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Existence Results for Some Fractional Differential Equations Related to $A \in \mathcal{B}(L^2[a,b])$

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Abstract. In this paper we define the (generalized) linear Volterra integral operator on $L^2[a,b]$. Then the problem of existence and uniqueness of solutions of the second kind Volterra integral equations, corresponding to this operator, will be answered. Finally, some applications of this work to the existence of solutions of some fractional differential equations, are given.

1. Introduction and Preliminaries

Systems of Volterra integral equations and their solutions are of great importance in science and engineering. Most physical problems, such as biological applications in population dynamics and genetics where impulses arise naturally or are caused by control, can be modeled by a differential equation, an integral equation, an integro-differential equation or a system of these equations.

In recent years, the systems of integral and integro-differential equations have been solved by various method [2, 5, 7–9]. One of the important ways to overcome the difficulty of the ODE and PDE problems, is to reformulate them to integral equation problems, which lead to bounded integral operators. This method will be used here for a class of fractional differential equations.

Throughout this paper we assume that $a, b \in \mathbb{R}, a < b$, $H := L^2[a,b]$, and use the following notations.

$R_{a,b} := [a,b] \times [a,b]$,

$\Delta_{a,b} := \{(s,t) : a \le s \le b, a \le t \le s\}$,

$\mathcal{B}(H)$: The set of all bounded linear operators on H.

Definition 1.1. *The Fredholm integral operator $T : H \to H$ with kernel $k \in L^2(R_{a,b})$ is defined as*

$$Tf(s) = \int_a^b k(s,t) f(t) dt, \qquad f \in H, \ s \in [a,b].$$

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Also, the Volterra integral operator $T : H \to H$ with kernel $k \in L^2(\Delta_{a,b})$ is of the form

$$Tf(s) = \int_a^s k(s,t)f(t)dt, \quad f \in H,\ s \in [a,b].$$

A bounded linear operator T on a Banach space X is said to be quasi-nilpotent if $\|T^n\|^{\frac{1}{n}} \to 0$ as $n \to \infty$. It is easy to see that every nilpotent operator is quasi-nilpotent. In [6], it is shown that any Volterra integral operator $T \in \mathcal{B}(H)$ is quasi-nilpotent. Moreover, a quasi-nilpotent operator has no non-zero eigenvalues.

Two types of fractional derivatives of Riemann-Liouville and Caputo derivatives, have been often used in fractional calculus. We briefly introduce these two definitions.

Definition 1.2. *The Riemann-Liouville integral of the function $f(t)$ with order $\alpha \in (0,\infty)$ is defined as*

$$J_a^\alpha f(t) := \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau)d\tau, \qquad t \in [a,b]$$

where Γ is the Gamma function. Also, we set $J_a^0 f(t) := f(t)$, and $J_a := J_a^1$.

Theorem 1.3. *[1] Let $f \in L^1[a,b]$ and $\alpha, \beta > 0$. Then the following assertions hold.*

(i) *$J_a^\alpha f(t)$ exists for almost every $t \in [a,b]$, and $J_a^\alpha f \in L^1[a,b]$.*

(ii) *$J_a^\alpha J_a^\beta f = J_a^{\alpha+\beta} f$.*

Definition 1.4. *The Riemann-Liouville derivative of function $f(t)$ with order $\alpha \in \mathbb{R}_+ := [0,\infty)$ is defined by*

$$D_a^\alpha f(t) := \frac{d^m}{dt^m} J_a^{m-\alpha} f(t),$$

where $m = \lceil \alpha \rceil := \min\{k \in \mathbb{Z} : k \geq \alpha\}$, is the ceiling of α.

Definition 1.5. *The Caputo derivative with order $\alpha \in \mathbb{R}_+$ of function $f(t)$ is defined by*

$${}_*D_a^\alpha f(t) := J_a^{m-\alpha} \frac{d^m}{dt^m} f(t),$$

where $m = \lceil \alpha \rceil$.

Theorem 1.6. *[1] Let $\alpha > 0$ and $m = \lceil \alpha \rceil$. Then for a function $f : [a,b] \to \mathbb{R}$, the following statements hold.*

(i) *If f is such that both $D_a^\alpha f$ and ${}_*D_a^\alpha f$ exist, then*

$${}_*D_a^\alpha f(t) = D_a^\alpha f(t) - \sum_{k=0}^{m-1} \frac{f^{(k)}(a)}{\Gamma(k-\alpha+1)}(t-a)^{k-\alpha}.$$

(ii) *If $f \in C^m[a,b]$, then*

$${}_*D_a^\alpha f \in C[a,b] \quad \text{and} \quad J_a^\alpha\, {}_*D_a^\alpha f(t) = \sum_{k=0}^{m-1} \frac{f^{(k)}(a)}{k!}(t-a)^k.$$

(iii) *If $f \in L^1[a,b]$, then $D_a^\alpha J_a^\alpha f(t) = f(t)$, for almost every $t \in [a,b]$.*

2. Generalized linear Volterra integral equations

Integral operator can be defined in various ways and in different spaces. As an example see [3]. Here we want to consider the combination of the (usual) Fredholm and Volterra operator to an arbitrary operator.

Definition 2.1. *Suppose that $A \in \mathcal{B}(H)$.*

(i) *For $k \in L^2(R_{a,b})$, the bounded linear operator $T : H \to H$, defined by*

$$Tf(s) = \int_a^b k(s,t)Af(t)dt, \quad f \in H,\ s \in [a,b]$$

is said the Fredholm operator (with kernel k), related to A.

(ii) *For $k \in L^2(\Delta_{a,b})$, the bounded linear operator $T : H \to H$, defined by*

$$Tf(s) = \int_a^s k(s,t)Af(t)dt, \quad f \in H,\ s \in [a,b]$$

is said the Volterra operator (with kernel k), related to A.

The corresponding integral equation of the form

$$Tf - \lambda f = g,$$

is said to be the Fredholm (Volterra) equation of the second kind, related to A, if T is a Fredholm (Volterra) operator, related to A, $\lambda \neq 0$, $g \in H$, and $f \in H$ is an unknown function.

It is clear that every Volterra operator, related to A, is a Fredholm operator, related to A. Also, if F is a Fredholm (Volterra) operator with kernel k and T is a Fredholm (Volterra) operator with kernel k, related to $A \in \mathcal{B}(H)$, then $T = FA$. Since V is a compact operator, this implies that T is also compact.

Remark 2.2. *Suppose that $A \in \mathcal{B}(H)$. Then an easy verification shows that*

$$|Af(t)| \leq \|A\| \cdot |f(t)|,$$

for almost every $t \in [a,b]$.

Theorem 2.3. *Let $k \in L^2(\Delta_{a,b})$ be a bounded map with bound M and $A \in \mathcal{B}(H)$. Then the Volterra operator with kernel k, related to A, is a quasi-nilpotent operator. Moreover, for each $n \in \mathbb{N}$ we have*

$$\|T^n\| \leq \frac{\big((b-a)M\|A\|\big)^n}{(n-1)!\,\sqrt{2n}}. \tag{1}$$

Proof. Suppose that $A \in \mathcal{B}(H)$ and $s \in [a,b]$. By using the previous remark and Holder's inequality, we have

$$\begin{aligned}
|Tf(s)| &\leq \int_a^s |k(s,t)|\,|Af(t)|\,dt \leq \|A\| \int_a^s |k(s,t)|\,|f(t)|\,dt \\
&\leq \|A\|\Big(\int_a^s |k(s,t)|^2\,dt\Big)^{\frac{1}{2}}\Big(\int_a^s |f(t)|^2\,dt\Big)^{\frac{1}{2}} \\
&\leq M\|A\|\,\|f\|(s-a)^{\frac{1}{2}}.
\end{aligned}$$

This shows that

$$\begin{aligned}|T^2 f(s)| &\leq \int_a^s |k(s,t)|\,|ATf(t)|\,dt \leq \|A\|\,\|T\| \int_a^s |k(s,t)|\,|f(t)|\,dt \\ &\leq \frac{1}{1+\frac{1}{2}} M^2 \|A\|^2 \|f\| (s-a)^{\frac{3}{2}}.\end{aligned}$$

By induction on n and a similar method we have

$$\begin{aligned}|T^n f(s)| &\leq \frac{1}{(1+\frac{1}{2})(2+\frac{1}{2})\cdots(n-1+\frac{1}{2})} M^n \|A\|^n \, \|f\| (s-a)^{\frac{2n-1}{2}} \\ &\leq \frac{(M\|A\|)^n}{(n-1)!} \|f\| (s-a)^{\frac{2n-1}{2}}.\end{aligned}$$

Thus for each $f \in H$

$$\begin{aligned}\|T^n f\| &= \Big(\int_a^b |T^n f(t)|^2\, dt \Big)^{\frac{1}{2}} \\ &\leq \Big\{ \int_a^b \Big(\frac{(M\|A\|)^n}{(n-1)!} \|f\| (t-a)^{\frac{2n-1}{2}} \Big)^2 dt \Big\}^{\frac{1}{2}} \\ &\leq \frac{((b-a)M\|A\|)^n}{(n-1)!\,\sqrt{2n}} \|f\|,\end{aligned}$$

which implies $\|T^n\| \leq \frac{\big((b-a)M\|A\|\big)^n}{(n-1)!\,\sqrt{2n}}$. This equation also shows that T is a quasi-nilpotent operator. $\square$

Theorem 2.4. *Let $\alpha > 0$, $A \in \mathcal{B}(H)$, and $T : H \to H$ be defined by*

$$Tf(x) = (J_a^\alpha A) f(t) \qquad f \in H, t \in [a,b].$$

Then T is a quasi-nilpotent operator. Moreover, for each $n \in \mathbb{N}$

$$\|T^n\| \leq \frac{\Big((b-a)^{2\alpha - \frac{1}{2n}} \|A\|\Big)^n}{(2n\alpha - 1)^{\frac{3}{2}} \Gamma(n\alpha)}. \tag{2}$$

Proof. Similar to the proof of the previous theorem, we have

$$\|Tf(s)\| \leq \|A\| \frac{1}{\Gamma(\alpha)} \int_a^s (s-t)^{\alpha-1} |f(t)|\, dt = \|A\| J_a^\alpha |f|(s)$$

Also, by induction on n, using Theorem 1.3, and Holder's inequality we can imply

$$\begin{aligned}|T^n f(s)| &\leq \|A\|^n \underbrace{J_a^\alpha J_a^\alpha \cdots J_a^\alpha}_{n} |f|(s) \\ &= \|A\|^n J_a^{n\alpha} |f|(s) \\ &= \|A\|^n \frac{1}{\Gamma(n\alpha)} \int_a^s (s-t)^{n\alpha-1} |f(t)|\, dt \\ &\leq \frac{1}{(2n\alpha-1)\Gamma(n\alpha)} \|A\|^n (s-t)^{2n\alpha-1} \|f\|.\end{aligned}$$

Thus

$$
\begin{aligned}
\|T^n f\| &\leq \frac{1}{(2n\alpha-1)\Gamma(n\alpha)}\|A\|^n\Big(\int_a^b (s-t)^{4n\alpha-2}\Big)^{\frac{1}{2}}\|f\| \\
&\leq \frac{1}{(2n\alpha-1)\sqrt{4n\alpha-1}\Gamma(n\alpha)}\|A\|^n(b-a)^{\frac{4n\alpha-1}{2}}\|f\|.
\end{aligned}
$$

Therefore,

$$
\|T^n\| \leq \frac{(b-a)^{2\alpha-\frac{1}{2n}}}{(2n\alpha-1)\sqrt{4n\alpha-1}\Gamma(n\alpha)} \leq \frac{\Big((b-a)^{2\alpha-\frac{1}{2n}}\|A\|\Big)^n}{(2n\alpha-1)^{\frac{3}{2}}\Gamma(n\alpha)}.
$$

To complete the proof, it is sufficient to use Stirling's formula in the last inequality. □

Remark 2.5. *Let T be an operator that satisfies in the conditions of Theorem 2.3 or Theorem 2.4. Then $\sigma(T)$, the spectrum of T, is equal to $\{0\}$, because T is a quasi-nilpotent operator. Furthermore, from (1) and (2), for each $\lambda \neq 0$, the Neumann series $R_\lambda(T) := \lambda^{-1}(Id + \frac{T}{\lambda} + \frac{T^2}{\lambda^2} + \cdots)$ converges absolutely. So, it converges. Therefore $T - \lambda$ is invertible and it is easy to see that $(T-\lambda)^{-1} = R_\lambda(T)$. Thus if $g \in H$, the Volterra equation $Tf - \lambda f = g$, related to A, has a unique solution $f = R_\lambda(T)g$.*

Theorem 2.6. *Let T be the Fredholm operator with kernel $k \in H$, related to $A \in \mathcal{B}(H)$. Then for each $n \geq 2$, T^n is a Fredholm operator with kernel k_n, related to A, where*

$$
k_n(s,t) = \underbrace{A\int_a^b \cdots A\int_a^b}_{n-1} k(s,t_1)k(t_1,t_2)\cdots k(t_{n-1},t)\, dt_1\cdots dt_{n-1}.
$$

Proof. Suppose that $f \in \mathcal{B}(H)$, $s \in [a,b]$. By definition of T,

$$
Tf(s) = \int_a^b k(s,t)Af(t)\,dt.
$$

This follows that

$$
\begin{aligned}
T^2 f(s) &= \int_a^b k(s,t_1)ATf(t_1)\,dt_1 \\
&= \int_a^b k(s,t_1)\Big(A\int_a^b k(t_1,t)Af(t)dt\Big)dt_1 \\
&= \int_a^b \Big(A\int_a^b k(s,t_1)k(t_1,t)dt_1\Big)Af(t)dt
\end{aligned}
$$

Thus by induction on n, we can get

$$
T^n f(s) = \int_a^b \Big(\underbrace{A\int_a^b \cdots A\int_a^b}_{n-1} k(s,t_1)k(t_1,t_2)\cdots k(t_{n-1},t)\, dt_1\cdots dt_{n-1}\Big)Af(t)dt,
$$

that completes the proof. □

Remark 2.7. *Here we mention to an important property of the Riemann-Liouville integral operator, required in the following theorem, that is this operator is one-to-one. Let $J_a^\alpha f = 0$, for some $f \in H$. Since $H \subseteq L^1[a,b]$, by Theorem 1.6, we have $0 = D_a^\alpha J_a^\alpha f = f$. Therefore, J_a^α is a one-to-one map.*

Theorem 2.8. *Suppose that $\alpha > 0$, $m = \lceil \alpha \rceil$, $g : [a, b] \to \mathbb{R}$ be a function, $A \in \mathcal{B}(H)$, $\lambda \neq 0$ and $c_0, \ldots, c_{m-1}$ are n arbitrary real numbers. Then the following statements hold.*

(i) *If $g \in H$, then the integral equation*

$$J_a^\alpha A f(t) = \lambda f(t) + g(t) \tag{3}$$

has exactly one solution in H.

(ii) *If $g \in H$, then the initial fractional differential equation*

$$_{C}D_a^\alpha f(t) = A f(t) + g(t) \qquad f(a) = c_0, \ldots, f^{(m-1)}(a) = c_{m-1} \tag{4}$$

has at most one solution $f \in C^m[a, b]$.

(iii) *If $h = g - \sum_{k=0}^{m-1} \frac{c_k}{\Gamma(k - \alpha + 1)} (\cdot - a)^{k-\alpha} \in H$, the initial fractional differential equation*

$$D_a^\alpha f(t) = A f(t) + g(t) \qquad f(a) = c_0, \ldots, f^{m-1}(a) = c_{m-1} \tag{5}$$

has at most one solution $f \in C^m[a, b]$.

Proof. (i) Set $T := J_a^\alpha A$. Then (3) converts to $Tf - \lambda f = g$. From Remark 2.5, T is quasi-nilpotent and hence $T - \lambda$ is invertible. So, $f = (T - \lambda)^{-1} g = R_\lambda(T) g$ is the unique solution of (3).

(ii) Suppose that $f \in C^m[a, b]$ and $g \in H$. By using Theorem 1.6, $_*D_a^\alpha f \in C[a, b]$. Therefore the both sides of the following equation

$$_*D_a^\alpha f = A f + g \tag{6}$$

belongs to H. According to Remark 2.7 and Theorem 1.6, (6) is equivalent to

$$f - \sum_{k=0}^{m-1} \frac{c_k}{k!} (\cdot - a)^k = J_a^\alpha \, {_*D_a^\alpha} f = J_a^\alpha A f + J_a^\alpha g. \tag{7}$$

Consider $T = J_a^\alpha A$ and $h(t) = -J_a^\alpha g(t) - \sum_{k=0}^{m-1} \frac{c_k}{k!} (t - a)^k$, for each $t \in [a, b]$. Then $h \in H$ and it can be rewrite (7) in the form $Tf - f = h$. Thus Remark 2.5 shows that the last equation has the only solution $f = R_1(T) h \in H$. So, it has at most one solution in $C^m[a, b]$.

(iii) Assume that $f \in C^m[a, b]$. By using Theorem 1.6 we have

$$D_a^\alpha f(t) = {_*D_a^\alpha} f(t) + \sum_{k=0}^{m-1} \frac{c_k}{\Gamma(k - \alpha + 1)} (t - a)^{k-\alpha}. \tag{8}$$

Substituting (8) in to (5), we obtain that

$$D_a^\alpha f(t) = A f(t) + \Big(g(t) - \sum_{k=0}^{m-1} \frac{c_k}{\Gamma(k - \alpha + 1)} (t - a)^{k-\alpha} \Big). \tag{9}$$

Noting that both sides of (9) belong to H. Thus by using Theorem 1.6 again, (9) follows that:

$$f(t) - \sum_{k=0}^{m-1} \frac{c_k}{k!} (t - a)^k = J_a^\alpha \, {_*D_a^\alpha} f(t) = J_a^\alpha A f(t) + J_a^\alpha h(t),$$

or

$$J_a^{\alpha} Af(t) - f(t) = -\Big(\sum_{k=0}^{m-1} \frac{c_k}{k!}(t-a)^k + J_a^{\alpha} h(t)\Big). \tag{10}$$

Now, a similar approach, used in (ii), shows that the equation (10) has at most one solution $f \in C^m[a,b]$ and (if exists) is of the form

$$f(t) = -R_1(T)\Big(J_a^{\alpha} h + \sum_{k=0}^{m-1} \frac{c_k}{k!}(\cdot - a)^k\Big)(t).$$

□

Theorem 2.9. *Let $c > 0$, $g \in H$, $\mathbf{m}$ be the Lebesgue measure on $\mathbb{R}$ and $\phi : [a,b] \to [a,b]$ satisfies in one of the following conditions:*

(a) *For some $c > 0$ and each measurable set $X \subseteq [a,b]$, $\mathbf{m}\Big(\phi^{-1}(X)\Big) \leq c\,\mathbf{m}(X)$,*
or

(b) *ϕ is differentiable, $\phi' \neq 0$ on $[a,b]$, and $c := \sup_{t\in[a,b]} \dfrac{1}{|\phi'(t)|} < \infty$.*
Then the following integral equation

$$J_a^{\alpha}\Big(f \circ \phi\Big)(s) = \frac{1}{\Gamma(\alpha)} \int_0^s (s-t)^{\alpha-1} f\Big(\phi(t)\Big)\, dt = f(s) + g(s), \tag{11}$$

has a unique solution $f \in H$.

Proof. For each $f \in H$ let $Af := f \circ \phi$. Assume that (a) satisfies and $f \in H$. According to Layer cake representation (see [4]) we have

$$\begin{aligned} \|Af\|^2 = \|f \circ \phi\|^2 &= 2\int_0^{\infty} x\,\mathbf{m}\Big(\{y : |f(\phi(y))| \geq x\}\Big)dx \\ &= 2\int_0^{\infty} x\,\mathbf{m}\Big(\phi^{-1}\{z : |f(z)| \geq x\}\Big)dx \\ &\leq 2c\int_0^{\infty} x\,\mathbf{m}\Big(\{z : |f(z)| \geq x\}\Big)dx \\ &= c\|f\|^2. \end{aligned}$$

Now assume that (b) is satisfied. Then by the mean value property of derivative, $\phi' > 0$ or $\phi' < 0$ on $[a,b]$. If $\phi' > 0$, then for each $f \in H$, by assumptions we have

$$\begin{aligned} \|Af\|^2 &= \int_a^b \Big|f(\phi(t))\Big|^2 dt = \int_a^b \Big(f(\phi(t))\Big)^2 \phi'(t)\frac{1}{\phi'(t)}\, dt \\ &\leq \int_a^b \Big(f(\phi(t))\Big)^2 \phi'(t)\frac{1}{\phi'(t)}\, dt \\ &\leq c\int_{\phi(a)}^{\phi(b)} \Big(f(u)\Big)^2\, du \\ &\leq c\|f\|^2. \end{aligned}$$

A similar method shows that $\|Af\|^2 \leq c\|f\|^2$ satisfies whenever $\phi' < 0$. So, in any way, $A : H \to H$ is a bounded linear operator with $\|A\| \leq \sqrt{c}$. Thus by using Theorem 2.8, the integral equation (11) has a unique solution $f \in H$. □

Example 2.10. *Consider the integral equation*

$$J_0^\alpha f(\sin t) = -f(t) + \chi_{[0,\frac{1}{2}]}(t), \tag{12}$$

where $0 \le t \le 1$. *For each* $f \in H = L^2[0,1]$ *and* $t \in [0,1]$, *let* $Af(t) := f(\sin t)$. *By using Theorem 2.9*(b), A *is a bounded linear operator, and so,* (12) *is simply written as* $J_0^\alpha Af = -f + \chi_{[0,\frac{1}{2}]}$, *has exactly a unique solution* $f \in H$, *by using Theorem 2.8. In addition,* $f(t) = R_{-1}(J_0^\alpha A)\chi_{[0,\frac{1}{2}]}(t)$. *We note that* (12) *has no solution* $f \in C[0,1]$ *since* $J_0^\alpha Af + f \in C[0,1]$, *but* $\chi_{[0,\frac{1}{2}]} \notin C[0,1]$.

Example 2.11. *Let* $g \in H = L^2[0,1]$ *and* $c_0, \ldots, c_{m-1} \in \mathbb{R}$, *where* $m = \lceil \alpha \rceil$ *and* $\alpha > 0$. *Consider the fractional initial value problem*

$$_*D_0^\alpha f(t) = f(t) + tf(1-t) + g(t), \qquad (1 \le t \le 1) \tag{13}$$

with initial conditions $_*D_0^\alpha f(0) = c_0, \ldots, {_*D_0^\alpha} f^{m-1}(0) = c_{m-1}$. *Then initial value problem* (13) *has at most one solution* $f \in C^m[0,1]$ *and if such a solution exists, then from Theorem 2.8 it is equal to* $f(t) = \Big(R_1(J_0^\alpha A)h\Big)(t)$, *where* $A : H \to H$ *is defined for each* $f \in H$ *and* $t \in [0,1]$ *by* $Af(t) = f(t) + tf(1-t)$, *and* $h(t) = -J_0^\alpha g(t) - \sum_{k=0}^{m-1} \frac{c_k}{k!} t^k$. *It should be noted that* (13) *may be has no solution* $f \in C^m[a,b]$. *For example if* $g \in H \setminus C[a,b]$, *then by Theorem 1.6,* $_*D_0^\alpha f \in C[0,1]$, *for each* $f \in C^m[0,1]$. *This implies* $_*D_0^\alpha f(t) - f(t) - tf(1-t) \in C[0,1]$, *and therefore,* (13) *has no solution in* $C^m[0,1]$.

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