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On a Solvable Three-Dimensional System of Difference Equations

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Abstract. In this paper, we show that the following three-dimensional system of difference equations

$$x_n = \frac{z_{n-2}x_{n-3}}{ax_{n-3} + by_{n-1}},\ y_n = \frac{x_{n-2}y_{n-3}}{cy_{n-3} + dz_{n-1}},\ z_n = \frac{y_{n-2}z_{n-3}}{ez_{n-3} + fx_{n-1}},\ n \in \mathbb{N}_0,$$

where the parameters a, b, c, d, e, f and the initial values x_{-i}, y_{-i}, z_{-i}, $i \in \{1, 2, 3\}$, are real numbers, can be solved, extending further some results in literature. Also, we determine the asymptotic behavior of solutions and the forbidden set of the initial values by using the obtained formulas.

1. Introduction and Preliminaries

Nonlinear difference equations and systems have attracted attention of many authors in recent years(see, e.g. [1, 41]). The domain trend in nonlinear difference equation and system is actually to find the equation or system which can be solved in closed-form. Almost all of them are various generalizations of solvable difference equations and systems. That is, when a solvable equation is found, generalizations such as solvability with parameters, solvability with increasing order, solvability with periodic coefficients, and solvability as two-dimensional or three-dimensional systems have been studied. For example, the following equations

$$x_{n+1} = \frac{x_n x_{n-1}}{x_n + x_{n-1}} \quad \text{and} \quad x_{n+1} = \frac{x_{n-1}x_{n-2}}{x_n + x_{n-2}},\ n \in \mathbb{N}_0, \tag{1}$$

was first presented, among other things, by Elmetwally et al. in [7]. Then, in [28], Eqs. (1) were generalized to the following equation

$$x_n = \frac{x_{n-k}x_{n-k-s}}{ax_{n-k-s} + bx_{n-s}},\ n \in \mathbb{N}_0, \tag{2}$$

where k, s fixed natural numbers, $a, b \in \mathbb{R} \setminus \{0\}$, and the initial values x_{-i}, $i = \overline{1, \tau}$, $\tau := \max\{k, s\}$ are real numbers. Also, in [11], the first equation in (1) was extended to the following two-dimensional system of difference equation

$$x_n = \frac{x_{n-1}y_{n-2}}{y_{n-2} \pm y_{n-1}},\ y_n = \frac{y_{n-1}x_{n-2}}{x_{n-2} \pm x_{n-1}},\ n \in \mathbb{N}_0. \tag{3}$$

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Some of their solution forms were proved by induction. Further, in [24], both the first equation in (1) and system (3) were extended to the following difference equations system

$$x_n = \frac{x_{n-1}y_{n-2}}{ay_{n-2} + by_{n-1}},\ y_n = \frac{y_{n-1}x_{n-2}}{cx_{n-2} + dx_{n-1}},\ n \in \mathbb{N}_0, \tag{4}$$

where parameters a, b, c, d, as well as the initial values are real numbers. The authors showed that system (3) is solvable in closed form and presented formulas for the solutions. They also studied the long-term behavior of the solutions of system (3). In [9], the second equation in (1) was generalized to the following three-dimensional system of difference equations

$$x_{n+1} = \frac{z_{n-1}x_{n-2}}{x_{n-2} \pm y_n},\ y_{n+1} = \frac{x_{n-1}y_{n-2}}{y_{n-2} \pm z_n},\ z_{n+1} = \frac{y_{n-1}z_{n-2}}{z_{n-2} \pm x_n},\ n \in \mathbb{N}_0. \tag{5}$$

Some of their solution forms were proved by induction. Motivated by aforementioned studies, in this study, we deal with the following system of difference equations

$$x_n = \frac{z_{n-2}x_{n-3}}{ax_{n-3} + by_{n-1}},\ y_n = \frac{x_{n-2}y_{n-3}}{cy_{n-3} + dz_{n-1}},\ z_n = \frac{y_{n-2}z_{n-3}}{ez_{n-3} + fx_{n-1}},\ n \in \mathbb{N}_0, \tag{6}$$

where the parameters a, b, c, d, e, f and the initial values x_{-i}, y_{-i}, z_{-i}, $i \in \{1, 2, 3\}$, are real numbers. We solve system (6) in closed form and determine the asymptotic behavior of solutions and the forbidden set of the initial values by using the obtained formulas. Also, we obtain the well-known Fibonacci numbers in the solutions of aforementioned system when $a = b = c = d = e = f = 1$. Note that system (6) is a natural generalization of both system (5) and the second equation in (1). Now, we should recall that the Fibonacci sequence $\{F_n\}_{n=0}^{\infty}$ is defined by

$$F_{n+2} = F_{n+1} + F_n,\ n \in \mathbb{N}_0, \tag{7}$$

with the initial values F_0 and F_1. Considering [16], it can be clearly obtained the characteristic equation of (7) as the form $x^2 - x - 1 = 0$ having the roots $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. Thus, the Binet's Formula for Fibonacci sequence, $F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$, can be thought as a solution of Fibonacci sequence. Also, it is obtained to extend negatively subscripted Fibonacci sequence as

$$F_{-n} = F_{-n+2} - F_{-n+1} = (-1)^{n+1} F_n,\ n \in \mathbb{N}_0. \tag{8}$$

In the analysing of solutions of a difference equation or a system, the matter of existence of solutions is of prime importance as such in differential equations. Therefore, the following definition gives us the set of all initial values which yields undefined solutions.

Definition 1.1. *[31] Consider the following system of difference equations*

$$\begin{aligned} x_n^{(1)} &= f_1\left(x_{n-1}^{(1)}, \ldots, x_{n-k}^{(1)}, x_{n-1}^{(2)}, \ldots, x_{n-k}^{(2)}, \ldots, x_{n-1}^{(m)}, \ldots, x_{n-k}^{(m)}\right), \\ x_n^{(2)} &= f_2\left(x_{n-1}^{(1)}, \ldots, x_{n-k}^{(1)}, x_{n-1}^{(2)}, \ldots, x_{n-k}^{(2)}, \ldots, x_{n-1}^{(m)}, \ldots, x_{n-k}^{(m)}\right), \\ &\vdots \\ x_n^{(m)} &= f_m\left(x_{n-1}^{(1)}, \ldots, x_{n-k}^{(1)}, x_{n-1}^{(2)}, \ldots, x_{n-k}^{(2)}, \ldots, x_{n-1}^{(m)}, \ldots, x_{n-k}^{(m)}\right), \end{aligned} \tag{9}$$

$n \in \mathbb{N}_0$, where $m, k \in \mathbb{N}$ and $x_{-j}^{(i)} \in \mathbb{R}$, $j = \overline{1, k}$, $i = \overline{1, m}$. The string of vectors $\left(x_j^{(1)}, x_j^{(2)}, \ldots, x_j^{(m)}\right)$, $-k \le j < n_0$ where $n_0 \ge -1$, is called an undefined solution of system (9) for $0 \le j < n_0 + 1$, and $x_{n_0+1}^{(i_0)}$ is not defined for an $i_0 \in \{1, \ldots, m\}$, that is the quantity $f_{i_0}\left(x_{n_0}^{(1)}, \ldots, x_{n_0-k+1}^{(1)}, x_{n_0}^{(2)}, \ldots, x_{n_0-k+1}^{(2)}, \ldots, x_{n_0}^{(m)}, \ldots, x_{n_0-k+1}^{(2)}\right)$ is not defined. The set of all initial values $x_{-j}^{(i)}$, $j = \overline{1, k}, i = \overline{1, m}$ which generate undefined solutions of system of difference equation (9) is called domain of undefineble solutions of system of difference equations.

2. Main Results

Let $(x_n, y_n, z_n)_{n\geq -3}$ be a solution of system (6). If at least one of the initial values x_{-i}, y_{-i}, z_{-i}, $i = 1, 2, 3$, is equal to zero, then the solutions of system (6) is not defined. For example, if $x_{-3} = 0$, then $x_0 = 0$, $y_2 = 0$, and so x_3, y_5 and z_4 can not be calculated. Similarly, if $y_{-3} = 0$ (or $z_{-3} = 0$), then $y_0 = 0$, $z_2 = 0$ (or $z_0 = 0$, $x_2 = 0$), and so x_4, y_3 and z_5 (or x_5, y_4 and z_3) are not calculated. For $i = 1, 2$, the other cases are similar. On the other hand, if $x_{n_0} = 0$ ($n_0 \in \mathbb{N}_0$), $x_n \neq 0$, for $-3 \leq n \leq n_0 - 1$, and x_k, y_k and z_k are defined for $-3 \leq k \leq n_0 - 1$, then according to the first equation in (6) we have that $z_{n_0-2} = 0$. If $n_0 - 2 \leq -1$, then $z_{-i_0} = 0$ for $i_0 \in \{1, 2\}$. If $n_0 > 1$, then according to the third equation in (6) we get that $y_{n_0-4} = 0$ or $z_{n_0-5} = 0$. If $2 \leq n_0 \leq 4$ and $y_{n_0-4} = 0$, then from this and the second equation in (6), we have that $y_{-i_1} = 0$ for $i_1 \in \{1, 2, 3\}$. If $n_0 > 4$ and $y_{n_0-4} = 0$, from this and equations in (6) we have that $x_{n_0-3} = 0$, which is a contradiction. If $n_0 = 2$, from this and equations in (6) we have that $z_{-3} = 0$. If $n_0 > 2$ and $z_{n_0-5} = 0$, then from this and the first equation in (6) we have that $x_{n_0-3} = 0$, which is a contradiction. The other cases ($y_{n_1} = 0$ and $z_{n_2} = 0$) can be similarly proved. Thus, for every well-defined solution of system (6), we get that $x_n y_n z_n \neq 0$, $n \geq -3$, if and only if $x_{-i}y_{-i}z_{-i} \neq 0$, $i \in \{1, 2, 3\}$. Note that the system (6) can be written in the form

$$\frac{x_n}{z_{n-2}} = \frac{1}{a + b\frac{y_{n-1}}{x_{n-3}}},\ \frac{y_n}{x_{n-2}} = \frac{1}{c + d\frac{z_{n-1}}{y_{n-3}}},\ \frac{z_n}{y_{n-2}} = \frac{1}{e + f\frac{x_{n-1}}{z_{n-3}}},\ n \in \mathbb{N}_0. \tag{10}$$

Next, by employing the change of variables

$$u_n = \frac{x_n}{z_{n-2}},\ v_n = \frac{y_n}{x_{n-2}},\ w_n = \frac{z_n}{y_{n-2}},\ n \geq -1, \tag{11}$$

system (10) is transformed into the following system

$$u_n = \frac{1}{a + bv_{n-1}},\ v_n = \frac{1}{c + dw_{n-1}},\ w_n = \frac{1}{e + fu_{n-1}},\ n \in \mathbb{N}_0, \tag{12}$$

which can be written as

$$u_n = \frac{cfu_{n-3} + ce + d}{(acf + bf)\, u_{n-3} + ace + ad + be},\ n \geq 2, \tag{13}$$

$$v_n = \frac{bev_{n-3} + ae + f}{(bce + bd)\, v_{n-3} + ace + ad + cf},\ n \geq 2, \tag{14}$$

$$w_n = \frac{adw_{n-3} + ac + b}{(ade + df)\, w_{n-3} + ace + cf + be},\ n \geq 2. \tag{15}$$

If we apply the decomposition of indices $n \to 3m + i$, $i \in \{-1, 0, 1\}$ and $m \in \mathbb{N}$, to (13)-(15), then they can be written as follows

$$u_m^{(i)} = \frac{cfu_{m-1}^{(i)} + ce + d}{(acf + bf)\, u_{m-1}^{(i)} + ace + ad + be}, \tag{16}$$

$$v_m^{(i)} = \frac{bev_{m-1}^{(i)} + ae + f}{(bce + bd)\, v_{m-1}^{(i)} + ace + ad + cf}, \tag{17}$$

$$w_m^{(i)} = \frac{adw_{m-1}^{(i)} + ac + b}{(ade + df)\, w_{m-1}^{(i)} + ace + cf + be}, \tag{18}$$

where $u_m^{(i)} = u_{3m+i}$, $v_m^{(i)} = v_{3m+i}$, $w_m^{(i)} = w_{3m+i}$, $m \in \mathbb{N}_0$, $i \in \{-1, 0, 1\}$. It is well-known that the substitutions

$$(acf + bf)\, u_{m-1}^{(i)} + ace + ad + be = \frac{r_m}{r_{m-1}},\ m \in \mathbb{N}, \tag{19}$$

$$(bce + bd)\, v_{m-1}^{(i)} + ace + ad + cf = \frac{s_m}{s_{m-1}},\ m \in \mathbb{N}, \tag{20}$$

$$(ade + df)\, w_{m-1}^{(i)} + ace + cf + be = \frac{t_m}{t_{m-1}},\ m \in \mathbb{N}, \tag{21}$$

transforms equations in (16)-(18) into the following second order linear difference equation, which represents one of the sequences $(r_m)_{m\in\mathbb{N}_0}$, $(s_m)_{m\in\mathbb{N}_0}$ and $(t_m)_{m\in\mathbb{N}_0}$,

$$q_{m+1} - (ace + ad + be + cf)\, q_m - bdf q_{m-1} = 0,\ m \in \mathbb{N}. \tag{22}$$

From (22), the general solutions of the sequences $(r_m)_{m\in\mathbb{N}_0}$, $(s_m)_{m\in\mathbb{N}_0}$ and $(t_m)_{m\in\mathbb{N}_0}$ are given by

$$r_m = \frac{\lambda_2 r_0 - r_1}{\lambda_2 - \lambda_1}\lambda_1^m + \frac{r_1 - \lambda_1 r_0}{\lambda_2 - \lambda_1}\lambda_2^m,\ m \in \mathbb{N}_0, \tag{23}$$

$$s_m = \frac{\lambda_2 s_0 - s_1}{\lambda_2 - \lambda_1}\lambda_1^m + \frac{s_1 - \lambda_1 s_0}{\lambda_2 - \lambda_1}\lambda_2^m,\ m \in \mathbb{N}_0, \tag{24}$$

$$t_m = \frac{\lambda_2 t_0 - t_1}{\lambda_2 - \lambda_1}\lambda_1^m + \frac{t_1 - \lambda_1 t_0}{\lambda_2 - \lambda_1}\lambda_2^m,\ m \in \mathbb{N}_0, \tag{25}$$

when $(ace + ad + be + cf)^2 + 4bdf \neq 0$, and

$$r_m = (r_1 m + r_0\lambda_1 (1 - m))\, \lambda_1^{m-1},\ m \in \mathbb{N}_0, \tag{26}$$

$$s_m = (s_1 m + s_0\lambda_1 (1 - m))\, \lambda_1^{m-1},\ m \in \mathbb{N}_0, \tag{27}$$

$$t_m = (t_1 m + t_0\lambda_1 (1 - m))\, \lambda_1^{m-1},\ m \in \mathbb{N}_0, \tag{28}$$

when $(ace + ad + be + cf)^2 + 4bdf = 0$, where $\lambda_1 = \frac{ace+ad+be+cf+\sqrt{(ace+ad+be+cf)^2+4bdf}}{2}$ and $\lambda_2 = \frac{ace+ad+be+cf-\sqrt{(ace+ad+be+cf)^2+4bdf}}{2}$. Note that λ_1 and λ_2 are roots of the characteristic equation of Eq. (22) as the form $\lambda^2 - (ace + ad + be + cf)\,\lambda - bdf = 0$. By substituting (23)-(25) and (26)-(28) into (19)-(21), respectively, we get that

$$\begin{aligned} u_{m-1}^{(i)} &= \frac{\left(\lambda_2 - (acf + bf)\, u_0^{(i)} - ace - ad - be\right)\lambda_1^m + \left((acf + bf)\, u_0^{(i)} + ace + ad + be - \lambda_1\right)\lambda_2^m}{(acf + bf)\left(\left(\lambda_2 - (acf + bf)\, u_0^{(i)} - ace - ad - be\right)\lambda_1^{m-1} + \left((acf + bf)\, u_0^{(i)} + ace + ad + be - \lambda_1\right)\lambda_2^{m-1}\right)} \\ &- \frac{ace + ad + be}{acf + bf}, \end{aligned} \tag{29}$$

$$v_{m-1}^{(i)} = \frac{\left(\lambda_2-(bce+bd)\,v_0^{(i)}-ace-ad-cf\right)\lambda_1^m+\left((bce+bd)\,v_0^{(i)}+ace+ad+cf-\lambda_1\right)\lambda_2^m}{(bce+bd)\left(\left(\lambda_2-(bce+bd)\,v_0^{(i)}-ace-ad-cf\right)\lambda_1^{m-1}+\left((bce+bd)\,v_0^{(i)}+ace+ad+cf-\lambda_1\right)\lambda_2^{m-1}\right)} - \frac{ace+ad+cf}{bce+bd}, \tag{30}$$

$$w_{m-1}^{(i)} = \frac{\left(\lambda_2-(ade+df)\,w_0^{(i)}-ace-be-cf\right)\lambda_1^m+\left((ade+df)\,w_0^{(i)}+ace+be+cf-\lambda_1\right)\lambda_2^m}{(ade+df)\left(\left(\lambda_2-(ade+df)\,w_0^{(i)}-ace-be-cf\right)\lambda_1^{m-1}+\left((ade+df)\,w_0^{(i)}+ace+be+cf-\lambda_1\right)\lambda_2^{m-1}\right)} - \frac{ace+be+cf}{ade+df}, \tag{31}$$

when $(ace+ad+be+cf)^2+4bdf \neq 0$, for $m \in \mathbb{N}$, $i \in \{-1,0,1\}$ and

$$u_{m-1}^{(i)} = \frac{\left(\left((acf+bf)\,u_0^{(i)}+ace+ad+be\right)m+\lambda_1(1-m)\right)\lambda_1^{m-1}}{(acf+bf)\left(\left((acf+bf)\,u_0^{(i)}+ace+ad+be\right)(m-1)+\lambda_1(2-m)\right)\lambda_1^{m-2}} - \frac{ace+ad+be}{acf+bf}, \tag{32}$$

$$v_{m-1}^{(i)} = \frac{\left(\left((bce+bd)\,v_0^{(i)}+ace+ad+cf\right)m+\lambda_1(1-m)\right)\lambda_1^{m-1}}{(bce+bd)\left(\left((bce+bd)\,v_0^{(i)}+ace+ad+cf\right)(m-1)+\lambda_1(2-m)\right)\lambda_1^{m-2}} - \frac{ace+ad+cf}{bce+bd}, \tag{33}$$

$$w_{m-1}^{(i)} = \frac{\left(\left((ade+df)\,w_0^{(i)}+ace+be+cf\right)m+\lambda_1(1-m)\right)\lambda_1^{m-1}}{(ade+df)\left(\left((ade+df)\,w_0^{(i)}+ace+be+cf\right)(m-1)+\lambda_1(2-m)\right)\lambda_1^{m-2}} - \frac{ace+be+cf}{ade+df}, \tag{34}$$

when $(ace+ad+be+cf)^2+4bdf = 0$, for $m \in \mathbb{N}$, $i \in \{-1,0,1\}$ and consequently

$$u_{3(m-1)+i} = \frac{\left(L_2-\frac{x_i}{z_{i-2}}\right)L_1\lambda_1^{m-1}+\left(\frac{x_i}{z_{i-2}}-L_1\right)L_2\lambda_2^{m-1}}{\left(L_2-\frac{x_i}{z_{i-2}}\right)\lambda_1^{m-1}+\left(\frac{x_i}{z_{i-2}}-L_1\right)\lambda_2^{m-1}}, \tag{35}$$

$$v_{3(m-1)+i} = \frac{\left(M_2-\frac{y_i}{x_{i-2}}\right)M_1\lambda_1^{m-1}+\left(\frac{y_i}{x_{i-2}}-M_1\right)M_2\lambda_2^{m-1}}{\left(M_2-\frac{y_i}{x_{i-2}}\right)\lambda_1^{m-1}+\left(\frac{y_i}{x_{i-2}}-M_1\right)\lambda_2^{m-1}}, \tag{36}$$

$$w_{3(m-1)+i} = \frac{\left(N_2-\frac{z_i}{y_{i-2}}\right)N_1\lambda_1^{m-1}+\left(\frac{z_i}{y_{i-2}}-N_1\right)N_2\lambda_2^{m-1}}{\left(N_2-\frac{z_i}{y_{i-2}}\right)\lambda_1^{m-1}+\left(\frac{z_i}{y_{i-2}}-N_1\right)\lambda_2^{m-1}}, \tag{37}$$

when $(ace+ad+be+cf)^2+4bdf \neq 0$, for $m \in \mathbb{N}$, $i \in \{-1,0,1\}$ and

$$u_{3(m-1)+i} = \frac{(acf+bf)^2L_1\left(\frac{x_i}{z_{i-2}}-L_1\right)m+\left(\frac{x_i}{z_{i-2}}-L_1-\frac{\lambda_1}{acf+bf}\right)(ace+ad+be)(acf+bf)+\lambda_1^2}{(acf+bf)^2\left(\frac{x_i}{z_{i-2}}-L_1\right)m+(acf+bf)^2\left(L_1-\frac{x_i}{z_{i-2}}\right)+(acf+bf)\lambda_1}, \tag{38}$$

$$v_{3(m-1)+i} = \frac{(bce+bd)^2 M_1 \left(\frac{y_i}{x_{i-2}} - M_1\right) m + \left(\frac{y_i}{x_{i-2}} - M_1 - \frac{\lambda_1}{bce+bd}\right)(ace+ad+cf)(bce+bd) + \lambda_1^2}{(bce+bd)^2 \left(\frac{y_i}{x_{i-2}} - M_1\right) m + (bce+bd)^2 \left(M_1 - \frac{y_i}{x_{i-2}}\right) + (bce+bd)\lambda_1}, \tag{39}$$

$$w_{3(m-1)+i} = \frac{(ade+df)^2 N_1 \left(\frac{z_i}{y_{i-2}} - N_1\right) m + \left(\frac{z_i}{y_{i-2}} - N_1 - \frac{\lambda_1}{ade+df}\right)(ace+be+cf)(ade+df) + \lambda_1^2}{(ade+df)^2 \left(\frac{z_i}{y_{i-2}} - N_1\right) m + (ade+df)^2 \left(N_1 - \frac{z_i}{y_{i-2}}\right) + (ade+df)\lambda_1}, \tag{40}$$

when $(ace+ad+be+cf)^2 + 4bdf = 0$, for $m \in \mathbb{N}$, $i \in \{-1,0,1\}$, where $L_k = \frac{\lambda_k - (ace+ad+be)}{acf+bf}$, $M_k = \frac{\lambda_k - (ace+ad+cf)}{bce+bd}$, $N_k = \frac{\lambda_k - (ace+be+cf)}{ade+df}$, for $k \in \{1,2\}$. From (11), we have that

$$x_{6m+l} = u_{6m+l} z_{6m+l-2} = u_{6m+l} w_{6m+l-2} y_{6m+l-4} = u_{6m+l} w_{6m+l-2} v_{6m+l-4} x_{6(m-1)+l}, \tag{41}$$

$$y_{6m+l} = v_{6m+l} x_{6m+l-2} = v_{6m+l} u_{6m+l-2} z_{6m+l-4} = v_{6m+l} u_{6m+l-2} w_{6m+l-4} y_{6(m-1)+l}, \tag{42}$$

and

$$z_{6m+l} = w_{6m+l} y_{6m+l-2} = w_{6m+l} v_{6m+l-2} x_{6m+l-4} = w_{6m+l} v_{6m+l-2} u_{6m+l-4} z_{6(m-1)+l}, \tag{43}$$

where $m \in \mathbb{N}$ and $l \in \{-3,-2,-1,0,1,2\}$, from which it follows that

$$x_{6m+3j+i+1} = x_{3j+i+1} \prod_{k=1}^{m} u_{6k+3j+i+1} w_{6k+3j+i-1} v_{6k+3j+i-3}, \tag{44}$$

$$y_{6m+3j+i+1} = y_{3j+i+1} \prod_{k=1}^{m} v_{6k+3j+i+1} u_{6k+3j+i-1} w_{6k+3j+i-3} \tag{45}$$

$$z_{6m+3j+i+1} = z_{3j+i+1} \prod_{k=1}^{m} w_{6k+3j+i+1} v_{6k+3j+i-1} u_{6k+3j+i-3} \tag{46}$$

where $m \in \mathbb{N}_0$, $j \in \{-1,0\}$ and $i \in \{-1,0,1\}$. By substituting the formulas in (35)-(37) into (44)-(46), we obtain the formulas for well-defined solutions of system (6) when $(ace+ad+be+cf)^2 + 4bdf \neq 0$. Similarly, by using the formulas in (38)-(40) into (44)-(46), we get the formulas for well-defined solutions of system (6) when $(ace+ad+be+cf)^2 + 4bdf = 0$.

Theorem 2.1. *Assume that $(ace+ad+be+cf)^2 + 4bdf > 0$, $ace+ad+be+cf \neq 0$, $\lambda_k \neq ace+ad+cf$, $\lambda_k \neq ace+be+cf$, $\lambda_k \neq ace+ad+be$, $L_k \neq \frac{x_i}{z_{i-2}}$, $M_k \neq \frac{y_i}{x_{i-2}}$, $N_k \neq \frac{z_i}{y_{i-2}}$, for $k \in \{1,2\}$ and $i \in \{-1,0,1\}$, and that $(x_n, y_n, z_n)_{n \geq -3}$ is a well-defined solution of system (6). Then the following results are true.*

(a) If $|\lambda_1| > |\lambda_2|$ and $|L_1M_1N_1| < 1$, then $x_n \to 0$, $y_n \to 0$ and $z_n \to 0$, as $n \to \infty$.

(b) If $|\lambda_1| > |\lambda_2|$ and $|L_1M_1N_1| > 1$, then $|x_n| \to \infty$, $|y_n| \to \infty$ and $|z_n| \to \infty$, as $n \to \infty$.

(c) If $|\lambda_1| > |\lambda_2|$ and $L_1M_1N_1 = 1$, then $(x_n)_{n\geq -3}$, $(y_n)_{n\geq -3}$ and $(z_n)_{n\geq -3}$ are convergent.

(d) If $|\lambda_1| > |\lambda_2|$ and $L_1M_1N_1 = -1$, then $x_{6m+3j+i+1}$, $y_{6m+3j+i+1}$ and $z_{6m+3j+i+1}$, for $m \in \mathbb{N}_0$, $j \in \{-1,0\}$, $i \in \{-1,0,1\}$, are convergent.

(e) If $|\lambda_1| < |\lambda_2|$ and $|L_2M_2N_2| < 1$, then $x_n \to 0$, $y_n \to 0$ and $z_n \to 0$, as $n \to \infty$.

(f) *If* $|\lambda_1| < |\lambda_2|$ *and* $|L_2M_2N_2| > 1$, *then* $|x_n| \to \infty$, $|y_n| \to \infty$ *and* $|z_n| \to \infty$, *as* $n \to \infty$.

(g) *If* $|\lambda_1| < |\lambda_2|$ *and* $L_2M_2N_2 = 1$, *then* $(x_n)_{n\geq-3}$, $(y_n)_{n\geq-3}$ *and* $(z_n)_{n\geq-3}$ *are convergent.*

(h) *If* $|\lambda_1| < |\lambda_2|$ *and* $L_2M_2N_2 = -1$, *then* $x_{6m+3j+i+1}$, $y_{6m+3j+i+1}$ *and* $z_{6m+3j+i+1}$, *for* $m \in \mathbb{N}_0$, $j \in \{-1,0\}$, $i \in \{-1,0,1\}$, *are convergent,*

where $L_k = \frac{\lambda_k-(ace+ad+be)}{acf+bf}$, $M_k = \frac{\lambda_k-(ace+ad+cf)}{bce+bd}$, $N_k = \frac{\lambda_k-(ace+be+cf)}{ade+df}$, *for* $k \in \{1,2\}$.

Proof. Firstly, in here we will just prove (a)-(d) since (e)-(h) can be thought in the same manner with them. Note that from (44), (45) and (46), the limits of $x_{6m+3j+i+1}$, $y_{6m+3j+i+1}$ and $z_{6m+3j+i+1}$, for every $m \in \mathbb{N}_0$, $j \in \{-1,0\}$ and $i \in \{-1,0,1\}$, depend on the limits of $u_{3(m-1)+i}$, $v_{3(m-1)+i}$ and $w_{3(m-1)+i}$, for every $m \in \mathbb{N}$ and $i \in \{-1,0,1\}$. From (35), (36) and (37), we have

$$\lim_{m\to\infty} u_{3(m-1)+i} = L_1,\ \lim_{m\to\infty} v_{3(m-1)+i} = M_1,\ \lim_{m\to\infty} w_{3(m-1)+i} = N_1, \tag{47}$$

for every $i \in \{-1,0,1\}$, when $|\lambda_1| > |\lambda_2|$, and

$$\lim_{m\to\infty} u_{3(m-1)+i} = L_2,\ \lim_{m\to\infty} v_{3(m-1)+i} = M_2,\ \lim_{m\to\infty} w_{3(m-1)+i} = N_2, \tag{48}$$

for every $i \in \{-1,0,1\}$, when $|\lambda_1| < |\lambda_2|$.
From (44)-(47), the results follow from the assumptions in (a) and (b).
Now, assume that $A_i = L_2 - \frac{x_i}{z_{i-2}}$, $B_i = \frac{x_i}{z_{i-2}} - L_1$, $C_i = M_2 - \frac{y_i}{x_{i-2}}$, $D_i = \frac{y_i}{x_{i-2}} - M_1$, $E_i = N_2 - \frac{z_i}{y_{i-2}}$ and $F_i = \frac{z_i}{y_{i-2}} - N_1$, for every $i \in \{-1,0,1\}$.
(c) : Using the Taylor expansion for $(1+x)^{-1}$, we have, for each $j \in \{-1,0\}$,

$$\begin{aligned}
x_{6m+3j} &= x_{3j}\prod_{k=1}^{m}\left(\frac{A_0L_1\lambda_1^{2k+j}+B_0L_2\lambda_2^{2k+j}}{A_0\lambda_1^{2k+j}+B_0\lambda_2^{2k+j}}\right)\left(\frac{E_1N_1\lambda_1^{2k+j-1}+F_1N_2\lambda_2^{2k+j-1}}{E_1\lambda_1^{2k+j-1}+F_1\lambda_2^{2k+j-1}}\right)\left(\frac{C_{-1}M_1\lambda_1^{2k+j-1}+D_{-1}M_2\lambda_2^{2k+j-1}}{C_{-1}\lambda_1^{2k+j-1}+D_{-1}\lambda_2^{2k+j-1}}\right)\\
&= x_{3j}C(m_0)\prod_{k=m_0}^{m}L_1M_1N_1\left(1+\frac{(L_2-L_1)B_0}{L_1A_0}\left(\frac{\lambda_2}{\lambda_1}\right)^{2k+j}+O\left(\frac{\lambda_2}{\lambda_1}\right)^{2k}\right)\left(1+\frac{(N_2-N_1)F_1}{N_1E_1}\left(\frac{\lambda_2}{\lambda_1}\right)^{2k+j-1}+O\left(\frac{\lambda_2}{\lambda_1}\right)^{2k}\right)\\
&\qquad\times\left(1+\frac{(M_2-M_1)D_{-1}}{M_1C_{-1}}\left(\frac{\lambda_2}{\lambda_1}\right)^{2k+j-1}+O\left(\frac{\lambda_2}{\lambda_1}\right)^{2k}\right)\\
&= x_{3j}C(m_0)\prod_{k=m_0}^{m}\left(1+\left(\frac{(L_2-L_1)B_0}{L_1A_0}+\frac{\lambda_1(N_2-N_1)F_1}{\lambda_2N_1E_1}+\frac{\lambda_1(M_2-M_1)D_{-1}}{\lambda_2M_1C_{-1}}\right)\left(\frac{\lambda_2}{\lambda_1}\right)^{2k+j}+O\left(\frac{\lambda_2}{\lambda_1}\right)^{2k}\right),
\end{aligned}\tag{49}$$

for sufficiently large m, $m \geq m_0$,

$$\begin{aligned}
x_{6m+3j+1} &= x_{3j+1}\prod_{k=1}^{m}\left(\frac{A_1L_1\lambda_1^{2k+j}+B_1L_2\lambda_2^{2k+j}}{A_1\lambda_1^{2k+j}+B_1\lambda_2^{2k+j}}\right)\left(\frac{E_{-1}N_1\lambda_1^{2k+j}+F_{-1}N_2\lambda_2^{2k+j}}{E_{-1}\lambda_1^{2k+j}+F_{-1}\lambda_2^{2k+j}}\right)\left(\frac{C_0M_1\lambda_1^{2k+j-1}+D_0M_2\lambda_2^{2k+j-1}}{C_0\lambda_1^{2k+j-1}+D_0\lambda_2^{2k+j-1}}\right)\\
&= x_{3j+1}C(m_1)\prod_{k=m_1}^{m}L_1M_1N_1\left(1+\frac{(L_2-L_1)B_1}{L_1A_1}\left(\frac{\lambda_2}{\lambda_1}\right)^{2k+j}+O\left(\frac{\lambda_2}{\lambda_1}\right)^{2k}\right)\\
&\qquad\times\left(1+\frac{(N_2-N_1)F_{-1}}{N_1E_{-1}}\left(\frac{\lambda_2}{\lambda_1}\right)^{2k+j}+O\left(\frac{\lambda_2}{\lambda_1}\right)^{2k}\right)\left(1+\frac{(M_2-M_1)D_0}{M_1C_0}\left(\frac{\lambda_2}{\lambda_1}\right)^{2k+j-1}+O\left(\frac{\lambda_2}{\lambda_1}\right)^{2k}\right)\\
&= x_{3j+1}C(m_1)\prod_{k=m_1}^{m}\left(1+\left(\frac{(L_2-L_1)B_1}{L_1A_1}+\frac{(N_2-N_1)F_{-1}}{N_1E_{-1}}+\frac{\lambda_1(M_2-M_1)D_0}{\lambda_2M_1C_0}\right)\left(\frac{\lambda_2}{\lambda_1}\right)^{2k+j}+O\left(\frac{\lambda_2}{\lambda_1}\right)^{2k}\right),
\end{aligned}\tag{50}$$

for sufficiently large m, $m \geq m_1$, and

$$\begin{aligned} x_{6m+3j+2} &= x_{3j+2}\prod_{k=1}^{m}\left(\frac{A_{-1}L_1\lambda_1^{2k+j+1}+B_{-1}L_2\lambda_2^{2k+j+1}}{A_{-1}\lambda_1^{2k+j+1}+B_{-1}\lambda_2^{2k+j+1}}\right)\left(\frac{E_0N_1\lambda_1^{2k+j}+F_0N_2\lambda_2^{2k+j}}{E_0\lambda_1^{2k+j}+F_0\lambda_2^{2k+j}}\right)\left(\frac{C_1M_1\lambda_1^{2k+j-1}+D_1M_2\lambda_2^{2k+j-1}}{C_1\lambda_1^{2k+j-1}+D_1\lambda_2^{2k+j-1}}\right) \\ &= x_{3j+2}C(m_2)\prod_{k=m_2}^{m}L_1M_1N_1\left(1+\frac{(L_2-L_1)B_{-1}}{L_1A_{-1}}\left(\frac{\lambda_2}{\lambda_1}\right)^{2k+j+1}+O\left(\frac{\lambda_2}{\lambda_1}\right)^{2k}\right) \\ &\quad\times\left(1+\frac{(N_2-N_1)F_0}{N_1E_0}\left(\frac{\lambda_2}{\lambda_1}\right)^{2k+j}+O\left(\frac{\lambda_2}{\lambda_1}\right)^{2k}\right)\left(1+\frac{(M_2-M_1)D_1}{M_1C_1}\left(\frac{\lambda_2}{\lambda_1}\right)^{2k+j-1}+O\left(\frac{\lambda_2}{\lambda_1}\right)^{2k}\right) \\ &= x_{3j+2}C(m_2)\prod_{k=m_2}^{m}\left(1+\left(\frac{\lambda_2(L_2-L_1)B_{-1}}{\lambda_1L_1A_{-1}}+\frac{(N_2-N_1)F_0}{N_1E_0}+\frac{\lambda_1(M_2-M_1)D_1}{\lambda_2M_1C_1}\right)\left(\frac{\lambda_2}{\lambda_1}\right)^{2k+j}+O\left(\frac{\lambda_2}{\lambda_1}\right)^{2k}\right), \end{aligned} \tag{51}$$

sufficiently large m, $m \geq m_2$, from which along with the assumptions $L_1M_1N_1 = 1$ and $|\lambda_1| > |\lambda_2|$, the results in (c) can be seen easily.

(d) : Employing the Taylor expansion for $(1+x)^{-1}$, we get, for each $j \in \{-1, 0\}$,

$$\begin{aligned} x_{6m+3j} &= x_{3j}\prod_{k=1}^{m}\left(\frac{A_0L_1\lambda_1^{2k+j}+B_0L_2\lambda_2^{2k+j}}{A_0\lambda_1^{2k+j}+B_0\lambda_2^{2k+j}}\right)\left(\frac{E_1N_1\lambda_1^{2k+j-1}+F_1N_2\lambda_2^{2k+j-1}}{E_1\lambda_1^{2k+j-1}+F_1\lambda_2^{2k+j-1}}\right)\left(\frac{C_{-1}M_1\lambda_1^{2k+j-1}+D_{-1}M_2\lambda_2^{2k+j-1}}{C_{-1}\lambda_1^{2k+j-1}+D_{-1}\lambda_2^{2k+j-1}}\right) \\ &= x_{3j}C(m_0)\prod_{k=m_0}^{m}L_1M_1N_1\left(1-\frac{(L_1-L_2)B_0}{L_1A_0}\left(\frac{\lambda_2}{\lambda_1}\right)^{2k+j}+O\left(\frac{\lambda_2}{\lambda_1}\right)^{2k}\right)\left(1-\frac{(N_1-N_2)F_1}{N_1E_1}\left(\frac{\lambda_2}{\lambda_1}\right)^{2k+j-1}+O\left(\frac{\lambda_2}{\lambda_1}\right)^{2k}\right) \\ &\quad\times\left(1-\frac{(M_1-M_2)D_{-1}}{M_1C_{-1}}\left(\frac{\lambda_2}{\lambda_1}\right)^{2k+j-1}+O\left(\frac{\lambda_2}{\lambda_1}\right)^{2k}\right) \\ &= x_{3j}C(m_0)(-1)^{m-m_0+1}\prod_{k=m_0}^{m}\left(1-\left(\frac{(L_1-L_2)B_0}{L_1A_0}+\frac{(N_1-N_2)F_1\lambda_1}{N_1E_1\lambda_2}+\frac{(M_1-M_2)D_{-1}\lambda_1}{M_1C_{-1}\lambda_2}\right)\left(\frac{\lambda_2}{\lambda_1}\right)^{2k+j}\right. \\ &\qquad\left.+O\left(\frac{\lambda_2}{\lambda_1}\right)^{2k}\right), \end{aligned} \tag{52}$$

for sufficiently large m, $m \geq m_0$,

$$\begin{aligned} x_{6m+3j+1} &= x_{3j+1}\prod_{k=1}^{m}\left(\frac{A_1L_1\lambda_1^{2k+j}+B_1L_2\lambda_2^{2k+j}}{A_1\lambda_1^{2k+j}+B_1\lambda_2^{2k+j}}\right)\left(\frac{E_{-1}N_1\lambda_1^{2k+j}+F_{-1}N_2\lambda_2^{2k+j}}{E_{-1}\lambda_1^{2k+j}+F_{-1}\lambda_2^{2k+j}}\right)\left(\frac{C_0M_1\lambda_1^{2k+j-1}+D_0M_2\lambda_2^{2k+j-1}}{C_0\lambda_1^{2k+j-1}+D_0\lambda_2^{2k+j-1}}\right) \\ &= x_{3j+1}C(m_1)\prod_{k=m_1}^{m}L_1M_1N_1\left(1-\frac{(L_1-L_2)B_1}{L_1A_1}\left(\frac{\lambda_2}{\lambda_1}\right)^{2k+j}+O\left(\frac{\lambda_2}{\lambda_1}\right)^{2k}\right) \\ &\quad\times\left(1-\frac{(N_1-N_2)F_{-1}}{N_1E_{-1}}\left(\frac{\lambda_2}{\lambda_1}\right)^{2k+j}+O\left(\frac{\lambda_2}{\lambda_1}\right)^{2k}\right)\left(1-\frac{(M_1-M_2)D_0}{M_1C_0}\left(\frac{\lambda_2}{\lambda_1}\right)^{2k+j-1}+O\left(\frac{\lambda_2}{\lambda_1}\right)^{2k}\right) \\ &= x_{3j+1}C(m_1)(-1)^{m-m_1+1}\prod_{k=m_1}^{m}\left(1-\left(\frac{(L_1-L_2)B_1}{L_1A_1}+\frac{(N_1-N_2)F_{-1}}{N_1E_{-1}}+\frac{(M_1-M_2)D_0\lambda_1}{M_1C_0\lambda_2}\right)\left(\frac{\lambda_2}{\lambda_1}\right)^{2k+j}\right. \\ &\qquad\left.+O\left(\frac{\lambda_2}{\lambda_1}\right)^{2k}\right), \end{aligned} \tag{53}$$

for sufficiently large m, $m \geq m_1$, and

$$\begin{aligned} x_{6m+3j+2} &= x_{3j+2}\prod_{k=1}^{m}\left(\frac{A_{-1}L_1\lambda_1^{2k+j+1}+B_{-1}L_2\lambda_2^{2k+j+1}}{A_{-1}\lambda_1^{2k+j+1}+B_{-1}\lambda_2^{2k+j+1}}\right)\left(\frac{E_0N_1\lambda_1^{2k+j}+F_0N_2\lambda_2^{2k+j}}{E_0\lambda_1^{2k+j}+F_0\lambda_2^{2k+j}}\right)\left(\frac{C_1M_1\lambda_1^{2k+j-1}+D_1M_2\lambda_2^{2k+j-1}}{C_1\lambda_1^{2k+j-1}+D_1\lambda_2^{2k+j-1}}\right) \\ &= x_{3j+2}C(m_2)\prod_{k=m_2}^{m}L_1M_1N_1\left(1-\frac{(L_1-L_2)B_{-1}}{L_1A_{-1}}\left(\frac{\lambda_2}{\lambda_1}\right)^{2k+j+1}+O\left(\frac{\lambda_2}{\lambda_1}\right)^{2k}\right) \\ &\qquad\times\left(1-\frac{(N_1-N_2)F_0}{N_1E_0}\left(\frac{\lambda_2}{\lambda_1}\right)^{2k+j}+O\left(\frac{\lambda_2}{\lambda_1}\right)^{2k}\right)\left(1-\frac{(M_1-M_2)D_1}{M_1C_1}\left(\frac{\lambda_2}{\lambda_1}\right)^{2k+j-1}+O\left(\frac{\lambda_2}{\lambda_1}\right)^{2k}\right) \\ &= x_{3j+2}C(m_2)(-1)^{m-m_2+1}\prod_{k=m_2}^{m}\left(1-\left(\frac{(L_1-L_2)\lambda_2B_{-1}}{L_1A_{-1}\lambda_1}+\frac{(N_1-N_2)F_0}{N_1E_0}+\frac{(M_1-M_2)D_1\lambda_1}{M_1C_1\lambda_2}\right)\left(\frac{\lambda_2}{\lambda_1}\right)^{2k+j}\right. \\ &\qquad\left.+O\left(\frac{\lambda_2}{\lambda_1}\right)^{2k}\right), \end{aligned} \tag{54}$$

for sufficiently large m, $m \geq m_2$, from which along with the assumptions $L_1M_1N_1 = -1$ and $|\lambda_1| > |\lambda_2|$, the results in (d) can be seen easily. Similarly, one can easily prove that $(y_n)_{n\geq-3}$ and $(z_n)_{n\geq-3}$ are convergent when $|\lambda_1| > |\lambda_2|$ and $L_1M_1N_1 = 1$, and $y_{6m+3j+i+1}$ and $z_{6m+3j+i+1}$ are convergent when $|\lambda_1| > |\lambda_2|$ and $L_1M_1N_1 = -1$, which completes the proof. □

Let

$$\begin{aligned} A_{1,i} &= (acf+bf)^2L_1\left(\frac{x_i}{z_{i-2}}-L_1\right), \\ B_{1,i} &= \left(\frac{x_i}{z_{i-2}}-L_1-\frac{\lambda_1}{acf+bf}\right)(ace+ad+be)(acf+bf)+\lambda_1^2, \\ C_{1,i} &= (acf+bf)^2\left(\frac{x_i}{z_{i-2}}-L_1\right), \\ D_{1,i} &= (acf+bf)^2\left(L_1-\frac{x_i}{z_{i-2}}\right)+(acf+bf)\lambda_1, \end{aligned}$$

$$\begin{aligned} A_{2,i} &= (bce+bd)^2M_1\left(\frac{y_i}{x_{i-2}}-M_1\right), \\ B_{2,i} &= \left(\frac{y_i}{x_{i-2}}-M_1-\frac{\lambda_1}{bce+bd}\right)(ace+ad+cf)(bce+bd)+\lambda_1^2, \\ C_{2,i} &= (bce+bd)^2\left(\frac{y_i}{x_{i-2}}-M_1\right), \\ D_{2,i} &= (bce+bd)^2\left(M_1-\frac{y_i}{x_{i-2}}\right)+(bce+bd)\lambda_1, \end{aligned}$$

$$\begin{aligned} A_{3,i} &= (ade+df)^2N_1\left(\frac{z_i}{y_{i-2}}-N_1\right), \\ B_{3,i} &= \left(\frac{z_i}{y_{i-2}}-N_1-\frac{\lambda_1}{ade+df}\right)(ace+be+cf)(ade+df)+\lambda_1^2, \\ C_{3,i} &= (ade+df)^2\left(\frac{z_i}{y_{i-2}}-N_1\right), \\ D_{3,i} &= (ade+df)^2\left(N_1-\frac{z_i}{y_{i-2}}\right)+(ade+df)\lambda_1, \end{aligned}$$

where $i \in \{-1, 0, 1\}$,

$$K_{j_1,i} := \frac{A_{j_1,i+1}}{C_{j_1,i+1}} \cdot \frac{A_{j_1+1,i+3}}{C_{j_1+1,i+3}} \cdot \frac{A_{j_1+2,i+2}}{C_{j_1+2,i+2}},$$

$$P_{j_1,i} := \frac{B_{j_1,i+1}}{A_{j_1,i+1}} - \frac{D_{j_1,i+1}}{C_{j_1,i+1}} + \frac{B_{j_1+1,i+3}}{A_{j_1+1,i+3}} - \frac{D_{j_1+1,i+3}}{C_{j_1+1,i+3}} + \frac{B_{j_1+2,i+2}}{A_{j_1+2,i+2}} - \frac{D_{j_1+2,i+2}}{C_{j_1+2,i+2}},$$

where $j_1 \in \{1, 2, 3\}$ and $i \in \{-1, 0, 1\}$. Throughout the manuscript, we assume that $X_{j_1+k_1,i+j_1} = X_{j_2,i_1}$, where X represents one of the A, B, C, D and $j_2 := \begin{cases} 3, & j_1 + k_1 \equiv 0 \bmod(3) \\ 1, & j_1 + k_1 \equiv 1 \bmod(3) \\ 2, & j_1 + k_1 \equiv 2 \bmod(3) \end{cases}$, $i_1 := \begin{cases} 0, & i + j_1 \equiv 0 \bmod(3) \\ 1, & i + j_1 \equiv 1 \bmod(3) \\ -1, & i + j_1 \equiv 2 \bmod(3) \end{cases}$ and $k_1 \in \{0, 1, 2\}$.

Theorem 2.2. *Assume that* $(ace + ad + be + cf)^2 + 4bdf = 0$, $abcdef \neq 0$, $A_{j_1,i+1}$, $C_{j_1,i+1}$, $A_{j_1+1,i+3}$, $C_{j_1+1,i+3}$, $A_{j_1+2,i+2}$, $C_{j_1+2,i+2} \in \mathbb{R} \setminus \{0\}$, *for* $i \in \{-1, 0, 1\}$ *and* $j_1 \in \{1, 2, 3\}$, *and that* $(x_n, y_n, z_n)_{n \geq -3}$ *is a well-defined solution of system (6). Then the following results are true.*

(a) If $|K_{1,i}| < 1$, *then* $x_n \to 0$ *as* $n \to \infty$.

(b) If $|K_{1,i}| > 1$, *then* $|x_n| \to \infty$ *as* $n \to \infty$.

(c) If $K_{1,i} = 1$ *and* $P_{1,i} < 0$, *then* $x_n \to 0$ *as* $n \to \infty$.

(d) If $K_{1,i} = 1$ *and* $P_{1,i} > 0$, *then* $|x_n| \to \infty$ *as* $n \to \infty$.

(e) If $K_{1,i} = 1$ *and* $P_{1,i} = 0$, *then the sequences* $\left(x_{6m+3j+i+1}\right)_{m \in \mathbb{N}_0}$, *for* $m \in \mathbb{N}_0$, $j \in \{-1, 0\}$, $i \in \{-1, 0, 1\}$, *are convergent.*

(f) If $K_{1,i} = -1$ *and* $P_{1,i} < 0$, *then* $x_n \to 0$ *as* $n \to \infty$.

(g) If $K_{1,i} = -1$ *and* $P_{1,i} > 0$, *then* $|x_n| \to \infty$ *as* $n \to \infty$.

(h) If $K_{1,i} = -1$ *and* $P_{1,i} = 0$, *then* x_{12m+j_1}, *for* $m \in \mathbb{N}_0$, $j_1 \in \{-3, -2, \ldots, 8\}$, *are convergent.*

(i) If $|K_{2,i}| < 1$, *then* $y_n \to 0$ *as* $n \to \infty$.

(j) If $|K_{2,i}| > 1$, *then* $|y_n| \to \infty$ *as* $n \to \infty$.

(k) If $K_{2,i} = 1$ *and* $P_{2,i} < 0$, *then* $y_n \to 0$ *as* $n \to \infty$.

(l) If $K_{2,i} = 1$ *and* $P_{2,i} > 0$, *then* $|y_n| \to \infty$ *as* $n \to \infty$.

(m) If $K_{2,i} = 1$ *and* $P_{1,i} = 0$, *then the sequences* $\left(y_{6m+3j+i+1}\right)_{m \in \mathbb{N}_0}$, *for* $m \in \mathbb{N}_0$, $j \in \{-1, 0\}$, $i \in \{-1, 0, 1\}$, *are convergent.*

(n) If $K_{2,i} = -1$ *and* $P_{2,i} < 0$, *then* $y_n \to 0$ *as* $n \to \infty$.

(o) If $K_{2,i} = -1$ *and* $P_{2,i} > 0$, *then* $|y_n| \to \infty$ *as* $n \to \infty$.

(p) If $K_{2,i} = -1$ *and* $P_{2,i} = 0$, *then* y_{12m+j_1}, *for* $m \in \mathbb{N}_0$, $j_1 \in \{-3, -2, \ldots, 8\}$, *are convergent.*

(q) If $|K_{3,i}| < 1$, *then* $z_n \to 0$ *as* $n \to \infty$.

(r) If $|K_{3,i}| > 1$, *then* $|z_n| \to \infty$ *as* $n \to \infty$.

(s) If $K_{3,i} = 1$ *and* $P_{3,i} < 0$, *then* $z_n \to 0$ *as* $n \to \infty$.

(t) If $K_{3,i} = 1$ *and* $P_{3,i} > 0$, *then* $|z_n| \to \infty$ *as* $n \to \infty$.

(u) If $K_{3,i} = 1$ and $P_{3,i} = 0$, then the sequences $\left(z_{6m+3j+i+1}\right)_{m\in\mathbb{N}_0}$, for $m \in \mathbb{N}_0$, $j \in \{-1,0\}$, $i \in \{-1,0,1\}$, are convergent.

(v) If $K_{3,i} = -1$ and $P_{3,i} < 0$, then $z_n \to 0$ as $n \to \infty$.

(w) If $K_{3,i} = -1$ and $P_{3,i} > 0$, then $|z_n| \to \infty$ as $n \to \infty$.

(y) If $K_{3,i} = -1$ and $P_{3,i} = 0$, then z_{12m+j_1}, for $m \in \mathbb{N}_0$, $j_1 \in \{-3,-2,\ldots,8\}$, are convergent.

Proof. $(a),(b)$: Employing the following facts

$$\lim_{m_1\to\infty}\left|\frac{A_{1,0}(2m_1+j+1)+B_{1,0}}{C_{1,0}(2m_1+j+1)+D_{1,0}}\cdot\frac{A_{3,1}(2m_1+j)+B_{3,1}}{C_{3,1}(2m_1+j)+D_{3,1}}\cdot\frac{A_{2,-1}(2m_1+j)+B_{2,-1}}{C_{2,-1}(2m_1+j)+D_{2,-1}}\right| = K_{1,-1}, \tag{55}$$

$$\lim_{m_1\to\infty}\left|\frac{A_{1,1}(2m_1+j+1)+B_{1,1}}{C_{1,1}(2m_1+j+1)+D_{1,1}}\cdot\frac{A_{3,-1}(2m_1+j+1)+B_{3,-1}}{C_{3,-1}(2m_1+j+1)+D_{3,-1}}\cdot\frac{A_{2,0}(2m_1+j)+B_{2,0}}{C_{2,0}(2m_1+j)+D_{2,0}}\right| = K_{1,0}, \tag{56}$$

$$\lim_{m_1\to\infty}\left|\frac{A_{1,-1}(2m_1+j+2)+B_{1,-1}}{C_{1,-1}(2m_1+j+2)+D_{1,-1}}\cdot\frac{A_{3,0}(2m_1+j+1)+B_{3,0}}{C_{3,0}(2m_1+j+1)+D_{3,0}}\cdot\frac{A_{2,1}(2m_1+j)+B_{2,1}}{C_{2,1}(2m_1+j)+D_{2,1}}\right| = K_{1,1}, \tag{57}$$

for every $j \in \{-1,0\}$, in (44), the results follow from the assumption in (a) and (b).
(c)-(e) : For each $j \in \{-1,0\}$ and sufficiently large m, we have

$$\begin{aligned}
x_{6m+3j} &= x_{3j}\prod_{k=1}^{m}\left(\frac{A_{1,0}(2k+j+1)+B_{1,0}}{C_{1,0}(2k+j+1)+D_{1,0}}\right)\left(\frac{A_{3,1}(2k+j)+B_{3,1}}{C_{3,1}(2k+j)+D_{3,1}}\right)\left(\frac{A_{2,-1}(2k+j)+B_{2,-1}}{C_{2,-1}(2k+j)+D_{2,-1}}\right)\\
&= x_{3j}\prod_{k=1}^{m}\left(1+\frac{\frac{1}{2k+j+1}\left(\frac{B_{1,0}}{A_{1,0}}-\frac{D_{1,0}}{C_{1,0}}\right)}{1+\frac{D_{1,0}}{(2k+j+1)C_{1,0}}}\right)\left(1+\frac{\frac{1}{2k+j}\left(\frac{B_{3,1}}{A_{3,1}}-\frac{D_{3,1}}{C_{3,1}}\right)}{1+\frac{D_{3,1}}{(2k+j)C_{3,1}}}\right)\left(1+\frac{\frac{1}{2k+j}\left(\frac{B_{2,-1}}{A_{2,-1}}-\frac{D_{2,-1}}{C_{2,-1}}\right)}{1+\frac{D_{2,-1}}{(2k+j)C_{2,-1}}}\right)\\
&= x_{3j}\prod_{k=1}^{m}\left(1+\left(\frac{B_{1,0}}{A_{1,0}}-\frac{D_{1,0}}{C_{1,0}}\right)\frac{1}{2k}+O\left(\frac{1}{k^2}\right)\right)\left(1+\left(\frac{B_{3,1}}{A_{3,1}}-\frac{D_{3,1}}{C_{3,1}}\right)\frac{1}{2k}+O\left(\frac{1}{k^2}\right)\right)\\
&\qquad\times\left(1+\left(\frac{B_{2,-1}}{A_{2,-1}}-\frac{D_{2,-1}}{C_{2,-1}}\right)\frac{1}{2k}+O\left(\frac{1}{k^2}\right)\right)\\
&= x_{3j}\prod_{k=1}^{m}\left(1+\left(\frac{B_{1,0}}{A_{1,0}}-\frac{D_{1,0}}{C_{1,0}}+\frac{B_{3,1}}{A_{3,1}}-\frac{D_{3,1}}{C_{3,1}}+\frac{B_{2,-1}}{A_{2,-1}}-\frac{D_{2,-1}}{C_{2,-1}}\right)\frac{1}{2k}+O\left(\frac{1}{k^2}\right)\right),
\end{aligned} \tag{58}$$

for every $j \in \{-1,0\}$,

$$\begin{aligned}
x_{6m+3j+1} &= x_{3j+1}\prod_{k=1}^{m}\left(\frac{A_{1,1}(2k+j+1)+B_{1,1}}{C_{1,1}(2k+j+1)+D_{1,1}}\right)\left(\frac{A_{3,-1}(2k+j+1)+B_{3,-1}}{C_{3,-1}(2k+j+1)+D_{3,-1}}\right)\left(\frac{A_{2,0}(2k+j)+B_{2,0}}{C_{2,0}(2k+j)+D_{2,0}}\right)\\
&= x_{3j+1}\prod_{k=1}^{m}\left(1+\frac{\frac{1}{2k+j+1}\left(\frac{B_{1,1}}{A_{1,1}}-\frac{D_{1,1}}{C_{1,1}}\right)}{1+\frac{D_{1,1}}{(2k+j+1)C_{1,1}}}\right)\left(1+\frac{\frac{1}{2k+j+1}\left(\frac{B_{3,-1}}{A_{3,-1}}-\frac{D_{3,-1}}{C_{3,-1}}\right)}{1+\frac{D_{3,-1}}{(2k+j+1)C_{3,-1}}}\right)\left(1+\frac{\frac{1}{2k+j}\left(\frac{B_{2,0}}{A_{2,0}}-\frac{D_{2,0}}{C_{2,0}}\right)}{1+\frac{D_{2,0}}{(2k+j)C_{2,0}}}\right)\\
&= x_{3j+1}\prod_{k=1}^{m}\left(1+\left(\frac{B_{1,1}}{A_{1,1}}-\frac{D_{1,1}}{C_{1,1}}\right)\frac{1}{2k}+O\left(\frac{1}{k^2}\right)\right)\left(1+\left(\frac{B_{3,-1}}{A_{3,-1}}-\frac{D_{3,-1}}{C_{3,-1}}\right)\frac{1}{2k}+O\left(\frac{1}{k^2}\right)\right)\\
&\qquad\times\left(1+\left(\frac{B_{2,0}}{A_{2,0}}-\frac{D_{2,0}}{C_{2,0}}\right)\frac{1}{2k}+O\left(\frac{1}{k^2}\right)\right)\\
&= x_{3j+1}\prod_{k=1}^{m}\left(1+\left(\frac{B_{1,1}}{A_{1,1}}-\frac{D_{1,1}}{C_{1,1}}+\frac{B_{3,-1}}{A_{3,-1}}-\frac{D_{3,-1}}{C_{3,-1}}+\frac{B_{2,0}}{A_{2,0}}-\frac{D_{2,0}}{C_{2,0}}\right)\frac{1}{2k}+O\left(\frac{1}{k^2}\right)\right),
\end{aligned} \tag{59}$$

for every $j \in \{-1, 0\}$, and

$$\begin{aligned} x_{6m+3j+2} &= x_{3j+2} \prod_{k=1}^{m} \left(\frac{A_{1,-1}(2k+j+2)+B_{1,-1}}{C_{1,-1}(2k+j+2)+D_{1,-1}} \right) \left(\frac{A_{3,0}(2k+j+1)+B_{3,0}}{C_{3,0}(2k+j+1)+D_{3,0}} \right) \left(\frac{A_{2,1}(2k+j)+B_{2,1}}{C_{2,1}(2k+j)+D_{2,1}} \right) \\ &= x_{3j+2} \prod_{k=1}^{m} \left(1 + \frac{\frac{1}{2k+j+2}\left(\frac{B_{1,-1}}{A_{1,-1}} - \frac{D_{1,-1}}{C_{1,-1}}\right)}{1+\frac{D_{1,-1}}{(2k+j+2)C_{1,-1}}} \right) \left(1 + \frac{\frac{1}{2k+j+1}\left(\frac{B_{3,0}}{A_{3,0}} - \frac{D_{3,0}}{C_{3,0}}\right)}{1+\frac{D_{3,0}}{(2k+j+1)C_{3,0}}} \right) \left(1 + \frac{\frac{1}{2k+j}\left(\frac{B_{2,1}}{A_{2,1}} - \frac{D_{2,1}}{C_{2,1}}\right)}{1+\frac{D_{2,1}}{(2k+j)C_{2,1}}} \right) \\ &= x_{3j+2} \prod_{k=1}^{m} \left(1 + \left(\frac{B_{1,-1}}{A_{1,-1}} - \frac{D_{1,-1}}{C_{1,-1}}\right)\frac{1}{2k} + O\left(\frac{1}{k^2}\right) \right) \left(1 + \left(\frac{B_{3,0}}{A_{3,0}} - \frac{D_{3,0}}{C_{3,0}}\right)\frac{1}{2k} + O\left(\frac{1}{k^2}\right) \right) \\ &\qquad \times \left(1 + \left(\frac{B_{2,1}}{A_{2,1}} - \frac{D_{2,1}}{C_{2,1}}\right)\frac{1}{2k} + O\left(\frac{1}{k^2}\right) \right) \\ &= x_{3j+2} \prod_{k=1}^{m} \left(1 + \left(\frac{B_{1,-1}}{A_{1,-1}} - \frac{D_{1,-1}}{C_{1,-1}} + \frac{B_{3,0}}{A_{3,0}} - \frac{D_{3,0}}{C_{3,0}} + \frac{B_{2,1}}{A_{2,1}} - \frac{D_{2,1}}{C_{2,1}}\right)\frac{1}{2k} + O\left(\frac{1}{k^2}\right) \right), \end{aligned} \tag{60}$$

for every $j \in \{-1, 0\}$. From (58), (59), (60) and the relations $\Sigma_{j_1=1}^{m}(1/j_1) \to \infty$ as $m \to \infty$, the results easily follow in these cases.

(f)-(h) : For each $j \in \{-1, 0\}$ and sufficiently large m, we obtain

$$\begin{aligned} x_{6m+3j} &= x_{3j} \prod_{k=1}^{m} \left(\frac{A_{1,0}(2k+j+1)+B_{1,0}}{C_{1,0}(2k+j+1)+D_{1,0}} \right) \left(\frac{A_{3,1}(2k+j)+B_{3,1}}{C_{3,1}(2k+j)+D_{3,1}} \right) \left(\frac{A_{2,-1}(2k+j)+B_{2,-1}}{C_{2,-1}(2k+j)+D_{2,-1}} \right) \\ &= x_{3j} \prod_{k=1}^{m} \left(-1 + \frac{\frac{1}{2k+j+1}\left(-\frac{B_{1,0}}{A_{1,0}} + \frac{D_{1,0}}{C_{1,0}}\right)}{1+\frac{D_{1,0}}{(2k+j+1)C_{1,0}}} \right) \left(-1 + \frac{\frac{1}{2k+j}\left(-\frac{B_{3,1}}{A_{3,1}} + \frac{D_{3,1}}{C_{3,1}}\right)}{1+\frac{D_{3,1}}{(2k+j)C_{3,1}}} \right) \left(-1 + \frac{\frac{1}{2k+j}\left(-\frac{B_{2,-1}}{A_{2,-1}} + \frac{D_{2,-1}}{C_{2,-1}}\right)}{1+\frac{D_{2,-1}}{(2k+j)C_{2,-1}}} \right) \\ &= x_{3j} \prod_{k=1}^{m} \left(-1 - \left(\frac{B_{1,0}}{A_{1,0}} - \frac{D_{1,0}}{C_{1,0}}\right)\frac{1}{2k} + O\left(\frac{1}{k^2}\right) \right) \left(-1 - \left(\frac{B_{3,1}}{A_{3,1}} - \frac{D_{3,1}}{C_{3,1}}\right)\frac{1}{2k} + O\left(\frac{1}{k^2}\right) \right) \\ &\qquad \times \left(-1 - \left(\frac{B_{2,-1}}{A_{2,-1}} - \frac{D_{2,-1}}{C_{2,-1}}\right)\frac{1}{2k} + O\left(\frac{1}{k^2}\right) \right) \\ &= x_{3j}(-1)^m \prod_{k=1}^{m} \left(1 + \left(\frac{B_{1,0}}{A_{1,0}} - \frac{D_{1,0}}{C_{1,0}} + \frac{B_{3,1}}{A_{3,1}} - \frac{D_{3,1}}{C_{3,1}} + \frac{B_{2,-1}}{A_{2,-1}} - \frac{D_{2,-1}}{C_{2,-1}}\right)\frac{1}{2k} + O\left(\frac{1}{k^2}\right) \right), \end{aligned} \tag{61}$$

$$
\begin{aligned}
x_{6m+3j+1} &= x_{3j+1}\prod_{k=1}^{m}\left(\frac{A_{1,1}(2k+j+1)+B_{1,1}}{C_{1,1}(2k+j+1)+D_{1,1}}\right)\left(\frac{A_{3,-1}(2k+j+1)+B_{3,-1}}{C_{3,-1}(2k+j+1)+D_{3,-1}}\right)\left(\frac{A_{2,0}(2k+j)+B_{2,0}}{C_{2,0}(2k+j)+D_{2,0}}\right)\\
&= x_{3j+1}\prod_{k=1}^{m}\left(-1+\frac{\frac{1}{2k+j+1}\left(-\frac{B_{1,1}}{A_{1,1}}+\frac{D_{1,1}}{C_{1,1}}\right)}{1+\frac{D_{1,1}}{(2k+j+1)C_{1,1}}}\right)\left(-1+\frac{\frac{1}{2k+j+1}\left(-\frac{B_{3,-1}}{A_{3,-1}}+\frac{D_{3,-1}}{C_{3,-1}}\right)}{1+\frac{D_{3,-1}}{(2k+j+1)C_{3,-1}}}\right)\\
&\quad\times\left(-1+\frac{\frac{1}{2k+j}\left(-\frac{B_{2,0}}{A_{2,0}}+\frac{D_{2,0}}{C_{2,0}}\right)}{1+\frac{D_{2,0}}{(2k+j)C_{2,0}}}\right)\\
&= x_{3j+1}\prod_{k=1}^{m}\left(-1-\left(\frac{B_{1,1}}{A_{1,1}}-\frac{D_{1,1}}{C_{1,1}}\right)\frac{1}{2k}+O\left(\frac{1}{k^2}\right)\right)\left(-1-\left(\frac{B_{3,-1}}{A_{3,-1}}-\frac{D_{3,-1}}{C_{3,-1}}\right)\frac{1}{2k}+O\left(\frac{1}{k^2}\right)\right)\\
&\quad\times\left(-1-\left(\frac{B_{2,0}}{A_{2,0}}-\frac{D_{2,0}}{C_{2,0}}\right)\frac{1}{2k}+O\left(\frac{1}{k^2}\right)\right)\\
&= x_{3j+1}(-1)^m\prod_{k=1}^{m}\left(1+\left(\frac{B_{1,1}}{A_{1,1}}-\frac{D_{1,1}}{C_{1,1}}+\frac{B_{3,-1}}{A_{3,-1}}-\frac{D_{3,-1}}{C_{3,-1}}+\frac{B_{2,0}}{A_{2,0}}-\frac{D_{2,0}}{C_{2,0}}\right)\frac{1}{2k}+O\left(\frac{1}{k^2}\right)\right),
\end{aligned}
\tag{62}
$$

and

$$
\begin{aligned}
x_{6m+3j+2} &= x_{3j+2}\prod_{k=1}^{m}\left(\frac{A_{1,-1}(2k+j+2)+B_{1,-1}}{C_{1,-1}(2k+j+2)+D_{1,-1}}\right)\left(\frac{A_{3,0}(2k+j+1)+B_{3,0}}{C_{3,0}(2k+j+1)+D_{3,0}}\right)\left(\frac{A_{2,1}(2k+j)+B_{2,1}}{C_{2,1}(2k+j)+D_{2,1}}\right)\\
&= x_{3j+2}\prod_{k=1}^{m}\left(-1+\frac{\frac{1}{2k+j+2}\left(-\frac{B_{1,-1}}{A_{1,-1}}+\frac{D_{1,-1}}{C_{1,-1}}\right)}{1+\frac{D_{1,-1}}{(2k+j+2)C_{1,-1}}}\right)\left(-1+\frac{\frac{1}{2k+j+1}\left(-\frac{B_{3,0}}{A_{3,0}}+\frac{D_{3,0}}{C_{3,0}}\right)}{1+\frac{D_{3,0}}{(2k+j+1)C_{3,0}}}\right)\\
&\quad\times\left(-1+\frac{\frac{1}{2k+j}\left(-\frac{B_{2,1}}{A_{2,1}}+\frac{D_{2,1}}{C_{2,1}}\right)}{1+\frac{D_{2,1}}{(2k+j)C_{2,1}}}\right)\\
&= x_{3j+2}\prod_{k=1}^{m}\left(-1-\left(\frac{B_{1,-1}}{A_{1,-1}}-\frac{D_{1,-1}}{C_{1,-1}}\right)\frac{1}{2k}+O\left(\frac{1}{k^2}\right)\right)\left(-1-\left(\frac{B_{3,0}}{A_{3,0}}-\frac{D_{3,0}}{C_{3,0}}\right)\frac{1}{2k}+O\left(\frac{1}{k^2}\right)\right)\\
&\quad\times\left(-1-\left(\frac{B_{2,1}}{A_{2,1}}-\frac{D_{2,1}}{C_{2,1}}\right)\frac{1}{2k}+O\left(\frac{1}{k^2}\right)\right)\\
&= x_{3j+2}(-1)^m\prod_{k=1}^{m}\left(1+\left(\frac{B_{1,-1}}{A_{1,-1}}-\frac{D_{1,-1}}{C_{1,-1}}+\frac{B_{3,0}}{A_{3,0}}-\frac{D_{3,0}}{C_{3,0}}+\frac{B_{2,1}}{A_{2,1}}-\frac{D_{2,1}}{C_{2,1}}\right)\frac{1}{2k}+O\left(\frac{1}{k^2}\right)\right).
\end{aligned}
\tag{63}
$$

From (61), (62), (63) and using the fact that $\Sigma_{j_1=1}^{m}(1/j_1)\to\infty$ as $m\to\infty$, then the statements easily follows. Proofs of the (i)-(y) are not given in here since they could be obtained similar with proofs of the (a)-(h). □

The following theorem gives us the forbidden set of the initial values for system (6).

Theorem 2.3. *The forbidden set of the initial values for system (6) is given by the set*

$$\begin{aligned}\mathcal{F} &= \bigcup_{m\in\mathbb{N}_0}\Big\{(x_{-3},x_{-2},x_{-1},y_{-3},y_{-2},y_{-1},z_{-3},z_{-2},z_{-1})\in\mathbb{R}^9:\ \frac{x_{-1}}{z_{-3}}=(h\circ g\circ f)^{-m}\left(-\frac{e}{f}\right)\ \ or\\
&\frac{x_{-1}}{z_{-3}}=(h\circ g\circ f)^{-m}\left(-\frac{ce+d}{cf}\right)\ \ or\ \ \frac{x_{-1}}{z_{-3}}=(h\circ g\circ f)^{-m}\left(-\frac{ace+ad+be}{acf+bf}\right)\ \ or\\
&\frac{y_{-1}}{x_{-3}}=(g\circ f\circ h)^{-m}\left(-\frac{a}{b}\right)\ \ or\ \ \frac{y_{-1}}{x_{-3}}=(g\circ f\circ h)^{-m}\left(-\frac{ae+f}{be}\right)\ \ or\\
&\frac{y_{-1}}{x_{-3}}=(g\circ f\circ h)^{-m}\left(-\frac{ace+ad+cf}{bce+bd}\right)\ \ or\ \ \frac{z_{-1}}{y_{-3}}=(f\circ h\circ g)^{-m}\left(-\frac{c}{d}\right)\ \ or\\
&\frac{z_{-1}}{y_{-3}}=(f\circ h\circ g)^{-m}\left(-\frac{ac+b}{ad}\right)\ \ or\ \ \frac{z_{-1}}{y_{-3}}=(f\circ h\circ g)^{-m}\left(-\frac{ace+be+cf}{ade+df}\right)\Big\}\\
&\bigcup\Big\{(x_{-3},x_{-2},x_{-1},y_{-3},y_{-2},y_{-1},z_{-3},z_{-2},z_{-1})\in\mathbb{R}^9: x_{-3}=0\ or\ x_{-2}=0\ or\ x_{-1}=0\ or\ y_{-3}=0\ or\\
&\quad y_{-2}=0\ or\ y_{-1}=0\ or\ z_{-3}=0\ or\ z_{-2}=0\ or\ z_{-1}=0\Big\}\end{aligned}\tag{64}$$

Proof. At the begininig of Section 2, we have obtained that the set

$$\Big\{(x_{-3},x_{-2},x_{-1},y_{-3},y_{-2},y_{-1},z_{-3},z_{-2},z_{-1})\in\mathbb{R}^9: x_{-3}=0 \text{ or } x_{-2}=0 \text{ or } x_{-1}=0 \text{ or } y_{-3}=0 \text{ or } y_{-2}=0 \text{ or } y_{-1}=0 \text{ or } z_{-3}=0 \text{ or } z_{-2}=0 \text{ or } z_{-1}=0\Big\}$$

belongs to the forbidden set of the initial values for system (6). If $x_{-i}\neq 0$, $y_{-i}\neq 0$ and $z_{-i}\neq 0$, $i\in\{1,2,3\}$, then system (6) is undefined if and only if

$$ax_{n-3}+by_{n-1}=0,\ cy_{n-3}+dz_{n-1}=0,\ ez_{n-3}+fx_{n-1}=0,\tag{65}$$

for some $n\in\mathbb{N}_0$. By taking into account the change of variables (11), we can write the corresponding conditions

$$u_{n-1}=-\frac{e}{f},\ v_{n-1}=-\frac{a}{b} \text{ and } w_{n-1}=-\frac{c}{d},\ n\in\mathbb{N}_0.\tag{66}$$

Therefore we can determine the forbidden set of the initial values for system (6) by using system (12). We know that the statements

$$u_{3m-1} = (h\circ g\circ f)^m(u_{-1})\tag{67}$$
$$u_{3m} = (h\circ g\circ f)^m\circ h(v_{-1})\tag{68}$$
$$u_{3m+1} = (h\circ g\circ f)^m\circ h\circ g(w_{-1})\tag{69}$$
$$v_{3m-1} = (g\circ f\circ h)^m(v_{-1})\tag{70}$$
$$v_{3m} = (g\circ f\circ h)^m\circ g(w_{-1})\tag{71}$$
$$v_{3m+1} = (g\circ f\circ h)^m\circ g\circ f(u_{-1})\tag{72}$$
$$w_{3m-1} = (f\circ h\circ g)^m(w_{-1})\tag{73}$$
$$w_{3m} = (f\circ h\circ g)^m\circ f(u_{-1})\tag{74}$$
$$w_{3m+1} = (f\circ h\circ g)^m\circ f\circ h(v_{-1})\tag{75}$$

where $f(x)=\frac{1}{e+fx}$, $g(x)=\frac{1}{c+dx}$ and $h(x)=\frac{1}{a+bx}$, characterize the solutions of system (12). By using the conditions (66) and the statements (67)-(75), we have, for some $m\in\mathbb{N}_0$,

$$u_{-1}=(h\circ g\circ f)^{-m}\left(-\frac{e}{f}\right),\tag{76}$$

$$v_{-1} = (g \circ f \circ h)^{-m} \circ h^{-1}\left(-\frac{e}{f}\right) = (g \circ f \circ h)^{-m}\left(-\frac{ae+f}{be}\right), \tag{77}$$

$$w_{-1} = (f \circ h \circ g)^{-m} \circ (h \circ g)^{-1}\left(-\frac{e}{f}\right) = (f \circ h \circ g)^{-m}\left(-\frac{ace+be+cf}{ade+df}\right), \tag{78}$$

$$v_{-1} = (g \circ f \circ h)^{-m}\left(-\frac{a}{b}\right), \tag{79}$$

$$w_{-1} = (f \circ h \circ g)^{-m} \circ g^{-1}\left(-\frac{a}{b}\right) = (f \circ h \circ g)^{-m}\left(-\frac{ac+b}{ad}\right), \tag{80}$$

$$u_{-1} = (h \circ g \circ f)^{-m} \circ (g \circ f)^{-1}\left(-\frac{a}{b}\right) = (h \circ g \circ f)^{-m}\left(-\frac{ace+ad+be}{acf+bf}\right), \tag{81}$$

$$w_{-1} = (f \circ h \circ g)^{-m}\left(-\frac{c}{d}\right), \tag{82}$$

$$u_{-1} = (h \circ g \circ f)^{-m} \circ f^{-1}\left(-\frac{c}{d}\right) = (h \circ g \circ f)^{-m}\left(-\frac{ce+d}{cf}\right), \tag{83}$$

$$v_{-1} = (g \circ f \circ h)^{-m} \circ (f \circ h)^{-1}\left(-\frac{c}{d}\right) = (g \circ f \circ h)^{-m}\left(-\frac{ace+ad+cf}{bce+bd}\right), \tag{84}$$

where $abcdef \neq 0$, $ade+df \neq 0$, $acf+bf \neq 0$ and $bce+bd \neq 0$. Also, let us indicate that the backward solutions of Eq. (12) are the forward solutions of the system

$$t_n = (h \circ g \circ f)^{-1}(t_{n-1}),\ \widetilde{t_n} = (g \circ f \circ h)^{-1}\left(\widetilde{t_{n-1}}\right),\ \widehat{t_n} = (f \circ h \circ g)^{-1}\left(\widehat{t_{n-1}}\right),\ n \in \mathbb{N}_0, \tag{85}$$

which corresponds the system

$$\begin{aligned} t_n &= \frac{-(ace+ad+be)\,t_{n-3}+ce+d}{(acf+bf)\,t_{n-3}-cf}, \\ \widetilde{t_n} &= \frac{-(ace+ad+cf)\,\widetilde{t_{n-3}}+ae+f}{(bce+bd)\,\widetilde{t_{n-3}}-be}, \\ \widehat{t_n} &= \frac{-(ace+cf+be)\,\widehat{t_{n-3}}+ac+b}{(ade+df)\,\widehat{t_{n-3}}-ad}, \end{aligned} \tag{86}$$

where $n \geq 2$. Using the procedure used to solve system (12), from (86), one can obtain the solution

$$t_{3m+i} = \frac{-A}{acf+bf}\frac{\left((acf+bf)\,t_i - cf + \lambda_2 A\right)\lambda_1^{m+1} + \left(-\lambda_1 A - (acf+bf)\,t_i + cf\right)\lambda_2^{m+1}}{\left((acf+bf)\,t_i - cf + \lambda_2 A\right)\lambda_1^{m} + \left(-\lambda_1 A - (acf+bf)\,t_i + cf\right)\lambda_2^{m}} + \frac{cf}{acf+bf}, \tag{87}$$

$$\widetilde{t}_{3m+i} = \frac{-A}{bce+bd}\frac{\left((bce+bd)\widetilde{t_i} - be + \lambda_2 A\right)\lambda_1^{m+1} + \left(-\lambda_1 A - (bce+bd)\widetilde{t_i} + be\right)\lambda_2^{m+1}}{\left((bce+bd)\widetilde{t_i} - be + \lambda_2 A\right)\lambda_1^{m} + \left(-\lambda_1 A - (bce+bd)\widetilde{t_i} + be\right)\lambda_2^{m}} + \frac{be}{bce+bd}, \tag{88}$$

$$\widehat{t}_{3m+i} = \frac{-A}{ade+df}\frac{\left((ade+df)\widehat{t_i} - ad + \lambda_2 A\right)\lambda_1^{m+1} + \left(-\lambda_1 A - (ade+df)\widehat{t_i} + ad\right)\lambda_2^{m+1}}{\left((ade+df)\widehat{t_i} - ad + \lambda_2 A\right)\lambda_1^{m} + \left(-\lambda_1 A - (ade+df)\widehat{t_i} + ad\right)\lambda_2^{m}} + \frac{ad}{ade+df}, \tag{89}$$

when $(ace+ad+be+cf)^2\, 4bdf \neq 0$, and

$$t_{3m+i} = \frac{-A}{acf+bf}\frac{-A + \left(A + 2\left(-cf + (acf+bf)\,t_i\right)\right)(m+1)}{-2A + \left(2A + 4\left(-cf + (acf+bf)\,t_i\right)\right)m} + \frac{cf}{acf+bf}, \tag{90}$$

$$\widetilde{t}_{3m+i} = \frac{-A}{bce+bd}\frac{-A+\left(A+2\left(-be+(bce+bd)\widetilde{t_i}\right)\right)(m+1)}{-2A+\left(2A+4\left(-be+(bce+bd)\widetilde{t_i}\right)\right)m} + \frac{be}{bce+bd}, \tag{91}$$

$$\widehat{t}_{3m+i} = \frac{-A}{ade+df}\frac{-A+\left(A+2\left(-ad+(ade+df)\widehat{t_i}\right)\right)(m+1)}{-2A+\left(2A+4\left(-ad+(ade+df)\widehat{t_i}\right)\right)m} + \frac{ad}{ade+df}, \tag{92}$$

when $(ace+ad+be+cf)^2+4bdf=0$, for $m \in \mathbb{N}_0$ and $i \in \{-1,0,1\}$, where $A = ace+ad+be+cf$. By employing (76)-(84) and the change of variables (11) to (87)-(92), we obtain the result in (64) □

3. The case a=b=c=d=e=f=1

In this section we will derive the solution forms of system (6) with $a=b=c=d=e=f=1$, that is, the system

$$x_n = \frac{z_{n-2}x_{n-3}}{x_{n-3}+y_{n-1}},\ y_n = \frac{x_{n-2}y_{n-3}}{y_{n-3}+z_{n-1}},\ z_n = \frac{y_{n-2}z_{n-3}}{z_{n-3}+x_{n-1}},\ n \in \mathbb{N}_0, \tag{93}$$

is given in [9], through analytical approach. Also, the general solutions of system (93) are expressed in terms of Fibonacci numbers. Now, to begin with, taking $a=b=c=d=e=f=1$ in (22), we have that

$$q_{m+1} - 4q_m - q_{m-1} = 0,\ m \in \mathbb{N}. \tag{94}$$

It can be clearly obtained from the roots λ_1 and λ_2 of characteristic equation of (94) as the form $\lambda^2-4\lambda-1=0$, where $\lambda_1 = 2+\sqrt{5} = \left(\frac{1+\sqrt{5}}{2}\right)^3 = \alpha^3$ and $\lambda_2 = 2-\sqrt{5} = \left(\frac{1-\sqrt{5}}{2}\right)^3 = \beta^3$. On the other hand, taking into account $L_1 = M_1 = N_1 = -\beta$, $L_2 = M_2 = N_2 = -\alpha$, $\alpha\beta = -1$ and the Binet Formula for Fibonacci numbers, then we can rewrite the equations in (35)-(37) as, for $m \in \mathbb{N}$ and $i \in \{-1,0,1\}$,

$$\begin{aligned} u_{3(m-1)+i} &= \frac{-\alpha^{3m-3}+\beta^{3m-3}-u_i\alpha^{3m-4}+u_i\beta^{3m-4}}{-\alpha^{3m-2}+\beta^{3m-2}-u_i\alpha^{3m-3}+u_i\beta^{3m-3}}, \\ &= \frac{F_{3m-3}+u_iF_{3m-4}}{F_{3m-2}+u_iF_{3m-3}}, \end{aligned} \tag{95}$$

$$\begin{aligned} v_{3(m-1)+i} &= \frac{-\alpha^{3m-3}+\beta^{3m-3}-v_i\alpha^{3m-4}+v_i\beta^{3m-4}}{-\alpha^{3m-2}+\beta^{3m-2}-v_i\alpha^{3m-3}+v_i\beta^{3m-3}}, \\ &= \frac{F_{3m-3}+v_iF_{3m-4}}{F_{3m-2}+v_iF_{3m-3}}, \end{aligned} \tag{96}$$

$$\begin{aligned} w_{3(m-1)+i} &= \frac{-\alpha^{3m-3}+\beta^{3m-3}-w_i\alpha^{3m-4}+w_i\beta^{3m-4}}{-\alpha^{3m-2}+\beta^{3m-2}-w_i\alpha^{3m-3}+w_i\beta^{3m-3}}, \\ &= \frac{F_{3m-3}+w_iF_{3m-4}}{F_{3m-2}+w_iF_{3m-3}}, \end{aligned} \tag{97}$$

where F_n is nth Fibonacci number, $u_i = \frac{x_i}{z_{i-2}}$, $v_i = \frac{y_i}{x_{i-2}}$ and $w_i = \frac{z_i}{y_{i-2}}$. From (11), (93) and (95), we get that, for $m \in \mathbb{N}$ and $i \in \{-1,0,1\}$,

$$\begin{aligned} u_{3(m-1)-1} &= \frac{F_{3m-3}+u_{-1}F_{3m-4}}{F_{3m-2}+u_{-1}F_{3m-3}} \\ &= \frac{z_{-3}F_{3m-3}+x_{-1}F_{3m-4}}{z_{-3}F_{3m-2}+x_{-1}F_{3m-3}}, \end{aligned} \tag{98}$$

$$\begin{aligned} u_{3(m-1)} &= \frac{F_{3m-3}+u_0F_{3m-4}}{F_{3m-2}+u_0F_{3m-3}} \\ &= \frac{x_{-3}F_{3m-2}+y_{-1}F_{3m-3}}{x_{-3}F_{3m-1}+y_{-1}F_{3m-2}}, \end{aligned} \tag{99}$$

$$\begin{aligned} u_{3(m-1)+1} &= \frac{F_{3m-3}+u_1F_{3m-4}}{F_{3m-2}+u_1F_{3m-3}} \\ &= \frac{y_{-3}F_{3m-1}+z_{-1}F_{3m-2}}{y_{-3}F_{3m}+z_{-1}F_{3m-1}}. \end{aligned} \tag{100}$$

Similarly, from (11), (93) and (96), we have that, for $m \in \mathbb{N}$ and $i \in \{-1, 0, 1\}$,

$$\begin{aligned} v_{3(m-1)-1} &= \frac{F_{3m-3}+v_{-1}F_{3m-4}}{F_{3m-2}+v_{-1}F_{3m-3}} \\ &= \frac{x_{-3}F_{3m-3}+y_{-1}F_{3m-4}}{x_{-3}F_{3m-2}+y_{-1}F_{3m-3}}, \end{aligned} \tag{101}$$

$$\begin{aligned} v_{3(m-1)} &= \frac{F_{3m-3}+v_0F_{3m-4}}{F_{3m-2}+v_0F_{3m-3}} \\ &= \frac{y_{-3}F_{3m-2}+z_{-1}F_{3m-3}}{y_{-3}F_{3m-1}+z_{-1}F_{3m-2}}, \end{aligned} \tag{102}$$

$$\begin{aligned} v_{3(m-1)+1} &= \frac{F_{3m-3}+v_1F_{3m-4}}{F_{3m-2}+v_1F_{3m-3}} \\ &= \frac{z_{-3}F_{3m-1}+x_{-1}F_{3m-2}}{z_{-3}F_{3m}+x_{-1}F_{3m-1}}. \end{aligned} \tag{103}$$

Similarly, from (11), (93) and (97), we obtain that, for $m \in \mathbb{N}$ and $i \in \{-1, 0, 1\}$,

$$\begin{aligned} w_{3(m-1)-1} &= \frac{F_{3m-3}+w_{-1}F_{3m-4}}{F_{3m-2}+w_{-1}F_{3m-3}} \\ &= \frac{y_{-3}F_{3m-3}+z_{-1}F_{3m-4}}{y_{-3}F_{3m-2}+z_{-1}F_{3m-3}}, \end{aligned} \tag{104}$$

$$\begin{aligned} w_{3(m-1)} &= \frac{F_{3m-3}+w_0F_{3m-4}}{F_{3m-2}+w_0F_{3m-3}} \\ &= \frac{z_{-3}F_{3m-2}+x_{-1}F_{3m-3}}{z_{-3}F_{3m-1}+x_{-1}F_{3m-2}}, \end{aligned} \tag{105}$$

$$\begin{aligned} w_{3(m-1)+1} &= \frac{F_{3m-3}+w_1F_{3m-4}}{F_{3m-2}+w_1F_{3m-3}} \\ &= \frac{x_{-3}F_{3m-1}+y_{-1}F_{3m-2}}{x_{-3}F_{3m}+y_{-1}F_{3m-1}}. \end{aligned} \tag{106}$$

By substituting the formulas in (98)-(106) into (44)-(46) and changing indices, we have the following results.

Theorem 3.1. *Assume that $(x_n, y_n, z_n)_{n\geq -3}$ is a well-defined solution of system (93). Then the following results are true.*

$$
\begin{aligned}
x_{6m-3} &= x_{-3}\prod_{k=0}^{m-1}\frac{(x_{-3}F_{6k+4}+y_{-1}F_{6k+3})(x_{-3}F_{6k+2}+y_{-1}F_{6k+1})(x_{-3}F_{6k}+y_{-1}F_{6k-1})}{(x_{-3}F_{6k+5}+y_{-1}F_{6k+4})(x_{-3}F_{6k+3}+y_{-1}F_{6k+2})(x_{-3}F_{6k+1}+y_{-1}F_{6k})},\\
x_{6m-2} &= x_{-2}\prod_{k=0}^{m-1}\frac{(y_{-3}F_{6k+5}+z_{-1}F_{6k+4})(y_{-3}F_{6k+3}+z_{-1}F_{6k+2})(y_{-3}F_{6k+1}+z_{-1}F_{6k})}{(y_{-3}F_{6k+6}+z_{-1}F_{6k+5})(y_{-3}F_{6k+4}+z_{-1}F_{6k+3})(y_{-3}F_{6k+2}+z_{-1}F_{6k+1})},\\
x_{6m-1} &= x_{-1}\prod_{k=0}^{m-1}\frac{(z_{-3}F_{6k+6}+x_{-1}F_{6k+5})(z_{-3}F_{6k+4}+x_{-1}F_{6k+3})(z_{-3}F_{6k+2}+x_{-1}F_{6k+1})}{(z_{-3}F_{6k+7}+x_{-1}F_{6k+6})(z_{-3}F_{6k+5}+x_{-1}F_{6k+4})(z_{-3}F_{6k+3}+x_{-1}F_{6k+2})},\\
x_{6m} &= x_{0}\prod_{k=0}^{m-1}\frac{(x_{-3}F_{6k+7}+y_{-1}F_{6k+6})(x_{-3}F_{6k+5}+y_{-1}F_{6k+4})(x_{-3}F_{6k+3}+y_{-1}F_{6k+2})}{(x_{-3}F_{6k+8}+y_{-1}F_{6k+7})(x_{-3}F_{6k+6}+y_{-1}F_{6k+5})(x_{-3}F_{6k+4}+y_{-1}F_{6k+3})},\\
x_{6m+1} &= x_{1}\prod_{k=0}^{m-1}\frac{(y_{-3}F_{6k+8}+z_{-1}F_{6k+7})(y_{-3}F_{6k+6}+z_{-1}F_{6k+5})(y_{-3}F_{6k+4}+z_{-1}F_{6k+3})}{(y_{-3}F_{6k+9}+z_{-1}F_{6k+8})(y_{-3}F_{6k+7}+z_{-1}F_{6k+6})(y_{-3}F_{6k+5}+z_{-1}F_{6k+4})},\\
x_{6m+2} &= x_{2}\prod_{k=0}^{m-1}\frac{(z_{-3}F_{6k+9}+x_{-1}F_{6k+8})(z_{-3}F_{6k+7}+x_{-1}F_{6k+6})(z_{-3}F_{6k+5}+x_{-1}F_{6k+4})}{(z_{-3}F_{6k+10}+x_{-1}F_{6k+9})(z_{-3}F_{6k+8}+x_{-1}F_{6k+7})(z_{-3}F_{6k+6}+x_{-1}F_{6k+5})},\\
y_{6m-3} &= y_{-3}\prod_{k=0}^{m-1}\frac{(y_{-3}F_{6k+4}+z_{-1}F_{6k+3})(y_{-3}F_{6k+2}+z_{-1}F_{6k+1})(y_{-3}F_{6k}+z_{-1}F_{6k-1})}{(y_{-3}F_{6k+5}+z_{-1}F_{6k+4})(y_{-3}F_{6k+3}+z_{-1}F_{6k+2})(y_{-3}F_{6k+1}+z_{-1}F_{6k})},\\
y_{6m-2} &= y_{-2}\prod_{k=0}^{m-1}\frac{(z_{-3}F_{6k+5}+x_{-1}F_{6k+4})(z_{-3}F_{6k+3}+x_{-1}F_{6k+2})(z_{-3}F_{6k+1}+x_{-1}F_{6k})}{(z_{-3}F_{6k+6}+x_{-1}F_{6k+5})(z_{-3}F_{6k+4}+x_{-1}F_{6k+3})(z_{-3}F_{6k+2}+x_{-1}F_{6k+1})},\\
y_{6m-1} &= y_{-1}\prod_{k=0}^{m-1}\frac{(x_{-3}F_{6k+6}+y_{-1}F_{6k+5})(x_{-3}F_{6k+4}+y_{-1}F_{6k+3})(x_{-3}F_{6k+2}+y_{-1}F_{6k+1})}{(x_{-3}F_{6k+7}+y_{-1}F_{6k+6})(x_{-3}F_{6k+5}+y_{-1}F_{6k+4})(x_{-3}F_{6k+3}+y_{-1}F_{6k+2})},\\
y_{6m} &= y_{0}\prod_{k=0}^{m-1}\frac{(y_{-3}F_{6k+7}+z_{-1}F_{6k+6})(y_{-3}F_{6k+5}+z_{-1}F_{6k+4})(y_{-3}F_{6k+3}+z_{-1}F_{6k+2})}{(y_{-3}F_{6k+8}+z_{-1}F_{6k+7})(y_{-3}F_{6k+6}+z_{-1}F_{6k+5})(y_{-3}F_{6k+4}+z_{-1}F_{6k+3})},\\
y_{6m+1} &= y_{1}\prod_{k=0}^{m-1}\frac{(z_{-3}F_{6k+8}+x_{-1}F_{6k+7})(z_{-3}F_{6k+6}+x_{-1}F_{6k+5})(z_{-3}F_{6k+4}+x_{-1}F_{6k+3})}{(z_{-3}F_{6k+9}+x_{-1}F_{6k+8})(z_{-3}F_{6k+7}+x_{-1}F_{6k+6})(z_{-3}F_{6k+5}+x_{-1}F_{6k+4})},\\
y_{6m+2} &= y_{2}\prod_{k=0}^{m-1}\frac{(x_{-3}F_{6k+9}+y_{-1}F_{6k+8})(x_{-3}F_{6k+7}+y_{-1}F_{6k+6})(x_{-3}F_{6k+5}+y_{-1}F_{6k+4})}{(x_{-3}F_{6k+10}+y_{-1}F_{6k+9})(x_{-3}F_{6k+8}+y_{-1}F_{6k+7})(x_{-3}F_{6k+6}+y_{-1}F_{6k+5})},\\
z_{6m-3} &= z_{-3}\prod_{k=0}^{m-1}\frac{(z_{-3}F_{6k+4}+x_{-1}F_{6k+3})(z_{-3}F_{6k+2}+x_{-1}F_{6k+1})(z_{-3}F_{6k}+x_{-1}F_{6k-1})}{(z_{-3}F_{6k+5}+x_{-1}F_{6k+4})(z_{-3}F_{6k+3}+x_{-1}F_{6k+2})(z_{-3}F_{6k+1}+x_{-1}F_{6k})},\\
z_{6m-2} &= z_{-2}\prod_{k=0}^{m-1}\frac{(x_{-3}F_{6k+5}+y_{-1}F_{6k+4})(x_{-3}F_{6k+3}+y_{-1}F_{6k+2})(x_{-3}F_{6k+1}+y_{-1}F_{6k})}{(x_{-3}F_{6k+6}+y_{-1}F_{6k+5})(x_{-3}F_{6k+4}+y_{-1}F_{6k+3})(x_{-3}F_{6k+2}+y_{-1}F_{6k+1})},\\
z_{6m-1} &= z_{-1}\prod_{k=0}^{m-1}\frac{(y_{-3}F_{6k+6}+z_{-1}F_{6k+5})(y_{-3}F_{6k+4}+z_{-1}F_{6k+3})(y_{-3}F_{6k+2}+z_{-1}F_{6k+1})}{(y_{-3}F_{6k+7}+z_{-1}F_{6k+6})(y_{-3}F_{6k+5}+z_{-1}F_{6k+4})(y_{-3}F_{6k+3}+z_{-1}F_{6k+2})},\\
z_{6m} &= z_{0}\prod_{k=0}^{m-1}\frac{(z_{-3}F_{6k+7}+x_{-1}F_{6k+6})(z_{-3}F_{6k+5}+x_{-1}F_{6k+4})(z_{-3}F_{6k+3}+x_{-1}F_{6k+2})}{(z_{-3}F_{6k+8}+x_{-1}F_{6k+7})(z_{-3}F_{6k+6}+x_{-1}F_{6k+5})(z_{-3}F_{6k+4}+x_{-1}F_{6k+3})},
\end{aligned}
$$

$$z_{6m+1} = z_1 \prod_{k=0}^{m-1} \frac{(x_{-3}F_{6k+8} + y_{-1}F_{6k+7})(x_{-3}F_{6k+6} + y_{-1}F_{6k+5})(x_{-3}F_{6k+4} + y_{-1}F_{6k+3})}{(x_{-3}F_{6k+9} + y_{-1}F_{6k+8})(x_{-3}F_{6k+7} + y_{-1}F_{6k+6})(x_{-3}F_{6k+5} + y_{-1}F_{6k+4})},$$

$$z_{6m+2} = z_2 \prod_{k=0}^{m-1} \frac{(y_{-3}F_{6k+9} + z_{-1}F_{6k+8})(y_{-3}F_{6k+7} + z_{-1}F_{6k+6})(y_{-3}F_{6k+5} + z_{-1}F_{6k+4})}{(y_{-3}F_{6k+10} + z_{-1}F_{6k+9})(y_{-3}F_{6k+8} + z_{-1}F_{6k+7})(y_{-3}F_{6k+6} + z_{-1}F_{6k+5})},$$

where F_n is nth Fibonacci number.

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