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Asymptotic Regularity, Fixed Points and Successive Approximations

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Abstract. Let (M, d) be a metric space. In this paper we survey some of the most relevant results which relate the three concepts involved in the title: a) the asymptotic regularity; b) the existence (and uniqueness) of fixed points and c) the convergence of the sequence of successive approximations to the fixed point(s), for a given operator $f : M \to M$ or for two operators $f, g : M \to M$ connected to each other in some sense.

1. Introduction

The concept of asymptotic regularity was introduced formally in 1966 by Browder and Petryshyn ([28], Definition 1, page 572) in connection with the study of fixed points of nonexpansive mappings. We present in the following the original definition of Browder and Petryshyn ([28], Definition 1, page 572): a (possibly) nonlinear mapping T of a Banach space X into itself is said to be *asymptotically regular* if for each x in X, $T^{n+1}x - T^n x \to 0$ strongly in X as $n \to \infty$.

This property was used in 1955 by Krasnosel'skiĭ [99], see also [100], to prove that if K is a compact convex subset of a uniformly convex Banach space and if $T : K \to K$ is nonexpansive, then, for any $x_0 \in K$, the sequence

$$x_{n+1} = \frac{1}{2}(x_n + Tx_n), \; n \geq 0, \tag{1}$$

converges to to a fixed point of T.

In proving his result, Krasnosel'skiĭ used the fact that, if T is nonexpansive, then the averaged mapping involved in (1), that is, $\frac{1}{2}I + \frac{1}{2}T$, is asymptotically regular. For the general averaged mapping

$$T_\lambda := (1 - \lambda)I + \lambda T, \; \lambda \in (0, 1),$$

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and in the setting of a Hilbert space, the corresponding result has been stated by Browder and Petryshyn ([29], Corollary to Theorem 5).

Ishikawa [87] proved in 1976 the following general result with no restriction on the geometry of the Banach space involved.

Theorem 1.1. *If C is a nonempty bounded closed convex subset of a Banach space X and $T : C \to C$ is nonexpansive, then the mapping T_λ is asymptotically regular, for each $\lambda \in (0, 1)$.*

Other important results on this topic are due to Edelstein and O'Brien [58], who proved in 1978 that T_λ is *uniformly* asymptotically regular over $x \in C$, and to Goebel and Kirk [66], who proved that the convergence is uniform with respect to all nonexpansive mappings from C into C.

For other examples of asymptotically regular mappings in a locally convex space, see the result of Anzai and Ishikawa [5]. We end this list by mentioning a very interesting result which makes use of the concept of asymptotic regularity in a concrete context [49].

From the large list of papers which attest the impact of the asymptotic regularity property in the fixed point theory of operators, mainly in Hilbert and Banach spaces, we mention the following [32], [33], [135], [166], [58], [19], [112], [137], [89], [130], [83], [118], [157], [129], [126], [156], [92], [9], [141], [101], [67], [56], [119], [53], [47], [60], [61], [17], [72], [10].

Let now (M, d) be a metric space and let $f, g : M \to M$ be two operators. A sequence $\{x_n\}_{n\in\mathbb{N}}$ in M is called *asymptotically regular* if,

$$d(x_n, x_{n+1}) \to 0 \text{ as } n \to \infty.$$

Clearly, any convergent sequence $\{x_n\}_{n\in\mathbb{N}}$ is asymptotically regular but the converse is not more true, as shown by the sequence of partial sums of the harmonic series, $x_n = \sum_{i=1}^{n} \frac{1}{i}$, $n \geq 1$. This example also illustrate a fundamental difference between the convergence property of sequences and the asymptotic regularity of sequences: the fact that $\{x_n\}_{n\in\mathbb{N}}$ is asymptotically regular does not imply that a subsequence of it is asymptotically regular as well.

The sequence $\{x_n\}_{n\in\mathbb{N}}$ is called *f-asymptotically regular* if,

$$d(x_n, f(x_n)) \to 0 \text{ as } n \to \infty.$$

The operator g is called *asymptotically regular* on M if the sequence of its iterates, $\{g^n(x)\}_{n\in\mathbb{N}}$, is asymptotically regular for all $x \in M$, that is,

$$d(g^n(x), g^{n+1}(x)) \to 0 \text{ as } n \to \infty,$$

for all $x \in M$. Similarly, the operator g is called *f-asymptotically regular* on M if the sequence of its iterates, $\{g^n(x)\}_{n\in\mathbb{N}}$, is f-asymptotically regular for all $x \in M$, that is,

$$d(g^n(x), f(g^n(x))) \to 0 \text{ as } n \to \infty,$$

for all $x \in M$.

The various hypostases in which asymptotic regularity appears in the fixed point theory are covered by the following problems formulated for a metric space (M, d) and an operator $f : M \to M$.

Problem 1. *Give metric conditions on f which imply that f is asymptotically regular.*

Problem 2. *In which conditions on M and f, the asymptotically regular property implies that the fixed point set of f, F_f, is nonempty?*

Problem 3. *Let f be asymptotically regular with $F_f \neq \emptyset$. In which conditions we have that*

$$f^n(x) \to x^*(x) \in F_f \text{ as } n \to \infty, \forall x \in M,$$

i.e., f is a weakly Picard operator ?

Let now $f, g : M \to M$ be two operators with $F_f = F_g$.

Problem 4. *Give conditions on f and g which imply that g is f-asymptotically regular.*

Problem 5. *In which conditions on f and g, the operator g is asymptotically regular ?*

Problem 6. *Let g be asymptotically regular. In which conditions on f and g, we have that $F_f \neq \emptyset$?*

Problem 7. *If the pair (f, g) is a solution of Problem 6, in which conditions g is is a weakly Picard operator ?*

The aim of this paper is to survey what is known on these problems and to give some new related results.

2. Preliminaries

2.1. Notations

Throughout this paper we shall use the following notations. Let (M, d) be a metric space and $\mathbb{K}$ be $\mathbb{R}$ or $\mathbb{C}$. Denote:

- $(E, +, \mathbb{K}, \tau)$:= a linear topological space;
- $(E, +, \mathbb{K}, \|\cdot\|)$:= a linear normed space;
- $(B, +, \mathbb{K}, \|\cdot\|)$:= a Banach space;
- $(H, +, \mathbb{K}, \langle\cdot,\cdot\rangle)$:= a Hilbert space.

Let (M, d) be a metric space. Denote:

- $\mathcal{P}(M)$:=$\{Y : Y \subset M\}$;
- $P(M)$:=$\{Y \in \mathcal{P}(M),\ Y \neq \emptyset\}$;
- $P_b(M)$:=$\{Y \in P(M) : Y \text{ is bounded}\}$;
- $P_{cl}(M)$:=$\{Y \in P(M) : Y \text{ is closed}\}$;
- $P_{cp}(M)$:=$\{Y \in P(M) : Y \text{ is compact}\}$;
- $P_{b,cl}(M)$:=$P_{cl}(M) \cap P_b(M)$;

Let $(E, +, \mathbb{K}, \tau)$ be a linear topological space. Denote:

- $P_{cv}(E)$:=$\{Y \in P(E) : Y \text{ is convex}\}$;
- $P_{cv,cp}(E)$:=$P_{cv}(E) \cap P_{cp}(E)$;
- $P_{cv,cl}(E)$:=$P_{cv}(E) \cap P_{cl}(E)$;

Let $(E, +, \mathbb{K}, \|\cdot\|)$ be a linear normed space. Denote:

- $P_{cv,b}(E)$:=$P_{cv}(E) \cap P_b(E)$;

Let (M, d) be a metric space and $f : M \to M$. Denote

- $O_f(x)$:=$\{x, f(x), \dots, f^n(x), \dots\}$;
- $\omega_f(x)$:=the set of limit (cluster) points $O_f(x)$;
- $\delta(A) := \sup\{d(x, y) : x, y \in A\}$, the diameter of A.

2.2. Some classes of operators on a metric space

Let (M, d) be a metric space and $f : M \to M$ be an operator. Then:

(1) f is an *l-contraction* if $0 < l < 1$ and
$$d(f(x), f(y)) \le l d(x, y),\ \forall\, x, y \in M;$$

(2) f is a *contractive* operator if,
$$d(f(x), f(y)) < d(x, y),\ \forall\, x, y \in M,\ x \neq y;$$

(3) f is a *graphic contraction* if $0 < l < 1$ and
$$d(f^2(x), f(x)) \le l d(x, f(x)),\ \forall\, x \in X;$$

(4) f is *nonexpansive* if,
$$d(f(x), f(y)) \le d(x, y),\ \forall\, x, y \in M;$$

(5) f is *quasinonexpansive* (see [162], [56], [126]) if $F_f \neq \emptyset$ and
$$d(f(x), x^*) \le d(x, x^*),\ \forall\, x \in M,\ \forall\, x^* \in F_f;$$

(6) f is *quasicontractive* if $F_f \neq \emptyset$ and
$$d(f(x), x^*) < d(x, x^*),\ \forall\, x \in M \setminus F_f,\ x^* \in F_f;$$

(7) f is *K-demicontractive* (see [111], [83], [43], [112], ...) if $K < 1$, $F_f \neq \emptyset$ and
$$(d(f(x), x^*))^2 \le (d(x, x^*))^2 + K(d(x, f(x)))^2,\ \forall\, x \in M,\ \forall\, x^* \in F_f;$$

(8) f is *demicompact* (see [121], [28], [15]) if:
$\{x_n\}_{n\in\mathbb{N}} \subset M$ bounded, $d(x_n, f(x_n)) \to 0$ as $n \to \infty \Rightarrow \exists$ a subsequence $\{x_{n_i}\}_{i\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ which is convergent;

(9) the fixed point problem for f is *well posed* if $F_f = \{x^*\}$ and
$\{x_n\}_{n\in\mathbb{N}} \subset M$, $d(x_n, f(x_n)) \to 0$ as $n \to \infty \Rightarrow x_n \to x^*$ as $n \to \infty$;

(10) the fixed point problem for f is *well posed in the generalized sense* if the following implication holds:
$\{x_n\}_{n\in\mathbb{N}} \subset M$, $d(x_n, f(x_n)) \to 0$ as $n \to \infty \Rightarrow \exists\, \{x_{n_i}\}_{i\in\mathbb{N}}$ a subsequence of $\{x_n\}_{n\in\mathbb{N}}$, which converges to a fixed point of f.

2.3. Measure of noncompactness and condensing operators

Let (M, d) be a metric space. By definition (see [132], [137],...) a functional $\alpha_{DP} : P_b(X) \to \mathbb{R}_+$ is called a *Daneš-Pasicki measure of noncompactness* if

(i) $\alpha_{DP}(Y) = 0 \Rightarrow \overline{Y} \in P_{cp}(X), \forall\, Y \in P_b(X)$;
(ii) $Y_1, Y_2 \in P_b(X), Y_1 \subset Y_2 \Rightarrow \alpha_{DP}(Y_1) \le \alpha_{DP}(Y_2)$;
(iii) $Y \in P_b(X), x \in X \Rightarrow \alpha_{DP}(Y \cup \{x\}) = \alpha_{DP}(Y)$.

For example, the Kuratowski's measure of noncompactness, α_K, and the Hausdorff's measure of noncompactness, α_H, are both Daneš-Pasicki measures of noncompactness.

Let (M, d) be a complete metric space. An operator $f : M \to M$ is called α_{DP}*-condensing* iff

(i) $A \in P_b(M) \Rightarrow f(A) \in P_b(M)$;
(ii) $A \in P_b(M), f(A) \subset A, \alpha_{DP}(A) \neq 0 \Rightarrow \alpha_{DP}(f(A)) < \alpha_{DP}(A)$.

The operator $f : M \to M$ is called *strong* α_{DP}*-condensing* iff

(i) $A \in P_b(M) \Rightarrow f(A) \in P_b(M)$;
(ii) $A \in P_b(M), \alpha_{DP}(A) \neq 0 \Rightarrow \alpha_{DP}(f(A)) < \alpha_{DP}(A)$.

For the above notions, see [94].

2.4. Equivalent fixed point equations: Examples

Example 2.1. (see [30], [132], [8], [22],...) *Let $X \subset M$, $\rho : M \to X$ be a set retraction (i.e., $\rho|_X = 1_X$) and $f : X \to M$ a nonself operator. By definition, f is retractible with respect to the retraction ρ iff*

$$F_f = F_{\rho \circ f},$$

i.e., the fixed point equations $x = f(x)$ and $x = (\rho \circ f)(x)$ are equivalent.

Remark 2.2. *We note that $\rho \circ f : X \to X$ is a self operator and that, if f is retractible with respect to ρ, then the fixed point equations*

$$x = f(x), \quad x = (\rho \circ f)(x)$$

are equivalent.

Example 2.3. (see [120]) *Let (M, d) be a metric space, $X \in P(M)$ and $f, g : X \to M$ be two operators. We suppose that*

$$d(f(x), g(x)) \leq d(x, g(x)), \forall x \in X,$$

for some $0 < l < 1$. Then the fixed point equations $x = f(x)$ and $x = g(x)$ are equivalent.

Example 2.4. *Let $(E, +, \mathbb{K})$ be a linear space, $X \subset E$ a linear subspace of E and $f : X \to E$ an operator. For each $\lambda \in \mathbb{K} \setminus \{0\}$ we consider the operator $f_\lambda : X \to E$ defined by*

$$f_\lambda(x) := (1 - \lambda)x + \lambda f(x),\ x \in X.$$

Then the fixed point equations $x = f(x)$ and $x = f_\lambda(x)$ are equivalent.

Example 2.5. (see [156], [157], [105], [15], [132],...) *Let (M, d) be a metric space with a convexity structure defined by the operator $W : M \times M \times [0, 1] \to M$ with the following properties:*
(a) for all $x, y \in M$ and any $\lambda \in [0, 1]$,

$$d(u, W(x, y; \lambda)) \leq \lambda d(u, x) + (1 - \lambda)d(u, y), \forall u \in M.$$

(b) $\lambda \in (0, 1)$, $x, y \in M$ and $W(x, y, \lambda) = x \Rightarrow y = x$.
Let $f : M \to M$ be an operator. For $\lambda \in (0, 1)$ we define the operator $f_{W,\lambda} : M \to M$ given by

$$f_{W,\lambda}(x) := W(x, f(x), \lambda),\ \forall x \in M.$$

Then the fixed point equations $x = f(x)$ and $x = f_{W,\lambda}(x)$ are equivalent.

Example 2.6. (Rus [133]*) Let M be a nonempty set and $G : M \times M \to M$ be an operator which satisfies:*
(A_1) $G(x, x) = x, \forall x \in M$;
(A_2) $x, y \in M$, $G(x, y) = x \Rightarrow y = x$.
Let $f : M \to M$ be a given operator and consider the operator $f_G : M \to M$ defined by

$$f_G(x) := G(x, f(x)),\ \forall x \in M.$$

Then the fixed point equations $x = f(x)$ and $x = f_G(x)$ are equivalent.

Basic problem of the equivalent fixed point equations

Let (M, d) be a metric space and $f : M \to M$ be an operator with $F_f \neq \emptyset$. The problem is to find an operator $g : M \to M$ such that:
(1) $F_f = F_g$;
(2) g is a weakly Picard operator.

A direct way to investigate this problem is to study the Problems 1-7 formulated in Introduction.

3. Problem 1

We start our considerations about solving Problem 1 with a general result which illustrates the relevance of asymptotic regularity in the theory of weakly Picard operators.

Theorem 3.1. (Theorem of equivalent statements, Rus [128], [130]) *Let X be a nonempty set and $g : X \to X$ be an operator. The following statements are equivalent:*

(*i*) $F_g = F_{g^n} \neq \emptyset$;

(*ii*) *there exists a metric d on X with respect to which g is WPO;*

(*iii*) *there exists a complete metric on X with respect to which g is a continuous graphic contraction;*

(*iv*) $F_g \neq \emptyset$ *and there exists a metric d on X with respect to which g is asymptotically regular.*

Another result in the same direction is the following one.

Theorem 3.2. (Belluce and Kirk, [14]) *Let (M, d) be a complete metric space and $g : M \to M$ be a nonexpansive operator. Then the following statements are equivalent:*

(1) *g is asymptotically regular on M;*

(2) *g has diminishing orbital diameters on M, i.e.,*

$$x \in M, \delta(O_g(x)) > 0 \Rightarrow \lim_{n\to\infty} \delta(O_g(g^n(x))) < \delta(O_g(x)).$$

Let $f \in C([a, b] \times \mathbb{R}^m, \mathbb{R}^m)$ and consider the following Cauchy problem:

$$y' = f(x, y),\ y(a) = y_0 \tag{2}$$

and the corresponding sequence of successive approximations associated to it,

$$y_{n+1}(x) := y_0 + \int_a^x f(s, y_n(s))ds, n = 0, 1, \ldots \tag{3}$$

The following result was given by Dieudonné in 1945 [49], see also [21].

Theorem 3.3. *We suppose that the Cauchy problem (2) has a unique solution. Then there exists $h \in]0, b-a[$ such that the sequence of successive approximations (3) converges to the unique solution of the Cauchy problem (2) uniformly on $[a, a + h]$, if and only if the sequence $\{y_n\}_{n\in\mathbb{N}}$ is uniformly asymptotically regular on $[a, a + h]$.*

Remark 3.4. *Let (M, d) be a metric space. Remind, see Rus [130], that an operator $g : M \to M$ is said to be a Picard operator if*

(1) $F_f = \{x^*\}$;

(2) The sequence of successive approximations associated to g, $\{g^n(x)\}_{n\in\mathbb{N}}$, converges to x^* as $n \to \infty$, for any $x \in M$.

So, Theorem 3.3 provides a characterization of the asymptotic regularity by means of the concept of Picard operator, see [21], [127], for more details.

The essence of Theorem 3.3 can be captured in the following context, see Hillam [84].

Theorem 3.5. (Hillam, [84]) *Let T be a continuous map of $[0, 1]$ into $[0, 1]$. The sequence $\{T^n x\}$ of successive approximations of T converges to a fixed point of T if and only if $\{T^n x\}$ is asymptotically regular.*

Remark 3.6. *Theorem 3.5 cannot be extended beyond the one dimensional case, as shown by Smart [153], who constructed an example of a continuous mapping T of the closed unit disc in the Euclidean plane such that the origin and points on the unit circle are fixed points of T and, for any other x, one has $d(T^n x, T^{n+1}x) \to 0$, but $\{T^n x\}$ is not convergent.*

This fact indicate how important is to study the connection between asymptotic regularity and convergence of a sequence in a more general setting.

We present in the following a result based on a metric condition which implies the asymptotic regularity in a metric space.

Theorem 3.7. (Rus [138]) *Let (M, d) be a complete metric space and $g : M \to M$ be an operator which satisfies the (α, β)-displacement condition, i.e., there exist $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ and $\beta : M \to \mathbb{R}_+$ such that*

(1) *$t_n \in \mathbb{R}_+$ and $\alpha(t_n) \to 0$ as $n \to \infty$ implies $t_n \to 0$ as $n \to \infty$;*

(2) *$\alpha(d(x, g(x))) \leq \beta(x) - \beta(g(x))$, $\forall\, x \in M$.*

Then g is asymptotically regular.

For relevant examples of operators which satisfy the (α, β)-displacement condition, see Rus [138].

As mentioned in Introduction, the asymptotical regularity property is related to many important results in the fixed point theory over metric spaces. All Banach contractions are (continuous) asymptotically regular operators. The Kannan operators and, in general, all almost contractions are important examples of discontinuous asymptotically regular operators, as shown by the next result.

Theorem 3.8. (Berinde [15], Theorem 2.11) *Let (M, d) be a metric space and $g : M \to M$ be a (θ, L)-almost contraction, i.e., an operator satisfying the condition*

$$d(g(x), g(y)) \leq \theta d(x, y) + L d(y, g(x)), \forall x, y \in M,$$

where $0 < \theta < 1$ and $L \geq 0$ are constants.

Then g is asymptotically regular.

Proof. Let $x_0 \in M$ be given and denote $x_n := g^n(x_0)$, $n \geq 1$. Then, by the above inequality we obtain

$$d(x_n, x_{n+1}) \leq \theta^n d(x_0, x_1), n \geq 1,$$

which proves the assertion. □

4. Problem 2

The asymptotic regularity of an operator T does not guarantee in general neither the existence of a fixed point of T nor the convergence of the sequence $\{T^n x\}$ of successive approximations of T to a fixed point of T. Some additional conditions are needed.

There exist some simple results in which asymptotic regularity implies the existence of a fixed point, like the following ([33], [58], [122], [81], [138],...). For some other related results, see also [26], [80], [82], [142].

Lemma 4.1. *Let (M, d) be a metric space and $g : M \to M$ be a continuous and asymptotically regular operator. Then $\omega_g(x) \subset F_g$. So, if $\omega_g(x) \neq \emptyset$, then $F_g \neq \emptyset$.*

Lemma 4.2. *Let (M, d) be a compact metric space and $g : M \to M$ be a continuous and asymptotically regular operator. Then $F_g \neq \emptyset$.*

Lemma 4.3. *Let (M, d) be a metric space and $g : M \to M$ be a continuous and asymptotically regular operator. If $\overline{g(M)} \in P_{cp}(M)$, then $F_g \neq \emptyset$.*

Theorem 4.4. *Let (M, d) be a bounded complete metric space and α_{DP} be a Daneš-Pasicki measure of noncompactness on M. If $g : M \to M$ is continuous, asymptotically regular and α_{DP}-condensing, then $F_g \neq \emptyset$.*

Proof. For $x \in M$, we have that

$$O_g(g(x)) = g(O_g(x)) \subset O_g(x)$$

and

$$\alpha_{DP}(g(O_g(x))) = \alpha_{DP}(O_g(x)).$$

Since g is α_{DP}-condensing, it follows that $\overline{O_g(x)}$ is compact.

This implies that there exists a subsequence $\{g^{n_i}(x)\}$ of $\{g^n(x)\}$ such that $g^{n_i}(x) \to x^*(x)$ as $i \to \infty$. From the continuity of g it follows that

$$g^{n_i+1}(x) \to g(x^*(x)) \text{ as } i \to \infty$$

and by the asymptotic regularity of g we get $g(x^*(x)) = x^*(x)$. □

Theorem 4.5. ([138], [121], [103]) *Let (M, d) be a metric space and $g : M \to M$ an operator. We suppose that*

(1) g is asymptotically regular;

(2) the fixed point problem for g is well posed in the generalized sense.

Then $F_g \neq \emptyset$.

Proof. From (1) we have that

$$d(g^n(x), g^{n+1}(x)) = d(g^n(x), g(g^n(x))) \to 0 \text{ as } n \to \infty.$$

By (2), there exists $\{g^{n_i}(x)\}$ such that $g^{n_i}(x) \to x^*(x) \in F_g$.

Hence $F_g \neq \emptyset$. □

Let (M, d) be a metric space and $g : M \to M$ be a Lipschitzian operator. Denote

$$\|g\|_{Lip} := \inf\{L > 0 | d(g(x), g(y)) \leq L d(x, y), \forall x, y \in M\}.$$

By the *Lipschitz constant of a metric space M* one understand the number

$$k(M) := \sup\{b > 0 | \exists a > 1, \forall x, y \in M, \forall r > 0$$

$$\Big[d(x, y) > r \Rightarrow \exists z \in M : \overline{B}(x, br) \cap \overline{B}(y, br) \subset \overline{B}(z, br)\Big]\}.$$

As an exotic result we mention the following one obtained by Górnicki [72].

Theorem 4.6. ([72]) *Let (M, d) be a complete metric space and $g : M \to M$ be an operator. If g is asymptotically regular,*

$$\liminf_{n \to \infty} \|g^n\|_{Lip} < k(M)$$

and, for some $x \in M$, $O_g(x)$ is bounded, then $F_g \neq \emptyset$.

We end this section with a result concerning operators which are not necessarily continuous, obtained by Guay and Singh [82], see also [44], [125], [142].

Theorem 4.7. (Guay and Singh [82]) *Let (M, d) be a complete metric space and $g : M \to M$ an operator satisfying the contractive condition*

$$d(g(x), g(y)) \leq a d(x, y) + b[d(x, g(x) + d(y, g(y))] + c[d(x, g(y)) + d(y, g(x))],$$

for all $x, y \in M$, where $0 \leq a, c$; $a + 2c < 1$ and $b + c < 1$.

If g is asymptotically regular at some point of M, then g has a unique fixed point.

Remark 4.8. *Note that the asymptotic regularity of T cannot be dropped in Theorem 4.7, as shown by the next example.*

Example 4.9. (Guay and Singh [82]) *Let $M = \{0\} \cup [1, \infty)$ be endowed with the usual norm and $g : M \to M$ be given by $g(x) = 0$, if $x \neq 0$, and $g(0) = 1$. Then g satisfies the contractive condition in Theorem 4.7 but is nowhere asymptotically regular in M. Clearly, T has no fixed points.*

For other references on Problem 2, see [33], [43], [105], [115], [122], [127], [140], [58], [168], [103], [120], [52], [60], [14], [10], [138], [74]-[79], [26],...

5. Problem 3

This problem is concerned with finding conditions in which an asymptotically regular operator $g : M \to M$ is a weakly Picard operator (WPO, for short).

In the case $F_g = \{x^*\}$, i.e., when g is a *Picard operator*, the problem was studied in [127] and [21], see also [24] and [80].

In order to present our basic results for this problem, we introduce a new concept.

Definition 5.1. *Let (M, d) be a metric space. An operator $g : M \to M$ is called orbitally quasinonexpansive iff the following implication holds:*

$$x \in M, g^{n_i}(x) \to x^*(x) \in F_g \Rightarrow d(g(u), g(x^*)) \leq d(u, x^*), \forall u \in O_g(x).$$

Example 5.2. *Any Banach contraction is a continuous orbitally quasinonexpansive operator.*

Example 5.3. *Any Kannan contraction is, in general, a discontinuous orbitally quasinonexpansive operator.*

Theorem 5.4. *Let (M, d) be a compact metric space and $g : M \to M$ be an operator. We suppose that*
(1) g is continuous;
(2) g is asymptotically regular;
(3) g is orbitally quasinonexpansive.
Then g is a WPO.

Proof. Let $x \in M$. Since M is compact, there exists a subsequence $\{g^{n_i}(x)\}$ of $\{g^n(x)\}$ such that

$$g^{n_i}(x) \to x^*(x) \text{ as } i \to \infty.$$

Conditions (1) and (2) imply that $x^*(x) \in F_g$. By condition (3),

$$d(g(u), x^*) \leq d(u, x^*), \forall u \in O_g(x),$$

which shows that the sequence $\{d(g^n(x), x^*(x))\}_{n \in \mathbb{N}}$ is decreasing. Denote

$$\lim_{n\to\infty} d(g^n(x), x^*(x)) := t \geq 0.$$

Since

$$d(g^{n_i}(x), x^*(x)) \to 0 \text{ as } i \to \infty$$

it follows that g is WPO. □

Theorem 5.5. *Let (M, d) be a metric space and $g : M \to M$ an operator. We suppose that*
(1) g is continuous and $\overline{g(M)} \in P_{cp}(M)$;
(2) g is asymptotically regular;
(3) g is orbitally quasinonexpansive.
Then g is a WPO.

Proof. Since $O_g(g(x)) \subset \overline{g(M)}$, the conclusion follows by Theorem 5.4. □

Theorem 5.6. *Let (M, d) be a bounded complete metric space, α_{DP} a Daneš-Pasicki measure of noncompactness and $g : M \to M$ an operator. We suppose that*
(1) g is continuous and α_{DP}-condensing;
(2) g is asymptotically regular;
(3) g is orbitally quasinonexpansive.
Then g is a WPO.

Proof. We use the arguments in the proof of Theorem 4.4 and apply Theorem 5.5. □

Theorem 5.7. *Let (M, d) be a metric space and $g : M \to M$ an operator. We suppose that*
(1) the fixed point problem for g is well posed in the generalized sense;
(2) g is asymptotically regular;
(3) g is orbitally quasinonexpansive.
Then g is a WPO.

Proof. Using (2) and (3), by Theorem 4.5 we get $F_g \neq \emptyset$. By (1) it follows that $\{g^n(x)\} \to x^*(x) \in F_g$. □

In a Banach space we have the following result.

Theorem 5.8. *Let B be a uniformly convex Banach space, $X \in P_{cl,cv}(B)$ and $g : M \to M$ a nonexpansive operator. We suppose that*
(i) $X = -X$;
(ii) g is odd;
(iii) g is asymptotically regular.
Then g is a WPO.

For other results on Problem 3, see Rus [135].

An interesting result in this direction, which does not assume the continuity of the operator and relies on a general principle involving the images of balls when their centers are not moved too far, was obtained in [81].

Theorem 5.9. (Granas and Dugundji [81], Theorem 5.1) *Let (X, d) be a complete metric space and $F : X \to X$ a map, not necessarily continuous. Assume*

$$\text{for each } \varepsilon > 0 \text{ there is a } \delta = \delta(\varepsilon) > 0 \text{ such that if } d(x, Fx) < \delta, \text{ then } F[B(x, \varepsilon)] \subset B(x, \varepsilon).$$

If F is asymptotically regular at some point $u \in X$, then the sequence $\{F^n u\}$ converges to a fixed point of F.

6. Problems 4 and 5

Let (M, d) be a metric space and $f, g : M \to M$ be two operators with $F_f = F_g$. The problem here is to find conditions on f and g which ensure that the operator g is asymptotically regular.

We start our considerations on Problem 4 with the following notion from Rus [138], see also Theorem 3.7.

Definition 6.1. *We say that g satisfies a (α, β, f)-displacement condition iff there exist $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ and $\beta : M \to \mathbb{R}_+$ such that*

(1) *$t_n \in \mathbb{R}_+$ and $\alpha(t_n) \to 0$ as $n \to \infty$ implies $t_n \to 0$ as $n \to \infty$;*

(2) *$\alpha(d(x, g(x))) \leq \beta(x) - \beta(g(x))$, $\forall\, x \in M$.*

We note that if g satisfies a (α, β, f)-displacement condition, then (see Theorem 3.2 in Rus [138]) g is f-asymptotically regular.

Theorem 6.2. *We suppose that*
(1) g satisfies an (α, β, f)-displacement condition;
(2) there exists $\theta : \mathbb{R}_+ \to \mathbb{R}_+$ such that:

(a) $t_n \in \mathbb{R}_+$, $\theta(t_n) \to 0 \Rightarrow t_n \to 0$;
(b) $d(x, f(x)) \geq \theta(d(x, g(x))), \forall x \in M$.

Then g is asymptotically regular.

Remark 6.3. *If $f, g : M \to M$ are such that there exists $0 < l < 1$, for which*

$$d(g(x), f(x)) \leq l d(x, f(x)), \forall x \in M,$$

then f and g satisfy conditions (2) in Theorem 6.2 with $\theta(t) = (l + 1)t$.

In the case of a normed linear space, for nonexpansive operators we have the following general result.

Theorem 6.4. (Ishikawa [138]; Edelstein-O'Brien [58]) *Let E be a linear normed space, $X \in P_{cv}(E)$ and $f : X \to X$ be a nonexpansive operator. For each $\lambda \in (0, 1)$ we consider the operator $f_\lambda : X \to X$ defined by*

$$f_\lambda(x) := (1 - \lambda)x + \lambda f(x),\ x \in X.$$

(i) If the set $\{f_\lambda^n(x)\}$ is bounded for some $x \in X$, then f_λ is asymptotically regular at x;
(ii) If X is a bounded subset of E, then f_λ is asymptotically regular on X.

In order to state the next result on Problem 4, we need some definitions taken from [133], [16] and [19].

Definition 6.5. (Rus [133]) Let X be a nonempty set. A mapping $G : X \times X \to X$ is called *admissible* if it satisfies the following two conditions:
(A_1) $G(x, x) = x$, for all $x \in X$;
(A_2) $G(x, y) = x$ implies $y = x$.

Definition 6.6. (Rus [133]) Let X be a nonempty set. If $f : X \to X$ is a given operator and $G : X \times X \to X$ is an admissible mapping, then the operator $f_G : X \to X$, defined by

$$f_G(x) = G(x, f(x)),\ \forall x \in X, \tag{4}$$

is called the *admissible perturbation* of f.

Definition 6.7. (Berinde [16]) *Let H be a Hilbert space and $f : H \to H$ be an operator with $F_f \neq \emptyset$. We say that the admissible mapping $G : H \times H \to H$ has the property (C) with respect to f if there exists $\lambda \in (0, 1)$ such that*

$$\left\|G(x, f(x)) - p\right\| \leq \lambda^2 \cdot \left\|x - p\right\|^2 + (1 - \lambda)^2 \cdot \left\|f(x) - p\right\|^2$$
$$+2\lambda(1 - \lambda) \left\langle f(x) - p, x - p\right\rangle,\ \text{ for all } x \in H \text{ and all } p \in F_f.$$

Remark 6.8. *Note that if $f : X \to X$ is a given operator and f_G is its admissible perturbation, then $F_f = F_{f_G}$.*
We also remark that the admissible mapping G corresponding to Theorem 6.4 is defined by

$$G(x, y) := \lambda x + (1 - \lambda) f(x),\ x \in X, \tag{5}$$

with $\lambda \in (0, 1)$.
In a Hilbert space H, the admissible mapping given by (5) *has the property (C) with respect to any operator $f : H \to H$ with $F_f \neq \emptyset$, see [16] for more details.*

Theorem 6.9. (Berinde [16]) *Let C be a bounded closed convex subset of a Hilbert space H and let $f : C \to C$ be a nonexpansive operator. If $G : H \times H \to H$ is an admissible mapping which has the property (C) with respect to f, then the sequence $\{x_{n+1} := G(x_n, T(x_n))\}$ with $x_0 \in C$ given is T-asymptotically regular.*

Proof. See the first part of the proof of Theorem 3.3 in [16]. □

7. Problems 6 and 7

In the case of problem 6, we are looking for conditions on f and g which guarantee that $F_f \neq \emptyset$.

Theorem 7.1. *We suppose that*
(1) g is f-asymptotically regular;
(2) the fixed point problem for f is well posed in the generalized sense.
Then $F_f \neq \emptyset$.

Proof. From (1) we have that

$$d(g^n(x), f(g^n(x))) \to 0 \text{ as } n \to \infty.$$

By (2) it follows that there exists a subsequence $\{g^{n_i}\}$ of $\{g^n(x)\}$ such that

$$g^{n_i} \to x^*(x) \in F_f.$$

□

Remark 7.2. *The conclusion of Theorem 7.1 is that, for ecah $x \in M$, there exists a subsequence $\{f^{n_i}\}$ of $\{f^n(x)\}$ such that*

$$f^{n_i} \to x^*(x) \in F_f.$$

Theorem 7.3. *We suppose that*
(1) g is asymptotically regular;
(2) there exists $\theta : \mathbb{R}_+ \to \mathbb{R}_+$ such that:

(a) $t_n \in \mathbb{R}_+, \theta(t_n) \to 0 \Rightarrow t_n \to 0;$
(b) $d(x, f(x)) \geq \theta(d(x, g(x))), \forall x \in M.$

(3) the fixed point problem for f is well posed in the generalized sense. Then $F_f \neq \emptyset$.

Proof. Conditions (1) and (2) imply that the operator g is f-asymptotically regular.
Now, the proof follows by Theorem 7.1. □

In the case of Problem 7, we seek for conditions on f and g which imply that g is WPO.

Theorem 7.4. *Let f and g be as in Theorem 7.1. In addition, we suppose that g is orbitally quasinonexpansive. Then g is WPO.*

Proof. Similarly to the proof of Theorem 7.1, by (1) we have that

$$d(g^n(x), f(g^n(x))) \to 0 \text{ as } n \to \infty.$$

By (2) it follows that there exists a subsequence $\{g^{n_i}\}$ of $\{g^n(x)\}$ such that

$$g^{n_i} \to x^*(x) \in F_f.$$

By orbitally quasinonexpansiveness,

$$d(g(u), x^*) \leq d(u, x^*), \forall u \in O_g(x),$$

which shows that the sequence $\{d(g^n(x), x^*(x))\}_{n \in \mathbb{N}}$ is decreasing. Denote

$$\lim_{n \to \infty} d(g^n(x), x^*(x)) := t \geq 0.$$

Since

$$d(g^{n_i}(x), x^*(x)) \to 0 \text{ as } i \to \infty$$

it follows that g is WPO. □

Theorem 7.5. *Let f and g be as in Theorem 7.3. In addition, we suppose that g is orbitally quasinonexpansive. Then g is WPO.*

Proof. Similar to the proof of Theorem 7.4. □

8. Conclusions

In this paper we surveyed some of the most relevant results in nonlinear analysis which relate three important concepts in the theory of fixed point problems:

(a) the asymptotic regularity and f-asymptotic regularity of an operator $g : M \to M$;

(a) the existence (and uniqueness) of the fixed points of g;

(a) the convergence of the sequence of successive approximations $\{g^n\}$ to the fixed point(s) of f.

The aspects we went on through this survey were grouped in the Problems 1-7, which were designed to cover the most significant results and connections involving the above notions.

There are many other aspects that were not covered in this paper for size reasons, like e.g., asymptotic regularity and semigroups in Banach spaces, asymptotic regularity and common fixed point problems, asymptotic regularity of multivalued operators etc.

A comprehensive but yet not complete list of references completes the material included in Sections 3-7 of the paper, see [1]-[170].

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