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Kato Decomposition Theorem for Linear Pencils

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Abstract. The aim of this paper is to give a Kato decomposition, associated to a pair of operators, which removes the jump at the origin. Exactly we will give the class of linear pencils having a constant jump as a generalization of upper semi-Fredholm pencils and we get a Kato decomposition related to this class..

1. Introduction

Let X and Y be infinite dimensional Banach spaces. Denote by $\mathcal{B}(X,Y)$ the set of all bounded linear operators from X into Y. For $T \in \mathcal{B}(X,Y)$, we write $\mathcal{D}(T) \subset X$ for the domain, $\mathcal{N}(T) = KerT$ for the null space and $\mathcal{R}(T) = RanT$ for the range of T. The nullity, $\alpha(T)$, of T is defined as the dimension of $\mathcal{N}(T)$ and the deficiency, $\beta(T)$, of T is defined as the codimension of $\mathcal{R}(T)$ in Y.

In this paper we will consider T and J two operators of $\mathcal{B}(X,Y)$ not equal to zero. We set $\mathcal{N}_1(T,J) = \mathcal{N}(T) \subset X$, and by iteration we define $\mathcal{N}_k(T,J) = T^{-1}(J(\mathcal{N}_{k-1}(T,J)))$
for all $k \geq 2$. Similarly, we define $\mathcal{R}_1(T,J) = \mathcal{R}(T)$ and by iteration $\mathcal{R}_k(T,J) = T(J^{-1}(\mathcal{R}_{k-1}(T,J)))$ for all $k \geq 2$. Clearly $\mathcal{N}_k(T,J)$ (respectively $\mathcal{R}_k(T,J)$) are linear subspaces of X (respectively Y).

We recall ([4],[5]) that $T \in \mathcal{B}(X,Y)$ is called upper semi-Fredholm if

$$T \text{ has a closed range and } \alpha(T) < \infty.$$

For $J \in \mathcal{B}(X,Y)$ write,

$\Psi_+(X,Y,J) = \{T \in \mathcal{B}(X,Y) : \mathcal{R}(T)$ is closed, $\mathcal{R}(T) \subset \mathcal{R}(J)$ and $\alpha(T - \lambda J)$
is constant for $0 < |\lambda| < \epsilon\}$.

In [9], West defined a jump of a semi-Fredholm operator. We extend this concept to the case of a larger class. If $T \in \Psi_+(X,Y,J)$ we define the upper jump, $j_+(T,J)$, associated to the couple (T,J) by setting

$$j_+(T,J) = \alpha(T) - \alpha(T - \lambda J), \quad 0 < |\lambda| < \epsilon.$$

With the understanding that for any real number r, $\infty - r = \infty$.

Kato's decomposition for linear operators, linear pencils and linear relations has been studied by many authors under different conditions, see ([1],[2],[3],[6],[9]).

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Kato's decomposition theorem [1, Theorem 4.5], says that if $T \in \Psi_+(X) = \Psi_+(X, X, I)$, and satisfying some additional conditions then $T = T_1 \oplus T_2$, where T_1 is nilpotent and $j_+(T_2) = 0$. The proof of this result is based on the special case of Kato's decomposition theorem given by [9, Theorem 7]. The purpose of this work is to pursue the investigation started in [1] and to extend it to the class $\Psi_+(X, Y, J)$. We shall show that if satisfies some conditions then the Kato's decomposition relative to J allows for T. Precisely, we prove that if $T \in \Psi_+(X, Y, J)$ be such that $\mathcal{N}(T)$ and $\mathcal{N}(T) + J^{-1}(\mathcal{R}(T))$ are complemented and $\mathcal{N}(T) \cap J^{-1}(\mathcal{R}(T))$ is finite dimensional then there exist closed subspaces X_1 and $X_2 \subset X$ with $T(X_i) \subset J(X_i)$ $i = 1, 2$ such that $X = X_1 \oplus X_2, dimX_1 < \infty$ and $X_1 \subset \mathcal{N}_k(T, J)$ for some $k \geq 1$ and $T/_{X_2}$ is upper semi-Fredholm with $j_+(T/_{X_2}, J/_{X_2}) = 0$.

The structure of this work is as follows. In section 2, we establish some preliminary results concerning the family of subspaces $\mathcal{N}_k(T, J)$ and $\mathcal{R}_k(T, J)$. The section 3, is devoted to the proof of the main result of the paper.

2. Preliminary results

Let X and Y be two Banach spaces and let $T, J \in \mathcal{B}(X, Y)$ be such that $\mathcal{R}(T) \subset \mathcal{R}(J)$. We recall in this section some definitions of subspaces related to (T, J) and some of their properties:

$$\mathcal{R}^\infty(T, J) = \bigcap_{n=1}^{\infty} \mathcal{R}_n(T, J)$$

and

$$\mathcal{N}^\infty(T, J) = \bigcup_{n=1}^{\infty} \mathcal{N}_n(T, J)$$

where $\mathcal{N}_k(T, J) = T^{-1}(J(\mathcal{N}_{k-1}(T, J)))$ for all $k \geq 2$. Similarly, $\mathcal{R}_k(T, J) = T(J^{-1}(\mathcal{R}_{k-1}(T, J)))$ for all $k \geq 2$.

Lemma 2.1. *Let T and $J \in \mathcal{B}(X, Y)$ be such that $\mathcal{R}(T) \subset \mathcal{R}(J)$. For $k \geq 2$, we have:*

$$\mathcal{N}_{k-1}(T, J) \subset \mathcal{N}_k(T, J) \text{ and } \mathcal{R}_k(T, J) \subset \mathcal{R}_{k-1}(T, J).$$

Lemma 2.2. *If $\lambda \neq 0$ and $T, J \in \mathcal{B}(X, Y)$ be such that $\mathcal{R}(T) \subset \mathcal{R}(J)$. Then*

$$\mathcal{N}(T - \lambda J) \subseteq J^{-1}(\mathcal{R}^\infty(T, J)).$$

Proof Let $x \in \mathcal{N}(T - \lambda J)$. Then $(T - \lambda J)(x) = 0$. Thus $T(x) = \lambda J(x)$ and so $J(x) = \lambda^{-1}T(x)$ hence $J(x) \in \mathcal{R}(T)$.

We must prove that $J(x) \in \mathcal{R}_n(T, J)$, for each $n \geq 2$.

First, we prove that

$$J(x) \in \mathcal{R}_2(T, J).$$

Indeed, we have $J(x) \in \mathcal{R}(T)$ then $x \in J^{-1}(\mathcal{R}(T))$ thus $Tx \in T(J^{-1}(\mathcal{R}(T))$ so $Tx \in \mathcal{R}_2(T, J)$ and $J(x) = \lambda^{-1}Tx \in \mathcal{R}_2(T)$. And by induction, we have

$$J(x) \in \mathcal{R}_n(T), \quad \text{for each } n \geq 2.$$

So, we prove that

$$x \in J^{-1}(\mathcal{R}^\infty(T, J)).$$

This gives

$$\mathcal{N}(T - \lambda J) \subseteq J^{-1}(\mathcal{R}^\infty(T, J)).$$

∎

We give here some useful notations for later. Let $A, B \in \mathcal{B}(X, Y)$. For $H \subset X$ and $K \subset Y$

$$(A^{-1}B)(H) = A^{-1}(B(H)), (AB^{-1})(K) = A(B^{-1}(K))$$

and by iteration we define for $n \geq 2$:

$$(A^{-1}B)^n(H) = A^{-1}(B((A^{-1}B)^{n-1}(H))), (AB^{-1})^n(K) = A(B^{-1}((AB^{-1})^{n-1}(K))).$$

Lemma 2.3. *Let T and $J \in \mathcal{B}(X, Y)$.*

(i) *Let $H \subset X, n \geq 1$ and $x \in X$. Then,*

$$x \in (T^{-1}J)^n(H) \text{ if and only if there exists } y \in H \text{ such that } y \in (J^{-1}T)^n(\{x\}).$$

(ii) *For all $n \geq 1, m \geq 2$*

$$(T^{-1}J)^n(\mathcal{N}(T)) = \mathcal{N}_{n+1}(T, J)$$

$$(TJ^{-1})^n(\mathcal{R}_m(T, J)) = \mathcal{R}_{n+m}(T, J) = T(J^{-1}T)^n(J^{-1}(\mathcal{R}_{m-1}(T, J))).$$

(iii) *Let $x, y \in X$. Then,*

$$\text{If } (J^{-1}T)^n(\{x\}) \cap (J^{-1}T)^n(\{y\}) \neq \emptyset, \text{ then } x - y \in \mathcal{N}_n(T, J).$$

(iv) *If $\mathcal{N}(T) \subset J^{-1}(\mathcal{R}_n(T, J))$ for all $n \geq 1$, then $\mathcal{N}_n(T, J) \subset J^{-1}(\mathcal{R}_m(T, J))$ for all $n, m \geq 1$.*

Proof

(i) We proceed by induction. The case $n = 1$ is trivial. Assume now, that the result is valid for the order n. Let $x \in (T^{-1}J)^{n+1}(H)$. Then $x \in T^{-1}(J(T^{-1}J)^n(H))$. So, $Tx \in J((T^{-1}J)^n(H))$. Therefore there exists $z \in (T^{-1}J)^n(H)$ such that $Tx = J(z)$. Then from induction assumption, there exists $y \in H$ such that:

$$y \in (J^{-1}T)^n(\{z\}) \subset (J^{-1}T)^n(J^{-1}T)(\{x\}) = (J^{-1}T)^{n+1}(\{x\}).$$

(ii) The first equality is proved by induction. The case $n = 1$ is trivial. Assume now that the equality is valid for the order n. We have:

$$\begin{aligned}(T^{-1}J)^{n+1}(\mathcal{N}(T)) &= T^{-1}(J(T^{-1}J)^n(\mathcal{N}(T))) \\ &= T^{-1}(J(\mathcal{N}_{n+1}(T, J))) \\ &= \mathcal{N}_{n+2}(T, J).\end{aligned}$$

For the second equality, the case $n = 1$ is trivial. Suppose that the result holds in the order n and for all $m \geq 2$,

$$\begin{aligned}(TJ^{-1})^{n+1}(\mathcal{R}_m(T, J)) &= (TJ^{-1})((TJ^{-1})^n(\mathcal{R}_m(T, J))) \\ &= (TJ^{-1})(\mathcal{R}_{n+m}(T, J)) \\ &= \mathcal{R}_{n+m+1}(T, J).\end{aligned}$$

For the third equality, the case $n = 1$ is clear. Suppose now that the result is valid for order n and for all $m \geq 2$, then we have

$$\begin{aligned}(TJ^{-1})^{n+1}(\mathcal{R}_m(T, J)) &= T(J^{-1}((TJ^{-1})^n(\mathcal{R}_m(T, J)))) \\ &= T(J^{-1}(T(J^{-1}T)^n(J^{-1}(\mathcal{R}_{m-1}(T, J))) \\ &= T((J^{-1}T)^{n+1}(J^{-1}(\mathcal{R}_{m-1}(T, J)))).\end{aligned}$$

(iii) We proceed by induction. For the case $n = 1$, let $z \in (J^{-1}T)(\{x\}) \cap (J^{-1}T)(\{y\})$. Then $J(z) = Tx$ and $J(z) = Ty$. Therefore $T(x - y) = 0$ and so, $x - y \in \mathcal{N}(T)$. Assume now that the result is valid for the order n. Let $z \in (J^{-1}T)^{n+1}\{x\} \cap (J^{-1}T)^{n+1}\{y\}$. Then, $z \in (J^{-1}T)^n((J^{-1}T)\{x\} \cap (J^{-1}T)^n((J^{-1}T)\{y\}$. Hence, there exists $\alpha_1 \in (J^{-1}T)\{x\}$ and $\alpha_2 \in (J^{-1}T)\{y\}$ such that $z \in (J^{-1}T)^n(\alpha_1) \cap (J^{-1}T)^n(\alpha_2)$. Then, by hypothesis of induction we have: $\alpha_1 - \alpha_2 \in \mathcal{N}_n(T, J)$. On the other hand, $J(\alpha_1) = Tx$ and $J(\alpha_2) = Ty$. So, $T(x - y) = J(\alpha_1 - \alpha_2) \in J(\mathcal{N}_n(T, J))$. Hence $x - y \in T^{-1}(J(\mathcal{N}_n(T, J))) = \mathcal{N}_{n+1}(T, J)$.

(*iv*) The inclusion is proved by induction. The case $n = 1$ is a direct consequence of the hypothesis. Assume now that the result is valid for the order n and for all $m \geq 1$. Let $x \in \mathcal{N}_{n+1}(T, J)$. Then by (*i*), there exists $z \in \mathcal{N}(T)$ such that $z \in (J^{-1}T)^n(\{x\})$. On the other hand, by hypothesis, we get $z \in \mathcal{N}(T) \subset J^{-1}(\mathcal{R}_{n+m}(T, J))$ $\forall\, m \geq 1$. Then, by (*ii*), $J(z) \in \mathcal{R}_{n+m}(T, J) = (TJ^{-1})^n(\mathcal{R}_m(T, J)) = T((J^{-1}T)^n(J^{-1}(\mathcal{R}_{m-1}(T, J)))$. So, $z \in J^{-1}(T((J^{-1}T)^n(J^{-1}(\mathcal{R}_{m-1}(T, J)))) = (J^{-1}T)^n(J^{-1}T)(J^{-1}(\mathcal{R}_{m-1}(T, J)))$. Then there exists $y_m \in (J^{-1}T)(J^{-1}(\mathcal{R}_{m-1}(T, J))) = J^{-1}(\mathcal{R}_m(T, J))$ such that, $z \in (J^{-1}T)^n(\{y_m\})$. Hence, $(J^{-1}T)^n(\{y\}) \cap (J^{-1}T)^n(\{x\}) \neq \emptyset$. Using (*iii*) we get $x - y_m \in J^{-1}(\mathcal{R}_m(T, J))$ and so, $x \in J^{-1}(\mathcal{R}_m(T, J))$. ■

In the sequel we need the following theorem wish is an immediate consequence of Theorem 1 and Theorem 7 in [6].

Theorem 2.1. *Let $J \in \mathcal{B}(X, Y)$ and T be an upper semi-Fredholm operator such that $\mathcal{R}(T) \subset \mathcal{R}(J)$. Then there exists $\epsilon > 0$ such that $\alpha(\lambda J + T) \leq \alpha(T)$ for all $|\lambda| < \epsilon$, and $\alpha(\lambda J + T)$ is constant for all $0 < |\lambda| < \epsilon$.*

According to this theorem we can see that

$$\{T \in \mathcal{B}(X, Y) : T \text{ is upper semi-Fredholm operator with } \mathcal{R}(T) \subset \mathcal{R}(J)\} \subseteq \Psi_+(X, Y, J).$$

Lemma 2.4. *Let T and $J \in \mathcal{B}(X, Y)$ be such that $\mathcal{R}(T) \subset \mathcal{R}(J)$. We have:*

(*i*) *$T(J^{-1}(\mathcal{R}^\infty(T, J)) \subset \mathcal{R}^\infty(T, J)$;*

(*ii*) *If $T \in \Phi_+(X, Y)$ then $\mathcal{R}^\infty(T, J)$ is closed.*

Proof (*i*) We have

$$\mathcal{R}^\infty(T, J) = \bigcap_{n=1}^{\infty} \mathcal{R}_n(T, J) \text{ and } \mathcal{R}_n(T, J) = T(J^{-1}(\mathcal{R}_{n-1}(T, J))) \text{ for all } n \geq 2.$$

So,

$$J^{-1}(\mathcal{R}^\infty(T, J)) = J^{-1}(\bigcap_{n=1}^{\infty} \mathcal{R}_n(T, J)) = \bigcap_{n=1}^{\infty} J^{-1}(\mathcal{R}_n(T, J)).$$

Then

$$T(J^{-1}(\mathcal{R}^\infty(T, J))) = T(\bigcap_{n=1}^{\infty} J^{-1}(\mathcal{R}_n(T, J))) \subset \bigcap_{n=1}^{\infty} T(J^{-1}(\mathcal{R}_n(T, J))) \subset \bigcap_{n=1}^{\infty} \mathcal{R}_{n+1}(T, J).$$

Then

$$T(J^{-1}(\mathcal{R}^\infty(T, J))) \subset \bigcap_{n=2}^{\infty} \mathcal{R}_n(T, J).$$

On the other hand we have

$$T(J^{-1}(\mathcal{R}^\infty(T, J))) \subset \mathcal{R}(T).$$

So, we have the result

$$T(J^{-1}(\mathcal{R}^\infty(T, J)) \subset \mathcal{R}^\infty(T, J).$$

(*ii*) Since $\mathcal{R}^\infty(T, J) = \bigcap_{n=1}^{\infty} \mathcal{R}_n(T, J)$, the result is obtained if we prove that $\mathcal{R}_n(T, J)$ is closed for all $n \geq 1$. We proceed by induction. For the case $n = 1$, we have $T \in \Phi_+(X, Y)$, then $\mathcal{R}_1(T, J) = \mathcal{R}(T)$ is closed. Assume now that $\mathcal{R}_n(T, J)$ is closed.
Define $T_1 := T/_{\mathcal{N}(T)+J^{-1}(\mathcal{R}_n(T,J))}$.

Since T is closed with finite dimensional null space and $J^{-1}(\mathcal{R}_n(T,J))$ is closed we obtain that T_1 is closed. We note that $\mathcal{N}(T_1) = \mathcal{N}(T)$ and hence $\gamma(T) \leq \gamma(T_1)$, where γ is the reduced minimum defined by

$$\gamma(T) := sup\{\epsilon \geq 0; \epsilon \ \ dist(x, \mathcal{N}(T)) \leq \|Tx\|, x \in \mathcal{D}(T)\}.$$

Applying [7, Theorem 2, page 97], we deduce that $\mathcal{R}(T_1)$ is closed. But $\mathcal{R}(T_1) = \mathcal{R}_{n+1}(T,J)$. Indeed, $\mathcal{R}(T_1) = T(\mathcal{N}(T) + J^{-1}(\mathcal{R}_n(T,J))) = T(J^{-1}(\mathcal{R}_n(T,J))) = \mathcal{R}_{n+1}(T,J)$. So, $\mathcal{R}_{n+1}(T,J)$ is closed. ∎

Let T and $J \in \mathcal{B}(X,Y)$ be such that $\mathcal{R}(T) \subset \mathcal{R}(J)$. Define

$$\widehat{T} : J^{-1}(\mathcal{R}^\infty(T,J)) \to \mathcal{R}^\infty(T,J)$$

the operator induced by T and

$$\widehat{J} : J^{-1}(\mathcal{R}^\infty(T,J)) \to \mathcal{R}^\infty(T,J)$$

the operator induced by J.

Proposition 2.1. *Let T and $J \in \mathcal{B}(X,Y)$ be such that $\mathcal{R}(T) \subset \mathcal{R}(J)$. If $\alpha(T) < \infty$ then $\beta(\widehat{T}) = 0$ and $\alpha(\widehat{T}) < \infty$.*

Proof We show that if $x \in \mathcal{R}^\infty(T,J)$ then $T^{-1}\{x\} \cap J^{-1}(\mathcal{R}^\infty(T,J)) \neq \emptyset$. Indeed, $x \in \mathcal{R}^\infty(T,J)$ then $x \in \mathcal{R}(T)$ and so, $T^{-1}\{x\} \neq \emptyset$. Let $w \in T^{-1}\{x\}$, then $T^{-1}\{x\} = w + \mathcal{N}(T)$ which, by hypothesis, is a finite dimensional hyperplane. Hence the decreasing sequence $T^{-1}\{x\} \cap J^{-1}(\mathcal{R}_n(T,J))$ terminates. Thus for some k

$$T^{-1}\{x\} \cap J^{-1}(\mathcal{R}^\infty(T,J)) = T^{-1}\{x\} \cap J^{-1}(\mathcal{R}_k(T,J)).$$

Now, $x \in \mathcal{R}^\infty(T,J)$ then $x \in \bigcap_k \mathcal{R}_k(T,J)$ so $x \in \mathcal{R}_{k+1}(T,J)$, thus $x \in T(J^{-1}(\mathcal{R}_k(T,J))$ then there exists $y \in J^{-1}(\mathcal{R}_k(T,J)$ such that $x = Ty$. So, $y \in T^{-1}\{x\}$.
Finally $y \in T^{-1}\{x\} \cap J^{-1}(\mathcal{R}_k(T,J) = T^{-1}\{x\} \cap J^{-1}(\mathcal{R}^\infty(T,J))$. So,

$$T^{-1}\{x\} \cap J^{-1}(\mathcal{R}^\infty(T,J)) \neq \emptyset.$$

Now,

$$\beta(\widehat{T}) = dim(\mathcal{R}^\infty(T,J)/T(J^{-1}(\mathcal{R}^\infty(T,J)))).$$

So, we prove that $\mathcal{R}^\infty(T,J) = T(J^{-1}(\mathcal{R}^\infty(T,J)))$. We have, $T(J^{-1}(\mathcal{R}^\infty(T,J))) \subset \mathcal{R}^\infty(T,J)$ is evident.

If $x \in \mathcal{R}^\infty(T,J)$, *then* $T^{-1}\{x\} \cap J^{-1}(\mathcal{R}^\infty(T,J)) \neq \emptyset$. Let $y \in T^{-1}\{x\} \cap J^{-1}(\mathcal{R}^\infty(T,J))$. Then

$$\begin{cases} y \in T^{-1}\{x\}; \\ y \in J^{-1}(\mathcal{R}^\infty(T,J)). \end{cases} \text{Then} \begin{cases} x = T(y); \\ y \in J^{-1}(\mathcal{R}^\infty(T,J)). \end{cases}$$

So $x \in T(J^{-1}(\mathcal{R}^\infty(T,J)))$, which concludes the proof and

$$\beta(\widehat{T}) = dim(\mathcal{R}^\infty(T,J)/T(J^{-1}(\mathcal{R}^\infty(T,J)))) = 0.$$

In the other hand, we have $\widehat{T} \subset T$ then $\mathcal{N}(\widehat{T}) \subset \mathcal{N}(T)$. So,

$$\alpha(\widehat{T}) \leq \alpha(T) < \infty.$$

∎

Remark 2.1. *From Proposition* 2.1 *we can conclude that if T is upper semi-Fredholm then $\widehat{T}$ is Fredholm and $\beta(\widehat{T}) = 0$.*

3. Main results

The following theorem gives a characterization of a constant neighborhood nullity linear pencils (T, J) with $j_+(T, J) = 0$.

Theorem 3.1. *If $T \in \Psi_+(X, Y, J)$ has a finite dimensional intersection $\mathcal{N}(T) \cap J^{-1}(\mathcal{R}_k(T, J))$ for some $k \in \mathbb{N}^*$ then*

$$j_+(T, J) = 0 \text{ if and only if } \mathcal{N}^\infty(T) \subset J^{-1}(\mathcal{R}^\infty(T, J)). \tag{3.1}$$

Proof

Suppose that $T \in \Psi_+(X, Y, J)$. If $j_+(T, J) = 0$ we claim that

$$\alpha(\widehat{T}) \leq \alpha(T) = \alpha(T - \lambda J) = \alpha(\widehat{T} - \lambda\widehat{J}) \leq \alpha(\widehat{T}) \quad \text{for } 0 < |\lambda| < \epsilon. \tag{3.2}$$

Indeed, the first inequality is evident. The second equality comes from the assumption. The third equality comes from Lemma 2.2. In fact, let $x \in \mathcal{N}(\widehat{T} - \lambda\widehat{J})$ then $(\widehat{T} - \lambda\widehat{J})(x) = 0$ and $x \in J^{-1}(\mathcal{R}^\infty(T, J))$. Thus, $\begin{cases} x \in J^{-1}(\mathcal{R}^\infty(T, J)) & ; \\ (T - \lambda J)(x) = 0 & . \end{cases}$

So we have $\mathcal{N}(\widehat{T} - \lambda\widehat{J}) = \mathcal{N}(T - \lambda J) \cap J^{-1}(\mathcal{R}^\infty(T, J))$ and by Lemma 2.2, we have $\mathcal{N}(T - \lambda J) \subset J^{-1}(\mathcal{R}^\infty(T, J))$. So $\alpha(T - \lambda J) = \alpha(\widehat{T} - \lambda\widehat{J})$. The last inequality comes from Theorem 2.1 . Thus (3.2) gives

$$\alpha(T) = \alpha(\widehat{T}).$$

Thus, we have $dim(\mathcal{N}(T)) = dim(\mathcal{N}(T)) \cap J^{-1}(\mathcal{R}^\infty(T, J))) < \infty$. It follows that $\mathcal{N}(T) \subseteq J^{-1}(\mathcal{R}^\infty(T, J))$ and hence using Lemma 2.3 (iv), we get $\mathcal{N}^\infty(T, J) \subseteq J^{-1}(\mathcal{R}^\infty(T, J))$.

Conversely, suppose that $\mathcal{N}^\infty(T, J) \subset J^{-1}(\mathcal{R}^\infty(T, J))$. Then $\mathcal{N}(T) = \mathcal{N}(T) \cap J^{-1}(\mathcal{R}^\infty(T, J)) \subseteq \mathcal{N}(T) \cap J^{-1}(\mathcal{R}_k(T, J))$ is finite dimensional. Thus T is upper semi-Fredholm. By Proposition (2.1), we have $\widehat{T}$ is Fredholm and $\beta(\widehat{T}) = 0$. Thus we have, by [8, Theorem 5.11],

$$\alpha(T) = \alpha(\widehat{T}) = \alpha(\widehat{T} - \lambda\widehat{J}) = \alpha(T - \lambda J) \quad \text{for } 0 < |\lambda| < \epsilon.$$

Which says that $j_+(T, J) = 0$.■

Our main theorem is an extension of Kato's decomposition theorem. The proof of this theorem is inspired essentially from the proof of [9, Theorem 7] and [1, Theorem 4.5].

Theorem 3.2. *If $T \in \Psi_+(X, Y, J)$ satisfies that*

$$\begin{cases} \mathcal{N}(T) \text{ and } \mathcal{N}(T) + J^{-1}(\mathcal{R}(T)) \text{ are complemented,} \\ \mathcal{N}(T) \cap J^{-1}(\mathcal{R}(T)) \text{ is finite dimensional} \end{cases}$$

then there exist closed subspaces X_1 and $X_2 \subset X$ with $T(X_i) \subset J(X_i)$ $i = 1, 2$ such that $X = X_1 \oplus X_2$, $dimX_1 < \infty$ and $X_1 \subset \mathcal{N}_k(T, J)$ for some $k \geq 1$ and $T/_{X_2}$ is upper semi-Fredholm with $j_+(T/_{X_2}, J/_{X_2}) = 0$.

Proof

If $j_+(T, J) = 0$, then by (3.1), we have $\mathcal{N}^\infty(T, J) \subset J^{-1}(\mathcal{R}^\infty(T, J))$ and so $\mathcal{N}(T) \subset J^{-1}(\mathcal{R}(T))$; thus our assumption says that $dim(\mathcal{N}(T)) < \infty$, then T is upper semi-Fredholm. Thus is nothing to prove.

If $j_+(T,J) \neq 0$ and therefore $\alpha(T) - \alpha(T-\lambda J) \neq 0$, then there is a smallest integer $v \geq 1$ such that $\mathcal{N}(T) \subseteq J^{-1}(\mathcal{R}_{v-1}(T))$ but $\mathcal{N}(T)) \nsubseteq J^{-1}(\mathcal{R}_v(T))$.

If $v \geq 2$ then by assumption:

$$\mathcal{N}(T) = \mathcal{N}(T) \cap J^{-1}(\mathcal{R}_{v-1}(T)) \subseteq \mathcal{N}(T) \cap J^{-1}(\mathcal{R}(T))$$

is finite dimensional and hence T is upper semi-Fredholm.

Let L_{v-1} be a finite-dimensional subspace such that

$$\mathcal{N}(T) = L_{v-1} \oplus (\mathcal{N}(T) \cap J^{-1}(\mathcal{R}_v(T,J))) \tag{3.3}$$

with $L_{v-1} \subset \mathcal{N}(T)$ and $dim\, J(L_{v-1}) = dim\, L_{v-1} = r \geq 1$.

We have

$$J(L_{v-1}) \cap \mathcal{R}_v(T,J) = \{0\}. \tag{3.4}$$

Indeed, let $z \in J(L_{v-1}) \cap \mathcal{R}_v(T,J)$. Then, there exist $y_1 \in L_{v-1}$ and $y_2 \in J^{-1}(\mathcal{R}_v(T,J))$ such that $z = J(y_1) = J(y_2)$. Hence, $J(y_1 - y_2) = 0 \in \mathcal{R}_v(T,J)$ and so, $y_1 - y_2 \in J^{-1}(\mathcal{R}_v(T,J))$. It follows that $y_1 \in J^{-1}(\mathcal{R}_v(T,J)) \cap \mathcal{N}(T) \cap L_{v-1}$. In combination with (3.3), this leads to $z = 0$ and (3.4) is proved. As, $L_{v-1} \subset \mathcal{N}(T) \subset J^{-1}(\mathcal{R}_{v-1}(T,J))$ and so $J(L_{v-1}) \subset T(J^{-1}(\mathcal{R}_{v-2}(T,J)))$ we can find a subspace $L_{v-2} \subset J^{-1}(\mathcal{R}_{v-2}(T,J))$ such that $T(L_{v-2}) = J(L_{v-1})$ with $dim\, L_{v-2} = dim\, L_{v-1} = r$. This implies, in particular, that $L_{v-2} \cap \mathcal{N}(T) = \{0\}$. Furthermore we have:

$$L_{v-2} \cap \mathcal{N}(J) = \{0\}. \tag{3.5}$$

$$L_{v-2} \cap J^{-1}(\mathcal{R}_{v-1}(T,J)) = \{0\}. \tag{3.6}$$

$$L_{v-2} \subset \mathcal{N}_2(T,J). \tag{3.7}$$

$$J(L_{v-2}) \cap \mathcal{R}_{v-1}(T,J) = \{0\}. \tag{3.8}$$

To prove (3.5), let $x \in L_{v-2} \cap \mathcal{N}(J)$. Then $J(x) = 0 \in \mathcal{R}_{v-1}(T,J)$ and so, $x \in J^{-1}(\mathcal{R}_{v-1}(T,J))$. This leads to $T(x) \in \mathcal{R}_v(T,J)$. On the other hand, $T(x) \in T(L_{v-2}) = J(L_{v-1})$. So, $T(x) \in J(L_{v-1}) \cap \mathcal{R}_v(T,J) = \{0\}$. Hence, $x = 0$ since $L_{v-2} \cap \mathcal{N}(T) = \{0\}$.

Let now, $z \in L_{v-2} \cap J^{-1}(\mathcal{R}_{v-1}(T,J))$. Then $T(z) \in J(L_{v-1})$ and $J(z) \in \mathcal{R}_{v-1}(T,J)$. Thus, $T(z) \in J(L_{v-1}) \cap \mathcal{R}_v(T,J) = \{0\}$. So, $z \in \mathcal{N}(T) \cap L_{v-2} = \{0\}$. Hence, $z = 0$. This proves (3.6).

For (3.7), we have $L_{v-1} \subset \mathcal{N}(T)$. Then $J(L_{v-1}) \subset J(\mathcal{N}(T))$. Therefore, $T(L_{v-2}) \subset J(\mathcal{N}(T))$ and so, $L_{v-2} \subset \mathcal{N}_2(T,J)$.

By iteration, we construct $L_j \subset J^{-1}\mathcal{R}_j(T,J)$ with $dim\, L_j = r$ such that $J(L_{j+1}) = T(L_j) \subset \mathcal{R}_{j+1}(T,J)$; $0 \leq j \leq v-2$ and satisfying:

$$L_j \cap \mathcal{N}(T) = \{0\} \qquad \text{for all } 0 \leq j \leq v-2. \tag{3.9}$$

$$L_j \cap \mathcal{N}(J) = \{0\} \quad \text{for all } 0 \le j \le v-2. \tag{3.10}$$

$$L_j \cap J^{-1}(\mathcal{R}_{j+1}(T,J)) = \{0\} \quad \text{for all } 0 \le j \le v-2. \tag{3.11}$$

$$L_j \subset \mathcal{N}_{v-j}(T,J) \quad \text{for all } 0 \le j \le v-2. \tag{3.12}$$

$$J(L_j) \cap \mathcal{R}_{j+1}(T,J) = \{0\} \quad \text{for all } 0 \le j \le v-2. \tag{3.13}$$

We claim that the subspaces

$$J(L_0), J(L_1), \dots, J(L_k) \text{ and } \mathcal{R}_{k+1}(T,J))$$

are linearly independent for every $0 \le k \le v-1$. Indeed, let $l'_i \in J(L_i)$ for $(0 \le i \le k)$ and $x'_k \in \mathcal{R}_{k+1}(T,J)$ such that $x'_k + l'_0 + \cdots + l'_k = 0$. There exist $l_i \in L_i$ $0 \le i \le k$ and $x_k \in J^{-1}(\mathcal{R}_{k+1}(T,J))$ such that $l'_i = J(l_i)$ for $0 \le i \le k$ and $x'_k = J(x_k)$. We have $J(l_0) + \cdots + J(l_k) + J(x_k) = 0$ and for all $0 \le i \le k$ $J(l_i) \in T(L_{i-1})$. Therefore, $J(l_0) \in \mathcal{R}(T) \cap J(L_0) = \{0\}$. Then $J(l_0) = 0$ and $J(l_1) + \cdots + J(l_k) + J(x_k) = 0$. Using (3.11) and continue the processus we get $l'_0 = l'_1 = \cdots = l'_k = x_k = 0$.

For $0 \le j \le v-2$, let $\{y_1^j, \dots, y_r^j\}$ be a basis of L_j such that $T(y_i^j) = J(y_i^{j+1})$ for all $0 \le j \le v-2$. Notice that since $dimL_j = dimJ(L_j)$, then $\{J(y_1^j), \dots, J(y_r^j)\}$ is a basis in $J(L_j)$.

Since $J(L_0)$ and $\mathcal{R}(T)$ are linearly independent and by Lemma 2.4 (*ii*), we have $\mathcal{R}(T)$ is closed, then by Hahn-Banach theorem we can find $f_i^0 \in Y^*$ for $1 \le i \le r$ such that $f_i^0(J(y_q^0)) = \delta_{iq}$ and $f_i^0 \in (\mathcal{R}(T))^\perp$ for $1 \le i, q \le r$.

We have $J^* f_i^0 \in (\mathcal{N}(T))^\perp$. Indeed, let $y \in \mathcal{N}(T)$.

$$J^* f_i^0(y) = f_i^0(J(y)).$$

But, we have $J(\mathcal{N}(T)) \subset \mathcal{R}_{v-1}(T,J) \subset \mathcal{R}(T)$. Then $J^* f_i^0(y) = 0$ and hence $J^* f_i^0 \in (\mathcal{N}(T))^\perp$ for all $1 \le i \le r$.

On the other hand, let $x \in T^{-1}(J(L_0) + \mathcal{R}_2(T,J))$. We claim that $J^* f_i^0(x) = 0$ for all $1 \le i \le r$. Indeed, we have $T(x) \in J(L_0) + T(J^{-1}(\mathcal{R}(T)))$, then there exists $y \in J^{-1}(\mathcal{R}(T))$ such that $T(x) - T(y) \in J(L_0)$. So $T(x-y) \in J(L_0) \cap \mathcal{R}(T) = \{0\}$ and then $T(x) = T(y)$. Therefore $T(x) \in \mathcal{R}_2(T,J)$. Thus $T(x) = T(z)$ where $z \in J^{-1}(\mathcal{R}(T))$ and so $x - z \in \mathcal{N}(T) \subset J^{-1}(\mathcal{R}(T))$. Thus $J^* f_i^0(x) = f_i^0(J(x)) = 0$. Applying [6, Lemma 3.3.4], there exist $f_1^i \in Y^*$, $1 \le i \le r$ such that $T^* f_i^1 = J^* f_i^0$ and satisfying $f_i^1 \in (J(L_0) + \mathcal{R}_2(T,J))^\perp$. Then we have $f_i^1(J(y_q^0)) = 0$ for all $1 \le i, q \le r$ and $f_i^1(J(y_q^1)) = f_i^1(T(y_q^0)) = J^* f_i^0(y_q^0) = f_i^0(J(y_q^0)) = \delta_{iq}$.

By iteration we construct $f_i^k \in Y^*$ for $1 \le i \le r$ and $1 \le k \le v-1$ such that:

$$\begin{cases} T^* f_i^k = J^* f_i^{k-1} ; \\ f_i^k(J(y_q^j)) = \delta_{iq}\delta_{kj} ; \\ f_i^k \in (\mathcal{R}_{k+1}(T,J))^\perp. \end{cases} \tag{3.14}$$

We now introduce:

$$P_1 = \sum_{i=1}^{r}\sum_{j=0}^{v-1} J^* f_i^j \otimes y_i^j$$

and

$$P_2 = \sum_{i=1}^{r}\sum_{j=0}^{v-1} f_i^j \otimes J(y_i^j).$$

We claim that P_1 is a projection in $\mathcal{B}(X)$ with range $X_1 = \bigoplus_{i=0}^{v-1} L_i$ and kernel $X_2 = \bigcap_{1\le i\le r}^{0\le j\le v-1} \mathcal{N}(J^* f_i^j)$, P_2 is a projection in $\mathcal{B}(Y)$ and $TP_1 = P_2T$. Indeed, for all $x \in X$,

$$\begin{aligned} TP_1(x) &= \textstyle\sum_{i=1}^{r}\sum_{j=0}^{v-1}(J^* f_i^j)(x)Ty_i^j \\ &= \textstyle\sum_{i=1}^{r}\sum_{j=0}^{v-1} f_i^j(J(x))Ty_i^j \end{aligned}$$

$$\begin{aligned} P_2T(x) &= \textstyle\sum_{i=1}^{r}\sum_{j=0}^{v-1} f_i^j(T(x))J(y_i^j) \\ &= \textstyle\sum_{i=1}^{r}\sum_{j=0}^{v-1} f_i^j(T(x))T(y_i^{j-1}) \\ &= \textstyle\sum_{i=1}^{r}\sum_{j=0}^{v-2} f_i^{j+1}(T(x))T(y_i^j) \\ &= \textstyle\sum_{i=1}^{r}\sum_{j=0}^{v-2} f_i^j(J(x))T(y_i^j). \end{aligned}$$

Let us now see that $T(X_1) \subset J(X_1)$ and $T(X_2) \subset J(X_2)$.

We have $X_1 = \mathcal{R}(P_1) = P_1(X)$. So $T(X_1) = TP_1(X) = P_2T(X) \subset \mathcal{R}(P_2) \subset J(X_1)$.

To prove $T(X_2) \subset J(X_2)$, let $z_2 \in T(X_2)$. Then $z_2 \in T(I - P_1)(X) = (I - P_2)T(X) \subset \mathcal{R}(I - P_2) = \mathcal{N}(P_2)$. Hence $z_2 \in \mathcal{N}(P_2) \cap \mathcal{R}(T) \subset \mathcal{N}(P_2) \cap \mathcal{R}(J)$. Then, $z_2 = J(x)$ for some $x \in X$ and $f_i^j(z_2) = 0$ for all i, j. Therefore, $x \in \mathcal{N}(f_i^j \circ J)$ for all i, j and so, $z_2 \in J(X_2)$.

Clearly we have $\dim X_1 < \infty$ and $X_1 \subset \mathcal{N}_v(T, J)$. Indeed, by (3.12) $X_1 = \bigoplus_{i=0}^{v-1} L_i \subset \bigoplus_{i=0}^{v-1} \mathcal{N}_{v-i}(T, J) = \mathcal{N}_v(T, J)$.

On the other hand $\alpha(T/_{X_1}) = \dim \mathcal{N}(T/_{X_1}) = \dim [\mathcal{N}(T) \cap X_1] = \dim L_{v-1} = r$ and $\alpha(J/_{X_1}) = 0$. Then for all $\lambda \neq 0$, $\alpha(T/_{X_1} - \lambda J/_{X_1}) = 0$. Indeed, let $x \in \mathcal{N}(T/_{X_1} - \lambda J/_{X_1})$. Then $T/_{X_1}(x) = \lambda J/_{X_1}(x)$. We have $x = x_0 + \cdots + x_{v-1}$ with $x_i \in L_i$.

$$T(x_0) + \cdots + T(x_{v-1}) = \lambda J(x_0) + \cdots + \lambda J(x_{v-1}).$$

On the other hand, we have $T(L_i) = J(L_{i+1})$ for all $0 \le i \le v - 2$. Then for x_i, there exists $x'_{i+1} \in L_{i+1}$ such that $T(x_i) = J(x'_{i+1})$. Hence we get,

$$\lambda J(x_0) = J(x'_1) - \lambda J(x_1) + \cdots + J(x'_{v-1}) - \lambda J(x_{v-1}).$$

And so $\begin{cases} J(x_0) = 0 \\ J(x'_i) = \lambda J(x_i) \quad \textit{for all } 1 \le i \le v - 1. \end{cases}$

Thus, $\begin{cases} x_0 \in L_0 \cap \mathcal{N}(J) = \{0\} \\ x'_i - \lambda x_i \in \mathcal{N}(J) \cap L_i = \{0\}. \end{cases}$

This processus can be continued and we prove that $x = 0$. Hence, $\alpha(T/_{X_1} - \lambda J/_{X_1}) = 0$ and $j_+(T/_{X_1}, J/_{X_1}) = r$.

Clearly we have $T/_{X_2}$ is upper semi-Fredholm and

$$j_+(T/_{X_2}, J/_{X_2}) = j_+(T, J) - j_+(T/_{X_1}, J/_{X_1}) = j_+(T, J) - r.$$

By Theorem 2.1, $j_+(T, J) \ge 0$, then continuing the processus a finite number of times reduces the jump of the residual operator to zero.

Now suppose

$$\mathcal{N}(T) \not\subseteq J^{-1}(\mathcal{R}(T)).$$

By assumption, we can find closed subspaces H, Z and W for wich

$$\mathcal{N}(T) = H \oplus \mathcal{N}(T) \cap J^{-1}(\mathcal{R}(T)) \tag{3.15}$$

$$J^{-1}(\mathcal{R}(T)) = \mathcal{N}(T) \cap J^{-1}(\mathcal{R}(T)) \oplus Z \tag{3.16}$$

and

$$X = \mathcal{N}(T) \oplus Z \oplus W. \tag{3.17}$$

Thus there are continuous projections $P \in \mathcal{B}(X)$ and $Q \in \mathcal{B}(\mathcal{N}(T))$ for which

$$\mathcal{R}(P) = \mathcal{N}(T) \text{ and } \mathcal{N}(P) = Z \oplus W$$

and

$$Q(\mathcal{N}(T)) = H \text{ and } \mathcal{N}(Q) = \mathcal{N}(T) \cap J^{-1}(\mathcal{R}(T)).$$

Then we have

$$QP = (QP)^2,$$

so that QP is a continuous projection on X with range H. Further,

$$T(QP(X)) = T(H) = \{0\},$$

$$QP(J^{-1}(\mathcal{R}(T))) = QP(Z \oplus \mathcal{N}(T) \cap J^{-1}(\mathcal{R}(T))) = Q(\mathcal{N}(T) \cap J^{-1}(\mathcal{R}(T)))) = \{0\}.$$

Thus T is reduced by the decomposition $X = \mathcal{R}(QP) \oplus \mathcal{N}(QP)$. Indeed, we have $T(QP(X)) = \{0\} \subset J(QP(X))$. On the other hand we should prove that $T(\mathcal{N}(QP)) \subset J(\mathcal{N}(QP))$. We have $\mathcal{N}(QP) = \mathcal{R}(I - QP)$. Let $x \in X$. Then by (3.15) and (3.17) there exist $x_H \in H, x_{\mathcal{N}(T)\cap J^{-1}(\mathcal{R}(T))} \in \mathcal{N}(T) \cap J^{-1}(\mathcal{R}(T)), x_Z \in Z$ and $x_W \in W$ such that $x = x_H + x_{\mathcal{N}(T)\cap J^{-1}(\mathcal{R}(T))} + x_Z + x_W$. Therefore, $T(I - QP)(x) = T(x_{\mathcal{N}(T)\cap J^{-1}(\mathcal{R}(T))} + x_Z + x_W) = T(x_Z + x_W)$. We claim that $T(I - QP)(x) \in J((I - QP)(J^{-1}(\mathcal{R}(T))))$. We have $\mathcal{R}(T) \subset \mathcal{R}(J)$, then there exists $y \in X$ such that $T(x_Z + x_W) = J(y)$. So $y \in J^{-1}(\mathcal{R}(T))$. Then, by (3.16) there exist $y_{\mathcal{N}(T)\cap J^{-1}(\mathcal{R}(T))} \in \mathcal{N}(T) \cap J^{-1}(\mathcal{R}(T))$ and $y_Z \in Z$ such that $y = y_{\mathcal{N}(T)\cap J^{-1}(\mathcal{R}(T))} + y_Z$ and $(I - QP)(y) = y - QP(y_{\mathcal{N}(T)\cap J^{-1}(\mathcal{R}(T))} + y_Z) = y$. Therefore, $T(I - QP)(x) = J((I - QP)y) \in J((I - QP)(J^{-1}(\mathcal{R}(T))))$. Hence $T(\mathcal{N}(QP)) \subset J(\mathcal{N}(QP))$.

We write

$$T = T_1 \oplus T_2,$$

where $T_1 = T\backslash\mathcal{R}(QP)$ and $T_2 = T\backslash\mathcal{N}(QP)$. Note that $T_1 = 0$. Since T_2 has a closed range and

$$\mathcal{N}(T_2) = \mathcal{N}(QP) \cap \mathcal{N}(T) = J^{-1}(\mathcal{R}(T)) \cap \mathcal{N}(T)$$

is finite dimensional. it follows that T_2 is upper semi Fredholm. Again, by an analogous construction of the first part of the proof we have the required result. In fact, we can find a decomposition $X = Y_1 \oplus Y_2$ such that $T_2(Y_i) \subset J(Y_i)$ $(i = 1, 2)$ and after a finite number of steps we obtain $j_+(T_2/_{Y_2}, J/_{Y_2}) = 0$. Thus we have $T_2 = T_2/Y_2$ and $Y_1 \subset \mathcal{N}_k(T, J)$. This completes the proof. ∎

References

[1] Y. B. Choi, Y. M. Han and I. S. Hwang, *On Kato's decomposition theorem,* Comn. Korean Math. Soc. 9 (1994), No. 2, pp. 317-325.
[2] Y. Chamkha, M. Mnif, *The class of B-Fredholm linear relations,* Complex Anal. Oper. Theor. DOI 10. 1007/511785-014-0438-3(2013).
[3] D. Gagnage, *Kato decomposition of linear pencils,* Stu. Math. 154(2) (2003).
[4] S. Goldberg, *Unbounded linear operators,* McGraw-Hill, New York, 1966.
[5] R. E. Harte, *Fredholm, Weyl and Browder theory,* Proc. Roy. Irish Acad. Sect. A 85 (1986), 151-176.
[6] T. Kato, *Perturbation theory for nullity, deficiency, and other quantities of linear operators,* J. Analyse Math. 6(1958), 261-322.
[7] V. Müller, *Spectral theory of linear operators and spectral system in Banach algebras,* Operator theory advance and application vol. **139**, (2003).
[8] M. Schechter, *Principles of Functional Analysis,* Graduate Studies in Mathematics, Vol. **36**.
[9] T. T. West, *Removing the jump-Kato's decomposition,* Rocky Mountain J. Math. 20 (1990), 603-612.