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The p_b-Cone Metric Spaces Over Banach Algebra With Applications

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Abstract. The aim of the article is to establish the structure of partial cone b-metric spaces over Banach algebras. Topological and structural properties are investigated of the new spaces. We also define generalized Lipschitz mappings and give their application in fixed point theory. The results presented in this paper substantially extend and strengthen the results of the literature. Few examples are provided to support our conclusions and as an application we establish the existence and uniqueness of a solution to a class of system of nonlinear integral equations.

1. Introduction

In 1994, Matthews [27] introduced the notion of a partial metric space. In this space, the usual metric is replaced by a partial metric with the interesting property that the self-distance of any point of space may not be zero. After introducing the idea of partial metric spaces, Matthews proved the partial metric version of Banach fixed point theorem. Many authors followed his idea and gave fixed point theorems in this space (see for example [2], [5], [20], [29]).

In 2007, the concept of cone metric spaces was introduced by Huang and Zhang [17] as a generalization of metric spaces. The distance $d(x, y)$ of two elements x and y in cone metric space X is defined to be a vector in a Banach space E whereas in metric spaces the distance of two elements is defined to be a non-negative real number.

Later in 2008, by omitting the assumption of normality of the cone, Rezapour and Hamlbarani [32], obtained generalizations of the results of [17] and presented few examples to support the existence of non-normal cones, which shows that the results in the setting of cone metric spaces are appropriate only if the underlying cone is not necessarily normal. In this direction several authors further established fixed point results in non-normal cone metric spaces (see, e.g., [4], [10], [13], [14], [15], [30], [31] and the references therein).

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In 2011, the notion of cone b-metric spaces as a generalization of b-metric spaces and cone metric spaces was studied by Hussain and Shah [19]. Afterwards, many authors have established fixed point results in cone b-metric spaces the reader may refer ([3], [6]).

In 2013 Shukla S. [35], introduced the concept of partial b-metric spaces as a generalization of partial metric and b-metric spaces. He also gave an analog to Banach contraction principle and Kannan type fixed point result in such space is proved. Mustafa et al. [28] has obtained a modified version of ordered partial b-metric spaces in the interest that each partial b-metric p_b generates a b-metric d_{p_b}. They studied some fixed point and common fixed point results for (ψ, φ)-weakly contractive mappings in the setup of ordered partial b-metric spaces. However, Ge and Lin [16] revised the definition of topology in partial b-metric space by giving a suitable counter example.

Very recently, Fernandez et al. [11] introduced the concept of partial b-cone metric space which is a generalization of cone b-metric space and partial metric spaces. They also proved some properties of these spaces and established some fixed point results for asymptotic regular maps in the setting of partial b-cone metric space.

In recent research some authors have raised a problem dealing with the existence of fixed points of the mappings in cone metric spaces and established an equivalence between some fixed point results in metric and in cone metric spaces (topological vector space-valued) see ([7], [8]) and also between cone b-metric spaces and b-metric spaces. They proved that any cone metric space (X, d) is equivalent to a usual metric space (X, d^*), if the real-valued metric function d^* is defined by a nonlinear scalarization function ξ_e (see [9]) or by a Minkowski functional q_e (see [24]).

In order to generalize and to overcome these problems Liu and Xu [26] in 2013 introduced the notion of cone metric space over Banach algebra by replacing the Banach space E by Banach algebra A which became a milestone in the study of fixed point theory since one can prove that cone metric spaces over Banach algebra are not equivalent to metric spaces in terms of the existence of the fixed points of the mappings. They proved some fixed point theorems of generalized Lipschitz mappings with weaker conditions on the generalized Lipschitz constant by restricting the contractive constants to be vectors and the relevant multiplications to be vector ones instead of usual real constants and scalar multiplications. Subsequently many authors established interesting and significant results in this space (see [11], [36]). Among these generalizations is partial cone metric spaces over Banach algebra introduced by Fernandez et al. [12] obtained by generalizing the partial metric spaces and cone metric spaces over Banach algebra which was **Selected for Young Scientist Award 2016**, M.P., India.

In 2015 Huang H. et al. [18] introduced the concept of cone b-metric space over Banach algebra and presented some common fixed point theorem of generalized Lipschitz mappings without normality of the underlying solid cones. They provided an example to explain the non-equivalence of the fixed point result between cone metric spaces over Banach algebra and cone b-metric space over Banach algebra. Reader may also refer [34].

On the other hand in 1986, the study of compatible maps for metric spaces was initiated by Jungck [22] and weakly compatible maps were firstly studied by Jungck and Rhoades in [23]. They proved some fixed point theorems for set valued noncontinuous functions. Using the ideas of Jungck for compatible maps and of weakly compatible maps of Jungck and Rhoades, from the setting of metric spaces Janković S. et al. [21] extended the definitions to the setting of cone metric spaces without the normality property on the cone.

Throughout this paper we propose a generalization of partial cone b-metric called partial cone b-metric spaces over Banach algebras, replacing Banach spaces by Banach algebras as the underlying spaces of partial cone b-metric spaces. Topological and structural properties are investigated in the new structure. We also define generalized Lipschitz mappings. With this modification, we shall obtain the existence and uniqueness of the fixed point for such mappings in the new setting without assumption of normality. Our results generalize and extend the recent result of [18] and [12]. Furthermore, an example is provided to support our conclusions. In addition, we show that our results establish the existence and uniqueness of a solution to a class of system of nonlinear integral equations.

2. Preliminaries

Let us recall some notions which will be needed in the sequel.
Let A always be a real Banach algebra. That is, A is a real Banach space in which an operation of multiplication is defined, subject to the following properties (for all $x, y, z \in A,\ \alpha \in R$)

1. $(xy)z = x(yz)$,
2. $x(y + z) = xy + xz$ and $(x + y)z = xz + yz$,
3. $\alpha(xy) = (\alpha x)y = x(\alpha y)$,
4. $\|xy\| \leq \|x\| \|y\|$.

Throughout this paper, we shall assume that a Banach algebra has a unit (i.e., a multiplicative identity) e such that $ex = xe = x$ for all $x \in A$. An element $x \in A$ is said to be invertible if there is an inverse element $y \in A$ such that $xy = yx = e$. The inverse of x is denoted by x^{-1}. For more details, we refer the reader to [33]. The following proposition is given in [33].

Proposition 2.1. *Let A be Banach algebra with a unit e, and $x \in A$. If the spectral radius $\rho(x)$ of x is less than 1, i.e.*

$$\rho(x) = \lim_{n\to+\infty} \|x^n\|^{\frac{1}{n}} = \inf \|x^n\|^{\frac{1}{n}} < 1.$$

then $e - x$ is invertible. Actually,

$$(e - x)^{-1} = \sum_{i=0}^{\infty} x^i.$$

Remark 2.2. *From [33] we see that the spectral radius $\rho(x)$ of x satisfies $\rho(x) \leq \|x\|$ for all $x \in A$, where A is a Banach algebra with a unit e.*

Remark 2.3. *(See [36]). In Proposition 2.1, if the condition '$\rho(x) < 1$' is replaced by $\|x\| \leq 1$, then the conclusion remains true.*

Remark 2.4. *(See [36]). If $\rho(x) < 1$ then $\|x^n\| \to 0 \quad (n \to +\infty)$.*

Now let us recall the concepts of cone over a Banach algebra A subset P of A is called a cone if

1. P is non-empty closed and $\{\theta, e\} \subset P$;
2. $\alpha P + \beta P \subset P$ for all non-negative real numbers α, β;
3. $P^2 = PP \subset P$;
4. $P \cap (-P) = \{\theta\}$,

where θ denotes the null of the Banach algebra A. For a given cone $P \subset A$, we can define a partial ordering $\preccurlyeq$ with respect to P by $x \preccurlyeq y$ if and only if $y - x \in P. x \prec y$ will stand for $x \preccurlyeq y$ and $x \neq y$, while $x \ll y$ will stand for $y - x \in intP$, where $intP$ denotes the interior of P. If $intP \neq \emptyset$ then P is called a solid cone.
The cone P is called normal if there is a number $M > 0$ such that, for all $x, y \in A$, $\theta \preccurlyeq x \preccurlyeq y$ implies $\|x\| \leq M \|y\|$. The least positive number satisfying the above is called the normal constant of P [17].
In the following we always assume that A is a Banach algebra with a unit e, P is a solid cone in A and $\preccurlyeq$ is the partial ordering with respect to P.

Definition 2.5. *([17,26]). Let X be a nonempty set. Suppose the mapping $d : X \times X \to A$ satisfies*

1. *$\theta \preccurlyeq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$,*

2. $d(x, y) = d(y, x)$ *for all* $x, y \in X$;

3. $d(x, y) \preccurlyeq d(x, z) + d(z, y)$ *for all* $x, y, z \in X$.

Then d is called a cone metric on X, and (X, d) *is called a cone metric space over Banach algebra A.*
For other definitions and related results on cone metric spaces with Banach algebra we refer to [26].

Definition 2.6. *([18]) Let X be a nonempty set and* $s \geq 1$ *be a constant. A mapping* $d : X \times X \to A$ *is said to be cone b-metric if and only if, for all* $x, y, z \in X$*, the following conditions are satisfied:*

1. $\theta \prec d(x, y)$ *for all* $x, y \in X$ *and* $d(x, y) = \theta$ *if and only if* $x = y$;

2. $d(x, y) = d(y, x)$ *for all* $x, y \in X$;

3. $d(x, y) \preccurlyeq s[d(x, z) + d(z, y)]$ *for all* $x, y, z \in X$.

Then d is called a cone b-metric on X, and (X, d) *is called a cone b-metric space over Banach algebra A. For more definitions and results on cone b-metric spaces, the reader may refer to [18].*

Definition 2.7. *([27]). A partial metric on a nonempty set X is a function* $p : X \times X \to R^+$ *such that for all* $x, y, z \in X$ *the following conditions hold:*

1. $x = y$ *if and only if* $p(x, x) = p(x, y) = p(y, y)$,

2. $p(x, x) \leq p(x, y)$,

3. $p(x, y) = p(y, x)$,

4. $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.

The pair (X, p) *is called a partial metric space. It is clear that, if* $p(x, y) = 0$*, then from (1) and (2)* $x = y$*. But if* $x = y, p(x, y)$ *may not be 0. To study the other details on partial metric spaces, refer to [27].*

Definition 2.8. *([35]). A partial b-metric on a nonempty set X is a function* $b : X \times X \to R^+$ *such that for all* $x, y, z \in X$,

1. $x = y$ *if and only if* $b(x, x) = b(x, y) = b(y, y)$;

2. $b(x, x) \leq b(x, y)$;

3. $b(x, y) = b(y, x)$;

4. *there exists a real number* $s \geq 1$ *such that* $b(x, y) \leq s[b(x, z) + b(z, x)] - b(z, z)$.

A partial b-metric space is a pair (X, b) *such that X is a nonempty set and b is a partial b-metric on X. The number s is called the coefficient of* (X, b).
For more details on partial b-metric spaces the reader may refer to [35].

Definition 2.9. *([10]). A partial cone b-metric on a nonempty set X is a function* $p_b : X \times X \to E$ *such that for all* $x, y, z \in X$,

1. $x = y$ *if and only if* $p_b(x, x) = p_b(x, y) = p_b(y, y)$;

2. $\theta \preccurlyeq p_b(x, x) \preccurlyeq p_b(x, y)$;

3. $p_b(x, y) = p_b(y, x)$;

4. $p_b(x, y) \preccurlyeq s[p_b(x, z) + p_b(z, y)] - p_b(z, z)$.

The pair (X, p_b) is called a partial cone b-metric space. The number $s \geq 1$ is called the coefficient of (X, p_b).
In a partial cone b-metric space (X, p_b), if $x, y \in X$ and $p_b(x, y) = \theta$, then $x = y$, but the converse may not be true.
It is clear that every partial cone b-metric space is a partial cone metric space with coefficient $s = 1$ and every cone b-metric space is a partial cone b-metric space with the same coefficient and zero self distance. However, the converse of these facts needs not to be hold.
The definitions and subsequent results on partial cone b-metric space are given in [10].

Definition 2.10. *([1]). Let X be a nonempty set and f, g be self maps on X and $x, z \in X$. Then x is called coincidence point of pair (f, g) if $fx = gx$, and z is called point of coincidence of pair (f, g) if $fx = gx = z$.*

Definition 2.11. *([1]). Let $f, g : X \to X$ be given self-mappings on X. The pair (f, g) is said to be weakly compatible if f and g commute at their coincidence points (i.e., $fgx = gfx$, whenever $fx = gx$).*

Lemma 2.12. *([1]). Let f and g be weakly compatible self maps of a set X. If f and g have a unique point of coincidence $w = fx = gx$, then w is the unique common fixed point of f and g.*

3. Partial cone *b*-metric spaces over Banach algebra

We now present the concept of partial cone *b*-metric spaces over Banach algebra *A* with appropriate examples and study some of its properties needed in the sequel.

Definition 3.1. *A partial cone b-metric on a nonempty set X is a function*
$p_b : X \times X \to A$ such that for all $x, y, z \in X$:
(p_{b1}) $x = y$ *if and only if* $p_b(x, x) = p_b(x, y) = p_b(y, y)$,
(p_{b2}) $\theta \preccurlyeq p_b(x, x) \preccurlyeq p_b(x, y)$,
(p_{b3}) $p_b(x, y) = p_b(y, x)$,
(p_{b4}) $p_b(x, y) \preccurlyeq s[p_b(x, z) + p_b(z, y)] - p_b(z, z)$.
The pair (X, p_b) is called a partial cone b-metric space over Banach algebra A. The number $s \geq 1$ is called the coefficient of (X, p_b).

Remark 3.2. *In a partial cone b-metric space over Banach algebra (X, p_b), if $x, y \in X$ and $p_b(x, y) = \theta$, then $x = y$, but the converse may not be true.*

Remark 3.3. *It is clear that every partial cone b-metric space over Banach algebra is a partial cone metric space over Banach algebra with coefficient $s = 1$ and every cone b-metric space is a partial cone b-metric space with the same coefficient and zero self distance. However, the converse of these facts needs not to be hold.*

Example 3.4. *Let $A = C[a, b]$ be the set of continuous functions on the interval $[a, b]$ with the norm $\|x\| = \|x\|_\infty + \left\|x'\right\|_\infty$. Define multiplication in the usual way. Then A is a Banach algebra with a unit 1. Set $P = \{x \in A : x(t) \geq 0,\ t \in [a, b]\}$ and $X = R^+$. Define a mapping $p_b : X \times X \to A$ by*

$$p_b(x, y)(t) = \Big([max\{x, y\}]^2 + |x - y|^2\Big)e^t$$

for all $x, y \in X$. This makes (X, p_b) into a partial cone b-metric space over Banach algebra A with the coefficient $s = 2$, but it is not a partial cone metric space over Banach algebra since the triangle inequality is not satisfied.

Example 3.5. *Let $A = C^1_R[0, 1]$ and define a norm on A by $\|x\| = \|x\|_\infty + \left\|x'\right\|_\infty$ for $x \in A$. Define multiplication in A as just point wise multiplication. Then A is a real unit Banach algebra with unit $e = 1$. Set $P = \{x \in A : x \geq 0\}$ is a cone in A. Moreover, P is not normal (see [32]). Let $X = [0, +\infty)$ and $a > 0$ be any constant. Define a mapping $p_b : X \times X \to A$ by*

$$p_b(x, y)(t) = \Big((max\{x, y\})^2 + a\Big)e^t$$

for all $x, y \in X$. Then (X, p_b) is a partial cone b-metric space over Banach algebra A with the coefficient $s \geq 1$ but it is not a cone b-metric space over Banach algebra A since for any $x > 0$ we have $p_b(x, x)(t) = (x^2 + a)e^t \neq \theta$.

Example 3.6. *Let $X = [0,1]$ and A be the set of all real valued function on X which also have continuous derivatives on X with the norm $\|x\| = \|x\|_\infty + \left\|x'\right\|_\infty$. Define multiplication in the usual way. Let $P = \{x \in A : x(t) \geq 0,\ t \in X\}$. It is clear that P is a nonnormal cone and A is a Banach algebra with a unit $e = 1$. Define a mapping $p_b : X \times X \to A$ by*

$$p_b(x,y)(t) = \begin{cases} x^p e^t, & x = y \\ (x+y)^p e^t, & \text{otherwise} \end{cases}$$

Then (X, p_b) is a partial cone b-metric space over Banach algebra A with the coefficient $s = 2^{p-1}$ but it is not a cone b-metric space over Banach algebra A since $p_b(x,x)(t) \neq \theta$ for each $x \in X$ with $x \neq 0$.

Definition 3.7. *Let (X, p_b) is a partial cone b-metric space over Banach algebra A. Then for an $x \in X$ a $\theta \ll c$, the p_b-ball with center x and radius $\theta \ll c$ is*

$$B_{p_b}(x,c) = \{y \in X : p_b(x,y) \ll p_b(x,x) + c\}$$

4. Topology on Partial cone *b*-metric spaces over Banach algebra

In this section, we define the topology in partial cone b-metric space over Banach algebra A.

Definition 4.1. *Let (X, p_b) be a partial cone b-metric space over Banach algebra A with coefficient $s \geq 1$. For each $x \in X$ and each $\theta \ll c$, put $B_{p_b}(x,c) = \{y \in X : p_b(x,y) \ll p_b(x,x) + c\}$ and put $\mathfrak{B} = \{B_{p_b}(x,c) : x \in X \text{ and } \theta \ll c\}$. Then $\mathfrak{B}$ is a subbase for some topology τ on X.*

Remark 4.2. *Let (X, p_b) be a partial cone b-metric space over Banach algebra A. In this paper, τ denotes the topology on X, $\mathfrak{B}$ denotes a subbase for the topology on τ and $B_{p_b}(x,c)$ denotes the p_b-ball in (X, p_b), which are described in Definition 4.1. In addition U denotes the base generated by the subbase $\mathfrak{B}$.*

Theorem 4.3. *Let (X, p_b) be a partial cone b-metric space over Banach algebra A. Then (X, p_b) is a T_0-space.*

Proof. Suppose $p_b : X \times X \to A$ be a partial cone b-metric and $x, y \in X$. If $x \neq y$ then $p_b(x,y) \succ \theta$. It follows from (p_1) and (p_2) that $p_b(x,x) \prec p_b(x,y)$ or $p_b(y,y) \prec p_b(x,y)$. Write $\left\|p_b(x,x) - p_b(x,y)\right\| = \delta_x$. Then $\delta_x > 0$. Hence there exists a $c_x \gg \theta$ with $\|c_x\| < \frac{\delta_x}{k}$.
Thus $x \in B_{p_b}(x, c_x)$ and $y \notin B_{p_b}(x, c_x)$.
For the case $p_b(y,y) \prec p_b(x,y)$, one can find a $c_y \gg \theta$ with $\left\|c_y\right\| < \frac{\delta_y}{k}$.
Thus $y \in B_{p_b}(y, c_y)$ and $x \notin B_{p_b}(y, c_y)$.
Consequently, we find that partial cone b-metric space over Banach algebra (X, p_b) is T_0.

Now, we define θ-Cauchy sequence and convergent sequence in partial cone b-metric space over Banach algebra A.

Definition 4.4. *Let (X, p_b) be a partial cone b-metric space over Banach algebra A and $\{x_n\}$ be a sequence in X. If for every $c \in intP$ there is a positive integer n_0 such that, $p_b(x_n, x) \ll c + p_b(x,x)$ for all $n > n_0$, then $\{x_n\}$ is said to be convergent and converges to x, and x is the limit of $\{x_n\}$ and we denote this by $x_n \to x$ as $n \to +\infty$ or $\lim_{n\to+\infty} x_n = x$.*

Definition 4.5. *Let (X, p_b) be a partial cone b-metric space over Banach algebra A. A sequence $\{x_n\}$ in (X, p_b) is called a θ-Cauchy sequence if $\{p_b(x_n, x_m)\}$ is a c-sequence in A, i.e. if for every $c \in intP$ there exists $n_0 \in N$ such that $p_b(x_n, x_m) \ll c$ for all $n, m > n_0$.*

Definition 4.6. *Let (X, p_b) be a partial cone b-metric space over Banach algebra A. Then (X, p_b) is said to be θ-complete if every θ-Cauchy sequence $\{x_n\}$ in X converges to a point $x \in X$; that is,*

$$\lim_{n,m\to+\infty} p_b(x_n, x_m) = \lim_{n\to+\infty} p_b(x_n, x) = p_b(x,x) = \theta.$$

Definition 4.7. *Let (X, p_b) and $(X', p_b{}')$ be a partial cone b-metric space over Banach algebra A. Then a function $f : X \to X'$ is said to be continuous at a point $x \in X$ if and only if it is sequentially continuous at x, that is, whenever $\{x_n\}$ is convergent to x we have $\{f x_n\}$ is convergent to $f(x)$.*

5. Generalized Lipschitz Mapping

In this section, we introduce the concept of generalized Lipschitz mapping on partial cone b-metric spaces over Banach algebras with an example.

Definition 5.1. *Let (X, p_b) be a partial cone b-metric space over a Banach algebra A. A mapping $T : X \to X$ is called a generalized Lipschitz mapping if there exists a vector $k \in P$ with $\rho(k) < 1$ and for all $x, y \in X$, one has*

$$p_b(Tx, Ty) \preccurlyeq k p_b(x, y).$$

Example 5.2. *Let $X = [0, +\infty)$ and let (X, p_b) be a partial cone b-metric space over Banach algebra A with the coefficient $s = 2$ and $p = 2$ as defined in Example 3.6. Define a self map T on X as follows*
$Tx = ln\Big(1 + \frac{x}{4}\Big)$. Since $ln(1 + u) \preccurlyeq u$ for each $u \in [0, \infty)$, for all $x, y \in X$, we have when $x \neq y$

$$\begin{aligned} p_b(Tx, Ty)(t) &= \left(ln\Big(1 + \frac{x}{4}\Big) + ln\Big(1 + \frac{x}{4}\Big)\right)^2 e^t \preccurlyeq \Big(\frac{x}{4} + \frac{y}{4}\Big)^2 e^t \\ &= \frac{1}{16}(x + y)^2 e^t = \frac{1}{16} p_b(x, y)(t) \end{aligned}$$

Now when $x = y$,

$$\begin{aligned} p_b(Tx, Tx)(t) &= \left(ln\Big(1 + \frac{x}{4}\Big)\right)^2 e^t \preccurlyeq \Big(\frac{x}{4}\Big)^2 e^t = \frac{1}{16}(x)^2 e^t \\ &= \frac{1}{16} p_b(x, x)(t) \end{aligned}$$

Therefore, $p_b(Tx, Ty)(t) \preccurlyeq \frac{1}{16} p_b(x, y)(t)$
Clearly, T is a generalized Lipschitz map in X.

Now we review some facts on c-sequence theory.

Definition 5.3. *([24]). Let P be a solid cone in a Banach space E. A sequence $\{u_n\} \subset P$ is said to be a c-sequence if for each $c \gg \theta$ there exists a natural number N such that $u_n \ll c$ for all $n > N$.*

Lemma 5.4. *([33]). Let P be a solid cone in a Banach algebra A. Suppose that $k \in P$ be an arbitrary vector and $\{u_n\}$ is a c-sequence in P. Then $\{ku_n\}$ is a c-sequence.*

Lemma 5.5. *([31]). Let A be a Banach algebra with a unit e, $k \in A$, then $\lim_{n\to+\infty} \|k^n\|^{\frac{1}{n}}$ exists and the spectral radius $\rho(k)$ satisfies*

$$\rho(k) = \lim_{n\to+\infty} \|k^n\|^{\frac{1}{n}} = inf \|k^n\|^{\frac{1}{n}}.$$

If $\rho(k) < |\lambda|$, then $(\lambda e - k)$ is invertible in A, moreover,

$$(\lambda e - k)^{-1} = \sum_{i=0}^{\infty} \frac{k^i}{\lambda^{i+1}}$$

where λ is a complex constant.

Lemma 5.6. *([31]). Let A be a Banach algebra with a unit $e, a, b \in A$. If a commutes with b, then*

$$\rho(a + b) \leq \rho(a) + \rho(b), \quad \rho(ab) \leq \rho(a)\rho(b).$$

Lemma 5.7. *([33]). Let A be a Banach algebra with a unit e and P be a solid cone in A. Let $a, k, l \in P$ hold $l \preccurlyeq k$ and $a \preccurlyeq la$. If $\rho(k) < 1$, then $a = \theta$.*

Lemma 5.8. *([33]). If E is a real Banach space with a solid cone P and $\{u_n\} \subset P$ be a sequence with $\|u_n\| \to 0$ $(n \to +\infty)$, then $\{u_n\}$ is a c-sequence.*

Lemma 5.9. *([33]). If E is a real Banach space with a solid cone P*

1. *If $a, b, c \in E$ and $a \preccurlyeq b \ll c$, then $a \ll c$.*
2. *If $a \in P$ and $a \ll c$ for each $c \gg \theta$, then $a = \theta$.*

Lemma 5.10. *([33]). Let A be a Banach algebra with a unit e and $k \in A$. If λ is a complex constant and $\rho(k) < |\lambda|$, then*

$$\rho\Big((\lambda e - k)^{-1}\Big) \leq \frac{1}{|\lambda| - \rho(k)}.$$

6. Application to Fixed Point Theorems

In this section, we present some fixed point theorems for generalized Lipschitz mappings in the framework of partial cone b-metric spaces over Banach algebra A over a non-normal solid cone with an illustrative example. We start with the following result.

Theorem 6.1. *Let (X, p_b) be a θ-complete partial cone b-metric space over Banach algebra A with the coefficient $s \geq 1$ and P be a solid cone in A. Let $k_i \in P(i = 1, 2, 3, 4, 5)$ be generalized Lipschitz constants with $(s+1)\rho(k) + 2s\rho(k_1) + s\rho(k_4) + s\rho(k_5) < 1$, where $k = \rho(k_2) + \rho(k_3) + s\rho(k_4) + s\rho(k_5) < 1$. Suppose the mappings $f, g : X \to X$ satisfy*

$$\begin{aligned} p_b(fx, fy) \preccurlyeq\ & k_1 p_b(gx, gy) + k_2 p_b(fx, gx) + k_3 p_b(fy, gy) + k_4 p_b(gx, fy) \\ & + k_5 p_b(fx, gy) \end{aligned} \tag{6.1}$$

for all $x, y \in X$. If the range of g contains the range of f and $g(X)$ is a complete subspace, then f and g have a unique point of coincidence in X. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point.

Proof. Let $x_0 \in X$ be an arbitrary point. Since $f(X) \subset g(X)$, there exists an $x_1 \in X$ such that $fx_0 = gx_1$. By induction, a sequence $\{fx_n\}$ can be chosen such that $fx_n = gx_{n+1} (n = 0, 1, 2, ...)$. Thus, by (6.1), for any natural number n, on the one hand, we have

$$\begin{aligned} p_b(gx_{n+1}, gx_n) &= p_b(fx_n, fx_{n-1}) \\ &\preccurlyeq k_1 p_b(gx_n, gx_{n-1}) + k_2 p_b(fx_n, gx_n) + k_3 p_b(fx_{n-1}, gx_{n-1}) \\ &\quad + k_4 p_b(gx_n, fx_{n-1}) + k_5 p_b(fx_n, gx_{n-1}) \\ &\preccurlyeq k_1 p_b(gx_n, gx_{n-1}) + k_2 p_b(gx_{n+1}, gx_n) + k_3 p_b(gx_n, gx_{n-1}) \\ &\quad + k_4 p_b(gx_n, gx_n) + k_5 p_b(gx_{n+1}, gx_{n-1}) \\ &\preccurlyeq k_1 p_b(gx_n, gx_{n-1}) + k_2 p_b(gx_{n+1}, gx_n) + k_3 p_b(gx_n, gx_{n-1}) \\ &\quad + k_4 p_b(gx_{n-1}, gx_n) + k_5[s p_b(gx_{n+1}, gx_n) + s p_b(gx_n, gx_{n-1}) \\ &\quad - p_b(gx_n, gx_n)] \ \text{[by } p_b 2 \text{ and } p_b 4] \\ &\preccurlyeq k_1 p_b(gx_n, gx_{n-1}) + k_2 p_b(gx_{n+1}, gx_n) + k_3 p_b(gx_n, gx_{n-1}) \\ &\quad + k_4 p_b(gx_{n-1}, gx_n) + k_5 s p_b(gx_{n+1}, gx_n) + k_5 s p_b(gx_n, gx_{n-1}) \\ &\preccurlyeq (k_1 + k_3 + k_4 + sk_5) p_b(gx_n, gx_{n-1}) + (k_2 + sk_5) p_b(gx_{n+1}, gx_n) \end{aligned}$$

which implies that

$$(e - k_2 - sk_5)p_b(gx_{n+1}, gx_n) \preccurlyeq (k_1 + k_3 + k_4 + sk_5)p_b(gx_n, gx_{n-1}) \tag{6.2}$$

On the other hand, we have

$$\begin{aligned}
p_b(gx_n, gx_{n+1}) &= p_b(fx_{n-1}, fx_n) \\
&\preccurlyeq k_1p_b(gx_{n-1}, gx_n) + k_2p_b(fx_{n-1}, gx_{n-1}) + k_3p_b(fx_n, gx_n) \\
&\quad + k_4p_b(gx_{n-1}, fx_n) + k_5p_b(fx_{n-1}, gx_n) \\
&\preccurlyeq k_1p_b(gx_{n-1}, gx_n) + k_2p_b(gx_n, gx_{n-1}) + k_3p_b(gx_{n+1}, gx_n) \\
&\quad + k_4p_b(gx_{n-1}, gx_{n+1}) + k_5p_b(gx_n, gx_n) \\
&\preccurlyeq k_1p_b(gx_{n-1}, gx_n) + k_2p_b(gx_n, gx_{n-1}) + k_3p_b(gx_{n+1}, gx_n) \\
&\quad + k_4[sp_b(gx_{n-1}, gx_n) + sp_b(gx_n, gx_{n+1}) - p_b(gx_n, gx_n)] \\
&\quad + k_5p_b(gx_{n-1}, gx_n) \\
&\preccurlyeq (k_1 + k_2 + sk_4 + k_5)p_b(gx_{n-1}, gx_n) + (k_3 + sk_4)p_b(gx_{n+1}, gx_n)
\end{aligned}$$

which implies that

$$(e - k_3 - sk_4)p_b(gx_{n+1}, gx_n) \preccurlyeq (k_1 + k_2 + sk_4 + k_5)p_b(gx_n, gx_{n-1}) \tag{6.3}$$

Add (6.2) and (6.3)

$$\begin{aligned}
(2e - k_2 - k_3 - sk_4 - sk_5)p_b(gx_{n+1}, gx_n) \preccurlyeq [2k_1 + k_2 + k_3 \\
+ (1 + s)k_4 + (1 + s)k_5]p_b(gx_{n-1}, gx_n)
\end{aligned} \tag{6.4}$$

Denote $k_2 + k_3 + sk_4 + sk_5 = k$ then (6.4) yields that

$$(2e - k)p_b(gx_{n+1}, gx_n) \preccurlyeq (k + 2k_1 + k_4 + k_5)p_b(gx_n, gx_{n-1}) \tag{6.5}$$

Note that

$$\rho(k) \leq (s + 1)\rho(k) \leq (s + 1)\rho(k) + 2s\rho(k_1) + s\rho(k_4) + s\rho(k_5) < 1 < 2$$

leads to $\rho(k) < 2$, then by Lemma 5.5 it follows that $(2e - k)$ is invertible. Furthermore,

$$(2e - k)^{-1} = \sum_{i=0}^{\infty} \frac{k_i}{2^{i+1}}$$

By multiplying in both sides of (6.5) by $(2e - k)^{-1}$, we arrive at

$$p_b(gx_{n+1}, gx_n) \preccurlyeq (2e - k)^{-1}(k + 2k_1 + k_4 + k_5)p_b(gx_n, gx_{n-1}) \tag{6.6}$$

Denote $h = (2e - k)^{-1}(k + 2k_1 + k_4 + k_5)$, then by (6.6) we get

$$p_b(gx_{n+1}, gx_n) \preccurlyeq hp_b(gx_n, gx_{n-1}) \preccurlyeq \cdots \preccurlyeq h^n p_b(gx_1, gx_0) = h^n p_b(fx_0, gx_0).$$

Note by Lemma 5.6 and Lemma 5.10 that

$$\begin{aligned}
\rho(h) &= \rho\Big((2e - k)^{-1} \cdot (k + 2k_1 + k_4 + k_5)\Big) \\
&\leq \rho\Big((2e - k)^{-1}\Big) \cdot \rho(k + 2k_1 + k_4 + k_5) \\
&\leq \frac{1}{2 - \rho(k)}\Big[\rho(k) + 2\rho(k_1) + \rho(k_4) + \rho(k_5)\Big] \\
&< \frac{1}{s}, \qquad \Big[\text{Since } (s + 1)\rho(k) + 2s\rho(k_1) + s\rho(k_4) + s\rho(k_5) < 1 < 2\Big]
\end{aligned}$$

which means that $(e - sh)$ is invertible and $\|h^m\| \to 0$ $(m \to \infty)$. Hence, for any $m \geq 1$, $p \geq 1$ and $h \in P$ with $\rho(h) < \frac{1}{s}$, we have that

$$\begin{aligned}
p_b(gx_m, gx_{m+p}) &\preccurlyeq s[p_b(gx_m, gx_{m+1}) + p_b(gx_{m+1}, gx_{m+p})] - p_b(gx_{m+1}, gx_{m+1}) \\
&\preccurlyeq sp_b(gx_m, gx_{m+1}) + sp_b(gx_{m+1}, gx_{m+p}) \\
&\preccurlyeq sp_b(gx_m, gx_{m+1}) + s[sp_b(gx_{m+1}, gx_{m+2}) + sp_b(gx_{m+2}, gx_{m+p})] \\
&\quad - p_b(gx_{m+2}, gx_{m+2}) \\
&\preccurlyeq sp_b(gx_m, gx_{m+1}) + s^2p_b(gx_{m+1}, gx_{m+2}) + s^2p_b(gx_{m+2}, gx_{m+p}) \\
&\preccurlyeq sp_b(gx_m, gx_{m+1}) + s^2p_b(gx_{m+1}, gx_{m+2}) + s^3p_b(gx_{m+2}, gx_{m+3}) \\
&\quad + \cdots + s^{p-1}p_b(gx_{m+p-2}, gx_{m+p-1}) + s^{p-1}p_b(gx_{m+p-1}, gx_{m+p}) \\
&\preccurlyeq sh^m p_b(fx_0, gx_0) + s^2h^{m+1}p_b(fx_0, gx_0) + s^3h^{m+2}p_b(fx_0, gx_0) \\
&\quad + \cdots + s^{p-1}h^{m+p-2}p_b(fx_0, gx_0) + s^p h^{m+p-1}p_b(fx_0, gx_0) \\
&= sh^m[e + sh + s^2h^2 + \cdots + (sh)^{p-1}]p_b(fx_0, gx_0) \\
&\preccurlyeq sh^m(e - sh)^{-1}p_b(fx_0, gx_0).
\end{aligned}$$

In view of Remark 2.4, $\left\|sh^m p_b(fx_0, gx_0)\right\| \leq \|sh^m\| \left\|p_b(fx_0, gx_0)\right\| \to 0$ $(m \to +\infty)$, by Lemma 5.8, we have $\left\{sh^m p_b(fx_0, gx_0)\right\}$ is a c-sequence. Next by using Lemma 5.4 and Lemma 5.9, we conclude that $\{gx_n\}$ is a θ-Cauchy sequence. Since $g(X)$ is θ-complete, there is $q \in g(X)$ such that $gx_n \to q$ $(n \to +\infty)$. Thus there exists $p \in X$ such that $gp = q$. Therefore

$$\lim_{n\to+\infty} p_b(gx_n, q) = \lim_{n,m\to+\infty} p_b(gx_n, gx_m) = p_b(q, q) = \theta.$$

We shall prove $fp = q$. In order to end this, for one thing,

$$\begin{aligned}
p_b(gx_n, fp) &= p_b(fx_{n-1}, fp) \\
&\preccurlyeq k_1p_b(gx_{n-1}, gp) + k_2p_b(fx_{n-1}, gx_{n-1}) + k_3p_b(fp, gp) \\
&\quad + k_4p_b(gx_{n-1}, fp) + k_5p_b(fx_{n-1}, gp) \\
&\preccurlyeq k_1p_b(gx_{n-1}, q) + k_2p_b(gx_n, gx_{n-1}) + k_3p_b(fp, gp) \\
&\quad + k_4p_b(gx_{n-1}, fp) + k_5p_b(fx_{n-1}, gp) \\
&\preccurlyeq k_1p_b(gx_{n-1}, q) + k_2p_b(gx_n, gx_{n-1}) + k_3p_b(fp, q) \\
&\quad + k_4p_b(gx_{n-1}, fp) + k_5p_b(fx_{n-1}, q) \\
&\preccurlyeq k_1p_b(gx_{n-1}, q) + k_2[sp_b(gx_n, q) + sp_b(q, gx_{n-1}) - p_b(q, q)] \\
&\quad + k_3[sp_b(fp, gx_n) + sp_b(gx_n, q) - p_b(gx_n, gx_n)] \\
&\quad + k_4[sp_b(gx_{n-1}, q) + sp_b(q, fp) \\
&\quad - p_b(q, q)] + k_5p_b(gx_n, q) \\
&\preccurlyeq k_1p_b(gx_{n-1}, q) + k_2sp_b(gx_n, q) + sk_2p_b(q, gx_{n-1}) + k_3sp_b(fp, gx_n) \\
&\quad + sk_3p_b(gx_n, q) + k_4sp_b(gx_{n-1}, q) + sk_4p_b(q, fp) + k_5p_b(gx_n, q) \\
&\preccurlyeq k_1p_b(gx_{n-1}, q) + k_2sp_b(gx_n, q) + sk_2p_b(q, gx_{n-1}) + k_3sp_b(fp, gx_n) \\
&\quad + sk_3p_b(gx_n, q) + sk_4p_b(gx_{n-1}, q) + k_4s[sp_b(q, gx_n) \\
&\quad + sp_b(gx_n, fp) - p_b(gx_n, gx_n)] \\
&\quad + k_5p_b(gx_n, q) \\
&\preccurlyeq k_1p_b(gx_{n-1}, q) + k_2sp_b(gx_n, q) + sk_2p_b(q, gx_{n-1}) + k_3sp_b(fp, gx_n) \\
&\quad + sk_3p_b(gx_n, q) + sk_4p_b(gx_{n-1}, q) \\
&\quad + k_4s^2p_b(q, gx_n) + k_4s^2p_b(gx_n, fp) + k_5p_b(gx_n, q)
\end{aligned}$$

which implies that

$$\begin{aligned}(e - sk_3 - s^2k_4)p_b(gx_n, fp) \preccurlyeq{}& (k_1 + k_2s + k_4s)p_b(gx_{n-1}, q) + (k_2s + k_3s + k_4s^2 \\ &+ k_5)p_b(gx_n, q)\end{aligned} \tag{6.7}$$

For another thing

$$\begin{aligned}p_b(gx_n, fp) &= p_b(fx_{n-1}, fp) = p_b(fp, fx_{n-1}) \\
&\preccurlyeq k_1p_b(gp, gx_{n-1}) + k_2p_b(fp, gp) + k_3p_b(fx_{n-1}, gx_{n-1}) \\
&\quad + k_4p_b(gp, fx_{n-1}) + k_5p_b(fp, gx_{n-1}) \\
&\preccurlyeq k_1p_b(gp, gx_{n-1}) + k_2[sp_b(fp, gx_n) + sp_b(gx_n, gp) \\
&\quad - p_b(gx_n, gx_n)] + k_3[sp_b(gx_n, q) + sp_b(q, gx_{n-1}) - p_b(q, q)] \\
&\quad + k_4p_b(q, gx_n) + k_5[sp_b(fp, q) + sp_b(q, gx_{n-1}) \\
&\quad - p_b(q, q)] \\
&\preccurlyeq k_1p_b(gp, gx_{n-1}) + k_2[sp_b(fp, gx_n) + sp_b(gx_n, gp) \\
&\quad - p_b(gx_n, gx_n)] + k_3[sp_b(gx_n, q) + sp_b(q, gx_{n-1}) - p_b(q, q)] \\
&\quad + k_4p_b(gx_n, q) + k_5[sp_b(fp, q) + sp_b(q, gx_{n-1}) - p_b(q, q)] \\
&\preccurlyeq k_1p_b(gx_{n-1}, q) + k_2sp_b(fp, gx_n) + sk_2p_b(gx_n, q) + k_3sp_b(gx_n, q) \\
&\quad + k_3sp_b(q, gx_{n-1}) + k_4p_b(gx_n, q) + k_5sp_b(fp, q) + k_5sp_b(q, gx_{n-1}) \\
&\preccurlyeq k_1p_b(gx_{n-1}, q) + k_2sp_b(fp, gx_n) + sk_2p_b(gx_n, q) + k_3sp_b(gx_n, q) \\
&\quad + k_3sp_b(q, gx_{n-1}) + k_4p_b(gx_n, q) + k_5s[sp_b(fp, gx_n) + sp_b(gx_n, q) \\
&\quad - p_b(gx_n, gx_n)] + k_5sp_b(q, gx_{n-1}) \\
&\preccurlyeq k_1p_b(gx_{n-1}, q) + k_2sp_b(fp, gx_n) + sk_2p_b(gx_n, q) + k_3sp_b(gx_n, q) \\
&\quad + k_3sp_b(q, gx_{n-1}) + k_4p_b(gx_n, q) + k_5s^2p_b(fp, gx_n) \\
&\quad + k_5s^2p_b(gx_n, q) + k_5sp_b(q, gx_{n-1})\end{aligned}$$

which means that

$$\begin{aligned}(e - sk_2 - s^2k_5)p_b(gx_n, fp) \preccurlyeq{}& (k_1 + k_3s + k_5s)p_b(gx_{n-1}, q) \\ &+ (k_2s + k_3s + k_4 + s^2k_5)p_b(gx_n, q)\end{aligned} \tag{6.8}$$

Combine (6.7) and (6.8), it follows that

$$\begin{aligned}(2e - sk)p_b(gx_n, fp) &= (2e - k_2s - k_3s - k_4s^2 - s^2k_5)p_b(gx_n, fp) \\
&\preccurlyeq (2k_1 + k_3s + k_5s + k_2s + k_4s)p_b(gx_{n-1}, q) \\
&\quad + (k_2s + k_3s + k_4 + k_5 + sk)p_b(gx_n, q)\end{aligned} \tag{6.9}$$

Now $\rho(sk) = s\rho(k) \leq (s + 1)\rho(k) + 2s\rho(k_1) + s\rho(k_4) + s\rho(k_5) < 1 < 2$, thus by Lemma 5.5, it concludes that $(2e - sk)$ is invertible. As a result, it follows immediately from (6.9) that

$$\begin{aligned}p_b(gx_n, fp) \preccurlyeq{}& (2e - sk)^{-1}[(2k_1 + k_3s + k_5s + k_2s + k_4s)p_b(gx_{n-1}, q) \\ &+ (k_2s + k_3s + k_4 + k_5 + sk)p_b(gx_n, q)]\end{aligned}$$

Since $\left\{p_b(gx_n, q)\right\}$ and $\left\{p_b(gx_{n-1}, q)\right\}$ are c-sequences, then by Lemma 5.4, we acquire that $\{p_b(gx_n, fp)\}$ is a c-sequence, thus $gx_n \to fp$ $(n \to +\infty)$. Hence $fp = gp = q$. In the following we shall prove f and g have a unique point of coincidence.

If there exists $p^{'} \neq p$ such that $fp^{'} = gp^{'} = u$. Then we get

$$\begin{aligned}
p_b(gp^{'}, gp) &= p_b(fp^{'}, fp) \\
&\preccurlyeq k_1 p_b(gp^{'}, gp) + k_2 p_b(fp^{'}, gp^{'}) + k_3 p_b(fp, gp) + k_4 p_b(gp^{'}, fp) \\
&\quad + k_5 p_b(fp^{'}, gp) \\
&\preccurlyeq k_1 p_b(gp^{'}, gp) + k_2 p_b(gp^{'}, gp^{'}) + k_3 p_b(gp, gp) + k_4 p_b(gp^{'}, fp) \\
&\quad + k_5 p_b(fp^{'}, gp) \\
&\preccurlyeq k_1 p_b(gp^{'}, gp) + k_2 p_b(gp, gp^{'}) + k_3 p_b(gp, gp^{'}) + k_4 p_b(gp^{'}, fp) \\
&\quad + k_5 p_b(fp^{'}, gp) \quad [by (p_b 2)] \\
&= (k_1 + k_2 + k_3 + k_4 + k_5) p_b(gp^{'}, gp)
\end{aligned}$$

Note the facts that $(s+1)\rho(k) + 2s\rho(k_1) + s\rho(k_4) + s\rho(k_5) < 1$ and $k_1 + k_2 + k_3 + k_4 + k_5 \leq (s+1)k + 2sk_1 + sk_4 + sk_5$, then by Lemma 5.7, we speculate that $gp^{'} = gp$. Finally, if (f, g) is weakly compatible, then by using Lemma 2.11, we claim that f and g have a unique common fixed point.

Corollary 6.2. *Let (X, p_b) be a θ-complete partial cone b-metric space over Banach algebra A with the coefficient $s \geq 1$ and P be a solid cone in A. Let $k \in P$ be a generalized Lipschitz constant with $\rho(k) < \frac{1}{2s}$. Suppose that the mappings $f, g : X \to X$ satisfy that*

$$p_b(fx, fy) \preccurlyeq k p_b(gx, gy)$$

for all $x, y \in X$. If the range of g contains the range of f and $g(X)$ is a complete subspace, then f and g have a unique point of coincidence in X. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point.

Proof. Choose $k_1 = k$ and $k_2 = k_3 = k_4 = k_5 = \theta$ in Theorem 6.1, the proof is valid.

Corollary 6.3. *Let (X, p_b) be a θ-complete partial cone b-metric space over Banach algebra A with the coefficient $s \geq 1$ and P be a solid cone in A. Let $k \in P$ be a generalized Lipschitz constant with $\rho(k) < \frac{1}{2(s+1)}$. Suppose that the mappings $f, g : X \to X$ satisfy that*

$$p_b(fx, fy) \preccurlyeq k[p_b(fx, gx) + p_b(fy, gy)]$$

for all $x, y \in X$. If the range of g contains the range of f and $g(X)$ is a complete subspace, then f and g have a unique point of coincidence in X. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point.

Proof. Putting $k_1 = k_4 = k_5 = \theta$ and $k_2 = k_3 = k$ in Theorem 6.1, we complete the proof.

Corollary 6.4. *Let (X, p_b) be a θ-complete partial cone b-metric space over Banach algebra A with the coefficient $s \geq 1$ and P be a solid cone in A. Let $k \in P$ be a generalized Lipschitz constant with $\rho(k) < \frac{1}{2s(s+2)}$. Suppose that the mappings $f, g : X \to X$ satisfy that*

$$p_b(fx, fy) \preccurlyeq k[p_b(fx, gy) + p_b(fy, gx)]$$

for all $x, y \in X$. If the range of g contains the range of f and $g(X)$ is a complete subspace, then f and g have a unique point of coincidence in X. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point.

Proof. Set $k_1 = k_2 = k_3 = \theta$ and $k_4 = k_5 = k$ in Theorem 6.1, the claim holds.
Finally, we add an example on non-normal θ-complete partial cone b-metric over Banach algebra that demonstrates Theorem 6.1.

Example 6.5. *Let A and P be the same ones as those in Example 3.6. We define a mapping $p_b : X \times X \to A$ by*

$$p_b(x, y)(t) = \left(\max\{x, y\}\right)^2 e^t$$

We make a conclusion that (X, p_b) is a θ-complete partial cone b-metric space over Banach algebra A with the coefficient $s = 2$. Now define the mappings $f, g : X \to X$ by

$$f(x) = \frac{x}{4} - \frac{x^2}{4}, \quad g(x) = \frac{x}{2}$$

$$\begin{aligned} p_b(fx, fy)(t) &= \left(\max\left\{\frac{x}{4} - \frac{x^2}{4}, \frac{y}{4} - \frac{y^2}{4}\right\}\right)^2 e^t \preceq \left(\max\left\{\frac{x}{4}, \frac{y}{4}\right\}\right)^2 e^t \\ &= \frac{1}{4}\left(\max\left\{x, y\right\}\right)^2 e^t \preceq \frac{1}{4} p_b(gx, gy)(t) \end{aligned}$$

Choose $k_1 = \frac{1}{4}$, $k_2 = k_3 = k_4 = k_5 = \theta$. Note that f and g commute at the coincidence point $x = 0$ of them, that is to say, the pair (f, g) is weakly compatible, it is easy to see that all the conditions of Theorem 6.1 holds trivially good and 0 is the unique common fixed point of f and g.

7. Application to integral equation

We start this section by giving the system of nonlinear integral equations. Furthermore, as an application of our results we establish the existence and uniqueness of solution to a class of system of nonlinear integral equations.

We will consider the following system of integral equations.

$$\begin{cases} x(t) = \int_a^t f\big(s, x(s)\big)ds \\ x(t) = \int_a^t x(s)ds \end{cases} \tag{7.1}$$

where $t \in [a, b]$ and $f : [a, b] \times R \to R$ is a continuous function.
Now we discuss the existence of a solution for the integral equations.

Theorem 7.1. *Let $L_p[a, b] = \left\{x = x(t) : \int_a^b |x(t)|^p < \infty\right\}$ $(0 < p < 1)$. For (7.1), assume that the following hypotheses hold:*

1. *If $f\big(s, x(s)\big) = x(s)$ for all $s \in [a, b]$, then*
$$f\left(s, \int_a^b x(w)dw\right) = \int_a^b f\big(w, x(w)\big)dw$$
for all $s \in [a, b]$.

2. *If there exists a constant $M \in (0, 2^{1-\frac{1}{p}})$ such that the partial derivative f_y of f with respect to y exists and $|f_y(x, y)| \le M$ for all the pairs $(x, y) \in [a, b] \times R$.*

Then the integral equation (7.1) has a unique common solution in $L_p[a, b]$.

Proof. Let $A = R^2$ with the norm $\|u_1, u_2\| = |u_1| + |u_2|$ and the multiplication by

$$uv = \big((u_1, u_2)(v_1, v_2)\big) = \big(u_1.v_1, u_1.v_2 + u_2.v_1\big).$$

Let $P = \big\{u = (u_1, u_2) \in A : u_1, u_2 \ge 0\big\}$. It is clear that P is a normal cone and A is a Banach algebra with a unit $e = (1, 0)$. Let $X = L_p[a, b]$. We endow X with the partial cone b-metric

$$p_b(x, y)(t) = \begin{cases} \left(\int_a^b \left\{|x(t)|^p\right\}^{\frac{1}{p}}, \ \int_a^b \left\{|x(t)|^p\right\}^{\frac{1}{p}}\right)e^t, & \text{when } x = y \\ \left(\int_a^b \left\{|x(t) + y(t)|^p\right\}^{\frac{1}{p}}, \ \int_a^b \left\{|x(t) + y(t)|^p\right\}^{\frac{1}{p}}\right)e^t, & \text{when } x \ne y \end{cases}$$

for all $x, y \in X$. It is clear that (X, p_b) is a θ-complete partial cone b-metric space over Banach algebra A with the coefficient $s = 2^{1-\frac{1}{p}}$ as defined in Example 3.6. Define the mappings $S, T : X \to X$ by

$$Sx(t) = \int_a^t x(s)ds, \ Tx(t) = \int_a^t f\Big(s, x(s)\Big)ds$$

for all $t \in [a, b]$. Then the existence of a solution to (7.1) is equivalent to the existence of common fixed point of S and T. Indeed, when $x \neq y$

$$\begin{aligned}
p_b(Tx, Ty) &= \Bigg(\Big\{\int_a^b \mid \int_a^t f\Big(s, x(s)\Big)ds + \int_a^t f\Big(s, y(s)\Big)ds \mid^p dt\Big\}^{\frac{1}{p}}, \\
&\qquad \Big\{\int_a^b \mid \int_a^t f\Big(s, x(s)\Big)ds + \int_a^t f\Big(s, y(s)\Big)ds\mid^p dt\Big\}^{\frac{1}{p}}\Bigg)e^t \\
&= \Bigg(\Big\{\int_a^b \mid \int_a^t \Big[f\Big(s, x(s)\Big) + f\Big(s, y(s)\Big)\Big]ds \mid^p dt\Big\}^{\frac{1}{p}}, \\
&\qquad \Big\{\int_a^b \mid \int_a^t \Big[f\Big(s, x(s)\Big) + f\Big(s, y(s)\Big)\Big]ds\mid^p dt\Big\}^{\frac{1}{p}}\Bigg)e^t \\
&\preccurlyeq \Bigg(M\Big\{\int_a^b \mid \int_a^t [x(s) + y(s)]ds \mid^p dt\Big\}^{\frac{1}{p}}, \\
&\qquad M\Big\{\int_a^b \mid \int_a^t [x(s) + y(s)]ds\mid^p dt\Big\}^{\frac{1}{p}}\Bigg)e^t \\
&= (M, 0)\Bigg(\Big\{\int_a^b \mid Sx(t) + Sy(t) \mid^p dt\Big\}^{\frac{1}{p}}, \\
&\qquad \Big\{\int_a^b \mid Sx(t) + Sy(t) \mid^p dt\Big\}^{\frac{1}{p}}\Bigg)e^t \\
&= (M, 0)\ p_b(Sx, Sy)(t).
\end{aligned}$$

Now when $x = y$

$$\begin{aligned}
p_b(Tx, Ty)(t) &= \Bigg(\Big\{\int_a^b \mid \int_a^t f\Big(s, x(s)\Big)ds \mid^p \Big\}^{\frac{1}{p}}, \Big\{\int_a^b \mid \int_a^t f\Big(s, x(s)\Big)ds\mid^p\Big\}^{\frac{1}{p}}\Bigg)e^t \\
&\preccurlyeq \Bigg(\Big\{M\int_a^b \mid \int_a^t x(s)ds \mid^p \Big\}^{\frac{1}{p}}, \Big\{M\int_a^b \mid \int_a^t x(s)ds\mid^p\Big\}^{\frac{1}{p}}\Bigg)e^t \\
&= (M, 0)\ p_b(Sx, Sx)(t) \\
&= (M, 0)\ p_b(Sx, Sy)(t).
\end{aligned}$$

Because $\|(M, 0)^n\|^{\frac{1}{n}} = \|(M^n, 0)\|^{\frac{1}{n}} \to M < 2^{1-\frac{1}{p}}$ $(n \to +\infty)$,
which means $\rho\Big((M, 0)\Big) < 2^{1-\frac{1}{p}}$. Now choose $k_1 = (M, 0)$ and $k_2 = k_3 = k_4 = k_5 = \theta$.
Note that by (i), it is easy to see that the mappings S and T are weakly compatible. Therefore, all conditions of Theorem 6.1 are satisfied. As a result, S and T have a unique common fixed point $x^* \in X$. That is, x^* is the unique common solution of the system of integral equation (7.1) .
We notice that the above mentioned application of fixed point theorem in cone b-metric space over Banach algebra was given by [18].

8. Conclusion

In this paper, we introduced the concept of partial cone b-metric space over Banach algebras which are not equivalent to metric spaces since all the coefficients are vectors and the multiplications are vector multiplications. Furthermore, we define generalized Lipschitz mapping in the new space. We have established common fixed point results for such maps. In addition as an application, we study the existence of solution to a class of system of integral equations. Our results may be the vision for other authors to extend and improve several results in such space.

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