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Ordinal Compactness

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Abstract. We introduce a new covering property, defined in terms of order types of sequences of open sets, rather than in terms of cardinalities. The most general form depends on two ordinal parameters.

Ordinal compactness turns out to be a much more varied notion than cardinal compactness. We prove many nontrivial results of the form "every $[\alpha, \beta]$-compact topological space is $[\alpha', \beta']$-compact", for ordinals α, β, α' and β', while only trivial results of the above form hold, if we restrict to regular cardinals. Counterexamples are provided showing that many results are optimal.

Many spaces satisfy the very same cardinal compactness properties, but have a broad range of distinct behaviors, as far as ordinal compactness is concerned. A much more refined theory is obtained for T_1 spaces, in comparison with arbitrary topological spaces. The notion of ordinal compactness becomes partly trivial for spaces of small cardinality.

1. Introduction

The nowadays standard notion of *compactness* for topological spaces is usually expressed in terms of cardinalities of open covers, and asserts that every open cover has a finite subcover. Compact spaces constitute a relatively special class; various weakenings have been extensively considered, the most notable being *Lindelöfness* ("any open cover has a countable subcover") and *countable compactness* ("any countable open cover has a finite subcover"). Still more generally, *final κ-compactness* asserts that any open cover has a subcover of cardinality $< \kappa$, and *initial κ-compactness* asserts that every open cover of cardinality $\leqslant \kappa$ has a finite subcover. A vast literature exists on the subject: see the surveys [12, 30, 35, 36], and, as a very subjective and partial choice, [3, 27, 28, 32] and references there for more recent lines of research. A more general form of compactness involving two cardinals shall be discussed in Subsection 1.2.

In this note we extend the above notions of compactness to ordinals, that is, we take into account order types of sequences of open sets, rather than just their cardinalities. As with cardinal compactness, these notions depend on certain parameters, this time chosen among ordinals. Assuming the Axiom of Choice, each cardinal can be seen as an ordinal, thus the ordinal notions are more general than the corresponding cardinal ones: when a sequence is cardinal-like ordered, we get back the more usual properties. On the contrary, and quite surprisingly, it turns out that the ordinal generalizations provide a much finer tuning of compactness properties of topological spaces.

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1.1. A first example: Lindelöf numbers

Before discussing the most general version of our notion, let us exemplify it in the particular case of Lindelöf numbers. Let us define the *Lindelöf*$^+$ *cardinal* of a topological space X as the smallest *cardinal* λ such that every open cover of X has a subcover of cardinality $< \lambda$ (the superscript $^+$ is a reminder that the more common definition asks just for a subcover of cardinality $\leqslant \lambda$. The present variant is more convenient here, since it distinguishes between compactness and Lindelöfness). In other words, the Lindelöf$^+$ cardinal of a topological space is the smallest cardinal λ such that the space is finally λ-compact.

As an ordinal generalization of the above notion, let us define the *Lindelöf ordinal* of a topological space X as the smallest *ordinal* α such that, for every open cover of X whose elements are indexed by some ordinal β, there exists some subset H of β such that H has order type $< \alpha$, and the set of elements with index in H still constitutes a cover of X. Thus we are dealing with covers taken in a certain (well) order and, when dealing with subcovers, we want the order of the original cover to be respected.

While the Lindelöf ordinal of a space clearly determines its Lindelöf$^+$ cardinal, on the contrary, there are spaces with the same Lindelöf$^+$ cardinal but with very different Lindelöf ordinals. As a simple example, if κ is a regular uncountable cardinal, then κ, both with the discrete topology, and with the order topology, has Lindelöf$^+$ cardinal κ^+. On the other hand, though κ^+ is also the Lindelöf ordinal of the former space, the latter space has a much smaller Lindelöf ordinal, that is, $\kappa + \omega$ (here and below $+$ denotes *ordinal sum*). Intermediate cases can occur: for example, the disjoint union of two copies of κ with the order topology has Lindelöf ordinal $\kappa + \kappa + \omega$. We can also have $\kappa + 1$, $\kappa + 2$, ... as Lindelöf ordinals, but only in some pathological cases, and only for spaces satisfying very few separation properties. More involved examples shall be presented in the body of the paper. Thus our ordinal generalization can be used to distinguish among spaces which appear to be quite similar, as far as the cardinal notion is considered.

Imposing further conditions on a space provides constraints on its Lindelöf ordinal. For example, the Lindelöf ordinal of a countable space is either ω_1, or is $\leqslant \omega \cdot \omega$ (here $\cdot$ denotes *ordinal product*). For spaces of cardinality κ there are similar limitations. Stronger restrictions are obtained by imposing mild separation axioms. For example, the Lindelöf ordinal of a T_1 space (of any cardinality) is either $\leqslant \omega$, or $\geqslant \omega_1$. Actually, only ordinals of a very special form can both have cofinality ω and be the Lindelöf ordinal of some T_1 space (Corollary 6.11). We also show that, for arbitrary spaces, the Lindelöf ordinal of a disjoint union is exactly determined by the Lindelöf ordinals of the summands.

Summing up, the Lindelöf ordinal of a topological space appears to be a quite fine measure of the compactness properties of the space. Moreover, there are interesting and deep connections between the possible values the Lindelöf ordinal can take, and cardinalities and separation properties of spaces.

1.2. $[\mu, \lambda]$-compactness (for cardinals)

Now we proceed by considering more general forms of compactness. All the (cardinal) compactness properties defined in the first paragraph of this introduction can be unified in a single framework by introducing the following two-cardinals property. For cardinals $\mu \leqslant \lambda$, a topological space is said to be $[\mu, \lambda]$-*compact* if and only if every open cover by at most λ sets has a subcover with $< \mu$ sets. Thus, for example, compactness is the same as $[\omega, \lambda]$-compactness, for every cardinal λ, and Lindelöfness is $[\omega_1, \lambda]$-compactness, for every cardinal λ. On the other hand, countable compactness is $[\omega, \omega]$-compactness, and, more generally, initial λ-compactness is $[\omega, \lambda]$-compactness.

In various equivalent forms, with a restriction on regular cardinals, $[\mu, \lambda]$-compactness has been introduced in 1929 by P. Alexandroff and P. Urysohn [1]. The very exact form of the above definition seems to have first appeared in [29]. It has been studied by many people, sometimes under different names and notations, and in several equivalent formulations. See a survey of further related notions and results in [34]. See also, e. g., [10, 20, 33] and references there for further information.

Apart from intrinsic interest, $[\mu, \lambda]$-compactness has proved useful in many cases. Besides providing a common generalization of countable compactness, Lindelöfness, and so on, it exhibits a very interesting feature: $[\mu, \lambda]$-compactness is equivalent to $[\nu, \nu]$-compactness, for every ν with $\mu \leqslant \nu \leqslant \lambda$. In particular, (full) compactness is equivalent to $[\nu, \nu]$-compactness for every infinite cardinal ν, and Lindelöfness is equivalent to $[\nu, \nu]$-compactness for every $\nu > \omega$. In other words, we can "slice" compactness into smaller

pieces. This fact has found many applications, mainly due to the fact that, for ν regular, $[\nu, \nu]$-compactness has many equivalent formulations, most notably in terms of the existence of complete accumulation points of sets of cardinality ν. See [1, 33, 34]. See also [25, Section 6] and [18], where the above mentioned "slicing" procedure has found other substantial applications.

By the way, let us also mention that the notion of $[\mu, \lambda]$-compactness has ostensibly inspired some further notions outside mainstream general topology. Most notably, some of the earliest definitions of both weakly and strongly compact cardinals were introduced as forms of $[\mu, \lambda]$-compactness for certain infinitary languages [14, Chapters 17 and 20]. The exact topological content of these definitions later clearly emerged: see [5, 23] for history, references, and for other notions in Model Theory and Logic which have apparently been inspired by $[\mu, \lambda]$-compactness. Also the notion of a (μ, λ)-regular ultrafilter, which played some role in the evolution of Set Theory [7], [15, Section 13], [14, p. 373], apparently originated in this stream of ideas.

1.3. The ordinal generalization

Motivated by the interest of (cardinal) $[\mu, \lambda]$-compactness, we started considering the possibility of an ordinal generalization. Though initially misled by the observation that "initial α-compactness" actually reduces to a cardinal notion (Corollary 2.8), we soon realized that the more general notion of "two ordinals compactness" is really new, as exemplified above in the particular case of Lindelöf-like properties or, put in other words, final α-compactness.

In detail, if β and α are ordinals, let us say that a space X is $[\beta, \alpha]$-compact if and only if every α-indexed open cover of X has a subcover indexed by a set of order type $< \beta$ (in the induced order).

Ordinal compactness, in the above sense, turns out to have some very particular features. As in the case of cardinals, we can show that, also for ordinals, $[\beta, \alpha]$-compactness is equivalent to $[\gamma, \gamma]$-compactness, for every ordinal γ with $\beta \leqslant \gamma \leqslant \alpha$. However, the similarities essentially end there. Indeed, for $\mu \neq \lambda$ infinite regular cardinals, $[\mu, \mu]$-compactness and $[\lambda, \lambda]$-compactness are independent properties. On the other hand, for ordinals, we have many results which tie together $[\beta, \alpha]$-compactness and $[\beta', \alpha']$-compactness, for various β, α, β' and α'. Just to state some of the simplest relations, we have that, for α and β infinite ordinals, the following statements hold (recall that $+$ and $\cdot$ denote ordinal sum and product, respectively).

(1) If $\beta \leqslant \alpha$, then $[\beta, \alpha]$-compactness implies $[\beta, \alpha + 1]$-compactness.

(2) $[\beta + \alpha, \beta + \alpha]$-compactness implies $[\beta + \alpha + \alpha, \beta + \alpha + \alpha]$-compactness.

(3) $[\alpha, \alpha]$-compactness implies both $[\beta + \alpha, \beta + \alpha]$-compactness and $[\beta \cdot \alpha, \beta \cdot \alpha]$-compactness.

However, not "everything" is provable, even for ordinals having the same cardinality. Indeed, still presenting simple examples:

(4) $[\alpha + 1, \alpha + 1]$-compactness does not imply $[\alpha, \alpha]$-compactness, in general.

(5) $[\kappa + \omega, \kappa + \omega]$-compactness does not imply $[\kappa, \kappa]$-compactness, in general.

(6) $[\kappa + \kappa, \kappa + \kappa]$-compactness does not imply $[\kappa \cdot \kappa, \kappa \cdot \kappa]$-compactness, in general.

Thus ordinal compactness is a highly nontrivial notion, in comparison with cardinal compactness. Moreover, we have quite sophisticated results which show that the ordinal compactness properties of a topological space are deeply affected both by its cardinality and its separation properties. For example, for κ an infinite regular cardinal, any counterexample to Clause (6) above must be of cardinality $> \kappa$. On the other hand, no T_1 space can be a counterexample to Clause (4). Furthermore, considering the compactness properties of disjoint unions involves some problems on ordinal arithmetic which are not entirely trivial.

T_1 spaces turn out to be a somewhat neat dividing line: many rather odd counterexamples, possible in spaces lacking separation properties, cannot be constructed using T_1 spaces. Thus we provide a rather refined theory for T_1 spaces. In particular, in this respect, countable ordinals behave very differently from uncountable ones: the compactness theory for T_1 spaces is trivial on countable ordinals; more generally, apart from a few exceptions, the ordinal properties of a T_1 space are "invariant" modulo intervals of

countable length. The compactness properties of T_1 spaces are completely determined by checking $[\beta, \beta]$-compactness both for ordinals of uncountable cofinality, and for a very small class of ordinals of cofinality ω (Corollary 6.10 and the comment below it).

Apparently, assuming stronger separation axioms does not seem to modify the theory a lot. At large, we get essentially the same results and counterexamples for T_1 and for normal spaces. However, there is still room for the possibility of a few finer results holding only for normal spaces; this is left as an open problem. See Section 7.

1.4. Synopsis of the paper

In summary, the paper is divided as follows. In Section 2 we introduce the main definition, together with some relatively simple properties and a couple of equivalent reformulations. Then we prove many results of the form "every $[\alpha, \beta]$-compact space is $[\alpha', \beta']$-compact"; most of these results shall be used in the rest of the paper. In Section 3 we then present a lot of examples, showing that $[\beta, \alpha]$-compactness, for α and β ordinals, provides a very fine tuning of properties of open coverings: there are many spaces which show a very differentiated behavior with respect to ordinals, but behave exactly the same way, when α and β are taken to vary only on cardinals. We also show that many of the results of Section 2 are the best possible ones. The most basic examples are presented in Subsection 3.1; then in Subsection 3.2 we discuss the behavior of ordinal compactness with respect to disjoint unions, and show that many more counterexamples can be obtained in this way. We also consider the Frolík sum, a generalized form of infinite disjoint union with a partial compactification. Compactness properties of disjoint unions are shown to be connected to some notions in ordinal arithmetics related to natural sums of ordinals. Such matters are clarified in detail in Subsection 3.3.

In Section 4 we show that many more implications between compactness properties hold for spaces of small cardinality; put in another way, certain counterexamples can be constructed only by means of spaces of sufficiently large cardinality. Such counterexamples are indeed provided in Section 5, where we give an exact characterization of those pairs of ordinals α and β such that $[\alpha, \alpha]$-compactness implies $[\beta, \beta]$-compactness. In Section 6 we then get a more refined theory, which holds for T_1 spaces. For such spaces, $[\beta, \alpha]$-compactness becomes trivial for countably infinite ordinals (Corollary 6.8). More generally, with a few exceptions, ordinal compactness for T_1 spaces is invariant modulo "translations" of countable length. Finally, Section 7 contains various quite disparate remarks and problems. In particular, it introduces further generalizations of ordinal compactness and also discusses the possibility of a variant in a model theoretical sense.

The present note by no means exhausts all that can be said about $[\beta, \alpha]$-compactness. Furthermore, as we mentioned, the notion of $[\beta, \alpha]$-compactness can be also generalized to different contexts.

2. Main definition and basic properties

In this section we introduce our main notion and state some simple properties. We compare it with the more usual notion which deals only with cardinals; then we start proving results of the form "every $[\beta, \alpha]$-compact space is $[\beta', \alpha']$-compact", for appropriate ordinals. For regular cardinals, only trivial results of the above kind hold. In the subsequent sections we shall present counterexamples showing that our results cannot be improved.

Throughout, let α, β and γ be nonzero ordinals, and λ, μ be nonzero (possibly finite) cardinals. As custom, we shall assume the Axiom of Choice, hence we identify cardinals with initial ordinals.

Definition 2.1. If X is a nonempty set (usually, but not necessarily, a topological space), and τ is a nonempty family of subsets of X, we say that (X, τ) is $[\beta, \alpha]$-*compact* if and only if the following condition holds.

Whenever $(O_\delta)_{\delta\in\alpha}$ is a sequence of members of τ such that $\bigcup_{\delta\in\alpha} O_\delta = X$, then there is $H \subseteq \alpha$ with order type $< \beta$ and such that $\bigcup_{\delta\in H} O_\delta = X$.

If there is no danger of confusion, we shall simply say X in place of (X, τ). As usual, a sequence $(O_\delta)_{\delta\in\alpha}$ of members of τ such that $\bigcup_{\delta\in\alpha} O_\delta = X$ shall be called a *cover* of X. A *subcover* of $(O_\delta)_{\delta\in\alpha}$ is a subsequence which is itself a cover.

By $[\beta, \alpha)$-*compactness* we mean $[\beta, \alpha']$-compactness for all $\alpha' < \alpha$. The notation is justified by Proposition 2.3(2)(4) below. Finally, $[\beta, \infty)$-*compactness* is $[\beta, \alpha]$-compactness for all ordinals $\alpha \geqslant \beta$.

When α and β are both cardinals and X is a topological space (τ being always understood to be the topology on X), we get back the classical cardinal compactness notion of Alexandroff, Urysohn and Smirnov [1, 29]. This is because, for λ a cardinal, having order type $< \lambda$ is the same as having cardinality $< \lambda$.

Notice that we allow repetitions in $(O_\delta)_{\delta\in\alpha}$, that is, we allow the possibility that $O_\delta = O_{\delta'}$, for $\delta \neq \delta'$. An equivalent and sometimes useful definition in which (among other things) repetitions are not allowed is given by Lemma 2.9. We have given the definition in the present form since it appears somewhat simpler.

Remark 2.2. In the definition of $[\beta, \alpha]$-compactness the assumption that the sequence is indexed by elements in the ordinal α is only for convenience. We get an equivalent definition by asking that, for every well ordered set J of order type α, if $(O_j)_{j\in J}$ is a cover of X, then there is $H \subseteq J$ such that the order type of H (under the order induced by the order on J) is $< \beta$, and such that $(O_j)_{j\in H}$ is a cover of X.

Of course, $[\beta, \alpha]$-compactness is equivalent to the following condition (just take complements!). Whenever $(C_\delta)_{\delta\in\alpha}$ is a sequence of complements of members of τ, and $\bigcap_{\delta\in H} C_\delta \neq \varnothing$, for every $H \subseteq \alpha$ with order type $< \beta$, then $\bigcap_{\delta\in\alpha} C_\delta \neq \varnothing$.

As we shall see below in Remark 3.4 and Example 3.11, ordinal compactness is actually a new notion, that is, it cannot be defined in terms of cardinal compactness.

We first list some simple but useful properties of $[\beta, \alpha]$-compactness.

Proposition 2.3. *Let α and β be nonzero ordinals.*

(1) If $\beta \leqslant \beta'$ and $\alpha' \leqslant \alpha$ then $[\beta, \alpha]$-compactness implies $[\beta', \alpha']$-compactness.

(2) $[\beta, \alpha]$-compactness is equivalent to $[\gamma, \gamma]$-compactness for every γ with $\beta \leqslant \gamma \leqslant \alpha$.

(3) If $\beta \leqslant \beta' \leqslant \alpha$, then X is $[\beta, \alpha]$-compact if and only if X is both $[\beta, \beta')$-compact and $[\beta', \alpha]$-compact.

(4) $[\beta, \alpha)$-compactness is equivalent to $[\gamma, \gamma]$-compactness for every γ with $\beta \leqslant \gamma < \alpha$.

Proof. (1) is trivial. If $\alpha' < \alpha$, add dummy elements at the top of the sequence, for example, by adding new occurrences of some element already in the sequence.

One implication in (2) is immediate from (1).

The converse is obtained by transfinite induction. Suppose that X is $[\gamma, \gamma]$-compact, for every γ with $\beta \leqslant \gamma \leqslant \alpha$. We shall prove $[\beta, \gamma]$-compactness, for every γ with $\beta \leqslant \gamma \leqslant \alpha$, by induction on γ. The induction basis $\gamma = \beta$ is true by assumption. As for the induction step, let $\beta < \gamma \leqslant \alpha$, and assume that X is $[\beta, \gamma']$-compact, for every γ' with $\beta \leqslant \gamma' < \gamma$. Let $(O_\delta)_{\delta\in\gamma}$ be a cover of X. By $[\gamma, \gamma]$-compactness, $(O_\delta)_{\delta\in\gamma}$ has a subcover $\mathcal{S}$ whose index set has order type $\gamma' < \gamma$. If $\gamma' < \beta$, we are done. Otherwise, by $[\beta, \gamma']$-compactness and Remark 2.2, we get a subcover of $\mathcal{S}$ whose index set has order type $< \beta$, and the item is proved.

(3) The only if condition is immediate from (1). For the converse, notice that, again by (1), $[\beta, \beta')$-compactness implies $[\gamma, \gamma]$-compactness for every γ with $\beta \leqslant \gamma < \beta'$, and that $[\beta', \alpha]$-compactness implies $[\gamma, \gamma]$-compactness for every γ with $\beta' \leqslant \gamma \leqslant \alpha$.

Thus we get $[\gamma, \gamma]$-compactness, for every γ with $\beta \leqslant \gamma \leqslant \alpha$, hence $[\beta, \alpha]$-compactness, by (2).

(4) is immediate from (2). □

Remark 2.4. When α, β, $\alpha' \ldots$ are restricted to vary only on cardinals, rather than ordinals, Proposition 2.3 still holds, with the same proof. In fact, for infinite cardinals, (1) and (2) are classical results about $[\mu, \lambda]$-compactness. Again for infinite cardinals, it is well known (and easy to prove) that, for topological spaces, $[\operatorname{cf}\mu, \operatorname{cf}\mu]$-compactness implies $[\mu, \mu]$-compactness. An ordinal generalization of the above fact will be given in Corollary 2.6(8).

For infinite regular cardinals, there is no nontrivial implication between $[\lambda, \lambda]$-compactness and $[\mu, \mu]$-compactness. Indeed, if λ is a regular infinite cardinal, then λ, with the order topology, is not $[\lambda, \lambda]$-compact, but it is $[\mu, \mu]$-compact for every infinite cardinal $\mu \neq \lambda$. By the cardinal analogue of Proposition 2.3(2),

if $\mu \leqslant \mu'$ are infinite cardinals, then λ with the order topology is $[\mu, \mu']$-compact if and only if λ does not belong to the interval $[\mu, \mu']$ of cardinals. More generally, the exact ordinal compactness properties of λ (with various topologies) shall be determined in Example 3.2.

Contrary to the case of cardinal compactness, and quite surprisingly, there are many nontrivial "transfer properties" for ordinal compactness, relating $[\beta, \alpha]$-compactness and $[\beta', \alpha']$-compactness, for various β, α, β' and α'. The next proposition and its corollary list some simple relations. More significant results along this line and some characterizations shall be proved in Section 5.

Proposition 2.5. *Suppose that β, α, β' and α' are nonzero ordinals and that there exists an injective function $f : \alpha' \to \alpha$ such that, for every $K \subseteq \alpha$ with order type $< \beta$, it happens that $f^{-1}(K)$ has order type $< \beta'$.*

Then $[\beta, \alpha]$-compactness implies $[\beta', \alpha']$-compactness.

The assumption that f is injective can be dropped in the case of topological spaces (or just assuming that τ is closed under unions).

Proof. Suppose that (X, τ) is $[\beta, \alpha]$-compact and let f be given satisfying the assumption. Let $(O_\delta)_{\delta \in \alpha'}$ be a cover of X and let $(U_\varepsilon)_{\varepsilon \in \alpha}$ be defined by $U_\varepsilon = O_\delta$, if $f(\delta) = \varepsilon$, and arbitrarily, if ε is not in the image of f. The definition is well posed, since f is injective. Let I be the image of f and let α'' be the order type of I.

$(U_\varepsilon)_{\varepsilon \in I}$ is still a cover of X, hence, by $[\beta, \alpha'']$-compactness (which follows from $[\beta, \alpha]$-compactness, by Proposition 2.3(1)) and by Remark 2.2, there is $K \subseteq \alpha$ of order type $< \beta$ and such that $(U_\varepsilon)_{\varepsilon \in K}$ still covers X. If we put $H = f^{-1}(K)$, then, by assumption, H has order type $< \beta'$; moreover, $(O_\delta)_{\delta \in H}$ is a cover of X, hence $[\beta', \alpha']$-compactness is proved.

In case τ is closed under unions and f is not injective, define $U_\varepsilon = \bigcup_{f(\delta) = \varepsilon} O_\delta$ and the same argument carries over. □

In what follows, if not otherwise specified, the operation $+$ will denote *ordinal sum*. That is, $\alpha + \beta$ is the order type of the order obtained by attaching a copy of β "at the top" of α. Similarly, $\cdot$ denotes *ordinal product*. The next corollary provides a sample of results that can be proved about the relationship between $[\beta, \alpha]$-compactness and $[\beta', \alpha']$-compactness, for various ordinals. Most of them shall be used in the rest of the paper.

Corollary 2.6. *Suppose that α, β and γ are nonzero ordinals, and λ and ν are cardinals.*

(1) If $\beta \leqslant \alpha$ and α is infinite, then $[\beta, \alpha]$-compactness implies $[\beta, \alpha + 1]$-compactness, hence also $[\beta, \alpha + n]$-compactness, for each $n < \omega$.

(2) If either γ or α is infinite, then $[\gamma + \alpha, \gamma + \alpha]$-compactness implies $[\gamma + \alpha + \alpha, \gamma + \alpha + \alpha]$-compactness, hence also $[\gamma + \alpha \cdot n, \gamma + \alpha \cdot n]$-compactness, for each $n < \omega$.

(3) If $\beta \leqslant \alpha$, α is infinite and $\lambda = \operatorname{cf} \alpha$, then $[\beta, \alpha]$-compactness implies $[\beta, \alpha + \lambda \cdot \omega)$-compactness.

(4) If $\beta \leqslant \lambda$, then $[\beta, \lambda]$-compactness implies $[\beta, \lambda^+)$-compactness.

(5) If $\beta \leqslant \alpha + \lambda$, and either $\operatorname{cf} \alpha > \lambda$, or α can be written as a limit of ordinals of cofinality $> \lambda$, then $[\beta, \alpha + \lambda]$-compactness implies $[\beta, \alpha + \lambda^+)$-compactness.

Suppose further that τ is closed under unions. Then:

(6) $[\alpha, \alpha]$-compactness implies $[\beta + \alpha, \beta + \alpha]$-compactness.

(7) $[\alpha, \alpha]$-compactness implies $[\beta \cdot \alpha, \beta \cdot \alpha]$-compactness.

(8) If $\operatorname{cf} \alpha = \nu$ is infinite, then $[\nu, \nu]$-compactness implies $[\alpha, \alpha]$-compactness.

Proof. (1) In view of Proposition 2.3(1) and since $\beta \leqslant \alpha$, then $[\beta, \alpha]$-compactness implies $[\alpha, \alpha]$-compactness. In view of Proposition 2.3(3), it is enough to show that $[\alpha, \alpha]$-compactness implies $[\alpha + 1, \alpha + 1]$-compactness.

The latter is proved by applying Proposition 2.5 to the function $f : \alpha + 1 \to \alpha$ defined as follows.

$$f(\varepsilon) = \begin{cases} 0 & \text{if } \varepsilon = \alpha, \\ \varepsilon + 1 & \text{if } \varepsilon < \omega, \\ \varepsilon & \text{if } \omega \leqslant \varepsilon < \alpha. \end{cases}$$

(2) If α is finite, then γ is infinite and the result follows from (1).

Otherwise, suppose that $\alpha = \alpha' + n$, with α' limit and $n < \omega$. Thus $\gamma + \alpha + \alpha = \gamma + \alpha' + \alpha' + n$. Consider the following function $f : \gamma + \alpha + \alpha \to \gamma + \alpha$.

$$f(\varepsilon) = \begin{cases} \varepsilon & \text{if } \varepsilon < \gamma, \\ \gamma + 2m & \text{if } \varepsilon = \gamma + m, \text{ with } m \in \omega, \\ \gamma + \alpha'' + 2m & \text{if } \varepsilon = \gamma + \alpha'' + m, \text{ with } \alpha'' \text{ limit} < \alpha', m \in \omega, \\ \gamma + 2m + 1 & \text{if } \varepsilon = \gamma + \alpha' + m, \text{ with } m \in \omega, \\ \gamma + \alpha'' + 2m + 1 & \text{if } \varepsilon = \gamma + \alpha' + \alpha'' + m, \text{ with } \alpha'' \text{ limit} < \alpha', m \in \omega, \\ \gamma + \alpha' + m & \text{if } \varepsilon = \gamma + \alpha' + \alpha' + m, \text{ with } m < n. \end{cases}$$

It is easy to see that f is injective.

Suppose that $K \subseteq \gamma + \alpha = \gamma + \alpha' + n$ and K has order type $< \gamma + \alpha$. Then either (a) $K \cap [\gamma + \alpha', \gamma + \alpha' + n)$ has order type $< n$, or (b) $K \cap [\gamma, \gamma + \alpha')$ has order type $\alpha^* < \alpha'$, or (c) $K \cap \gamma$ has order type $\gamma^* < \gamma$.

If (a) holds, then $f^{-1}([\gamma+\alpha', \gamma+\alpha'+n))$ has order type $< n$, hence $f^{-1}(K)$ has order type $< \gamma+\alpha'+\alpha'+n = \gamma+\alpha+\alpha$. In case (b), $f^{-1}([\gamma, \gamma+\alpha'))$ has order type $\leqslant \alpha^*+\alpha^*$, hence $f^{-1}(K)$ has order type $\leqslant \gamma+\alpha^*+\alpha^*+n$, which is strictly smaller than $\gamma + \alpha' + \alpha' + n$, since $\alpha^* < \alpha'$. Finally, we can suppose that we are in case (c), and both (a) and (b) fail. Since K has order type $< \gamma + \alpha = \gamma + \alpha' + n$ and $K \cap \gamma$ has order type $\gamma^* < \gamma$, then $\gamma^* + \alpha < \gamma + \alpha$. This easily implies that $\gamma^* + \alpha + \alpha < \gamma + \alpha + \alpha$ (for example, by expressing γ^*, γ and α in Cantor normal form). Since f is injective and, restricted to γ, is the identity, then $f^{-1}(K)$ has order type $\leqslant \gamma^* + \alpha + \alpha < \gamma + \alpha + \alpha$.

We have proved that $f^{-1}(K)$ has order type $< \gamma + \alpha + \alpha$ in all cases, hence Proposition 2.5 can be applied.

(3) If $\operatorname{cf}\alpha = 1$, this follows from (1), hence let us suppose that $\operatorname{cf}\alpha \geqslant \omega$.

By Proposition 2.3, it is enough to prove that if $\delta < \lambda \cdot \omega$, then $[\alpha, \alpha]$-compactness implies $[\alpha + \delta, \alpha + \delta]$-compactness. Refining further, it is enough to prove that

$$\text{if } \delta \leqslant \lambda, \text{ then } [\alpha, \alpha]\text{-compactness implies } [\alpha + \delta, \alpha + \delta]\text{-compactness,} \tag{*}$$

since then $[\alpha, \alpha]$-compactness implies $[\alpha + \lambda, \alpha + \lambda]$-compactness and then we can proceed inductively, by applying the result with $\alpha + \lambda$ in place of α, and then with $\alpha + \lambda + \lambda$ in place of α, and so on.

Hence, suppose that $\delta \leqslant \lambda = \operatorname{cf}\alpha$ and that $[\alpha, \alpha]$-compactness holds. If $\delta + \alpha > \alpha$, then necessarily $\delta = \operatorname{cf}\alpha$ and $\alpha = (\operatorname{cf}\alpha) \cdot m$, for some $m < \omega$, and (*) follows from (2) with $\gamma = 0$. Otherwise, $\delta + \alpha = \alpha$, hence we can define the following injective function $f : \alpha + \delta \to \alpha$.

$$f(\varepsilon) = \begin{cases} \delta + \varepsilon & \text{if } \varepsilon < \alpha, \\ \eta & \text{if } \varepsilon = \alpha + \eta, \text{ for } \eta < \delta. \end{cases}$$

Now, if $K \subseteq \alpha$ has order type $\zeta < \alpha$, then $f^{-1}(K)$ has order type $\leqslant \zeta + \delta$, which is necessarily $< \alpha + \delta$, since $\delta \leqslant \operatorname{cf}\alpha$. Hence Proposition 2.5 can be applied in order to get (*).

(4) Again by Proposition 2.3, it is enough to prove that $[\lambda, \lambda]$-compactness implies $[\alpha, \alpha]$-compactness, for every α with $|\alpha| = \lambda$. This is accomplished by Proposition 2.5, letting f be any injection from α to λ.

(5) As above, it is sufficient to prove that $[\alpha + \lambda, \alpha + \lambda]$-compactness implies $[\alpha + \gamma, \alpha + \gamma]$-compactness, for every γ with $|\gamma| = \lambda$. Let g be any injection from γ to λ and apply Proposition 2.5 to the following function $f : \alpha + \gamma \to \alpha + \lambda$.

$$f(\varepsilon) = \begin{cases} \varepsilon & \text{if } \varepsilon < \alpha, \\ \alpha + g(\eta) & \text{if } \varepsilon = \alpha + \eta, \text{ with } \eta < \gamma. \end{cases}$$

If $K \subseteq \alpha + \lambda$ has order type $< \alpha + \lambda$, then either $K \cap \alpha$ has order type $< \alpha$, or $K \cap [\alpha, \alpha + \lambda)$ has order type $< \lambda$. In the latter case, and since λ is a cardinal, we have that $f^{-1}(K)$ has order type $\leqslant \alpha + \gamma'$, for some γ' with $|\gamma'| < \lambda$, hence $f^{-1}(K)$ has order type $< \alpha + \gamma$, since $|\gamma| = \lambda$.

On the other hand, if $K \cap \alpha$ has order type $< \alpha$, then $f^{-1}(K) \cap \alpha$ has order type $< \alpha$, since f is the identity on α. The assumptions on α, and $|\gamma| = \lambda$ then imply that $f^{-1}(K)$ has order type $< \alpha + \gamma$.

(6) Apply the last statement in Proposition 2.5 to the function $f : \beta + \alpha \to \alpha$ defined by

$$f(\varepsilon) = \begin{cases} 0 & \text{if } \varepsilon < \beta, \\ \eta & \text{if } \varepsilon = \beta + \eta, \text{ with } \eta < \alpha. \end{cases}$$

(7) Apply the last statement in Proposition 2.5 to the function $f : \beta \cdot \alpha \to \alpha$ defined by $f(\varepsilon) = \zeta$ if $\varepsilon = \beta \cdot \zeta + \eta$, for some $\eta < \beta$.

(8) Let $(\gamma_\eta)_{\eta \in \nu}$ be a sequence cofinal in α of order type ν. Define $f : \alpha \to \nu$ by $f(\varepsilon) = \inf\{\eta \in \nu \mid \varepsilon < \gamma_\eta\}$ and apply Proposition 2.5. □

Example 2.7. As suggested by Corollary 2.6 (6)-(8), the relationships between various ordinal compactness properties change according to whether τ is required or not to be closed under unions. For example, if $\lambda > \mu$ are infinite cardinals, then every $[\mu, \mu]$-compact topological space is $[\lambda + \mu, \lambda + \mu]$-compact, by Corollary 2.6(6). On the other hand, if $\lambda > \mu$ are regular and $X = (\lambda + \mu, \tau)$, where $\tau = \{[0, \beta) \mid \beta \in \lambda\} \cup \{[\lambda, \lambda + \gamma) \mid \gamma \in \mu\}$, then X is trivially $[\mu, \mu]$-compact (since it has no cover of cardinality μ), but it is not $[\lambda + \mu, \lambda + \mu]$-compact. This is an example of a more general fact: see Corollary 5.6. See also Example 4.4.

We shall see in Sections 3 and 5 that $[\beta, \alpha]$-compactness is very far from being a trivial notion. However, Corollary 2.6(4) implies that $[\beta, \alpha]$-compactness becomes partly trivial for intervals containing a cardinal.

Corollary 2.8. *If α is infinite and $\beta \leqslant |\alpha|$, then the following properties are equivalent.*

(1) *$[\beta, |\alpha|]$-compactness.*

(2) *$[\beta, |\alpha|^+)$-compactness.*

(3) *$[\beta, \alpha]$-compactness.*

In particular, if $\mu \leqslant \lambda$ are infinite cardinals, then $[\mu, \lambda]$-compactness is equivalent to $[\mu, \lambda^+)$-compactness.

Proof. (1) ⇒ (2) is from Corollary 2.6(4).

(2) ⇒ (3) and (3) ⇒ (1) are immediate from Proposition 2.3(1). □

In particular, "initial α-compactness", that is, $[\omega, \alpha]$-compactness, does become trivial, in the sense that it actually reduces to cardinal compactness, in fact, to $[\omega, |\alpha|]$-compactness.

The next Lemma gives a somewhat useful equivalent formulation of $[\beta, \alpha]$-compactness. It states that it is enough to take into account only covers which are made of "irredundant" elements.

Lemma 2.9. *Let X be a nonempty set, τ be a nonempty family of subsets of X and β, α be nonzero ordinals.*

Then (X, τ) is $[\beta, \alpha]$-compact if and only if the following condition holds.

Whenever $\alpha^* \leqslant \alpha$, and $(O_\delta)_{\delta \in \alpha^*}$ is a sequence of members of τ such that

(1) *$\bigcup_{\delta \in \alpha^*} O_\delta = X$, and*

(2) *for every $\delta < \alpha^*$, O_δ is not contained in $\bigcup_{\varepsilon < \delta} O_\varepsilon$,*

then there is $H \subseteq \alpha^*$ with order type $< \beta$ and such that $\bigcup_{\delta \in H} O_\delta = X$.

Proof. The "only if" part follows trivially from Proposition 2.3(1).

Conversely, suppose that $(O_\delta)_{\delta \in \alpha}$ is a cover of X. Let $K = \{\delta \in \alpha \mid O_\delta \text{ is not contained in } \bigcup_{\varepsilon < \delta} O_\varepsilon\}$. Clearly, $(O_\delta)_{\delta \in K}$ is still a cover of X. Let α^* be the order type of K, and let $f : \alpha^* \to K$ be the order preserving bijection. Applying the assumption to the sequence $(O_{f(\gamma)})_{\gamma \in \alpha^*}$, we get $H \subseteq \alpha^*$ with order type $< \beta$, such that $\bigcup_{\gamma \in H} O_{f(\gamma)} = X$. This means that $(O_\delta)_{\delta \in f(H)}$ is a cover of X indexed by a set of order type $< \beta$. In particular, it is a subcover of $(O_\delta)_{\delta \in \alpha}$, thus $[\beta, \alpha]$-compactness is proved. □

3. First examples

In this section we provide many examples showing that ordinal compactness is not a "trivial" notion. In particular, it cannot be reduced to cardinal compactness. We also show that many of the results proved in Corollary 2.6 are the best possible ones, in the general case. On the contrary, we shall show in Section 6 that certain results can be improved if we just assume that we are dealing with a T_1 topological space.

In subsection 3.1 we endow cardinals with several topologies and characterize exactly the ordinal compactness properties they share. Then in Subsection 3.2 we give detailed results about compactness properties of disjoint unions and show that taking disjoint unions is a very flexible way to get more counterexamples. Examples of a different kind shall be presented in Section 5.

Finally, in Subsection 3.3 we discuss the technical notion of a shifted sum of two ordinals, introduced in connection with compactness properties of disjoint unions.

3.1. Basic examples

Definition 3.1. We shall endow cardinals with several topologies.

As usual, the *discrete* topology $\mathfrak{d}$ (on any set) is the topology in which every subset is open.

The *left order topology* $\mathfrak{lo}$ on some cardinal λ is the topology whose open sets are the intervals of the form $[0, \beta)$, with $\beta \leqslant \lambda$.

The *order topology* $\mathfrak{ord}$ on some cardinal λ is the more usual topology; a base for this topology is given by the intervals (α, β) ($\alpha < \beta \leqslant \lambda$), and $[0, \beta)$ ($\beta \leqslant \lambda$).

Examples 3.2. Let λ be any nonzero cardinal, and κ be an infinite regular cardinal.

(1) $(\lambda, \mathfrak{d})$ is $[\lambda^+, \infty)$-compact and not $[\alpha, \alpha]$-compact, for every nonzero $\alpha < \lambda^+$.

(2) $(\kappa, \mathfrak{lo})$ is not $[\kappa, \kappa]$-compact, but it is $[\kappa + 1, \infty)$-compact, and $[2, \kappa)$-compact.

(3) If $\kappa > \omega$, then $(\kappa, \mathfrak{ord})$ is a normal topological space which is $[\kappa + \omega, \infty)$-compact, $[\omega, \kappa)$-compact, and not $[\kappa + n, \kappa + n]$-compact, for each $n \in \omega$.

Proof. (1) is trivial.

(2) The sequence $[0, \beta)_{\beta<\kappa}$ itself proves $[\kappa, \kappa]$-incompactness, since κ is an infinite regular cardinal.

On the other hand, let $(O_\delta)_{\delta\in\alpha}$ be a cover of $(\kappa, \mathfrak{lo})$. If $O_\delta = \kappa$, for some $\delta \in \alpha$, then clearly $\{O_\delta\}$ itself is a one-element subcover.

Suppose otherwise. Since κ is regular, then necessarily $\alpha \geqslant \kappa$, and our aim is to extract a subcover of order type $\leqslant \kappa$. In fact, the subcover will turn out to be of order type exactly κ.

By Lemma 2.9, the result follows from the particular case in which the cover $(O_\delta)_{\delta\in\alpha}$ has the additional property that, for every $\delta < \alpha$, O_δ is not contained in $\bigcup_{\varepsilon<\delta} O_\varepsilon$. Suppose that the above condition is satisfied. Since each O_δ has the form $[0, \beta_\delta)$, for some $\beta_\delta < \kappa$, then, by the above condition, $\beta_\delta < \beta_{\delta'}$, for all pairs $\delta < \delta' < \alpha$. Since $(O_\delta)_{\delta\in\alpha}$ is a cover of κ, then $\sup_{\delta<\alpha} \beta_\delta = \kappa$. Thus the sequence $(\beta_\delta)_{\delta<\alpha}$ is strictly increasing, and cofinal in κ, hence has order type κ, since κ is a regular cardinal.

(3) Let $(O_\delta)_{\delta\in\alpha}$ be a cover of $(\kappa, \mathfrak{ord})$.

First, consider the case when some $O_{\bar\delta}$ contains an interval of the form (ε, κ), for some $\varepsilon < \kappa$. Since $[0, \varepsilon]$ is compact, it is covered by a finite number of the O_δ's. If we add $O_{\bar\delta}$ to these, we get a finite subcover of κ, since $\kappa = [0, \varepsilon] \cup (\varepsilon, \kappa)$, hence the conclusion holds in this case.

So we can suppose that no O_δ contains an interval of the form (ε, κ), thus necessarily $\alpha \geqslant \kappa$, since κ is regular. Since $(O_\delta)_{\delta\in\alpha}$ is a cover and each O_δ is a union of intervals, we have that, for every $\beta \in \kappa$ with $\beta \neq 0$, there is an interval $I_\beta = (\varepsilon_\beta, \phi_\beta)$ with $\varepsilon_\beta < \phi_\beta < \kappa$, such that $\beta \in I_\beta$ and $I_\beta \subseteq O_{\delta(\beta)}$, for some $\delta(\beta) \in \alpha$. For every nonzero $\beta \in \kappa$, choose some I_β and some $\delta(\beta) \in \alpha$ as above. The function $f : \kappa\backslash\{0\} \to \kappa$ defined by $f(\beta) = \varepsilon_\beta$ is regressive, hence, by Fodor's Theorem, constant on a set S stationary in κ, say, $f(\beta) = \bar\varepsilon$, for $\beta \in S$.

Let $D = \{\delta \in \alpha \mid \delta = \delta(\beta), \text{ for some } \beta \in S\}$. For $\delta \in D$, let $\eta_\delta = \sup\{\eta < \kappa \mid O_\delta \supseteq (\bar\varepsilon, \eta)\}$, and let $J_\delta = (\bar\varepsilon, \eta_\delta)$. Thus $O_\delta \supseteq J_\delta$, for $\delta \in D$. Moreover, $I_\beta \subseteq J_{\delta(\beta)}$, for $\beta \in S$, since if $\beta \in S$, then $(\bar\varepsilon, \phi_\beta) = I_\beta \subseteq O_{\delta(\beta)}$.

We now show that $(J_\delta)_{\delta\in D}$ is a cover of $(\bar\varepsilon,\kappa)$. Indeed, since S is stationary, in particular, cofinal, then, for every β' with $\bar\varepsilon<\beta'<\kappa$, there is $\beta>\beta'$, such that $\beta\in S$, thus $\beta'\in I_\beta$, since $\bar\varepsilon<\beta'<\beta\in I_\beta=(\varepsilon_\beta,\phi_\beta)$, hence $\beta'\in J_{\delta(\beta)}$, since $I_\beta\subseteq J_{\delta(\beta)}$.

Since $(\bar\varepsilon,\kappa)$ is order-isomorphic to κ and, through this isomorphism, the J_δ's correspond to open sets in the $\mathfrak{lo}$ topology, we can apply (2) in order to get a subset $E\subseteq D\subseteq\alpha$ such that E has order type $\leqslant\kappa$ and $(J_\delta)_{\delta\in E}$ covers $(\bar\varepsilon,\kappa)$. Hence also $(O_\delta)_{\delta\in E}$ covers $(\bar\varepsilon,\kappa)$.

Since $\kappa=[0,\bar\varepsilon]\cup(\bar\varepsilon,\kappa)$ and $[0,\bar\varepsilon]$ is compact, it is enough to add to E a finite number of elements from the original sequence $(O_\delta)_{\delta\in\alpha}$ in order to get a cover of the whole κ. Since we have added a finite number of elements to a sequence of order type $\leqslant\kappa$, we get a cover of κ which has order type $<\kappa+\omega$ and which is a subsequence of the original sequence. Thus we have proved $[\kappa+\omega,\alpha]$-compactness.

In order to finish the proof, we have to show that, for each $n\in\omega$, $(\kappa,\mathfrak{ord})$ is not $[\kappa+n,\kappa+n]$-compact. An easy counterexample is given by the sequence $(O_\delta)_{\delta\in\kappa+n}$ defined by

$$O_\delta=\begin{cases}[n,n+\delta) & \text{if } \delta<\kappa,\\ \{m\} & \text{if } \delta=\kappa+m, \text{ with } m<n.\end{cases}\quad\square$$

The situation appears in a clearer light if we introduce an ordinal variant of the Lindelöf number of a space.

Definition 3.3. The *Lindelöf ordinal* of (X,τ) is the smallest ordinal α such that (X,τ) is $[\alpha,\infty)$-compact.

Compare the above definition with the classical notion of the *Lindelöf number* of a topological space X, which is the smallest *cardinal* μ such that X is $[\mu^+,\infty)$-compact (the *Lindelöf number* is a distinct notion from the *Lindelöf*$^+$ *cardinal* defined in the introduction.)

Thus the Lindelöf number μ of X is determined by its Lindelöf ordinal α. Indeed, μ is the predecessor of α, if α is a successor cardinal, and $\mu=|\alpha|$ otherwise. On the other hand, in general, the Lindelöf ordinal cannot be determined by the Lindelöf number, as shown by Examples 3.2. Indeed, taking $\lambda=\kappa$ regular and uncountable, all the spaces in Examples 3.2 have Lindelöf number equal to κ, however, their Lindelöf ordinals are, respectively, κ^+, $\kappa+1$, and $\kappa+\omega$. Other possibilities for the Lindelöf ordinal are presented in Examples 3.10, 3.11 and 3.12. On the other hand, restrictions on the possible values Lindelöf ordinals can assume are given in Corollary 4.8 for spaces of small cardinality and in Corollary 6.11 for T_1 spaces.

Remark 3.4. Examples 3.2 also show that ordinal compactness cannot be determined exclusively by the cardinal compactness properties enjoyed by some space. For example, $X_1=(\omega,\mathfrak{lo})$ is $[\omega+1,\infty)$-compact, hence, by Proposition 2.3(1), it is $[\alpha,\alpha]$-compact, for every ordinal $\alpha>\omega$. On the other hand, $X_2=(\omega,\mathfrak{d})$ is $[\omega_1,\infty)$-compact, but not $[\alpha,\alpha]$-compact, for every countable ordinal α. Thus X_1 and X_2 are $[\lambda,\mu]$-compact exactly for the same pairs of infinite cardinals λ and μ, but there are many ordinals α for which X_1 is $[\alpha,\alpha]$-compact, but X_2 is not.

Example 3.11 below furnishes two normal topological spaces which are $[\lambda,\mu]$-compact exactly for the same pairs of cardinals λ and μ, no matter whether finite or infinite, but not $[\alpha,\alpha]$-compact for the same ordinals.

3.2. Disjoint unions

In order to refine Examples 3.2, we need some definitions.

Definition 3.5. If X_1 and X_2 are sets and τ_1, τ_2 are respective families of subsets, the *disjoint union* $(X_1\cup X_2,\tau)$ of (X_1,τ_1) and (X_2,τ_2) is a set $X_1\cup X_2$ obtained by taking the union of disjointed copies of X_1 and X_2, with τ being the family of all subsets of $X_1\cup X_2$ which either belong to (the copy of) τ_1, or belong to (the copy of) τ_2, or are the union of a set in τ_1 and a set in τ_2. Of course, in the case when X_1 and X_2 are topological spaces, we get back the usual notion of disjoint union in the topological sense.

Definition 3.6. If α and β are ordinals, we say that some ordinal γ is a *shifted sum* of α and β if and only if $\gamma=I\cup J$, for some (not necessarily disjoint) subsets $I,J\subseteq\gamma$ such that I has order type α and J has order type β, under the induced order.

Trivially, both $\alpha + \beta$ and $\beta + \alpha$ are shifted sums of α and β. The (Hessenberg) natural sum $\alpha \oplus \beta$ is the largest possible shifted sum of α and β. This is immediate from [6, Theorem 1, I, II], where the Hessenberg natural sum is denoted by $\sigma(\alpha, \beta)$, and follows also from Proposition 3.16 below.

However, there are other possibilities for shifted sums. For example, $\omega_1 + \omega$ is a shifted sum of ω_1 and $\omega + \omega$. A quite involved formula for determining all the possible shifted sums of α and β shall be obtained in Proposition 3.16, by expressing ordinals in additive normal form. The complication arises from the fact that, say, though both $\omega^3 + \omega + 1$ and $\omega^3 + \omega^2 + 1$ are shifted sums of $\alpha = \omega^3 + \omega$ and $\beta = \omega^2 + 1$, on the contrary $\omega^3 + 1$ is not a shifted sum of α and β.

If α and β are ordinals, we denote by $\alpha +^* \beta$ the smallest ordinal δ strictly larger than all the shifted sums of α' and β', for $\alpha' < \alpha$ and $\beta' < \beta$. Alternatively, $\alpha +^* \beta$ can be defined as $\sup\{\alpha' \oplus \beta' + 1 \mid \alpha' < \alpha, \beta' < \beta\}$.

We shall also need the following lemma.

Lemma 3.7. *Suppose that γ is a shifted sum of α and β, that is, $\gamma = I \cup J$, with I having order type α and J having order type β. Then the following additional property is satisfied. Whenever $I^* \subseteq I$ has still order type α, and $J^* \subseteq J$ has still order type β, then $I^* \cup J^*$ has still order type γ.*

Proof. Express γ in additive normal form as

$$\gamma = \omega^{\eta_h} + \omega^{\eta_{h-1}} + \cdots + \omega^{\eta_1} + \omega^{\eta_0},$$

for some integer $h \geqslant 0$ and ordinals $\eta_h \geqslant \eta_{h-1} \geqslant \cdots \geqslant \eta_1 \geqslant \eta_0$. Put $\gamma_{h+1} = 0$ and, for $i = 0, \dots, h$, put

$$\gamma_i = \omega^{\eta_h} + \omega^{\eta_{h-1}} + \cdots + \omega^{\eta_{i+1}} + \omega^{\eta_i}.$$

Consider the intervals $K_i = [\gamma_{i+1}, \gamma_i)$, for $i = 0, \dots, h$. Clearly, each K_i has order type ω^{η_i}. Moreover, γ is the disjoint union of the K_i's.

Fix some $\bar{\imath}$. Since $\gamma = I \cup J$, then $K_{\bar{\imath}} = (I \cap K_{\bar{\imath}}) \cup (J \cap K_{\bar{\imath}})$. Since $K_{\bar{\imath}}$ has order type $\omega^{\eta_{\bar{\imath}}}$, then, by an easy property of such exponents, either $I \cap K_{\bar{\imath}}$ or $J \cap K_{\bar{\imath}}$ has order type $\omega^{\eta_{\bar{\imath}}}$ (the proof is similar to, e. g., Hilfssatz 1 in [16]). Suppose that, say, $I_{\bar{\imath}} = I \cap K_{\bar{\imath}}$ has order type $\omega^{\eta_{\bar{\imath}}}$. Let $I_\bullet = I \cap (K_h \cup \cdots \cup K_{\bar{\imath}+1})$ and $I_\blacktriangle = I \cap (K_{\bar{\imath}-1} \cup \cdots \cup K_0)$, and let $\alpha_\bullet$, $\alpha_\blacktriangle$ be, respectively, the order types of $I_\bullet$ and of $I_\blacktriangle$. Since γ is the union of the K_i's, then $I = I_\bullet \cup I_{\bar{\imath}} \cup I_\blacktriangle$. Because of the relative way the elements of the K_i's are ordered in γ, we have that $\alpha = \alpha_\bullet + \omega^{\eta_{\bar{\imath}}} + \alpha_\blacktriangle$. Notice that $\alpha_\blacktriangle < \omega^{\eta_{\bar{\imath}}+1}$, since the order type of $K_{\bar{\imath}-1} \cup \cdots \cup K_0$ is $\omega^{\eta_{\bar{\imath}-1}} + \cdots + \omega^{\eta_0} < \omega^{\eta_{\bar{\imath}}} \cdot \omega = \omega^{\eta_{\bar{\imath}}+1}$. Since $I^* \subseteq I$, then the order types of, respectively, $I^* \cap I_\bullet$, $I^* \cap I_{\bar{\imath}}$ and $I^* \cap I_\blacktriangle$ are $\leqslant$ than, respectively, $\alpha_\bullet$, $\omega^{\eta_{\bar{\imath}}}$ and $\alpha_\blacktriangle$. However, since both I^* and I have order type α, then necessarily $I^* \cap I_{\bar{\imath}} = I^* \cap K_{\bar{\imath}}$ has order type $\omega^{\eta_{\bar{\imath}}}$, since otherwise the order type of I^* would be strictly smaller than $\alpha = \alpha_\bullet + \omega^{\eta_{\bar{\imath}}} + \alpha_\blacktriangle$, since, as we mentioned, $\alpha_\blacktriangle < \omega^{\eta_{\bar{\imath}}+1}$.

In a similar way, if $J \cap K_{\bar{\imath}}$ has order type $\omega^{\eta_{\bar{\imath}}}$, then also $J^* \cap K_{\bar{\imath}}$ has order type $\omega^{\eta_{\bar{\imath}}}$. Since the above argument works for every i, we get that, for each $i = 0, \dots, h$, either $I^* \cap K_i$ or $J^* \cap K_i$ contribute to K_i with order type ω^{η_i}, that is, $(I^* \cup J^*) \cap K_i$ has order type ω^{η_i}. This, together with the definition of the K_i's, implies that $I^* \cup J^*$ has order type γ. □

In the next lemma we characterize the compactness properties of disjoint unions. The lemma has not the most general form possible, but it is quite good for our purposes.

Lemma 3.8. *Assume the notation in Definitions 3.5 and 3.6.*

(1) Suppose that X_1 is not $[\alpha, \alpha]$-compact and X_2 is not $[\beta, \beta]$-compact.

If γ is a shifted sum of α and β, then $X_1 \cup X_2$ is not $[\gamma, \gamma]$-compact.

In particular, $X_1 \cup X_2$ is neither $[\alpha + \beta, \alpha + \beta]$-compact, nor $[\beta + \alpha, \beta + \alpha]$-compact, nor $[\alpha \oplus \beta, \alpha \oplus \beta]$-compact.

(2) If X_1 is $[\beta_1, \alpha]$-compact and X_2 is $[\beta_2, \alpha]$-compact, then $X_1 \cup X_2$ is $[\beta_1 +^* \beta_2, \alpha]$-compact.

Proof. (1) Represent γ as $I \cup J$ as in the definition of a shifted sum, with I of order type α and J of order type β, and let $f_1 : I \to \alpha$ and $f_2 : J \to \beta$ be the order preserving bijections.

Let $(O_\delta)_{\delta\in\alpha}$ be a cover of X_1 witnessing $[\alpha, \alpha]$-incompactness and let $(P_\varepsilon)_{\varepsilon\in\beta}$ be a cover of X_2 witnessing $[\beta, \beta]$-incompactness. For $\phi \in \gamma$, let $Q_\phi \subseteq X_1 \cup X_2$ be defined by

$$Q_\phi = \begin{cases} O_\delta & \text{if } \phi \in I\setminus J \text{ and } \delta = f_1(\phi)\,, \\ P_\varepsilon & \text{if } \phi \in J\setminus I \text{ and } \varepsilon = f_2(\phi)\,, \\ O_\delta \cup P_\varepsilon & \text{if } \phi \in I \cup J,\ \delta = f_1(\phi), \text{ and } \varepsilon = f_2(\phi). \end{cases}$$

By the definition of disjoint union, $(Q_\phi)_{\phi\in\gamma}$ is a cover of $X_1 \cup X_2$ with elements in τ. Suppose that $H \subseteq \gamma$ and that $(Q_\phi)_{\phi\in H}$ is still a cover of $X_1 \cup X_2$. Then it is easy to see that $(O_\delta)_{\delta\in f_1(H\cap I)}$ is a cover of X_1. Since $(O_\delta)_{\delta\in\alpha}$ witnesses the $[\alpha, \alpha]$-incompactness of X_1, then $f_1(H \cap I)$ has order type α, hence also $I^* = H \cap I$ has order type α, since f_1 is an order preserving bijection. Similarly, $J^* = H \cap J$ has order type β.

By Lemma 3.7, $H = H \cap \gamma = H \cap (I \cup J) = (H \cap I) \cup (H \cap J) = I^* \cup J^*$ has order type γ. Thus $(Q_\phi)_{\phi\in\gamma}$ is a counterexample to the $[\gamma, \gamma]$-compactness of $X_1 \cup X_2$.

The last statement in (1) follows from the remarks in Definition 3.5.

(2) Let $(O_\delta)_{\delta\in\alpha}$ be a cover of $X_1 \cup X_2$. Let I be the set of all $\delta \in \alpha$ such that either $O_\delta = P_\delta$, for some $P_\delta \in \tau_1$, or $O_\delta = P_\delta \cup Q_\delta$, for some (unique pair) $P_\delta \in \tau_1$ and $Q_\delta \in \tau_2$. Similarly, let J be the set of all $\delta \in \alpha$ such that either $O_\delta = Q_\delta$, for some $Q_\delta \in \tau_2$, or $O_\delta = P_\delta \cup Q_\delta$, for some $P_\delta \in \tau_1$ and $Q_\delta \in \tau_2$. Notice that $I \cup J = \alpha$, because of the definition of disjoint union.

Since $(O_\delta)_{\delta\in\alpha}$ is a cover of $X_1 \cup X_2$, then $(P_\delta)_{\delta\in I}$ is a cover of X_1 and, since I has order type $\leqslant \alpha$, then, by Remark 2.2, Proposition 2.3(1) and the $[\beta_1, \alpha]$-compactness of X_1, there is $I^* \subseteq I$ such that I^* has order type $\beta_1' < \beta_1$ and $(P_\delta)_{\delta\in I^*}$ is still a cover of X_1. Similarly, there is $J^* \subseteq J$ such that J^* has order type $\beta_2' < \beta_2$ and $(Q_\delta)_{\delta\in J^*}$ is a cover of X_2.

Let γ be the order type of $I^* \cup J^*$. Then γ is a shifted sum of β_1' and β_2', thus $\gamma < \beta_1 +^* \beta_2$. Since $(O_\delta)_{\delta\in I^*\cup J^*}$ turns out to be a cover of $X_1 \cup X_2$, the conclusion follows. □

Corollary 3.9. *Suppose that the Lindelöf ordinal of X_1 is β_1 and that the Lindelöf ordinal of X_2 is β_2. Then the Lindelöf ordinal of $X_1 \cup X_2$ is $\beta_1 +^* \beta_2$.*

Proof. The Lindelöf ordinal of $X_1 \cup X_2$ is $\leqslant \beta_1 +^* \beta_2$, as an immediate consequence of Lemma 3.8(2).

Hence, to prove equality, and in view of Proposition 2.3(2), we have to show that, for every $\gamma < \beta_1 +^* \beta_2$, there is γ'' with $\gamma \leqslant \gamma'' < \beta_1 +^* \beta_2$ and such that $X_1 \cup X_2$ is not $[\gamma'', \gamma'']$-compact. Let $\gamma < \beta_1 +^* \beta_2$. By the definition of $\beta_1 +^* \beta_2$, there are $\beta_1' < \beta_1$, $\beta_2' < \beta_2$ and $\gamma' < \beta_1 +^* \beta_2$ such that $\gamma \leqslant \gamma'$ and γ' is a shifted sum of β_1' and β_2'. By assumption, X_1 is not $[\beta_1', \infty)$-compact, hence, by Proposition 2.3(2), there is $\beta_1'' \geqslant \beta_1'$ such that X_1 is not $[\beta_1'', \beta_1'']$-compact, and necessarily $\beta_1'' < \beta_1$. Similarly, there is β_2'' such that X_2 is not $[\beta_2'', \beta_2'']$-compact and $\beta_2' \leqslant \beta_2'' < \beta_2$.

It follows trivially from the definition of a shifted sum, from $\beta_1' \leqslant \beta_1''$ and $\beta_2' \leqslant \beta_2''$ that there is some shifted sum γ'' of β_1'' and β_2'' such that $\gamma' \leqslant \gamma''$. By Lemma 3.8(1), $X_1 \cup X_2$ is not $[\gamma'', \gamma'']$-compact. Since $\beta_1'' < \beta_1$ and $\beta_2'' < \beta_2$, then $\gamma'' < \beta_1 +^* \beta_2$. Thus γ'' is an ordinal as wanted. □

We are now ready to present many improvements of Examples 3.2.

Examples 3.10. Let κ be an infinite regular cardinal and $n \in \omega$, $n \geqslant 2$.

(1) If X is the disjoint union of two copies of κ with the left order topology $\mathfrak{lo}$ of Definition 3.1, then X is not $[\kappa, \kappa]$-compact, not $[\kappa + 1, \kappa + 1]$-compact and not $[\kappa + \kappa, \kappa + \kappa]$-compact, but it is $[\kappa + \kappa + 1, \infty)$-compact, $[\kappa + 2, \kappa + \kappa)$-compact and $[3, \kappa)$-compact. Thus X has Lindelöf ordinal (Definition 3.3) $\kappa + \kappa + 1$.

(2) More generally, if X is the disjoint union of n copies of κ with the left order topology, then X is not $[\kappa, \kappa]$-compact, not $[\kappa + \kappa, \kappa + \kappa]$-compact, ..., not $[\kappa \cdot n, \kappa \cdot n]$-compact, but it is $[\kappa \cdot n + 1, \infty)$-compact, $[\kappa + n, \kappa + \kappa)$-compact, $[\kappa + \kappa + n - 1, \kappa + \kappa + \kappa)$-compact, ..., $[\kappa \cdot (n - 1) + 2, \kappa \cdot n)$-compact and $[n + 1, \kappa)$-compact. Its Lindelöf ordinal is $\kappa \cdot n + 1$.

Example 3.11. Suppose that κ is regular and $> \omega$, let $X_1 = (\kappa, \mathfrak{ord})$ and let X_2 be the disjoint union of two copies of X_1.

Then both X_1 and X_2 are $[\mu, \lambda]$-compact, for every pair of infinite cardinals μ and λ such that either $\kappa < \mu \leqslant \lambda$, or $\omega \leqslant \mu \leqslant \lambda < \kappa$; furthermore, X_1 and X_2 are not $[\kappa, \kappa]$-compact and not $[n, n]$-compact, for every positive integer n. Thus X_1 and X_2 are $[\mu, \lambda]$-compact exactly for the same pairs of cardinals μ and λ, whether finite or not.

However, X_1 is $[\kappa + \omega, \infty)$-compact, while X_2 is not even $[\kappa + \kappa, \kappa + \kappa]$-compact. Actually, X_2 is not $[\kappa + \kappa + n, \kappa + \kappa + n]$-compact, for every $n < \omega$, but it is $[\kappa + \omega, \kappa + \kappa)$-compact, and $[\kappa + \kappa + \omega, \infty)$-compact. Its Lindelöf ordinal is $\kappa + \kappa + \omega$.

Example 3.12. Suppose that X_1 is a nonempty set and τ is a nonempty family of subsets of X_1. Suppose that X_2 is a discrete topological space of cardinality μ and that X is the disjoint union of X_1 and X_2. Then the following statements hold.

(1) If X_1 is not $[\alpha, \alpha]$-compact, $|\beta| \leqslant \mu$, and γ is a shifted sum of α and β, then X is not $[\gamma, \gamma]$-compact.

(2) If X_1 is $[\beta, \alpha]$-compact, then X is $[\beta +^* \mu^+, \alpha]$-compact.

In particular, by adding a discrete finite set to Example 3.2(2), we get a $[\kappa + m + 1, \infty)$-compact space which is not $[\kappa + m, \kappa + m]$-compact. Thus we have $\kappa + m + 1$ as a Lindelöf ordinal of some space. In a similar way, by starting with Example 3.10, we have $\kappa \cdot n + m + 1$ as a Lindelöf ordinal.

Proof. Almost everything in Examples 3.10, 3.11 and 3.12 follows from Proposition 2.3, Examples 3.2 and Lemma 3.8.

An exception is $[\kappa + 2, \kappa + \kappa)$-compactness in Example 3.10(1), which is proved as follows. Let X be the disjoint union of two copies of $(\kappa, \mathfrak{lo})$ and consider an ordinal-indexed cover $\mathcal{C}$ of X. By Example 3.2(2), there is a subsequence of $\mathcal{C}$ which is a cover of the first copy of $(\kappa, \mathfrak{lo})$ and either has order type κ, or consists of a single element, that is, has order type 1. Similarly, there is a subsequence of $\mathcal{C}$ which is a cover of the second copy and has the same possible order types. By joining the above two partial subcovers, we get a cover of the whole of X, whose order type is a shifted sum of β_1 and β_2, where the possible values β_1 and β_2 are either κ or 1. Any such shifted sum, if $< \kappa + \kappa$, must necessarily be $\leqslant \kappa + 1$, from which $[\kappa + 2, \alpha]$-compactness follows, for every α with $\kappa + 2 \leqslant \alpha < \kappa + \kappa$.

The proofs of $[\kappa + n, \kappa + \kappa)$-compactness, $[\kappa + \kappa + n - 1, \kappa + \kappa + \kappa)$-compactness, ... in 3.10(2), and of $[\kappa + \omega, \kappa + \kappa)$-compactness of X_2 in 3.11 are similar. □

Many other similar examples can be obtained by combining in various ways the examples in 3.2 with Lemma 3.8. Further counterexamples can be obtained by applying disjoint unions to the examples we shall introduce in Definition 5.1(2).

Example 3.13. It is trivial to show that, for $\mu \leqslant \lambda$ infinite cardinals, the disjoint union of two topological spaces is $[\mu, \lambda]$-compact if and only if the two spaces are both $[\mu, \lambda]$-compact (this also follows from Lemma 3.8).

The spaces constructed in Example 3.11 show that, for ordinals, the disjoint union of two $[\beta, \alpha]$-compact spaces is not necessarily $[\beta, \alpha]$-compact. Just take $\alpha = \beta = \kappa + \kappa$, for some regular $\kappa > \omega$, and consider the union of two disjoint copies of $(\kappa, \mathfrak{ord})$.

One can also deal with the obviously defined notion of the disjoint union of an infinite family. It appears to be promising also the possibility of considering a partial compactification of an infinite disjoint union. This can be accomplished as follows (see Frolík [9] and Juhász and Vaughan [13] in the case of topological spaces).

Definition 3.14. Suppose that $(X_i)_{i \in I}$ is a family of nonempty sets and, for each $i \in I$, τ_i is a nonempty family of subsets of X_i. Suppose, for sake of simplicity, that each τ_i contains the empty set.

The *Frolík sum* (X, τ) of $(X_i, \tau_i)_{i \in I}$ is defined as follows.

Set theoretically, $X = \{x\} \cup \bigcup_{i\in I} X_i$ is the union of (disjoint copies) of the X_i's, plus a new element x which belongs to no X_i. The members of τ are those subsets O of X which have one of the following two forms.

$$O = \bigcup_{i\in I} O_i,$$

where $O_i \in \tau_i$, for every $i \in I$, or

$$O = \{x\} \cup \bigcup_{i\in I} O_i,$$

where, for some finite set $F \subseteq I$, it happens that $O_i \in \tau_i$, for $i \in F$, and $O_i = X_i$, for $i \in I\setminus F$.

The Frolík sum is sometimes called the *one-point countable-compactification*. We have called it the *Fréchet disjoint union* in some former works.

The above definition appears to be interesting, in the present context, since, as in Example 3.13, $[\beta, \alpha]$-compactness of a Frolík sum is not necessarily preserved. However, (infinite) cardinal compactness and many other topological properties are preserved, as asserted by the next proposition.

Proposition 3.15. *If $(X_i)_{i\in I}$ is a family of topological spaces, then their Frolík sum $X = \{x\} \cup \bigcup_{i\in I} X_i$ is a topological space. It is, respectively, T_0, T_1, Hausdorff, regular, normal, $[\lambda, \mu]$-compact (for given infinite cardinals λ and μ), has a base of clopen sets if and only if so is (has) each X_i.*

Proof. Straightforward. We shall comment only on regularity and normality. For these, just observe that if C is closed in X and C has nonempty intersection with infinitely many X_i's, then $x \in C$. □

Notice that the spaces in Examples 3.2(2) and 3.10 satisfy very few separation axioms. Indeed, just assuming that X is a T_1 topological space, it is impossible to construct such counterexamples. See Section 6.

Curiously enough, Counterexample 3.10 cannot be generalized in a simple way in order to get some (X, τ) which is not $[\kappa\cdot\kappa, \kappa\cdot\kappa]$-compact, but which is, say, $[\kappa\cdot\kappa+\kappa, \kappa\cdot\kappa+\kappa]$-compact. Such a counterexample exists (Remark 5.5), but we need a much more involved construction. Indeed, if X is such a counterexample, then $|X| > \kappa$, as we shall show in the next section.

3.3. A note on shifted sums and mixed sums

We now give the promised characterization of those ordinals γ which can be realized as a shifted sum of two ordinals α and β.

Recall that every ordinal $\gamma > 0$ can be expressed in a unique way in additive normal form as

$$\gamma = \omega^{\eta_h} + \omega^{\eta_{h-1}} + \cdots + \omega^{\eta_1} + \omega^{\eta_0},$$

for some integer $h \geqslant 0$ and ordinals $\eta_h \geqslant \eta_{h-1} \geqslant \cdots \geqslant \eta_1 \geqslant \eta_0$. Hence to any ordinal γ we can uniquely associate the finite string $\sigma(\gamma)$ of ordinals in decreasing order $\eta_h * \eta_{h-1} * \cdots * \eta_1 * \eta_0$. Here and below $*$ denotes string juxtaposition. We are allowing the empty string, which is associated to the ordinal 0. Moreover, through the present subsection, *decreasing* is always intended in the non strict sense, namely, it means "decreasing but not necessarily strictly decreasing".

To every finite string of ordinals $\sigma = \eta_h * \eta_{h-1} * \cdots * \eta_1 * \eta_0$ we can associate the ordinal $\delta(\sigma) = \omega^{\eta_h} + \omega^{\eta_{h-1}} + \cdots + \omega^{\eta_1} + \omega^{\eta_0}$. We are not necessarily assuming that the ordinals in σ are in decreasing order. However, an arbitrary string σ can be *reduced* to a string σ^r whose elements are in decreasing order, by taking out from σ all those elements which are followed by some strictly larger element. Notice that, anyway, $\delta(\sigma^r) = \delta(\sigma)$, since, for example, $\omega^\xi + \omega^{\xi'} = \omega^{\xi'}$, if $\xi < \xi'$. In particular, if $\gamma = \delta(\sigma)$, then $\sigma(\gamma) = \sigma^r$, since the correspondence between ordinals and strings consisting of decreasing ordinals is bijective.

Proposition 3.16. *Suppose that α, β, γ are ordinals and $\sigma(\gamma) = \eta_h * \eta_{h-1} * \cdots * \eta_1 * \eta_0$. Then the following conditions are equivalent.*

(1) γ *is a shifted sum of* α *and* β.

(2) There are (possibly empty) strings $\sigma_h \dots \sigma_0$ *and* $\sigma'_h \dots \sigma'_0$ *such that*

(a) $\alpha = \delta(\sigma_h * \sigma_{h-1} * \dots * \sigma_0)$,

(b) $\beta = \delta(\sigma'_h * \sigma'_{h-1} * \dots * \sigma'_0)$,

(c) for each $i = 0, \dots, h$, *either* $\sigma_i = \eta_i$, *or* σ_i *is empty, or every element of* σ_i *is* $< \eta_i$,

(d) for each $i = 0, \dots, h$, *either* $\sigma'_i = \eta_i$, *or* σ'_i *is empty, or every element of* σ'_i *is* $< \eta_i$,

(e) for each $i = 0, \dots, h$, *either* $\sigma_i = \eta_i$, *or* $\sigma'_i = \eta_i$.

(3) Same as (2) with conditions (a) and (b) replaced by

(a′) $\sigma(\alpha) = \sigma_h * \sigma_{h-1} * \dots * \sigma_0$,

(b′) $\sigma(\beta) = \sigma'_h * \sigma'_{h-1} * \dots * \sigma'_0$.

Proof. For $i = 0, \dots, h$, define the intervals K_i as in the proof of Lemma 3.7. Recall that each K_i has order type ω^{η_i}, that γ is the disjoint union of the K_i's, and that, for every $i > i'$, each element of K_i precedes every element of $K_{i'}$ in the ordering induced by the ordering on γ.

(1) $\Rightarrow$ (2) By (1), $\gamma = I \cup J$, for some I and J of order types, respectively, α and β. For $i = 0, \dots, h$, let α_i be the order type of $I \cap K_i$, thus $\alpha = \alpha_h + \dots + \alpha_0$, by the above properties of the K_i's. Put $\sigma_i = \sigma(\alpha_i)$. Then (a) is satisfied, since $\delta(\sigma_h * \dots * \sigma_0) = \delta(\sigma_h) + \dots + \delta(\sigma_0)$ and since $\delta(\sigma(\varepsilon)) = \varepsilon$, for every ordinal ε. Moreover, (c), too, holds, since the order type of α_i is $\leqslant$ the order type of K_i, that is, ω^{η_i}. Similarly, letting β_i be the order type of $J \cap K_i$ and $\sigma'_i = \sigma(\beta_i)$, we have that (b) and (d) hold. Finally, as remarked in the proof of Lemma 3.7, since $K_i = (I \cap K_i) \cup (J \cap K_i)$ has order type ω^{η_i}, then either α_i or β_i has order type ω^{η_i}, thus (e) holds.

(2) $\Rightarrow$ (3) Observe that $(\sigma_h * \dots * \sigma_0)^r = \sigma_h^- * \dots * \sigma_0^-$, for appropriate strings σ_i^- such that each σ_i^- is a substring of σ_i (however, it is not necessarily the case that $\sigma_i^- = \sigma_i^r$). Then, by the last remark before the statement of the proposition, $\sigma(\alpha) = (\sigma_h * \dots * \sigma_0)^r = \sigma_h^- * \dots * \sigma_0^-$. Moreover, if the σ_i's satisfy (c), then also the σ_i^-'s satisfy (c), since we are just taking out elements. Furthermore, if $\sigma_i = \eta_i$ and (c) holds, then this occurrence of η_i is not deleted in σ_i^-, since $\eta_i \geqslant \eta_{i'}$, for $i > i'$. By taking further strings $\sigma_i'^-$ such that $(\sigma'_h * \dots * \sigma'_0)^r = \sigma_h'^- * \dots * \sigma_0'^-$, and arguing as before, we get that the σ_i^-'s and the $\sigma_i'^-$'s witness (3).

(3) $\Rightarrow$ (2) is trivial, since $\delta(\sigma(\varepsilon)) = \varepsilon$, for every ordinal ε.

(2) $\Rightarrow$ (1) For $i = 0, \dots, h$, put $\alpha_i = \delta(\sigma_i)$ and $\beta_i = \delta(\sigma'_i)$. By Clauses (c)-(d), α_i and β_i are both $\leqslant \omega^{\eta_i}$. Let I_i be the initial segment of K_i of order type α_i, and J_i be the initial segment of K_i of order type β_i. The definition is well posed, since the order type of K_i is ω^{η_i}. By Clause (e), $I_i \cup J_i = K_i$. If we put $I = I_0 \cup \dots \cup I_h$ and $J = J_0 \cup \dots \cup J_h$, then $I \cup J = K_0 \cup \dots \cup K_h = \gamma$. Notice that, by the properties of the K_i's, I has order type $\alpha_h + \dots + \alpha_0 = \delta(\sigma_h) + \dots + \delta(\sigma_0) = \delta(\sigma_h * \dots * \sigma_0) = \alpha$, by Clause (a). Similarly, by Clause (b), J has order type β, thus we are done. □

Notice that, given α and β, there is only a finite number of ordinals γ which are shifted sums of α and β. Indeed, by Proposition 3.16, the elements of $\sigma(\gamma)$ are a (possibly proper) subset of the union of the sets of the elements of $\sigma(\alpha)$ and of $\sigma(\beta)$ (counting multiplicities), and this can be accomplished only in a finite number of ways.

On the other hand, given γ, it might be the case that γ can be realized in infinitely many ways as a shifted sum. For example, for every $n < \omega$, $\omega^\omega + 1$ can be realized as the shifted sum of ω^ω and $\omega^n + 1$.

Notice that γ is the natural sum $\alpha \oplus \beta$ of α and β if and only if a representation as in Proposition 3.16 exists in such a way that, for each $i = 0, \dots, h$, either $\sigma_i = \eta_i$ and σ'_i is empty, or $\sigma'_i = \eta_i$ and σ_i is empty.

The notion of a shifted sum is related to a known similar notion, usually called mixed sum (*Mischsumme*, [16, 24]) or *shuffle*. In our notation, γ is a *mixed sum* of α and β if γ can be realized as a shifted sum of α and β as in Definition 3.6, with the additional assumption that $I \cap J = \varnothing$.

Proposition 3.17. *Under the assumptions in Proposition 3.16, we have that* γ *is a mixed sum of* α *and* β *if and only if Condition (2) (equivalently, Condition (3)) in 3.16 holds with the following additional clause*

(f) For each $i = 0, \dots, h$, if $\eta_i = 0$, then either σ_i or σ'_i is the empty string.

Proof. If γ is a mixed sum of α and β, then, in particular, it is a shifted sum, hence the conditions in Proposition 3.16(2)(3) hold. In order to prove (f), notice that, if γ is a mixed sum of α and β and $\eta_i = 0$, then $|K_i| = 1$, hence either I_i or J_i is empty, since, in the present situation, they are disjoint and contained in K_i, thus (f) follows.

It remains to show how to get disjoint I_i and J_i, for each i, in the proof of (2) $\Rightarrow$ (1) (hence we get disjoint I and J, since the K_i's are pairwise disjoint). If $\eta_i = 0$, this follows from Clause (f). Otherwise, observe that any set of order type ω^{η_i} can always be expressed as the union of two disjoint subsets having prescribed order types α_i and β_i, provided that α_i and β_i are both $\leqslant \omega^{\eta_i}$ and their maximum is ω^{η_i}. □

A somewhat similar characterization of those ordinals γ which can be expressed as a mixed sum of α and β has been given in [16]. Actually, [16] deals with mixed sums with possibly more than two summands. Also the results presented here can be easily generalized to the case of more than two summands. We leave details to the reader.

We now discuss in more detail the relationship between the notions of a shifted sum and of a mixed sum. It turns out that the only difference is made by the "finite tail" of γ, that is, if $\gamma = \gamma^\bullet + m$, with $\gamma^\bullet$ limit, then the ways $\gamma^\bullet$ can be realized as a shifted sum determine the ways γ can be realized as a mixed sum.

Corollary 3.18. *Let α, β and γ be ordinals.*

(1) Suppose that γ is a limit ordinal. Then γ is a mixed sum of α and β if and only if γ is a shifted sum of α and β (and, if this is the case, then either α or β is limit, but not necessarily both).

(2) More generally, suppose that $\gamma = \gamma^\bullet + m$, with $\gamma^\bullet$ limit, and $\omega > m \geqslant 0$. Then γ is a mixed sum of α and β if and only if there are integers $n, p \geqslant 0$ such that $n + p = m$, α has the form $\alpha^\bullet + n$, β has the form $\beta^\bullet + p$, and $\gamma^\bullet$ is a shifted sum of $\alpha^\bullet$ and $\beta^\bullet$ (one of $\alpha^\bullet$ and $\beta^\bullet$ must thus be limit, but not necessarily both).

Proof. (1) If γ is limit and $\sigma(\gamma) = \eta_h * \eta_{h-1} * \cdots * \eta_1 * \eta_0$, then all the η_i's are > 0, thus Clause (f) in Proposition 3.17 is automatically satisfied.

(2) If $\gamma = \gamma^\bullet + m$, then $\eta_i = 0$ exactly for $i = 0, \dots, m-1$, thus $\sigma(\gamma^\bullet) = \eta_h * \eta_{h-1} * \cdots * \eta_m$. The conclusion now follows easily from (1) and Propositions 3.16 and 3.17. □

Notice that the notions of a shifted sum and of a mixed sum are distinct. Indeed, it follows easily from Proposition 3.16 that the smallest shifted sum of α and β is $\sup\{\alpha, \beta\}$. However, the smallest mixed sum of, say, $\omega + 1$ and $\omega + 2$ is $\omega + 3 > \sup\{\omega + 1, \omega + 2\}$. In general, as a corollary of Proposition 3.17, we obtain a result by Neumer [24]: for $\alpha = \alpha^\bullet + n$ and $\beta = \beta^\bullet + p$, where $\alpha^\bullet$ and $\beta^\bullet$ are limit ordinals, the smallest mixed sum of α and β is $\alpha^\bullet + n + p$, if $\alpha^\bullet = \beta^\bullet$, and $\sup\{\alpha, \beta\}$, if $\alpha^\bullet \neq \beta^\bullet$.

4. Some indispensability arguments and spaces of small cardinality

As we mentioned, a discrete space of cardinality λ is not $[\alpha, \alpha]$-compact, for every ordinal α of cardinality $\leqslant \lambda$. In a more general way, we can exhibit plenty of spaces which behave as discrete spaces, that is, for which ordinal (in-)compactness reduces to cardinal (in-)compactness. This is the theme of the first proposition in the present section. Then we proceed to prove a more sophisticated result, Theorem 4.5, which implies that, if we restrict ourselves to spaces of cardinality κ, then $[\alpha, \alpha]$-compactness is equivalent to $[\beta, \beta]$-compactness, for a large set of limit ordinals α and β of cardinality κ. In particular, for countable spaces, Corollary 4.7 shows that $[\alpha, \alpha]$-compactness becomes trivial above $\omega \cdot \omega$. The above mentioned results imply that the relatively simple examples introduced in the previous section are really far from exhausting all possible kinds of counterexamples. Indeed, further and more involved counterexamples shall be constructed in the next section. In fact, in the next section we shall prove some equivalences which show that Proposition 2.5 cannot be improved, in the general case when τ is an arbitrary family.

In order to carry on the proof of the next proposition, we need a definition.

Definition 4.1. If $(O_\delta)_{\delta\in\alpha}$ is a cover of X, let us say that some $O_{\bar\delta}$ is *indispensable* if and only if every subcover of $(O_\delta)_{\delta\in\alpha}$ must contain $O_{\bar\delta}$. Equivalently, $O_{\bar\delta}$ is indispensable if and only if there is $x \in O_{\bar\delta}$ such that $x \notin \bigcup_{\delta\in\alpha,\delta\neq\bar\delta} O_\delta$.

For example, if X is a topological space with the discrete topology and $(O_\delta)_{\delta\in\alpha}$ is a cover of X consisting of (all) singletons (without repetitions), then each element of this cover is indispensable.

Proposition 4.2. *Suppose that α is a nonzero ordinal, λ is an infinite cardinal and (X,τ) has some cover $(O_\delta)_{\delta\in\alpha}$ having at least λ indispensable elements.*

(1) If $|\alpha| = \lambda$, then X is not $[\beta,\beta]$-compact, for every ordinal β with $|\beta| = \lambda$.

(2) If τ is closed under unions, then X is not $[\beta,\beta]$-compact, for every nonzero ordinal β with $|\beta| \leqslant \lambda$.

Proof. (1) Let $|\beta| = \lambda$. Rearrange the sequence $(O_\delta)_{\delta\in\alpha}$ as $(O'_\varepsilon)_{\varepsilon\in\beta}$ in such a way that, in this latter sequence, the subsequence of the indispensable elements has order type β. This is always possible, since λ is an infinite cardinal, $|\beta| = \lambda$ and there are λ-many indispensable elements in the original sequence. For example, if μ is the cardinality of the set of non indispensable elements (it may happen that $\mu = 0$), choose a subset $Z \subseteq \lambda$ with $|Z| = \mu$ and such that $|\lambda\backslash Z| = \lambda$, assign to non indispensable elements only positions in Z and assign all the other positions in $\beta\backslash Z$ to all indispensable elements.

Every subcover of $(O'_\varepsilon)_{\varepsilon\in\beta}$ must contain all of its indispensable elements, thus has order type β. This implies that X is not $[\beta,\beta]$-compact.

(2) Let $|\beta| \leqslant \lambda$, say $|\beta| = \nu$. Consider a new cover of X obtained by choosing ν-many indispensable O_δ's and joining all the remaining O_δ's into one of them (the union is still in τ, since τ is closed under unions). If ν is finite, then the result is trivial. Otherwise, it is obtained by applying (1), with ν in place of λ, to this new cover. □

In Section 6 we shall use arguments similar to those used in the proof of Proposition 4.2 in order to prove results about compactness properties of T_1 spaces.

Theorem 4.5 below is a far more sophisticated result than Propositions 4.2. Recall that $+$ and $\cdot$ denote, respectively, ordinal sum and product. Moreover, also exponentiation, if not otherwise specified, will denote *ordinal exponentiation*.

Lemma 4.3. *Suppose that κ is an infinite regular cardinal and α is an ordinal of the form $\alpha_1 + \kappa^\varepsilon$, for some ordinals $\alpha_1 \geqslant 0$ and $\varepsilon > 1$ such that ε is either a successor ordinal, or $\operatorname{cf}\varepsilon = \kappa$. Suppose further that $|X| = \kappa$ and that (X,τ) is not $[\alpha,\alpha]$-compact.*

Then (X,τ) is not $[\alpha',\alpha']$-compact, for every limit ordinal α' having the form $\alpha' = \kappa\cdot\alpha'_1$, for some $\alpha'_1 > 0$ with $|\alpha'_1| \leqslant \kappa$.

If, in addition, τ is closed under unions (in particular, if τ is a topology on X), then (X,τ) is not $[\alpha',\alpha']$-compact, for every ordinal α' with $|\alpha'| \leqslant \kappa$.

Proof. Suppose that $(O_\delta)_{\delta\in\alpha}$ is a counterexample to $[\alpha,\alpha]$-compactness. In particular, for every $\beta < \alpha$, we have $\bigcup_{\delta<\beta} O_\delta \subset X$ properly. We shall show a little more.

Claim. *For every $\beta < \alpha$, there are $x \in X\backslash\bigcup_{\delta<\beta} O_\delta$ and $\gamma_x < \alpha$ such that $x \notin \bigcup_{\delta\geqslant\gamma_x} O_\delta$ (hence $x \in \bigcup_{\beta\leqslant\delta<\gamma_x} O_\delta$ since $(O_\delta)_{\delta\in\alpha}$ is a cover of X; in particular, $\beta < \gamma_x$).*

Proof. Suppose by contradiction that the statement in the claim fails. Then, for some given $\beta < \alpha$, we have that, for every $x \in X\backslash\bigcup_{\delta<\beta} O_\delta$, there are arbitrarily large indices $\delta < \alpha$ such that $x \in O_\delta$. Fix some β as above and enumerate the elements in $X\backslash\bigcup_{\delta<\beta} O_\delta$ as $(x_\gamma)_{\gamma\in\kappa'}$ with $\kappa' \leqslant \kappa$ (here we are using the assumption that $|X| \leqslant \kappa$).

We shall define by transfinite induction a strictly increasing sequence $(\delta_\gamma)_{\gamma\in\kappa'}$ such that $x_\gamma \in O_{\delta_\gamma}$ for every $\gamma \in \kappa'$. First, choose some $\delta_0 < \alpha$ such that $x_0 \in O_{\delta_0}$ and $\beta < \delta_0$.

Suppose that $\gamma < \kappa'$ and that $(\delta_{\gamma'})_{\gamma'<\gamma}$ has already been defined. Notice that, by the assumption on ε, the cofinality of $\alpha = \alpha_1 + \kappa^\varepsilon$ is κ. Since $\gamma < \kappa' \leqslant \kappa$ and κ is regular, then $\sup_{\gamma'<\gamma}\delta_{\gamma'} < \alpha$. Hence, by the

first paragraph in the proof, there is some $\delta_\gamma > \sup_{\gamma'<\gamma}\delta_{\gamma'}$ such that $x_\gamma \in O_{\delta_\gamma}$. We have constructed the sequence $(\delta_\gamma)_{\gamma\in\kappa'}$.

Notice that $\{\delta_\gamma \mid \gamma \in \kappa'\}$ has order type $\kappa' \leqslant \kappa$. Hence, if we put $D = [0,\beta) \cup \{\delta_\gamma \mid \gamma \in \kappa'\}$, then D has order type $\beta + \kappa'$. Notice that $\beta + \kappa' < \alpha$, since α is of the form $\alpha_1 + \kappa^\varepsilon$ with $\varepsilon > 1$, hence each final subset of α has order type $\kappa^\varepsilon > \kappa$.

However, by construction, $\bigcup_{\delta\in D} O_\delta = X$, hence we have found a subcover of $(O_\delta)_{\delta\in\alpha}$ of order type $< \alpha$, and this contradicts the assumption that $(O_\delta)_{\delta\in\alpha}$ witnesses the failure of $[\alpha,\alpha]$-compactness of X.

We have reached a contradiction, thus the claim is proved. □ *Claim*

Proof of Lemma 4.3 (continued) Now we are going to construct by transfinite induction two sequences $(x_\xi)_{\xi\in\alpha''}$ and $(\gamma_\xi)_{\xi\in\alpha''}$, for some ordinal $\alpha'' \leqslant \alpha$, such that

(1) x_ξ belongs to X, for every $\xi < \alpha''$,

(2) $\gamma_{\xi'} < \gamma_\xi < \alpha$, for every $\xi' < \xi < \alpha''$,

(3) $\gamma_0 = 0$, $(\gamma_\xi)_{\xi\in\alpha''}$ is continuous and $\sup_{\xi\in\alpha''}\gamma_\xi = \alpha$,

(4) $x_\xi \in \bigcup\{O_\delta \mid \gamma_\xi \leqslant \delta < \gamma_{\xi+1}\}$, for every $\xi < \alpha''$,

(5) $x_\xi \notin \bigcup\{O_\delta \mid \delta \in [0,\gamma_\xi) \cup [\gamma_{\xi+1}\alpha)\}$, for every $\xi < \alpha''$.

Put $\gamma_0 = 0$. By applying the claim to $\beta = \gamma_0 = 0$, we get $x_0 \in X$ and $\gamma_1 < \alpha$ such that $x_0 \in \bigcup_{\delta<\gamma_1} O_\delta$ and $x_0 \notin \bigcup_{\delta\geqslant\gamma_1} O_\delta$.

Suppose that x_ξ and $\gamma_{\xi+1}$ have been already defined, for some ξ. Apply the claim to $\beta = \gamma_{\xi+1}$, in order to obtain $x_{\xi+1}$ and $\gamma_{\xi+2} < \alpha$.

Now suppose that ξ is a limit ordinal and that $x_{\xi'}$ and $\gamma_{\xi'}$ have already been defined, for all $\xi' < \xi$. If $\sup_{\xi'<\xi}\gamma_{\xi'} = \alpha$, take $\alpha'' = \xi$ and terminate the induction. Otherwise, let $\gamma_\xi = \sup_{\xi'<\xi}\gamma_{\xi'}$. Then apply the claim with $\beta = \gamma_\xi$ in order to obtain x_ξ and $\gamma_{\xi+1}$.

It is immediate to show that the sequences constructed in this way satisfy (1)-(5) above.

Notice that, since $|X| = \kappa$ and X is not $[\alpha,\alpha]$-compact, then necessarily $|\alpha| \leqslant \kappa$. On the other hand, $\alpha \geqslant \kappa$, since $\alpha = \alpha_1 + \kappa^\varepsilon$, for $\varepsilon > 1$. Hence $|\alpha| = \kappa$. Moreover, by (2), (3) and since $\operatorname{cf}\alpha = \kappa$, we also get $\operatorname{cf}\alpha'' = \kappa$, thus $|\alpha''| = \kappa$, since $\alpha'' \leqslant \alpha$.

If we assume that τ is closed under unions, then the proof can be concluded in a rather simple way. Indeed, by letting $U_\xi = \bigcup\{O_\delta \mid \gamma_\xi \leqslant \delta < \gamma_{\xi+1}\}$, for $\xi < \alpha''$, we have that $x_\xi \in U_\eta$ if and only if $\xi = \eta$. Thus $(U_\xi)_{\xi<\alpha''}$ is a cover, by (3) and since $(O_\delta)_{\delta\in\alpha}$ is a cover. Moreover, $(U_\xi)_{\xi<\alpha''}$ consists of $|\alpha''| = \kappa$ indispensable elements, hence we are done by Proposition 4.2(2).

It remains to prove the lemma without the assumption that τ is closed under unions and this involves some technical computations. Hence suppose that $\alpha' = \kappa\cdot\alpha'_1$, for some $\alpha'_1 > 0$ with $|\alpha'_1| \leqslant \kappa$.

Partition α'' into α'_1-many classes $(Z_\eta)_{\eta<\alpha'_1}$ in such a way that $|Z_\eta| = \kappa$, for every $\eta < \alpha'_1$. This is possible, since $|\alpha''| = \kappa$ and $|\alpha'_1| \leqslant \kappa$. For $\eta < \alpha'_1$, put $I_\eta = [\kappa\cdot\eta, \kappa\cdot(\eta+1))$ and $W_\eta = \bigcup_{\xi\in Z_\eta}[\gamma_\xi, \gamma_{\xi+1})$. Notice that $|W_\eta| = \kappa$, for every $\eta < \alpha'_1$. For each η, let f_η be a bijection from I_η onto W_η. Notice that $\alpha' = \bigcup_{\eta<\alpha'_1} I_\eta$ and that each I_η has order type κ. Rearrange the original cover $(O_\delta)_{\delta\in\alpha}$ as $(O'_\zeta)_{\zeta\in\alpha'}$ according to the following rule.

If $\zeta \in \alpha'$, then put $O'_\zeta = O_{f_\eta(\zeta)}$ where η is the unique ordinal $< \alpha'_1$ such that $\zeta \in I_\eta$.

We shall show that $(O'_\zeta)_{\zeta\in\alpha'}$ witnesses $[\alpha',\alpha']$-incompactness of X. Indeed, since $(Z_\eta)_{\eta<\alpha'_1}$ is a partition of α'', then, by Condition (3) above and by the definition of the W_η's, we get that $\bigcup_{\eta<\alpha'_1} W_\eta = \alpha$. Since each f_η is a bijection and $\alpha' = \bigcup_{\eta<\alpha'_1} I_\eta$ then we get that $(O'_\zeta)_{\zeta\in\alpha'}$ is actually a rearrangement of $(O_\delta)_{\delta\in\alpha}$, thus it is still a cover of X.

Let $Y \subseteq \alpha'$ and suppose that $(O'_\zeta)_{\zeta\in Y}$ is a cover of X. We have to show that Y has order type α'. It is enough to show that for every $\eta < \alpha'_1$ we have $|Y \cap I_\eta| = \kappa$, thus $Y \cap I_\eta$ and I_η have the same order type $(= \kappa)$. Hence Y and α' have the same order type, since $\alpha' = \bigcup_{\eta<\alpha'_1} I_\eta$.

So, fix $\eta < \alpha'_1$. For every $\xi \in Z_\eta$ we have, by Condition (5) above, that $x_\xi \notin O_\delta$ for every $\delta \in [0, \gamma_\xi) \cup [\gamma_{\xi+1}, \alpha)$. Since $(O'_\zeta)_{\zeta \in Y}$ is a cover, there is $\zeta \in Y$ such that $x_\xi \in O'_\zeta$. Necessarily $O'_\zeta = O_\delta$ for some $\delta \in [\gamma_\xi, \gamma_{\xi+1})$, thus $\delta \in W_\eta$ hence $\delta = f_\eta(\zeta)$ and $\zeta \in Y \cap I_\eta$. By construction, $|Z_\eta| = \kappa$. Since, for $\xi \neq \xi'$, the intervals $[\gamma_\xi, \gamma_{\xi+1})$ and $[\gamma_{\xi'}, \gamma_{\xi'+1})$ are disjoint, then, for each $\xi \in Z_\eta$ we get a distinct $\delta \in \alpha$, hence a distinct $\zeta \in Y \cap I_\eta$ since f_η is injective, thus $|Y \cap I_\eta| = \kappa$.

Since the above argument works for each $\eta < \alpha'_1$ we get that $(O'_\zeta)_{\zeta \in \alpha'}$ is indeed a counterexample to $[\alpha', \alpha']$-compactness. □

Example 4.4. If τ is not supposed to be closed under unions, the conclusion in the second statement in Lemma 4.3 might fail.

Indeed, let κ be an infinite regular cardinal, let $X = \kappa \cdot \kappa$ and let τ consist of the sets of the form $[\kappa \cdot \gamma, \kappa \cdot \gamma + \delta]$, for $\gamma, \delta < \lambda$. Then (X, τ) is trivially not $[\kappa \cdot \kappa, \kappa \cdot \kappa]$-compact, but it is $[\kappa + 1, \kappa + 1]$-compact, since any cover of X always remains a cover if we take off any single member of the cover.

Actually, if $|\alpha| = \kappa$, then X is $[\alpha, \alpha]$-compact if and only if α has not the form $\kappa \cdot \alpha_1$, for some ordinal α_1.

The example also shows that the assumption that τ is closed under unions is necessary in Condition (5) in Theorem 4.5 below, as well as in Condition (4) in Corollary 4.7.

As a consequence of Lemma 4.3, for spaces of cardinality κ, the theory of $[\alpha, \alpha]$-compactness becomes trivial on a large class of limit ordinals, as explicitly stated in the next theorem. More strikingly, for countable spaces, the theory of $[\alpha, \alpha]$-compactness is nontrivial only for ordinals $\leqslant \omega \cdot \omega$ (Corollary 4.7 below).

Theorem 4.5. *Suppose that κ is an infinite regular cardinal and let K be the class of all ordinals of the form $\alpha_1 + \kappa^\varepsilon$, for some ordinals $\alpha_1 \geqslant 0$ and $\varepsilon > 1$ such that ε is either a successor ordinal, or $\operatorname{cf} \varepsilon = \kappa$.*

If $|X| = \kappa$, then the following conditions are equivalent.

(1) X is $[\kappa \cdot \kappa, \kappa \cdot \kappa]$-compact.

(2) X is $[\alpha, \alpha]$-compact, for some limit ordinal α of the form $\alpha = \kappa \cdot \alpha_1$, for some $\alpha_1 > 0$ with $|\alpha_1| \leqslant \kappa$.

(3) X is $[\alpha, \alpha]$-compact, for every $\alpha \in K$.

(4) X is $[\alpha, \alpha + \kappa \cdot \omega)$-compact, for every $\alpha \in K$.

If τ is closed under unions, then the preceding conditions are also equivalent to:

(5) X is $[\alpha, \alpha]$-compact, for some nonzero ordinal α such that $|\alpha| \leqslant \kappa$.

Proof. (1) ⇒ (2) and (1) ⇒ (5) are trivial.

(2) ⇒ (3) and, for τ closed under unions, (5) ⇒ (3) follow from Lemma 4.3.

(3) ⇒ (4) is from Corollary 2.6(3).

(4) ⇒ (1) is immediate from Proposition 2.3(1), taking $\alpha = \alpha_1 + \kappa^\varepsilon$ in (4) with $\alpha_1 = 0$ and $\varepsilon = 2$. □

Corollary 4.6. *Suppose that κ is an infinite regular cardinal, $|X| = \kappa$ and let A be the set of all ordinals $\alpha < \kappa^+$ of the form $\kappa \cdot (\alpha^\bullet + n)$, with $\operatorname{cf} \alpha^\bullet = \kappa$ and $0 \leqslant n < \omega$.*

Then X is $[\alpha, \alpha]$-compact, for some $\alpha \in A$, if and only if X is $[\alpha, \alpha]$-compact, for all $\alpha \in A$.

Proof. Suppose that X is $[\alpha', \alpha']$-compact, for some $\alpha' \in A$. Since α' is of the form given in Clause 4.5(2), then all the equivalent conditions in Theorem 4.5 hold.

Now let $\alpha = \kappa \cdot (\alpha^\bullet + n) \in A$ be arbitrary. Since $\operatorname{cf} \alpha^\bullet = \kappa$, then $\alpha^\bullet = \kappa \cdot \alpha^*$, where α^* is either successor or has cofinality κ itself. In both cases, $\alpha = \kappa \cdot (\alpha^\bullet + n)$ is of the form $\alpha_1 + \kappa^\varepsilon + \kappa \cdot n$, with $\varepsilon > 1$ either successor, or of cofinality κ. Thus X is $[\alpha, \alpha]$-compact, in force of Clause 4.5(4) and of Proposition 2.3(1). □

Corollary 4.7. *If $|X| = \omega$, then the following conditions are equivalent.*

(1) X is $[\omega \cdot \omega, \omega \cdot \omega]$-compact.

(2) X is $[\alpha, \alpha]$-compact, for some countable limit ordinal α.

(3) X is $[\omega \cdot \omega, \infty)$-compact.

If τ is closed under unions, then the preceding conditions are also equivalent to:

(4) X is $[\alpha, \alpha]$-compact, for some nonzero ordinal $\alpha < \omega_1$.

Proof. The equivalence of (1), (2), and (4) is a particular case of Theorem 4.5 (Conditions (1), (2) and (5) there).

(3) $\Rightarrow$ (1) is immediate from Proposition 2.3.

In order to finish the proof, suppose that (2) holds. Then, by Theorem 4.5 (2) $\Rightarrow$ (4), X is $[\delta, \delta]$-compact, for every ordinal δ of the form $\alpha_1 + \omega^\varepsilon + m$, for $\varepsilon > 1$, that is, for every countable ordinal $\delta \geqslant \omega \cdot \omega$. Since X, being countable, is trivially $[\delta, \delta]$-compact for every uncountable δ, we get $[\omega \cdot \omega, \infty)$-compactness from Proposition 2.3(2). Hence (3) holds. □

A result similar to Corollary 4.7 holds for T_1 spaces (of arbitrary cardinality): see Corollary 6.8.

Corollary 4.8. *If $|X| = \omega$, then the Lindelöf ordinal of X is either ω_1 or is $\leqslant \omega \cdot \omega$.*

If κ is regular and $|X| = \kappa$, then the Lindelöf ordinal of X cannot have the form $\alpha_1 + \kappa^\varepsilon + \gamma$, with $0 < \gamma < \kappa \cdot \omega$ and $\varepsilon > 1$ such that ε is either a successor ordinal or $\operatorname{cf} \varepsilon = \kappa$.

Proof. The first statement is immediate from Corollary 4.7 (2) $\Rightarrow$ (3).

As for the second statement, if the Lindelöf ordinal of X is $< \kappa^+$, then X is $[\alpha, \alpha]$-compact, for some α as in item (2) in Theorem 4.5. The conclusion now follows from Proposition 2.3 and item (4) in Theorem 4.5. □

5. An exact characterization of transfer properties

In this section we introduce some further examples, more involved than those presented in Examples 3.2. This is necessary in order to avoid the limitations given by Theorem 4.5 and Corollaries 4.6 and 4.7. The examples introduced in this section are optimal, in the sense that they provide an exact characterization of those ordinals α and β such that $[\alpha, \alpha]$-compactness implies $[\beta, \beta]$-compactness. Here *optimal* is intended relative to results provable for (X, τ) when τ is an arbitrary family of subsets of X. Optimal results for topologies have still to be worked out.

Definitions 5.1. As usual, we denote by ${}^\alpha 2$ the set of all the functions from α to $2 = \{0, 1\}$.

If $f \in {}^\alpha 2$, the *support* of f is $\{\delta \in \alpha \mid f(\delta) = 1\}$.

For nonzero ordinals $\beta \leqslant \alpha$, we now define $S_\beta(\alpha) = \{f \in {}^\alpha 2 \mid \text{the support of } f \text{ has order type} < \beta\}$.

$S_\beta(\alpha)$ is in a one-to one correspondence, via characteristic functions, with the set of all subsets of α which have order type $< \beta$. The S in our notation is a reminder for *Subset*. However, in the present note, we shall mainly deal with elements of ${}^\alpha 2$, rather than with subsets of α, since it will be more convenient for our purposes.

We shall mainly deal with the case $\beta = \alpha$ and we shall consider various families of subsets of $S_\beta(\alpha)$.

We put $X(\beta, \alpha) = (S_\beta(\alpha), \tau_0)$, where the elements of τ_0 are all the subsets of $S_\beta(\alpha)$ having the form $Z(\varepsilon) = \{f \in S_\beta(\alpha) \mid f(\varepsilon) = 0\}$, ε varying in α.

We also let $X_U(\beta, \alpha) = (S_\beta(\alpha), \tau_u)$, where τ_u is the smallest family of subsets of $S_\beta(\alpha)$ which contains τ_0 above and is closed under unions. In other words, a generic element of τ_u has the form $\bigcup_{\varepsilon \in H} Z(\varepsilon) = \{f \in S_\beta(\alpha) \mid f(\varepsilon) = 0, \text{ for some } \varepsilon \in H\}$, for some $H \subseteq \alpha$.

For $\alpha, \beta \geqslant 2$, neither τ_0 nor τ_u are topologies, since they are not closed under finite intersections. However, if we take the closure of τ_u under finite intersections, we do get a topology τ on $S_\beta(\alpha)$. For $\boldsymbol{\varepsilon} = \{\varepsilon_0, \varepsilon_1, \dots, \varepsilon_{n-1}\} \subseteq \alpha$, let $Z(\boldsymbol{\varepsilon}) = Z(\varepsilon_0, \varepsilon_1, \dots, \varepsilon_{n-1}) = Z(\varepsilon_0) \cap Z(\varepsilon_1) \cap \cdots \cap Z(\varepsilon_{n-1}) = \{f \in S_\beta(\alpha) \mid f(\varepsilon_0) = f(\varepsilon_1) = \cdots = f(\varepsilon_{n-1}) = 0\}$. Members of τ have then the form $\bigcup_{\boldsymbol{\varepsilon} \in H} Z(\boldsymbol{\varepsilon})$, H varying among subsets of the family of finite subsets of α. We let $X_\tau(\beta, \alpha) = (S_\beta(\alpha), \tau)$.

The above topology τ is T_0, but not even T_1. A topology satisfying stronger separation axioms can be introduced as follows. We let $X_T(\beta, \alpha) = (S_\beta(\alpha), \tau_T)$, where τ_T is the (Tychonoff) topology inherited by the

product topology on ${}^{\alpha}2$, where 2 is given the discrete topology. Notice that if β has the form ω^η, for some $\eta > 0$, then $X_T(\beta, \alpha)$ inherits from ${}^{\alpha}2$ also the structure of a topological group. This is because, if $\beta', \beta'' < \omega^\eta$, then also $\beta' \oplus \beta'' < \omega^\eta$, hence $S_\beta(\alpha)$ is closed under the group operation inherited from ${}^{\alpha}2$.

We shall write $X(\beta)$ in place of $X(\beta, \beta)$ and similarly for $X_U(\beta)$, $X_\tau(\beta)$ and $X_T(\beta)$. The subscript τ is a reminder for *topology*, the subscript U is a reminder for (closed under) *Unions* and the subscript T is a reminder for *Tychonoff*.

Remark 5.2. Similar constructions, when restricted to cardinal numbers, have sometimes been considered in the literature. See, e. g., [2, Example 4.1], [17] and [30, Example 4.2].

Lemma 5.3. *Suppose $0 < \beta \leqslant \alpha$ and assume the notations in Definition 5.1.*

If $H \subseteq \alpha$, then the sequence $(Z(\varepsilon))_{\varepsilon \in H}$ is a cover of $X(\beta, \alpha)$ if and only if H has order type $\geqslant \beta$. In particular, $X(\beta, \alpha)$ is not $[\beta, \beta]$-compact, hence neither $X_U(\beta, \alpha)$, nor $X_\tau(\beta, \alpha)$, nor $X_T(\beta, \alpha)$ are $[\beta, \beta]$-compact.

Proof. If H has order type $< \beta$, define $f : \alpha \to 2$ by $f(\delta) = 1$ if and only if $\delta \in H$. Then $f \in X(\beta, \alpha)$ but f belongs to no $Z(\varepsilon)$ ($\varepsilon \in H$).

On the contrary, suppose by contradiction that H has order type $\geqslant \beta$, but there is $f \in X(\beta, \alpha)$ such that f belongs to no $Z(\varepsilon)$ ($\varepsilon \in H$). If $f \notin Z(\varepsilon)$, then $f(\varepsilon) = 1$, thus the support of f contains H, which has order type $\geqslant \beta$, and this contradicts $f \in X(\beta, \alpha)$.

In order to show that $X(\beta, \alpha)$ is not $[\beta, \beta]$-compact, it is enough to choose some $H \subseteq \alpha$ of order type β. Then, by above, $(Z(\varepsilon))_{\varepsilon \in H}$ is a cover of $X(\beta, \alpha)$, but if $K \subseteq H$ has order type $< \beta$, then $(Z(\varepsilon))_{\varepsilon \in K}$ is not a cover of $X(\beta, \alpha)$. The same argument works for $X_U(\beta, \alpha)$, $X_\tau(\beta, \alpha)$, and $X_T(\beta, \alpha)$. □

Theorem 5.4. *Let α and β be nonzero ordinals and assume the notations in Definition 5.1. Then the following conditions are equivalent.*

(a) $X(\beta)$ is not $[\alpha, \alpha]$-compact.

(b) There exists an injective function $f : \beta \to \alpha$ such that, for every $K \subseteq \alpha$ with order type $< \alpha$, it happens that $f^{-1}(K)$ has order type $< \beta$.

(c) For arbitrary (X, τ), $[\alpha, \alpha]$-compactness implies $[\beta, \beta]$-compactness.

Proof. (a) ⇒ (b) Suppose that (a) holds. Then $X(\beta)$ has a cover $(O_\delta)_{\delta \in \alpha}$ such that, whenever $H \subseteq \alpha$ has order type $< \alpha$, then $(O_\delta)_{\delta \in H}$ is not a cover of $X(\beta)$. By Lemma 2.9, we can suppose that $O_\delta \neq O_{\delta'}$ for $\delta \neq \delta' \in \alpha$. Because of the definition of τ_0, for each $\delta \in \alpha$, there is $\varepsilon \in \beta$ such that $O_\delta = Z(\varepsilon)$. Let $W = \{\varepsilon \in \beta \mid Z(\varepsilon) = O_\delta \text{ for some } \delta \in \alpha\}$. Since $(O_\delta)_{\delta \in \alpha}$ is a cover of $X(\beta)$, then also $(Z(\varepsilon))_{\varepsilon \in W}$ is a cover of $X(\beta)$. By Lemma 5.3, W has order type β.

Let $g : W \to \alpha$ be defined by $g(\varepsilon) = \delta$ if and only if $Z(\varepsilon) = O_\delta$. Such a δ exists because of the definition of W and is unique because of the property $(O_\delta)_{\delta \in \alpha}$ is assumed to satisfy.

If $K \subseteq \alpha$ has order type $< \alpha$, then, by $[\alpha, \alpha]$-incompactness, $(O_\delta)_{\delta \in K}$ is not a cover of $X(\beta)$. Hence $(Z(\varepsilon))_{\varepsilon \in g^{-1}(K)}$ is not a cover of $X(\beta)$. By Lemma 5.3, $g^{-1}(K)$ has order type $< \beta$.

Thus the counterimage by g of a subset of α of order type $< \alpha$ has order type $< \beta$. Since W has order type β, then, by composing g with an isomorphism between W and β, we get a function f satisfying the required property. Notice that g (hence also f) is injective, since $Z(\varepsilon) \neq Z(\varepsilon')$, for $\varepsilon \neq \varepsilon'$.

(b) ⇒ (c) is a particular case of Proposition 2.5.

(c) ⇒ (a) If (c) holds, then $X(\beta)$ is not $[\alpha, \alpha]$-compact, since, by Lemma 5.3, it is not $[\beta, \beta]$-compact. □

Remark 5.5. Thus, for example, for every pair $\nu \leqslant \kappa$ of infinite regular cardinals, $[\kappa + \kappa, \kappa + \kappa]$-compactness does not imply $[\kappa \cdot \nu, \kappa \cdot \nu]$-compactness, since there is no function $f : \kappa \cdot \nu \to \kappa + \kappa$ satisfying Condition (b) in Theorem 5.4.

Similarly, $[\kappa^2 + \kappa, \kappa^2 + \kappa]$-compactness does not imply $[\kappa \cdot \nu, \kappa \cdot \nu]$-compactness.

Thus Corollary 2.6(2)(3) cannot be improved. Notice that, because of Theorem 4.5(2) ⇒ (1), if X is $[\kappa + \kappa, \kappa + \kappa]$-compact and not $[\kappa^2, \kappa^2]$-compact, then $|X| > \kappa$.

Corollary 5.6. *Suppose that α and β are nonzero ordinals and $|\alpha| \neq |\beta|$. Then the following statements hold.*

(1) $X(\beta)$ is $[\alpha, \alpha]$-compact.

(2) There is some (X, τ) which is $[\alpha, \alpha]$-compact and not $[\beta, \beta]$-compact.

Proof. If $f : \beta \to \alpha$ is an injective function, then $|\alpha| > |\beta|$, since $|\alpha| \neq |\beta|$. Hence $K = f(\beta) \subseteq \alpha$ has order type $< \alpha$, but $f^{-1}(K) = \beta$ has order type β. Hence Condition (b) in Theorem 5.4 fails, hence also the equivalent Conditions (a) and (c) fail. □

Of course, Corollary 5.6(2) does not hold in the case when τ is requested to be closed under unions. See, e. g., Corollary 2.6(6)-(8). The next Theorem is the analogue of Theorem 5.4 in the case when τ is asked to be closed under unions.

Theorem 5.7. *Let α, β be nonzero ordinals and assume the notations in Definition 5.1. Then the following conditions are equivalent.*

(a) $X_U(\beta)$ is not $[\alpha, \alpha]$-compact.

(b) There exists a function $f : \beta \to \alpha$ such that, for every $K \subseteq \alpha$ with order type $< \alpha$, it happens that $f^{-1}(K)$ has order type $< \beta$.

(c) For every X and τ, if τ is closed under unions, then $[\alpha, \alpha]$-compactness of (X, τ) implies $[\beta, \beta]$-compactness of (X, τ).

Proof. (a) ⇒ (b) Suppose that (a) holds and that $(O_\delta)_{\delta\in\alpha}$ is a counterexample to the $[\alpha, \alpha]$-compactness of $X_U(\beta)$. By the definition of τ_u, each O_δ has the form $\bigcup_{\varepsilon\in W_\delta} Z(\varepsilon)$, for some $W_\delta \subseteq \beta$.

For $\delta \in \alpha$, let $W^*_\delta = W_\delta \setminus \bigcup_{\gamma<\delta} W_\gamma$ and let $O^*_\delta = \bigcup_{\varepsilon\in W^*_\delta} Z(\varepsilon)$. Notice that $(O^*_\delta)_{\delta\in\alpha}$ is still a cover of $X_U(\beta)$, hence it is still a counterexample to the $[\alpha, \alpha]$-compactness of $X_U(\beta)$, since $O^*_\delta \subseteq O_\delta$, for every $\delta \in \alpha$.

Since $(O^*_\delta)_{\delta\in\alpha}$ covers $X_U(\beta)$, we have that $\bigcup\{Z(\varepsilon) \mid \varepsilon \in W^*_\delta \text{ for some } \delta \in \alpha\} = \bigcup\{Z(\varepsilon) \mid \varepsilon \in \bigcup_{\delta\in\alpha} W^*_\delta\} = X(\beta)$, hence, by Lemma 5.3, the order type of $W = \bigcup_{\delta\in\alpha} W^*_\delta = \bigcup_{\delta\in\alpha} W_\delta$ equals β.

Let $g : W \to \alpha$ be defined by $g(\varepsilon) =$ the unique $\delta \in \alpha$ such that $\varepsilon \in W^*_\delta$. If $K \subseteq \alpha$ has order type $< \alpha$, then, by $[\alpha, \alpha]$-incompactness, $(O^*_\delta)_{\delta\in K}$ is not a cover of $X(\beta)$. Hence $(Z(\varepsilon))_{\varepsilon\in g^{-1}(K)}$ is not a cover of $X(\beta)$. By Lemma 5.3, $g^{-1}(K)$ has order type $< \beta$.

We have proved that the counterimage by g of a subset of α of order type $< \alpha$ has order type $< \beta$, thus, arguing as in corresponding part of the proof of Theorem 5.4 and since W has order type β, we get a function f as desired.

(b) ⇒ (c) follows from the last statement in Proposition 2.5.

(c) ⇒ (a) If (c) holds, then $X_U(\beta)$ is not $[\alpha, \alpha]$-compact, since, by Lemma 5.3, it is not $[\beta, \beta]$-compact and since τ_u is closed under unions. □

6. $[\alpha, \beta]$-compactness of T_1 spaces

The counterexamples presented in Examples 3.2(2) and 3.10 satisfy very few separation axioms. In fact, we are going to show that more results about $[\beta, \alpha]$-compactness can be proved just under the assumption that we are dealing with T_1 topological spaces.

Since in this note we have kept the greatest possible generality, we mention that we do not actually need a T_1 topological space, in order to prove the results in the present section. The following weaker notion is enough.

Definition 6.1. If X is a nonempty set and τ is a nonempty family of subsets of X, we say that (X, τ) is T_1 if and only if, for every $O \in \tau$ and every $x \in O$, it happens that $O\setminus\{x\} \in \tau$.

Clearly, the above condition is equivalent to asking that, for every $O \in \tau$ and every finite $F \subseteq X$, $O\setminus F \in \tau$. Trivially, if τ is a topology on X, then (X, τ) is T_1 in the above sense if and only if it is T_1 in the ordinary topological theoretical sense.

It is convenient to introduce some notation, in order to state the next Proposition more concisely.

Definition 6.2. If β is an infinite ordinal, we let β^ℓ be the largest limit ordinal $\leqslant \beta$. Thus $\beta^\ell = \beta - n$, for an appropriate $n \in \omega$.

Proposition 6.3. *Suppose that X is T_1 and let α be an infinite ordinal.*

(1) X is $[\alpha, \alpha]$-compact if and only if X is $[\alpha + 1, \alpha + 1]$-compact.

(2) For every $n \in \omega$ and infinite $\beta \leqslant \alpha$, X is $[\beta, \alpha]$-compact if and only X is $[\beta^\ell, \alpha + n]$-compact.

(3) For every infinite $\beta \leqslant \alpha$, X is $[\beta, \alpha]$-compact if and only if X is $[\beta^\ell, \alpha + \omega)$-compact.

(4) If $\beta \leqslant \alpha$ and β is infinite, then X is $[\beta, \alpha]$-compact if and only if X is $[\gamma, \gamma]$-compact, for every limit ordinal γ with $\beta^\ell \leqslant \gamma \leqslant \alpha$.

Proof. (1) One implication follows from Corollary 2.6(1) and Proposition 2.3(1).

On the other hand, suppose that X is $[\alpha + 1, \alpha + 1]$-compact and let $(O_\delta)_{\delta\in\alpha}$ be a cover of X. Without loss of generality, e. g., by Lemma 2.9, we can suppose that $O_0 \neq \varnothing$. Let $x \in O_0$, and, for $\delta \in \alpha$ with $\delta > 0$, let $O'_\delta = O_\delta \backslash \{x\}$. Since (X, τ) is assumed to be T_1, each O'_δ still belongs to τ. Moreover, $(O'_\delta)_{\delta\in\alpha}$ is still a cover of X. Notice that every subcover of $(O'_\delta)_{\delta\in\alpha}$ must contain O_0, which is the only element of the cover containing x.

Rearrange $(O'_\delta)_{\delta\in\alpha}$ as $(U_\delta)_{\delta\in\alpha+1}$ by letting $U_\delta = O'_{f(\delta)}$ where $f : \alpha + 1 \to \alpha$ is the bijection defined by

$$f(\delta) = \begin{cases} \delta + 1 & \text{if } \delta < \omega, \\ \delta & \text{if } \omega \leqslant \delta < \alpha, \\ 0 & \text{if } \delta = \alpha. \end{cases}$$

By applying $[\alpha+1, \alpha+1]$-compactness to $(U_\delta)_{\delta\in\alpha+1}$, we get $H \subseteq \alpha+1$ such that H has order type $< \alpha+1$ and $(U_\delta)_{\delta\in H}$ is a cover. Since $U_\alpha = O'_0$ and O'_0 is the only element of the cover containing x, we have that U_α belongs to the subcover, that is, $\alpha \in H$. Since H has order type $< \alpha + 1$, then necessarily $H \cap \alpha$ has order type $< \alpha$. Since $f_{|\alpha}$ is order-preserving, then also $f^{-1}(H \cap \alpha)$ has order type $< \alpha$. Hence $K = f^{-1}(H)$, too, has order type $< \alpha$, since α is infinite and we are adding to $f^{-1}(H \cap \alpha)$ just one element "at the beginning".

Then $(O'_\delta)_{\delta\in K}$ is a cover of X indexed by a set of order type $< \alpha$, and also $(O_\delta)_{\delta\in K}$ is a cover, since $O'_\delta \subseteq O_\delta$ for every $\delta \in \alpha$. Hence $(O_\delta)_{\delta\in K}$ is a subcover of order type $< \alpha$ of our original cover $(O_\delta)_{\delta\in\alpha}$ and we have proved $[\alpha, \alpha]$-compactness.

(2) - (4) are immediate from (1) and Proposition 2.3. □

Of course, item 1 in Proposition 6.3 is false without the assumption that α is infinite. Indeed, the discrete space with exactly n elements is $[n + 1, n + 1]$-compact but not $[n, n]$-compact.

The next Lemma captures a very useful consequence of being T_1.

Lemma 6.4. *Suppose that α is an ordinal, $\operatorname{cf}\alpha = \omega$ and $(\alpha_n)_{n\in\omega}$ is a strictly increasing sequence such that $\sup_{n\in\omega} \alpha_n = \alpha$.*

If X is T_1 and not $[\alpha, \alpha]$-compact, then there is a counterexample $(O_\delta)_{\delta\in\alpha}$ to the $[\alpha, \alpha]$-compactness of X with the property that, for every $n \in \omega$, O_{α_n} is indispensable (Definition 4.1).

Proof. Let α and the α_n's be given. Suppose that $(O_\delta)_{\delta\in\alpha}$ is a counterexample to $[\alpha, \alpha]$-compactness. By Lemma 2.9, we can also suppose that, for every $\delta < \alpha$, O_δ is not contained in $\bigcup_{\varepsilon<\delta} O_\varepsilon$. In particular, for every $n \in \omega$, we can choose $x_n \in O_{\alpha_n}$ such that $x_n \notin \bigcup_{\varepsilon<\alpha_n} O_\varepsilon$. Define $(O'_\delta)_{\delta\in\alpha}$ as follows.

$$O'_\delta = \begin{cases} O_\delta & \text{if } \delta \leqslant \alpha_0, \\ O_\delta \backslash \{x_0, \dots, x_n\} & \text{if } \alpha_n < \delta \leqslant \alpha_{n+1}. \end{cases}$$

Since X is T_1, each O'_δ still belongs to τ. Moreover, $(O'_\delta)_{\delta\in\alpha}$ is still a cover of X. Indeed, for every $n \in \omega$, $x_n \in O'_{\alpha_n}$. If x is not one of the x_n's, then $x \in O_\delta$, for some $\delta \in \alpha$, and also $x \in O'_\delta$. Since $O'_\delta \subseteq O_\delta$, for every $\delta \in \alpha$, we have that $(O'_\delta)_{\delta\in\alpha}$, too, is a counterexample to $[\alpha, \alpha]$-compactness and it is easy to see that $(O'_{\alpha_n})_{n\in\omega}$ is a set of indispensable elements. Thus $(O'_\delta)_{\delta\in\alpha}$ is a cover as wanted. □

Many results on T_1 spaces will be obtained by rearranging the indispensable elements given by Lemma 6.4. The following notation shall be useful in the proof of the forthcoming Theorem 6.6.

Definition 6.5. If β is any ordinal, let β^* be the smallest ordinal $\leqslant \beta$ such that $|[\beta^*, \beta]| \leqslant \omega$. Thus β^* is the largest ordinal $\leqslant \beta$ which is either 0, or has uncountable cofinality, or has cofinality ω but can be written as a limit of ordinals of uncountable cofinality.

Theorem 6.6. *Suppose that X is T_1 and β is an ordinal of cofinality ω. Then the following conditions are equivalent.*

(1) X is $[\beta, \beta]$-compact.

(2) X is $[\beta + \alpha, \beta + \alpha]$-compact, for every ordinal α with $|\alpha| \leqslant \omega$.

(3) X is $[\beta + \alpha, \beta + \alpha]$-compact, for some ordinal α with $|\alpha| \leqslant \omega$.

(4) X is $[\beta, \beta + \omega_1)$-compact.

Proof. (2) ⇔ (4) follows from Proposition 2.3(4), hence it is enough to prove the equivalence of (1) - (3).

We shall first prove the theorem in some particular cases.

Claim 1. *Conditions (1) - (3) are equivalent in case $\beta = \beta^* + \omega$.*

Proof. In case $\beta^* = 0$, (1) ⇒ (2) follows from Proposition 2.3(4) and Corollary 2.6(4) with $\beta = \lambda = \omega$.

In case $\beta^* > 0$, (1) ⇒ (2) follows from Proposition 2.3(4) and Corollary 2.6(5), by taking there $\alpha = \beta^*$, $\lambda = \omega$ and $\beta = \beta^* + \omega$.

(2) ⇒ (3) is trivial.

We shall prove (3) ⇒ (1) by proving the contrapositive form.

So suppose that X is not $[\beta, \beta]$-compact and $|\alpha| \leqslant \omega$. We want to show that X is not $[\beta+\alpha, \beta+\alpha]$-compact. For $n < \omega$, let $\alpha_n = \beta^* + n$. Since $\beta = \beta^* + \omega$, then, by Lemma 6.4, there is some cover $(O_\delta)_{\delta\in\beta}$ witnessing $[\beta, \beta]$-incompactness and such that each O_{α_n} is indispensable. If $\beta^* = 0$, then $[\beta + \alpha, \beta + \alpha]$-incompactness follows from Proposition 4.2(1), hence in what follows let us suppose $\beta^* > 0$.

For every $H \subseteq \beta = \beta^* + \omega$, if $(O_\delta)_{\delta\in H}$ is a cover of X, then the order type of H is $\beta = \beta^* + \omega$, hence the order type of $H \cap \beta^*$ is β^*, since β^* is a limit ordinal. Moreover, $H \cap [\beta^*, \beta) = [\beta^*, \beta)$, since O_δ is indispensable, for every $\delta \in [\beta^*, \beta)$.

Let $f : \beta^* + \omega + \alpha \to \beta^* + \omega$ be a bijection which is the identity on β^* and let $(U_\varepsilon)_{\varepsilon\in\beta^*+\omega+\alpha}$ be defined by $U_\varepsilon = O_{f(\varepsilon)}$. We claim that $(U_\varepsilon)_{\varepsilon\in\beta^*+\omega+\alpha}$ witnesses that X is not $[\beta^* + \omega + \alpha, \beta^* + \omega + \alpha]$-compact, and this is what we want, since $\beta^* + \omega + \alpha = \beta + \alpha$. Indeed, if $K \subseteq \beta^* + \omega + \alpha$ and $(U_\varepsilon)_{\varepsilon\in K}$ is a cover of X, then $(O_\delta)_{\delta\in H}$, with $H = f(K)$, is a cover of X. Since f is the identity on β^*, then, by the above mentioned properties of H, we get that the order type of $K \cap \beta^*$ equals the order type of $H \cap \beta^*$, that is, β^*; moreover, $K \cap [\beta^*, \beta^* + \omega + \alpha) = [\beta^*, \beta^* + \omega + \alpha)$, thus K has order type $\beta^* + \omega + \alpha$, hence $[\beta^* + \omega + \alpha, \beta^* + \omega + \alpha]$-incompactness is proved. □ *Claim* 1

Claim 2. *Conditions (1) - (3) are equivalent in the case when β^* has cofinality ω and $\beta = \beta^*$.*

Proof. In view of Claim 1 and of Proposition 6.3(1), it is enough to show that if $\operatorname{cf}\beta^* = \omega$, then $[\beta^*, \beta^*]$-compactness is equivalent to $[\beta^*+\omega, \beta^*+\omega]$-compactness. The former implies the latter because of Corollary 2.6(3) (taking $\beta = \alpha = \beta^*$ there), by Proposition 2.3(4) and since we have assumed that $\operatorname{cf}\beta^* = \omega$. We shall prove the reverse implication by contraposition. Suppose that X is not $[\beta^*, \beta^*]$-compact. We want to show that X is not $[\beta^* + \omega, \beta^* + \omega]$-compact. Choose some strictly increasing sequence $(\alpha_n)_{n\in\omega}$ cofinal in β^*. This is possible, since $\operatorname{cf}\beta^* = \omega$. By Lemma 6.4, there is a counterexample $(O_\delta)_{\delta\in\beta^*}$ to $[\beta^*\beta^*]$-compactness such

that each O_{α_n} is indispensable. Thus if $H \subseteq \beta^*$ and $(O_\delta)_{\delta\in H}$ is a cover of X, then H has order type β^*, and moreover $\alpha_n \in H$, for every $n \in \omega$.

Let $A = (\beta^* + \omega)\backslash\{\alpha_n \mid n \in \omega\}$. A has order type $\beta^* + \omega$, since β^* is expressible as a limit of ordinals of uncountable cofinality, hence taking off a sequence of order type ω does not alter the order type of β^*. Let $(O'_\delta)_{\delta\in A}$ be defined by $O'_\delta = O_\delta$, if $\delta \in \beta^*\backslash\{\alpha_n \mid n \in \omega\}$, and by $O'_{\beta^*+n} = O_{\alpha_n}$ for $n \in \omega$. Since these latter elements of the cover are indispensable, it is easy to see that $(O'_\delta)_{\delta\in A}$ is a counterexample to $[\beta^* + \omega, \beta^* + \omega]$-compactness. $\square_{\text{Claim 2}}$

Proof of Theorem 6.6 (continued). Summing up, we have proved the theorem in the case when either

(I) $\beta = \beta^* + \omega$, or

(II) $\beta = \beta^*$ and $\operatorname{cf}\beta^* = \omega$.

Now let β be arbitrary. By definition, $\beta \geqslant \beta^*$ and, since we have assumed $\operatorname{cf}\beta = \omega$, we have further that if $\operatorname{cf}\beta^* > \omega$, then $\beta \geqslant \beta^* + \omega$. Notice also that, by definition, there is γ with $|\gamma| \leqslant \omega$ such that $\beta = \beta^* + \gamma$ and, if $\operatorname{cf}\beta^* > \omega$, then, by above, there is γ' with $|\gamma'| \leqslant \omega$ such that $\beta = \beta^* + \omega + \gamma'$.

Now observe that if the statement of the theorem holds for some given ordinal β' in place of β, and β'' is another ordinal such that $\beta'' = \beta' + \gamma$, for some γ with $|\gamma| \leqslant \omega$, then the statement of the theorem holds for β'' in place of β, too.

The above observations show that the two already proved particular cases (I) and (II) imply the statement of the theorem in its full generality. $\square$

Remark 6.7. (a) The assumption that β has cofinality ω in Theorem 6.6 is necessary. By Example 3.2(3), if κ is regular and uncountable, then $(\kappa, \mathfrak{ord})$ is $[\kappa + \omega, \kappa + \omega]$-compact, but not $[\kappa, \kappa]$-compact, hence the implication (3) $\Rightarrow$ (1) in the statement of Theorem 6.6 fails, for $\beta = \kappa$ and $\alpha = \omega$.

(b) On the other hand, for $\beta \geqslant \omega$ and T_1 spaces, the implication (1) $\Rightarrow$ (2) in Theorem 6.6 always holds, even without the assumption that β has cofinality ω. Indeed, by Proposition 6.3(4), $[\beta, \beta]$-compactness implies $[\beta^\ell, \beta^\ell]$-compactness, thus, without loss of generality, we can suppose that β is limit. Then, for every α with $|\alpha| \leqslant \omega$, we get $[\beta + \alpha, \beta + \alpha]$-compactness: this follows from Theorem 6.6 itself, if $\operatorname{cf}\beta = \omega$, and from Corollary 2.6(3) and Proposition 2.3(1), if $\operatorname{cf}\beta > \omega$.

(c) On the contrary, the implication (1) $\Rightarrow$ (2) in the statement of Theorem 6.6 fails, in general, when X is not assumed to be T_1. See, for example, the first example in Remark 5.5, with $\kappa = \nu = \omega$.

(d) Also the implication (3) $\Rightarrow$ (1) in the statement of Theorem 6.6 fails, in general, for non T_1 spaces. Just consider Example 3.2(2), and take $\beta = \kappa = \omega$ and arbitrary $\alpha > 1$.

Corollary 6.8. *Suppose that X is T_1. Then X is $[\omega, \omega]$-compact if and only if X is $[\alpha, \alpha]$-compact, for some (equivalently, every) countably infinite ordinal α, if and only if X is $[\omega, \omega_1)$-compact.*

Proof. The corollary follows by taking $\beta = \omega$ in Theorem 6.6. $\square$

Theorem 6.6 can be used to strengthen Proposition 6.3.

Definition 6.9. Recall from Definition 6.5 the definition of β^*. For an ordinal β, define β^{**} as follows:

$$\beta^{**} = \begin{cases} \beta^* & \text{if either } \operatorname{cf}\beta^* = \omega, \text{ or } \beta = \beta^* + n, \text{ for some } n < \omega, \\ \beta^* + \omega & \text{otherwise.} \end{cases}$$

Notice that $\beta^{**} \leqslant \beta$, for every ordinal β.

Corollary 6.10. *Suppose that X is T_1 and $\beta \leqslant \alpha$ are infinite ordinals. Then the following conditions are equivalent.*

(1) X is $[\beta, \alpha]$-compact.

(2) X is $[\beta^{**}, \alpha + \omega_1)$-compact.

(3) X is both $[\beta^{**}, \beta^{**}]$-compact, and $[\gamma, \gamma]$-compact, for every γ such that $\beta \leqslant \gamma \leqslant \alpha$ and $\gamma = \gamma^*$.

Proof. (1) $\Rightarrow$ (3) From Proposition 2.3(1) we get $[\beta, \beta]$-compactness. If $\operatorname{cf}\beta^{**} = \omega$, then $[\beta^{**}, \beta^{**}]$-compactness follows from Theorem 6.6(3) $\Rightarrow$ (1), with β^{**} in place of β, and since, by the definitions of β^* and of β^{**}, we have that $\beta = \beta^{**} + \alpha'$, for some α' with $|\alpha'| \leqslant \omega$. If $\operatorname{cf}\beta^{**} \neq \omega$, then $\beta = \beta^{**} + n$, for some $n < \omega$, and $[\beta^{**}, \beta^{**}]$-compactness follows from Proposition 6.3(1), since β is assumed to be infinite. Finally, $[\gamma, \gamma]$-compactness, for every γ such that $\beta \leqslant \gamma \leqslant \alpha$, is trivial, by Proposition 2.3(1).

In order to prove (3) $\Rightarrow$ (2), in view of Proposition 2.3(4), it is enough to prove $[\varepsilon, \varepsilon]$-compactness, for every ε such that $\beta^{**} \leqslant \varepsilon < \alpha + \omega_1$. Let us fix some ε as above and let $\gamma = \varepsilon^*$. Notice that $\gamma = \gamma^*$ and that $\gamma \leqslant \alpha$, since the cardinality of the interval $[\alpha, \varepsilon]$ is $\leqslant \omega$. If $\gamma \geqslant \beta$, then, by assumption, we have $[\gamma, \gamma]$-compactness, which implies $[\varepsilon, \varepsilon]$-compactness, by Theorem 6.6 and Corollary 2.6(3), as remarked in Remark 6.7(b). On the other hand, if $\gamma < \beta$, then $\varepsilon^* = \beta^*$, since $\beta^* \leqslant \beta^{**} \leqslant \varepsilon$ and $\varepsilon^* = \gamma < \beta$. Then $[\beta^{**}, \beta^{**}]$-compactness implies $[\varepsilon, \varepsilon]$-compactness, again by Remark 6.7(b).

(2) $\Rightarrow$ (1) follows from Proposition 2.3(1), since $\beta^{**} \leqslant \beta$. □

In particular, the compactness properties of T_1 spaces are completely determined by checking $[\beta, \beta]$-compactness for

(1) β finite,

(2) $\beta = \omega$,

(3) β of uncountable cofinality,

(4) $\beta = \gamma + \omega$, for γ of uncountable cofinality, and

(5) β of cofinality ω, but expressible as a limit of ordinals of uncountable cofinality.

The above statement, and the next corollary as well, follow from Corollary 6.10 and the fact that, for infinite β, both β^* and β^{**} have necessarily one among the forms (2)-(5).

Corollary 6.11. *If X is T_1 and β is the Lindelöf ordinal of X, then β has one of the above forms (1)-(5). In particular, if $\beta < \omega_1$, then $\beta \leqslant \omega$.*

Remark 6.12. It follows from Example 3.2(3) that the behavior of countable ordinals in Theorem 6.6 and Corollary 6.8 constitutes an exceptional case. The situation is radically different for larger cardinals and ordinals, even for normal topological spaces. Indeed, if κ is a regular and uncountable cardinal, then $(\kappa, \mathfrak{ord})$ is $[\kappa + \kappa, \kappa + \kappa]$-compact but not $[\kappa, \kappa]$-compact. Thus 6.6 and 6.8 do not hold when ω is replaced by an uncountable cardinal.

As another example, the disjoint union of two copies of $(\kappa, \mathfrak{ord})$ is $[\kappa + \kappa + \kappa, \kappa + \kappa + \kappa]$-compact, but not $[\kappa + \kappa, \kappa + \kappa]$-compact (see Example 3.11).

However, Theorem 6.6 does admit a generalization to larger cardinals, but only under a somewhat stronger assumption.

Definition 6.13. If λ is an infinite cardinal, we say that (X, τ) is λ-T_1 if and only if, for every $O \in \tau$, and every $Z \subseteq X$ with $|Z| < \lambda$, $O \setminus Z \in \tau$. Thus T_1 is the same as ω-T_1.

If (X, τ) is a T_1 topological space and the intersection of $< \lambda$ open sets of X is still an open set of X, then (X, τ) is λ-T_1 in the above sense.

Proposition 6.14. *Suppose that X is λ-T_1 and β is a limit ordinal of cofinality $\leqslant \lambda$. Then the following conditions are equivalent.*

(1) X is $[\beta, \beta]$-compact.

(2) X is $[\beta + \alpha, \beta + \alpha]$-compact, for every ordinal α with $|\alpha| \leqslant \lambda$.

(3) X is $[\beta + \alpha, \beta + \alpha]$-compact, for some ordinal α with $|\alpha| \leqslant \lambda$.

(4) X is $[\beta, \beta + \lambda^+)$-compact.

The next lemma is proved as Lemma 6.4.

Lemma 6.15. *Suppose that λ is an infinite cardinal, α and γ are limit ordinals, $\gamma \leqslant \lambda$, $\operatorname{cf}\gamma = \operatorname{cf}\alpha$ and $(\alpha_\zeta)_{\zeta\in\gamma}$ is a strictly increasing sequence such that $\sup_{\zeta\in\gamma}\alpha_\zeta = \alpha$.*

If X is λ-T_1 and not $[\alpha, \alpha]$-compact, then there is a counterexample $(O_\delta)_{\delta\in\alpha}$ to the $[\alpha, \alpha]$-compactness of X with the property that, for every $\zeta \in \gamma$, O_ζ is indispensable.

Proof of Proposition 6.14. If β is any ordinal, let $\beta^{*\lambda}$ be the smallest ordinal $\leqslant \beta$ such that $|[\beta^{*\lambda}, \beta]| \leqslant \lambda$. Thus $\beta^{*\lambda}$ is the largest ordinal $\leqslant \beta$ which is either 0, or has cofinality $> \lambda$, or can be written as a limit of ordinals of cofinality $> \lambda$.

The proof now follows the lines of the proof of Theorem 6.6: prove first the result in the case when $\beta = \beta^{*\lambda} + \lambda$, and then when $\beta = \beta^{*\lambda}$ and $\omega \leqslant \operatorname{cf}\beta^{*\lambda} \leqslant \lambda$. □

7. Related notions and problems

The spaces introduced in Examples 3.2(3) and 3.11 are normal topological spaces with a base of clopen sets and they thus provide certain limits to provable results for $[\beta, \alpha]$-compactness of normal spaces. However, the theory developed so far appears to be not sharp enough to deal with such spaces.

As a very rough hypothesis, we conjecture that there is not very much difference in the theory of $[\beta, \alpha]$-compactness for, say, T_1 spaces and Tychonoff spaces. We also conjecture that we can get some more theorems under the additional assumption of normality. All the above rough hypotheses need to be verified; the present note appears to be already long enough, thus we postpone the discussion of such matters to a subsequent work.

Problem 7.1. Give characterizations (possibly similar to the ones given in Theorems 5.4 and 5.7 for arbitrary families) for those pairs of ordinals α and β such that $[\alpha, \alpha]$-compactness implies $[\beta, \beta]$-compactness, for general topological spaces and, respectively, for topological spaces satisfying some given separation axiom. Of course, the spaces introduced in Examples 3.2, 3.10, 3.11, 3.12, as well as the spaces $X_\tau(\beta, \alpha)$ and $X_T(\beta, \alpha)$ of Definitions 5.1 might be relevant to the solution of this problem.

Remark 7.2. For normal spaces, some problems might be open even when restricted to cardinal compactness. For example, it is easy to see that X is a linearly Lindelöf not Lindelöf space (see, e. g., [2]) if and only if X is $[\kappa, \kappa]$-compact, for every regular uncountable cardinal κ, but there is some uncountable cardinal λ (necessarily singular of cofinality ω) such that X is not $[\lambda, \lambda]$-compact.

Problem 7.3. Study the behavior of $[\beta, \alpha]$-compactness of topological spaces with respect to products. This problem might have some interest, since nontrivial results about cardinal compactness of products of topological spaces are already known. See, e. g., [5, 11, 17, 18, 25, 26, 30, 31, 35, 36] for results and references.

Problem 7.4. Study the mutual relationships among $[\beta, \alpha]$-compactness and other compactness properties, either defined in terms of covering properties or not.

Definition 7.5. We can also generalize the present notion of ordinal compactness to the relativized notion introduced in [19].

If X is a topological space and $\mathcal{F}$ is a family of subsets of X, let us say that X is $\mathcal{F}$-$[\beta, \alpha]$-*compact* if and only if the following condition holds.

For every sequence $(C_\delta)_{\delta\in\alpha}$ of closed sets of X, if, for every $H \subseteq \alpha$ with order type $< \beta$, there exists $F \in \mathcal{F}$ such that $\bigcap_{\delta\in H} C_\delta \supseteq F$, then $\bigcap_{\delta\in\alpha} C_\delta \neq \varnothing$.

In the above notation, $[\beta, \alpha]$-compactness turns out to be the particular case of $\mathcal{F}$-$[\beta, \alpha]$-compactness when $\mathcal{F}$ is the set of all singletons of X.

The particular case when $\mathcal{F}$ is the set of all nonempty open sets of X might have particular interest. The corresponding notion when both α and β are cardinals has been studied in [20].

Still another generalization is suggested by [19]. If $\mathcal{F}$ is a family of subsets of X, let us say that X is $[\beta, \alpha]$-*compact relative to* $\mathcal{F}$ if and only if the following condition holds.

For every sequence $(F_\delta)_{\delta\in\alpha}$ of elements of $\mathcal{F}$, if, for every $H \subseteq \alpha$ of order type $< \beta$, $\bigcap_{\delta\in H} F_\delta \neq \emptyset$, then $\bigcap_{\delta\in\alpha} F_\alpha \neq \emptyset$.

For a topological space X, $[\beta, \alpha]$-compactness is the same as $[\beta, \alpha]$-compactness relative to the family of all closed subsets of X.

Problem 7.6. A definition corresponding to ordinal compactness can be given for *abstract logics*. See [8] for definitions and background about logics.

Let us say that a logic $\mathcal{L}$ is (α, β)-*compact* if and only if, for every α-indexed set $(\sigma_\delta)_{\delta\in\alpha}$ of $\mathcal{L}$-sentences, if, for every $H \subseteq \alpha$ with order type $< \beta$, $\{\sigma_\delta \mid \delta \in H\}$ has a model, then $\{\sigma_\delta \mid \delta \in \alpha\}$ has a model.

Notice the reversed order of α and β, to be consistent with the standard notation used in the literature about compactness of logics.

We do not know whether ordinal compactness for logics is really a new notion, that is, whether or not it can be expressed in terms of cardinal compactness only. See, e. g., [23] for notions of cardinal compactness for logics.

Definition 7.7. We can define an even more general notion of compactness. If Z is any set and W is a subset of the power set of Z, let us say that a topological space X is $[W, Z]$-*compact* if and only if, whenever $(O_z)_{z\in Z}$ is an open cover of X, then there is $w \in W$ such that $(O_z)_{z\in w}$ is still a cover of X.

The usual notion of $[\mu, \lambda]$-compactness is the particular case when Z has cardinality λ and W is the set of all subsets of Z of cardinality $< \mu$.

More generally, the present notion of $[\beta, \alpha]$-compactness is the particular case when $Z = \alpha$ and W is the set of all subsets of α of order type $< \beta$.

Also the notion of D-compactness, for D an ultrafilter, can be showed to be a particular case of $[W, Z]$-compactness [20, Corollary 34].

Meanwhile, this general notion of $[W, Z]$-compactness has been extensively studied in [21], with equivalent characterizations explicitly stated in [22].

The idea of defining $[\beta, \alpha]$-compactness came to us after reading the definition of an (α, κ)-regular ultrafilter in [4, p. 237].

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