation (4) does not hold, in general.

Example 2.1. Let $\{e_i\}_{i=1}^\infty$ and $\{f_i\}_{i=1}^3$ be orthonormal bases of $\mathcal{H}$ and $\mathcal{K}$, respectively. Define one rank operators A and B by $A = -f_1 \otimes e_2$ and $B = f_1 \otimes e_1$, respectively. Then it is easy to see that $A^*A = e_2 \otimes e_2$, $A^*B = -e_2 \otimes e_1$, and $B^*B = e_1 \otimes e_1$. Thus

$$\widetilde{A} = \begin{pmatrix} I - A^*A & I - B^*A \\ I - A^*B & I - B^*B \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & I & 0 & 0 & I \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & I & 0 & 0 & I \end{pmatrix} : \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3,$$

where $\mathcal{H}_1 = \vee\{e_1\}, \mathcal{H}_2 = \vee\{e_1\}, \mathcal{H}_3 = \vee\{e_i : i \geq 3\}$.

Denoting a unit vector $x = (\frac{1}{\sqrt{2}}e_1, 0, 0, 0, \frac{1}{\sqrt{2}}e_2, 0)$, we conclude that

$$\|\widetilde{A}\| \geq \|\widetilde{A}x\| = \|(\sqrt{2}e_1, \frac{1}{\sqrt{2}}e_2, 0, \frac{1}{\sqrt{2}}e_1, \sqrt{2}e_2, 0)\| = \sqrt{5} > 2.$$

Theorem 2.1. Let $\mathcal{K}$ be an infinite dimensional Hilbert space. Then
(a) $\widetilde{H}^{(m)} \subseteq B(\mathcal{H}^m)$ is a convex set.
(b) $\widetilde{H}^{(m)}$ is a compact subset in the ω^* topology.
(c) $\widetilde{H}^{(m)}$ is a compact subset in the weak operator topology.

Proof. (a) For any $\widetilde{A}, \widetilde{B} \in \widetilde{H}^{(m)}$, we assume that

$$\widetilde{A} = (I - A_j^* A_i)_{i,j=1}^m \text{ and } \widetilde{B} = (I - B_j^* B_i)_{i,j=1}^m,$$

where $A_i \in \mathcal{B}(\mathcal{H}, \mathcal{K}^s)$ and $B_i \in \mathcal{B}(\mathcal{H}, \mathcal{K}^l)$ satisfy $\|\sum_{i=1}^m A_i A_i^*\| \leq 1$ and $\|\sum_{i=1}^m B_i B_i^*\| \leq 1$. Then for $0 \leq t \leq 1$, it is clear that

$$t\widetilde{A} + (1-t)\widetilde{B} = (I - (tA_j^* A_i + (1-t)B_j^* B_i))_{i,j=1}^m.$$

Defining $C_j \in \mathcal{B}(\mathcal{H}, \mathcal{K}^{s+l})$ as the following operator matrices for $j = 1, 2, \cdots m$,

$$C_j = \begin{pmatrix} \sqrt{t}A_j \\ \sqrt{1-t}B_j \end{pmatrix} : \mathcal{H} \to \mathcal{K}^s \oplus \mathcal{K}^l,$$

we have

$$C_j^* C_i = \begin{pmatrix} \sqrt{t}A_j^* & \sqrt{1-t}B_j^* \end{pmatrix} \begin{pmatrix} \sqrt{t}A_i \\ \sqrt{1-t}B_i \end{pmatrix} = tA_j^* A_i + (1-t)B_j^* B_i,$$

so

$$t\widetilde{A} + (1-t)\widetilde{B} = (I - C_j^* C_i)_{i,j=1}^m.$$

Moreover,

$$\begin{aligned} \|\sum_{i=1}^m C_i C_i^*\| &= \|(C_i^* C_j)_{i,j=1}^m\| \\ &= \|(tA_i^* A_j + (1-t)(B_i^* B_j))_{i,j=1}^m\| \\ &\leq t\|(A_i^* A_j)_{i,j=1}^m\| + (1-t)\|(B_i^* B_j)_{i,j=1}^m\| \\ &= t\|\sum_{i=1}^m A_i A_i^*\| + (1-t)\|\sum_{i=1}^m B_i B_i^*\| \\ &\leq 1. \end{aligned}$$

Thus

$$t\widetilde{A} + (1-t)\widetilde{B} = (I - C_j^* C_i)_{i,j=1}^m \in \widetilde{H}^{(m)}.$$

which says that $\widetilde{H}^{(m)}$ is a convex set.

(b) By Proposition 2.3, we conclude that $\widetilde{H}^{(m)} \subseteq (2m-1)\mathcal{B}_1(\mathcal{H}^m)$. It is well known that Alaoglu's theorem implies that $\mathcal{B}_1(\mathcal{H}^m)$ is a compact set in the ω^* topology. So $(2m-1)\mathcal{B}_1(\mathcal{H}^m)$ is also a compact set in the ω^* topology. Thus, it is enough to show that $\widetilde{H}^{(m)}$ is a closed subset in the ω^* topology. Let $\widetilde{A_\tau}$ be a net of $\widetilde{H}^{(m)}$ with $\omega^* \lim_\tau \widetilde{A_\tau} = \widetilde{A}$. We may assume that for all index τ, $\mathcal{K}_\tau = \mathcal{K}$, as $\mathcal{K}$ is infinite dimensional and

$$\widetilde{A_\tau} = (I - A_{\tau j}^* A_{\tau i})_{i,j=1}^m, \text{ where } A_{\tau i} \in \mathcal{B}(\mathcal{H}, \mathcal{K}).$$

Also, set

$$\widetilde{A} = (M_{ij})_{i,j=1}^m, \text{ where } M_{ij} \in \mathcal{B}(\mathcal{H}), \text{ for } i, j = 1, 2, \cdots m.$$

From Lemma 2.3 (b), it is easy to observe that $\omega^* \lim_\tau \widetilde{A_\tau} = \widetilde{A}$ implies that

$$\omega^* \lim_\tau (I - A_{\tau j}^* A_{\tau i}) = M_{ij}.$$

so

$$\omega^* \lim_\tau (I - A_{\tau i}^* A_{\tau j}) = M_{ji}.$$

Using Lemma 2.3 (b) again, we get

$$\omega^* \lim_\tau (I - A_{\tau i}^* A_{\tau j}) = M_{ij}^*.$$

Thus $M_{ji} = M_{ij}^*$ and

$$\omega^* \lim_\tau (A_{\tau i}^* A_{\tau j})_{i,j=1}^m = (I - M_{ji})_{i,j=1}^m,$$

which yields

$$(I - M_{ji})_{i,j=1}^m \in \mathcal{B}(\mathcal{H}^m)^+,$$

since $\|\sum_{i=1}^m A_{\tau i} A_{\tau i}^*\| \leq 1$ implies $(I - A_{\tau i}^* A_{\tau j})_{i,j=1}^m \in \mathcal{B}(\mathcal{H}^m)^+$ for all τ. Applying Lemma 2.2, we get that $C_i \in \mathcal{B}(\mathcal{H}, \mathcal{H}^m)$ such that

$$(I - M_{ji})_{i,j=1}^m = (C_i^* C_j)_{i,j=1}^m,$$

Hence

$$\widetilde{A} = (M_{ij})_{i,j=1}^m = (I - C_j^* C_i)_{i,j=1}^m,$$

As two Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ are infinite dimensional, there exists the unitary operator U from $\mathcal{H}^m$ onto $\mathcal{K}$. Let $D_i = UC_i$ for $i = 1, 2 \cdots m$. Then $D_i \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ and

$$\widetilde{A} = (M_{ij})_{i,j=1}^m = (I - C_j^* C_i)_{i,j=1}^m = (I - D_j^* D_i)_{i,j=1}^m.$$

Furthermore, Lemma 2.3 (a) implies

$$\|\sum_{i=1}^m D_i D_i^*\| = \|\sum_{i=1}^m C_i C_i^*\| = \|(C_i^* C_j)_{i,j=1}^m\| = \|(I - M_{ji})_{i,j=1}^m\| \leq 1,$$

since

$$\|(A_{\tau i}^* A_{\tau j})_{i,j=1}^m\| \leq 1, \quad \text{for all index} \quad \tau.$$

So $\widetilde{A} \in \widetilde{H}^{(m)}$, which says that $\widetilde{H}^{(m)}$ is a ω^*−closed set.

(c) follows from (b) and the fact that the ω^* topology is equivalent to the weak operator topology on the closed ball $(2m-1)\mathcal{B}_1(\mathcal{H}^m)$. □

Theorem 2.2. Let $\mathcal{K}$ be an infinite dimensional Hilbert space. Then the set of all extreme points of $\widetilde{H}^{(m)}$ is $\widetilde{S}^{(m)}$.

Proof. Firstly, we show that $\widetilde{A} \in \widetilde{S}^{(m)}$ is an extreme point of $\widetilde{H}^{(m)}$. Let $\widetilde{A} \in \widetilde{S}^{(m)}$ satisfy

$$\widetilde{A} = t\widetilde{A}_1 + (1-t)\widetilde{A}_2,$$

where $\widetilde{A}_1, \widetilde{A}_2 \in \widetilde{H}^{(m)}$ and $0 < t < 1$. By the definitions of $\widetilde{H}^{(m)}$ and $\widetilde{S}^{(m)}$, we assume that

$$\widetilde{A} = (I - A_j^* A_i)_{i,j=1}^m, \quad \widetilde{A_1} = (I - B_j^* B_i)_{i,j=1}^m \text{ and } \widetilde{A_2} = (I - C_j^* C_i)_{i,j=1}^m.$$

Then

$$(A_i^* A_j)_{i,j=1}^m \in P(\mathcal{H}^m), \ \|\sum_{i=1}^m B_i B_i^*\| \leq 1 \quad \text{and} \quad \|\sum_{i=1}^m C_i C_i^*\| \leq 1.$$

Obviously,

$$(I - A_i^* A_j)_{i,j=1}^m = (I - (tB_i^* B_j + (1-t)C_i^* C_j))_{i,j=1}^m,$$

which implies

$$(A_i^* A_j)_{i,j=1}^m = t(B_i^* B_j)_{i,j=1}^m + (1-t)(C_i^* C_j)_{i,j=1}^m.$$

Moreover,

$$\|(B_i^* B_j)_{i,j=1}^m\| = \|\sum_{i=1}^m B_i B_i^*\| \le 1 \text{ and } \|(C_i^* C_j)_{i,j=1}^m\| = \|\sum_{i=1}^m C_i C_i^*\| \le 1.$$

Using lemma 2.4, we get

$$(A_i^* A_j)_{i,j=1}^m = (B_i^* B_j)_{i,j=1}^m = (C_i^* C_j)_{i,j=1}^m,$$

which induces $\widetilde{A} = \widetilde{A_1} = \widetilde{A_2}$. Thus $\widetilde{A}$ is an extreme point of $\widetilde{H}^{(m)}$.

In the following, we shall show that $\widetilde{E} \in \widetilde{H}^{(m)} \bigcap \widetilde{S}^{(m)c}$ is not the extreme point of $\widetilde{H}^{(m)}$, where $\widetilde{S}^{(m)c}$ is the complement set of $\widetilde{S}^{(m)}$. Without loss of generality, we assume that

$$\widetilde{E} = (I - E_j^* E_i)_{i,j=1}^m, \tag{8}$$

so

$$\|(E_i^* E_j)_{i,j=1}^m\| = \|\sum_{i=1}^m E_i E_i^*\| \le 1,$$

which implies from Lemma 2.4 that

$$(E_i^* E_j)_{i,j=1}^m = t\widetilde{C} + (1-t)\widetilde{D}, \tag{9}$$

where $\widetilde{C}, \widetilde{D} \in \mathcal{B}_1(\mathcal{H}^m) \cap \mathcal{B}(\mathcal{H}^m)^+$, $0 < t < 1$ and $\widetilde{C} \ne \widetilde{D}$, as $(E_i^* E_j)_{i,j=1}^m \notin P(\mathcal{H}^m)$. Using again Lemma 2.2 and the fact that $\mathcal{H}$ and $\mathcal{K}$ are infinite dimensional, we can get that

$$\widetilde{C} = (C_i^* C_j)_{i,j=1}^m, \quad \text{and} \quad \widetilde{D} = (D_i^* D_j)_{i,j=1}^m, \tag{10}$$

where $C_i, D_i \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, for $i = 1, 2, \cdots m$.

Furthermore, using inequalities $\|\widetilde{C}\| \le 1$ and $\|\widetilde{D}\| \le 1$, we conclude that $\|\sum_{i=1}^m C_i C_i^*\| \le 1$ and $\|\sum_{i=1}^m D_i D_i^*\| \le 1$, so

$$(I - C_j^* C_i)_{i,j=1}^m \in \widetilde{H}^{(m)} \quad \text{and} \quad (I - D_j^* D_i)_{i,j=1}^m \in \widetilde{H}^{(m)}.$$

According to equations (8),(9) and (10), we get

$$\widetilde{E} = t(I - C_j^* C_i)_{i,j=1}^m + (1-t)(I - D_j^* D_i)_{i,j=1}^m. \tag{11}$$

Clearly, $\widetilde{C} \ne \widetilde{D}$ implies

$$(I - C_j^* C_i)_{i,j=1}^m \ne (I - D_j^* D_i)_{i,j=1}^m.$$

Thus equation (11) yields $\widetilde{E}$ isn't an extreme point of $\widetilde{H}^{(m)}$. □

Proposition 2.4. Let $\mathcal{K}$ be an infinite dimensional Hilbert space. Then

$$\widetilde{H}^{(m)} = \overline{co(\widetilde{S}^{(m)})}^{\omega^*}, \text{ where } co(\widetilde{S}^{(m)}) = \{\sum_{i=1}^n t_i x_i | n \ge 1, t_i \ge 0, \sum_{i=1}^n t_i = 1, x_i \in \widetilde{S}^{(m)}\}.$$

Proof. By Theorem 2.1, $\widetilde{H}^{(m)}$ is convex and compact in the ω^* topology. Then Theorem 2.2 and Krein-Milman theorem imply $\widetilde{H}^{(m)} = \overline{co(\widetilde{S}^{(m)})}^{\omega^*}$. □

Theorem 2.3. Let $\mathcal{K}$ be an infinite dimensional Hilbert space. Then $sup\{\|\widetilde{A}\| : \widetilde{A} \in \widetilde{H}^{(2)}\} \leq \frac{5}{2}$.

Proof. For $\widetilde{A} \in co(\widetilde{S}^{(2)})$, we can write $\widetilde{A} = \sum_{i=1}^{n} t_i \widetilde{A}_i$, where $\widetilde{A}_i \in \widetilde{S}^{(2)}$, so Lemma 2.5 yields that

$$\|\widetilde{A}\| = \| \sum_{i=1}^{n} t_i \widetilde{A}_i \| \leq \sum_{i=1}^{n} t_i \|\widetilde{A}_i\| \leq \frac{5}{2}.$$

Using again Lemma 2.3 (a), we conclude that $\|\widetilde{B}\| \leq \frac{5}{2}$, for any $\widetilde{B} \in \widetilde{H}^{(2)} = \overline{co(\widetilde{S}^{(2)})}^{\omega^*}$. □

Remark 2.3. If Hilbert space $\mathcal{K}$ is infinite dimensional, then we conjecture

$$sup\{\|\widetilde{A}\| : \widetilde{A} \in \widetilde{H}^{(2)}\} = \frac{5}{2}.$$

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