Filomat 34:3 (2020), 945–964
https://doi.org/10.2298/FIL2003945Y

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Convergence Theorems for Generalized Contractions and Applications

Mudasir Younis[a], Deepak Singh[b], Stojan Radenović[c], Mohammad Imdad[d]

[a]*Department of Applied Mathematics, UIT-Rajiv Gandhi Technological University (University of Technology of Madhya Pradesh), Bhopal, 462033, India.*
[b]*Department of Applied Sciences, NITTTR, Under Ministry of HRD, Govt. of India, Bhopal, 462002, (M.P.), India.*
[c]*Faculty of Mechanical Engineering, University of Belgrade, Kraljice Marije 16, 11120 Beograd 35, Serbia.*
[d]*Department of Mathematics, Aligarh Muslim University, Aligarh, 202002, India.*

Abstract. The principal results in this article deal with the existence of fixed points of a new class of generalized F-contraction. In our approach, by visualizing some non-trivial examples we will obtain better geometrical interpretation. Our main results substantially improve the theory of F-contraction mappings and the related fixed point theorems. In section-4, application to graph theory is entrusted and proved results are endorsed by an example through graph. The presented new techniques give the possibility to justify the existence problems of the solutions for some class of integral equations. For the future aspects of our study, an open problem is suggested regarding discretized population balance model, whose solution may be derived from the established techniques.
Keywords: Fixed point, parial b-metric space, graphic contraction, F-contraction, Suzuki-Geraghty type generalized (F, ψ)-contraction, integral equation.

1. Introduction and Preliminaries:

The celebrated Banach contraction principle theorem is one of the keystones in the development of fixed point theory and has been improved and extended in numerous ways (see eg., [21, 34]). In this connection the classical results of Geraghty [12] and Suzuki [28] have been the source of motivation for frequent researchers working in the area of nonlinear analysis. Bakhtin [3] introduced the notion of b-metric spaces, which was extensively extended by Czerwik [6] in 1993. The notion of partial metric spaces was introduced by Matthews [18] as a part of the study of denotational semantics of data flow networks. Shukla [26] generalized both the concepts of partial metric spaces and b-metric spaces and introduced partial b-metric spaces. This concept was further modified by Mustafa [19] in order to find that each partial b-metric p_b generates a b-metric d_{p_b}. Readers interested in aforementioned spaces may switch to [2, 10, 11, 15, 25].
In recent investigations, Wardowski [29] considered a new type of contractions (the so called F-contractions) and proves some fixed point results in a very general setting. Piri et al. [22] refined the result of Wardowski [29] by launching some weaker conditions on the self mapping regarding a complete metric space and over the mapping F (for more details see, eg., [31, 32] and the related references therein). From last few years

2010 *Mathematics Subject Classification*. 47H10, 54H25
Keywords. Fixed point, parial b-metric space, graphic contraction, F-contraction, Suzuki-Geraghty type generalized (F, ψ)-contraction, integral equation
Received: 12 January 2019; Revised: 30 December 2019; Accepted: 03 January 2020
Communicated by Adrian Petrusel
Email addresses: mudasiryouniscuk@gmail.com (Mudasir Younis), dk.singh1002@gmail.com (Deepak Singh), radens@beotel.rs (Stojan Radenović), mhimdad@gmail.com (Mohammad Imdad)

fixed point theory in graph structure has been the center of intensive research for many authors, where the authors aim to bridge the gap between metric fixed point theory and graph theory. Recent work in this structure along with some noteworthy applications can be seen in [20, 27, 33–35].
In this paper, we extend and revamp the work done in Altun and Sadarangani [1], Dung and Hang [9], Karapinar et al. [16], Rosa and Vetro [17], Piri and Kumam [22, 23] and Wardowski and Dung [30]. Using concrete forms of F-contraction it is possible to obtain other known types of variety of contractions; e.g. taking $F(\xi) = ln(\xi), \xi > 0$, we get a Banach contraction (for details, see [29]). Inspired by the ideas given in ([12], [28], [29],), we propose a new contraction called Suzuki-Geraghty type generalized (F, ψ)-contraction and graphic contraction in the setting of partial b-metric spaces. The proposed type of contraction is not a special case of generalized contractions as can be seen in the examples of the subsequent study.
Within the paper, $\mathbb{N}$, $\mathbb{R}^+$, $\mathbb{R}$ denote the set of natural numbers, the set of all non-negative real numbers and the set real numbers respectively.

In the subsequent part we enumerate some basic definitions and handy results that are constructive tools in succeeding analysis and will be deployed in the rest of this paper.
As a generalization and unification of metric spaces, Shukla [26] introduced the concept of partial b-metric spaces as follows:

Definition 1.1. *[26] Let X be a nonempty set and $s \geq 1$ be a given real number. A function $p_b : X \times X \to [0, \infty)$ is called a partial b-metric if for all $x, y, z \in X$ the following conditions are satisfied:*

(p_{b1}) *$x = y$ iff $p_b(x, x) = p_b(x, y) = p_b(y, y)$;*

(p_{b2}) *$p_b(x, x) \leq p_b(x, y)$;*

(p_{b3}) *$p_b(x, y) = p_b(y, x)$;*

(p_{b4}) *$p_b(x, y) \leq s[p_b(x, z) + p_b(z, y)] - p_b(z, z)$.*

The pair (X, p_b) is called a partial b-metric space. The number $s \geq 1$ is called the coefficient of (X, p_b).

Observe that, if $s = 1$ in Definition 1.1, the pair (X, p_b) is called a partial metric and denoted by (X, p) (see[18]).
In order to find that each partial b-metric p_b generates a b-metric d_{p_b}, Mustafa et al. [19] modified the Definition 1.1 and replaced condition (p_{b4}) by (p'_{b4}) as follows:
(p'_{b4}) $p_b(x, y) \leq s(p_b(x, z) + p_b(z, y) - p_b(z, z)) + (\frac{1-s}{2})(p_b(x, x) + p_b(y, y))$.

Remark 1.2. *The class of partial b-metric space (X, p_b) is effectively larger than the class of partial metric space and b-metric space as well.*
If $p_b(x, y) = 0$, then from (p_{b1}) and (p_{b2}) it follows that $x = y$. But, if $x = y$, then $p_b(x, y)$ may not be 0.

Proposition 1.3. *[19] Every partial b-metric p_b defines a b-metric d_{p_b}, where*
$d_{p_b}(x, y) = 2p_b(x, y) - p_b(x, x) - p_b(y, y) \quad$ for all $\quad x, y \in X$.

For covergence , Cauchy sequence and completeness, in the context of partial b-metric spaces, we refer [19].

Lemma 1.4. *[19] Let (X, p_b) be a partial b-metric space. Then*

1. *A sequence $\{x_n\}$ is a p_b-Cauchy sequence in (X, p_b) if and only if it is a b-Cauchy sequence in the b-metric space (X, d_{p_b});*
2. *(X, p_b) is p_b-complete if and only if the b-metric space (X, d_{p_b}) is complete. Moreover, $\lim_{n\to\infty} d_{p_b}(x_n, x) = 0$ if and only if $p_b(x, x) = \lim_{n\to\infty} p_b(x_n, x) = \lim_{n,m\to\infty} p_b(x_n, x_m)$.*

On the other hand, Wardowski [29] described a new type of contraction mapping, called F-contraction defined by

$$\text{for all } x, y \in X \Big(d(Tx, Ty) > 0 \ \text{ implies } \ \tau + F(d(Tx, Ty)) \leq F(d(x, y)) \Big),$$

and obtained a fixed point result as a generalization of Banach contraction principle. Later, Secelean et al. [24], Piri and Kumam [22] extended and refined above definition of Wardowski [29] by establishing some equivalent conditions.
Throughout our succeeding discussion, we denote the set of all functions satisfying $(F1)$ of [29] , $(F2')$ of [24] and $(F3')$ of [22] by Δ_F.

Definition 1.5. *[4] Let X be a non empty set, $T : X \to X$ and $\alpha, \beta : X \times X \to [0,\infty)$. We say that T is an α,β-admissible mapping if*

$$x, y \in X, \quad \alpha(x,y) \geq 1 \text{ and } \beta(x,y) \geq 1 \quad \text{implies} \quad \alpha(Tx,Ty) \geq 1, \text{ and } \beta(Tx,Ty) \geq 1.$$

Definition 1.6. *[12] Let Θ denote the family of all functions $\theta : [0,\infty) \to [0,1)$ such that for any bounded sequence $\{t_n\}$ of positive reals, $\theta(t_n) \to 1 \Rightarrow t_n \to 0$.*

For our further discussion following family of functions will be utilized.
Let Ψ be a family of functions $\psi : [0,\infty) \to [0,\infty)$ such that ψ is continuous and $\psi(p) = 0$ if and only if $p = 0$.

Remark 1.7. *Note that in the paper [14], the authors showed that some fixed point generalizations to partial metric spaces can be obtained from the corresponding results in metric spaces. In this paper we strictly confine for $s > 1$, therefore our generalizations are useful and generalizations in real sense. For the role of θ we refer [4], [5] and the references therein.*

2. Main Results:

We begin this section by inaugurating the following definition.

Definition 2.1. *Let (X, p_b) be a partial b-metric space and $T : X \to X$ be a mapping. Then T is said to be a Suzuki-Geraghty type generalized (F,ψ)-contraction on X, if there exists $F \in \Delta_F$, $\theta \in \Theta$ and $\psi \in \Psi$ such that for all $x, y \in X$ and $s > 1$,*

$$\frac{1}{2s} p_b(x,Tx) < p_b(x,y) \Rightarrow F(s^\epsilon p_b(Tx,Ty)) \leq \theta\Big(M_s^T(x,y)\Big) F\Big(M_s^T(x,y)\Big) - \psi\Big(N_s^T(x,y)\Big), \tag{1}$$

where

$$M_s^T(x,y) = \max\Big\{p_b(x,y), p_b(x,Tx), p_b(y,Ty), \frac{p_b(x,Ty) + p_b(y,Tx)}{2s}\Big\},$$
$$N_s^T(x,y) = \max\Big\{p_b(x,y), p_b(x,Tx), p_b(y,Ty)\Big\}$$

and $\epsilon > 1$ is a constant.

Following example is worked out to illustrate above definition.

Example 2.2. *Let $X = [0,2]$ be equipped with partial b-metric $p_b : X \times X \to [0,\infty)$ defined by $p_b(x,y) = [\max\{x,y\}]^2$, for all $x, y \in X$. It is obvious that, (X, p_b) is a complete partial b-metric space with $s = 2$.*
Let the mapping $T : X \to X$ is defined by

$$Tx = \frac{x}{\sqrt{9 + \sqrt{x}}}.$$

Define $\theta : [0,\infty) \to [0,1)$ by $\theta(k) = \frac{99}{100+\log(2^k)}$ and let $\psi : [0,\infty) \to [0,\infty)$ be given by $\psi(k) = \log(2^k)$, $F(k) = k + \log(k)$ for all $k \in \mathbb{R}^+$. Without loss of generality we take $x, y \in X$ with $x > y$.
One can easily check that, $\frac{1}{2s} p_b(x,Tx) < p_b(x,y)$. In order to verify inequality (1), we have

$$\begin{aligned} F(s^\epsilon p_b(Tx,Ty)) &= F\Big(s^\epsilon \max\{Tx,Ty\}^2\Big) \\ &= 2^\epsilon\Big(\frac{x^2}{9 + \sqrt{x}}\Big) + \log\bigg(2^\epsilon\Big(\frac{x^2}{9 + \sqrt{x}}\Big)\bigg). \end{aligned} \tag{2}$$

On the other hand, one can easily verify that $M_s^T(x,y) = x^2$ and $N_s^T(x,y) = x^2$. *Further,*

$$\theta\big(x^2\big)F\big(x^2\big) - \psi\big(x^2\big) = \frac{99(x^2 + logx^2)}{100 + \log(2^{x^2})} - \log(2^{x^2}). \tag{3}$$

for all $x, y \in X = [0,2]$. *For* $\epsilon = (1, 2.878)$, *one can see that (3) dominates (2) as shown in Figure*(1).

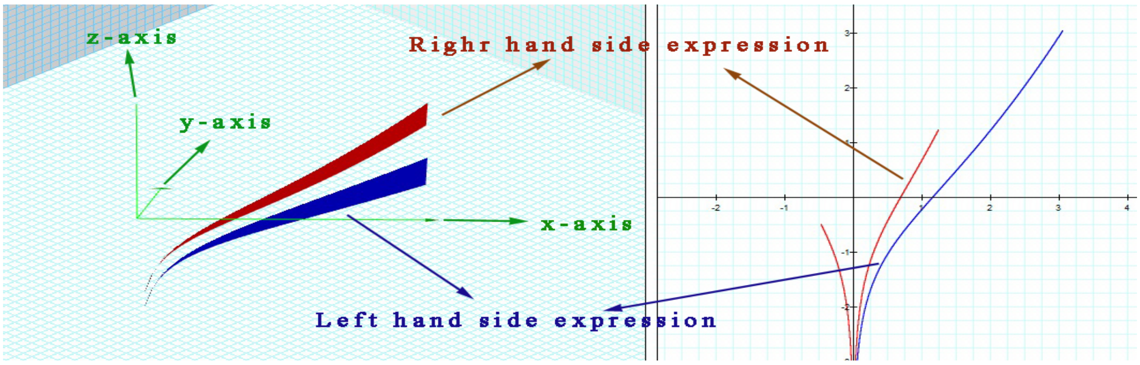

Figure 1: Plot of inequality with $\epsilon = 1.6$, 3D and 2D view.

Now we enunciate a fixed point result concerning generalized F-contractions as follows:

Theorem 2.3. *Let* (X, p_b) *be a complete partial b-metric space and* $T : X \to X$ *be Suzuki-Geraghthy type generalized* (F, ψ)*−contraction. If* T *is continuous, then* T *has a unique fixed point* $u \in X$.

Proof. Let $x_0 \in X$ be arbitrary point. We construct a sequence $\{x_n\}$ in X such that

$$x = x_0 \text{ and } x_n = Tx_{n-1}, \quad \text{for all } n \in \mathbb{N}.$$

Suppose $p_b(x_{n-1}, x_n) = p_b(x_{n-1}, Tx_{n-1}) = 0$ for some $n \in \mathbb{N}$, then x_{n-1} is the required fixed point and we are done in this case.
Consequently, we assume that $p_b(x_{n-1}, Tx_{n-1}) > 0$ for all $n \in \mathbb{N}$.
Hence, we have

$$\frac{1}{2s} p_b(x_{n-1}, Tx_{n-1}) < p_b(x_{n-1}, x_n), \quad \text{for all } n \in \mathbb{N}.$$

So by the hypothesis of our theorem, we have

$$\begin{aligned} F(p_b(x_n, x_{n+1})) &\le F(s^\epsilon p_b(Tx_{n-1}, Tx_n)) \\ &\le \theta\Big(M_s^T(x_{n-1}, x_n)\Big)F\Big(M_s^T(x_{n-1}, x_n)\Big) - \psi\Big(N_s^T(x_{n-1}, x_n)\Big), \end{aligned} \tag{4}$$

where

$$\begin{aligned} M_s^T(x_{n-1}, x_n) &= \max\Big\{p_b(x_{n-1}, x_n), p_b(x_{n-1}, x_n), p_b(x_n, x_{n+1}), \frac{p_b(x_{n-1}, x_{n+1}) + p_b(x_n, x_n)}{2s}\Big\} \\ &= \max\Big\{p_b(x_{n-1}, x_n), p_b(x_n, x_{n+1})\Big\} \end{aligned}$$

and

$$N_s^T(x_{n-1}, x_n) = \max\Big\{p_b(x_{n-1}, x_n), p_b(x_n, x_{n+1})\Big\}.$$

Note that $\max\Big\{p_b(x_{n-1},x_n),p_b(x_n,x_{n+1})\Big\}=p_b(x_n,x_{n+1})$ is impossible due to the definitions of θ and ψ. Indeed,

$$\begin{aligned}F(p_b(x_n,x_{n+1}))&\le F(s^{\epsilon}p_b(Tx_{n-1},Tx_n))\\&\le\theta\Big(p_b(x_n,x_{n+1})\Big)F\Big(p_b(x_n,x_{n+1})\Big)-\psi\Big(p_b(x_n,x_{n+1})\Big)\\&<F\Big(p_b(x_n,x_{n+1})\Big).\end{aligned}$$

It follows that $\max\Big\{p_b(x_{n-1},x_n),p_b(x_n,x_{n+1})\Big\}=p_b(x_{n-1},x_n)$.
Again from (4) and by the hypothesis of θ and ψ, we have

$$\begin{aligned}F(p_b(x_n,x_{n+1}))&\le F(s^{\epsilon}p_b(Tx_{n-1},Tx_n))\\&\le\theta\Big(p_b(x_{n-1},x_n)\Big)F\Big(p_b(x_{n-1},x_n)\Big)-\psi\Big(p_b(x_{n-1},x_n)\Big)\\&<F\Big(p_b(x_{n-1},x_n)\Big).\end{aligned}\tag{5}$$

Therefore $\{p_b(x_n,x_{n+1})\}$ is a non negative decreasing sequence of real numbers and is bounded below. This amounts to say that it is convergent to some point, say $\alpha\ge 0$. i.e.

$$\lim_{n\to\infty}p_b(x_n,x_{n+1})=\alpha.$$

Letting $n\to\infty$ in (5), we obtain

$$F(\alpha)\le F(\alpha)-\psi(\alpha).$$

This implies that $\psi(\alpha)=0$ and thus $\alpha=0$. Consequently, we have

$$\lim_{n\to\infty}p_b(x_n,Tx_n)=\lim_{n\to\infty}p_b(x_n,x_{n+1})=0.\tag{6}$$

Further, by property (p_{b_2}) of partial $b-$metric space, we have the following equality

$$\lim_{n\to\infty}p_b(x_n,x_n)=0.\tag{7}$$

Next, we will maintain that $\{x_n\}$ is a p_b-Cauchy sequence in X. From Lemma 1.4 we need to prove that $\{x_n\}$ is a b-Cauchy sequence in the b-metric space (X,d_{p_b}). Suppose to the contrary that, there exists $\delta>0$ such that for an integer k there exist integer $m'(k)>m(k)\ge k$ such that

$$d_{p_b}(x_{m(k)},x_{m'(k)})\ge\delta.\tag{8}$$

For every integer k, let $m(k)$ is the least positive integer satisfying (8) and such that

$$d_{p_b}(x_{m(k)},x_{m'(k)-1})<\delta.\tag{9}$$

Due to triangle inequality and from (8), we get

$$\delta\le d_{p_b}(x_{m(k)},x_{m'(k)})\le sd_{p_b}(x_{m(k)},x_{m'(k)-1})+sd_{p_b}(x_{m'(k)-1},x_{m'(k)}).\tag{10}$$

Which on passing limit $k\to\infty$ and using (9) give rise to

$$\frac{\delta}{s}\le\lim_{k\to\infty}\inf d_{p_b}(x_{m(k)},x_{m'(k)-1})\le\lim_{k\to\infty}\sup d_{p_b}(x_{m(k)},x_{m'(k)-1})\le\delta.\tag{11}$$

Also from (9), (10) and (11), we have

$$\delta\le\lim_{k\to\infty}\sup d_{p_b}(x_{m(k)},x_{m'(k)})\le s\delta.\tag{12}$$

Furthermore,

$$d_{p_b}(x_{m(k)+1}, x_{m'(k)-1}) \leq s d_{p_b}(x_{m(k)+1}, x_{m(k)}) + s d_{p_b}(x_{m(k)}, x_{m'(k)-1}).$$

Which yields

$$\limsup_{k\to\infty} d_{p_b}(x_{m(k)+1}, x_{m'(k)-1}) \leq s\delta. \tag{13}$$

From triangle inequality, we have

$$\begin{aligned} \delta &\leq d_{p_b}(x_{m'(k)}, x_{m(k)}) \\ &\leq s d_{p_b}(x_{m'(k)}, x_{m(k)+1}) + s d_{p_b}(x_{m(k)+1}, x_{m(k)}) \end{aligned} \tag{14}$$

and

$$d_{p_b}(x_{m'(k)}, x_{m(k)+1}) \leq s d_{p_b}(x_{m'(k)}, x_{m(k)}) + s d_{p_b}(x_{m(k)}, x_{m(k)+1}). \tag{15}$$

It follows from (6), (12), (14), and (15) that

$$\frac{\delta}{s} \leq \limsup_{k\to\infty} d_{p_b}(x_{m'(k)}, x_{m(k)+1}) \leq s^2\delta. \tag{16}$$

Utilizing Proposition (1.3) in (11), (12), (13) and (16), one will get

$$\frac{\delta}{2s} \leq \limsup_{k\to\infty} p_b(x_{m(k)}, x_{m'(k)-1}) \leq \frac{\delta}{2}. \tag{17}$$

$$\frac{\delta}{2} \leq \limsup_{k\to\infty} p_b(x_{m(k)}, x_{m'(k)}) \leq \frac{s\delta}{2}. \tag{18}$$

$$\limsup_{k\to\infty} p_b(x_{m(k)+1}, x_{m'(k)-1}) \leq \frac{s\delta}{2}. \tag{19}$$

$$\frac{\delta}{2s} \leq \limsup_{k\to\infty} p_b(x_{m'(k)}, x_{m(k)+1}) \leq \frac{s^2\delta}{2}. \tag{20}$$

From (6), we can choose a positive integer $k_1 \in \mathbb{N}$ such that

$$\frac{1}{2s} p_b(x_{m(k)}, Tx_{m(k)}) < p_b(x_{m(k)}, x_{m'(k)-1}), \text{ for all } k \geq k_1.$$

Therefore by the assumption of the theorem for every $k \geq k_1$, we have

$$\begin{aligned} F(p_b(x_{m(k)+1}, x_{m'(k)})) &\leq F\Big(s^\epsilon p_b(Tx_{m(k)}, Tx_{m'(k)-1})\Big) \\ &\leq \theta\Big(M_s^T(x_{m(k)}, x_{m'(k)-1})\Big) F\Big(N_s^T(x_{m(k)}, x_{m'(k)-1})\Big) \\ &\qquad - \psi\Big(N_s^T(x_{m(k)}, x_{m'(k)-1})\Big). \end{aligned} \tag{21}$$

Utilizing the definition of $M_s^T(x, y)$ and $N_s^T(x, y)$ along with inequalities (17), (18) and (19), give rise

$$\limsup_{k\to\infty} M_s^T(x_{m(k)}, x_{m'(k)-1}) \leq \frac{\delta}{2}. \tag{22}$$

And

$$\limsup_{k\to\infty} N_s^T(x_{m(k)}, x_{m'(k)-1}) \leq \frac{\delta}{2}. \tag{23}$$

Indeed,

$$M_s^T(x_{m(k)}, x_{m'(k)-1}) = \max\Big\{p_b(x_{m(k)}, x_{m'(k)-1}), p_b(x_{m(k)}, x_{m(k)+1}), p_b(x_{m'(k)-1}, x_{m'(k)}), \frac{p_b(x_{m(k)}, x_{m'(k)}) + p_b(x_{m'(k)-1}, x_{m(k)+1})}{2s}\Big\}.$$

So that

$$\lim_{k\to\infty} \sup M_s^T(x_{m(k)}, x_{m'(k)-1}) \le \max\Big\{\frac{\delta}{2}, 0, 0, \frac{1}{2s}[\frac{s\delta}{2} + \frac{s\delta}{2}]\Big\} \le \frac{\delta}{2}.$$

By repeating the above technique, one can easily arrive at

$$\lim_{k\to\infty} \sup N_s^T(x_{m(k)}, x_{m'(k)-1}) \le \max\Big\{\frac{\delta}{2}, 0, 0\Big\} \le \frac{\delta}{2}.$$

Taking $\lim\sup$ as $n \to \infty$ in (21) and using (20), (22) and (23), we get

$$\begin{aligned}
F(\frac{\delta}{2}) = F\Big(s\frac{\delta}{2s}\Big) &\le \lim_{k\to\infty} \sup F\Big(s^{\epsilon} p_b(Tx_{m(k)}, Tx_{m'(k)-1})\Big) \\
&\le \lim_{k\to\infty} \sup \theta\Big(M_s^T(x_{m(k)}, x_{m'(k)-1})\Big)F\Big(M_s^T(x_{m(k)}, x_{m'(k)-1})\Big) \\
&\qquad - \lim_{k\to\infty} \sup \psi\Big(N_s^T(x_{m(k)}, x_{m'(k)-1})\Big) \\
&\le \lim_{k\to\infty} \sup F\Big(M_s^T(x_{m(k)}, x_{m'(k)-1})\Big) - \lim_{k\to\infty} \sup \psi\Big(N_s^T(x_{m(k)}, x_{m'(k)-1})\Big) \\
&\le F(\frac{\delta}{2}) - \psi(\frac{\delta}{2}),
\end{aligned}$$

which is absurd, since $\delta > 0$. Thus we have maintained that $\{x_n\}$ is a b-Cauchy sequence in the b-metric space (X, d_{p_b}), then from Lemma 1.4, $\{x_n\}$ is a p_b-Cauchy sequence in the partial b-metric space (X, p_b). (X, p_b) being complete, Lemma 1.4 assures that b-metric space (X, d_{p_b}) is b-complete. Therefore, the sequence $\{x_n\}$ converges to some point $u \in X$, that is, $\lim_{n\to\infty} d_{p_b}(x_n, u) = 0$. Again, from Lemma 1.4

$$\lim_{n\to\infty} p_b(x_n, u) = \lim_{n,m\to\infty} p_b(x_n, x_m) = p_b(u, u) = 0. \tag{24}$$

Next, We show that u is a fixed point of T.
We assert that for every $n \in \mathbb{N}$,

$$\frac{1}{2s} p_b(x_n, Tx_n) < p_b(x_n, u) \quad \text{or} \quad \frac{1}{2s} p_b(Tx_n, T^2x_n) < p_b(Tx_n, u). \tag{25}$$

Arguing by contradiction, we assume that there exists $m \in \mathbb{N}$ such that

$$\frac{1}{2s} p_b(x_m, Tx_m) \ge p_b(x_m, u) \quad \text{and} \quad \frac{1}{2s} p_b(Tx_m, T^2x_m) \ge p_b(Tx_m, u). \tag{26}$$

Which gives

$$\begin{aligned}
2sp_b(x_m, u) &\le p_b(x_m, Tx_m) \\
&\le sp_b(x_m, u) + sp_b(u, Tx_m) - p_b(u, u)
\end{aligned}$$

$$\Rightarrow p_b(x_m, u) \le p_b(u, Tx_m). \tag{27}$$

Further

$$\begin{aligned}
p_b(Tx_m, T^2x_m) &< p_b(x_m, Tx_m) \\
&\le sp_b(x_m, u) + sp_b(u, Tx_m) - p_b(u, u) \\
&\le 2sp_b(u, Tx_m).
\end{aligned}$$

(28)

It follows from (26) and (28) that $p_b(Tx_m, T^2x_m) < p_b(Tx_m, T^2x_m)$, a contradiction. Thus (25) holds. Consider, if part first of (25) is true ,then one has by inequality (1)

$$\begin{aligned} F(p_b(x_{n+1}, Tu)) &\leq F(s^{\epsilon} p_b(Tx_n, Tu)) \\ &\leq \theta\Big(M_s^T(x_n, u)\Big)F\Big(M_s^T(x_n, u)\Big) - \psi\Big(N_s^T(x_n, u)\Big). \end{aligned} \tag{29}$$

In which

$$M_s^T(x_n, u) = \max\Big\{p_b(x_n, u), p_b(x_n, Tx_n), p_b(u, Tu), \frac{p_b(x_n, Tu) + p_b(u, Tx_n)}{2s}\Big\}$$

and

$$N_s^T(x_n, u) = \max\Big\{p_b(x_n, u), p_b(x_n, Tx_n), p_b(u, Tu)\Big\}.$$

Which further asserts that

$$\begin{aligned} \lim_{n\to\infty} M_s^T(x_n, u) &= \max\Big\{p_b(u, u), p_b(u, u), p_b(u, Tu), \frac{p_b(u, Tu) + p_b(u, Tu)}{2s}\Big\} \\ &= p_b(u, Tu). \end{aligned} \tag{30}$$

Similarly

$$\lim_{n\to\infty} N_s^T(x_n, u) = p_b(u, Tu). \tag{31}$$

Taking limit as $n \longrightarrow \infty$ in (29) and employing inequalities (30), (31), hypothesis of function F, θ together with continuity of T, we get

$$\begin{aligned} F(p_b(u, Tu)) &\leq \lim_{n\longrightarrow\infty} \theta\Big(M_s^T(x_n, u)\Big)F(p_b(u, Tu)) - \psi(p_b(u, Tu)) \\ &\leq F(p_b(u, Tu)) - \psi(p_b(u, Tu)), \end{aligned}$$

which implies $\psi(p_b(u, Tu)) = 0$. This yields $u = Tu$, i.e., u is a fixed point of T.
If part second of (25) is true , employing a similar approach as above , we conclude that $u = Tu$. Hence u is the fixed point of T.
For the uniqueness of fixed point, suppose u and v are two fixed points of T, such that $u \neq v$, then we have $\frac{1}{2s}p_b(u, Tu) < p_b(u, v)$, and by assumption of theorem, we obtain

$$\begin{aligned} F(p_b(u, v)) &= F(p_b(Tu, Tv)) \\ &\leq F(s^{\epsilon} p_b(p_b(Tu, Tv))) \\ &\leq \theta\Big(M_s^T(u, v)\Big)F(M_s^T(u, v)) - \psi(N_s^T(u, v)) \\ &\leq F(p_b(u, v)) - \psi(p_b(u, v)). \end{aligned}$$

Which yields $\psi(p_b(u, v)) = 0$, that is, $u = v$. Thus fixed point of T is unique. □

To show the substantiation of our findings, we expound an example which demonstrates the superiority of our results.

Example 2.4. *Let $X = [0, 10]$ be equipped with partial metric $p_b : X \times X \to [0, \infty)$ defined by*

$$p_b(x, y) = [\max\{x, y\}]^2,$$

for all $x, y \in X$. It is obvious that, (X, p_b) is a complete partial b-metric space with $s = 2$.
Let the mapping $T : X \to X$ is defined by

$$Tx = \frac{x}{\sqrt{3 + x^2}}.$$

In order to check condition 1, let $F(k) = \log k$. Define $\theta : [0, \infty) \to [0, 1)$ by $\theta(k) = \frac{500}{k+501}$ and $\psi : [0, \infty) \to [0, \infty)$ by $\psi(k) = \frac{k}{1000}$ for all $k \in \mathbb{R}^+$.
Without loss of generality we may take $x, y \in X$ such that $x > y$.
First observe that $\frac{1}{2s} p_b(x, Tx) < p_b(x, y)$. In order to verify inequality (1), we have

$$\begin{aligned} F(s^\epsilon p_b(Tx, Ty)) &= F\Big(s^\epsilon \max\{Tx, Ty\}^2\Big) \\ &= \log\left(2^\epsilon\Big(\frac{x^2}{3 + x^2}\Big)\right). \end{aligned} \tag{32}$$

On the other hand, one can easily verify that $M_s^T(x, y) = x^2$ and $N_s^T(x, y) = x^2$. Further, for all $x, y \in X = [0, 10]$, we have

$$\theta\big(x^2\big)F\big(x^2\big) - \psi\big(x^2\big) = \frac{500(\log x^2)}{x^2 + 501} - \frac{x^2}{1000}. \tag{33}$$

For $\epsilon = 1.1$, one can see that (33) dominates (32) as shown in Figure(2).

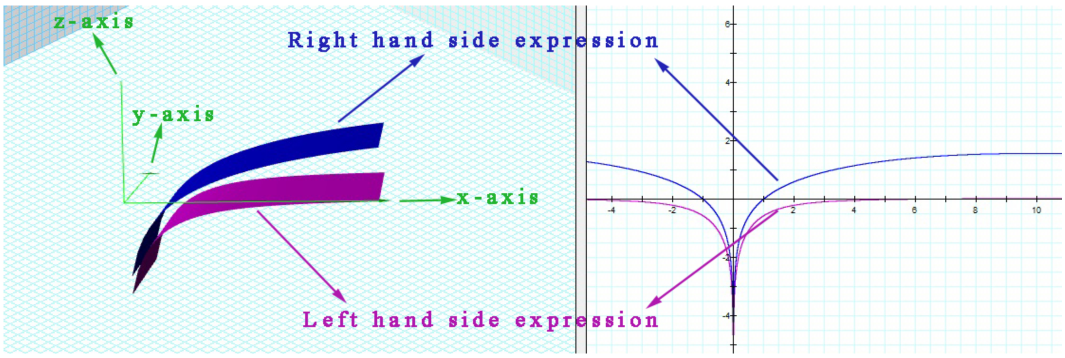

Figure 2: Domination of R.H.S. over L.H.S. with $\epsilon = 1.1.$, 3D and 2D view

Thus all the conditions of Theorem 2.3 are fulfilled and $0 \in X$ is the unique fixed point of the involved mapping T.
Note that by taking $\tau = \psi(t)$ in Karapinar et al. [16], T does not satisfy the contractive condition of [16] with $d(x, y) = [\max\{x, y\}]^2$. So the result of [16] can not be applied on T.

Following remarks make our findings worth mentioning.

Remark 2.5. *Theorem 2.3 generalizes Theorem* 2.2 *of Piri and Kumam [23] and Theorem* 2.2 *of Karapinar et al. [16] in the context of parial b-metric space along with Geraghty type contraction.*

Remark 2.6. *Theorem 2.3 generalizes and extend F-contraction version of main result of Altun and Sadarangani [1] in the setting of complete partial b-metric space along with Suzuki type contraction.*

Remark 2.7. *Theorem 2.3 generalizes Theorem* 2.4 *of Wardowski and Dung [30] and Theorem* 3 *of Dung and Hang [9] for parial b-metric space along with Suzuki-Geraghty type contraction.*

Remark 2.8. *Theorem 2.3 generalizes Theorem* 2.1 *of Piri and Kumum [22] in the setting of parial b-metric space along with Suzuki-Geraghty type contraction by taking $\psi(N_s^T(x, y)) = \tau$ and $M_s^T(x, y) = p_b(x, y)$.*

2.1. Some Consequences:

Corollary 2.9. *Theorem 2.3 remains true if the assumption embodied in (1) is replaced by the following (besides retaining the rest of the hypotheses)*

$$\frac{1}{2s}p_b(x,Tx) < p_b(x,y) \Rightarrow F(s^{\epsilon}p_b(Tx,Ty)) \leq \theta\Big(M_s^T(x,y)\Big)F\Big(N_s^T(x,y)\Big) - \psi\Big(N_s^T(x,y)\Big).$$

Another version of Theorem 2.3 is the following

Theorem 2.10. *Let (X, p_b) be a complete partial metric space with $s > 1$. Let T be a continuous self mapping on X. If there exist $F \in \Delta_F$, $\theta \in \Theta$, $\psi \in \Psi$ such that for all $x, y \in X$,*

$$\frac{1}{2s}p_b(x,Tx) < p_b(x,y) \Rightarrow F(s^{\epsilon}p_b(Tx,Ty)) \leq F\Bigg(\theta\Big(M_s^T(x,y)\Big)\Big(M_s^T(x,y)\Big)\Bigg) - \psi\Big(N_s^T(x,y)\Big),$$

where $M_s^T(x,y)$ and $N_s^T(x,y)$ are defined as in the Theorem 2.3 and $\epsilon > 1$. Then T has a unique fixed point in X.

Proof. The proof can be completed on the similar lines as done in Theorem 2.3, hence we skip it. □

Taking $\psi(t) = \tau$ in Theorem 2.10, we acquire the following

Corollary 2.11. *Theorem 2.10 remains true if the assumption embodied in (**??**) is replaced by the following (besides retaining the rest of the hypotheses):*

$$\frac{1}{2s}p_b(x,Tx) < p_b(x,y) \Rightarrow F(s^{\epsilon}p_b(Tx,Ty)) \leq F\Bigg(\theta\Big(M_s^T(x,y)\Big)\Big(M_s^T(x,y)\Big)\Bigg) - \tau.$$

As an independent result, one can obtain following Corollary in the setting of partial metric space by setting $s = 1$.

Corollary 2.12. *Let (X, p) be a complete partial metric space. Let T be a continuous self mapping on X. If there exist $F \in \Delta_F$, $\theta \in \Theta$ and $\psi \in \Psi$ such that for all $x, y \in X$,*

$$\frac{1}{2}p(x,Tx) < p(x,y) \Rightarrow F(p(Tx,Ty)) \leq \theta\Big(M_s^T(x,y)\Big)F\Big(M_s^T(x,y)\Big) - \psi\Big(N_s^T(x,y)\Big),$$

where

$$M_s^T(x,y) = \max\Big\{p(x,y), p(x,Tx), p(y,Ty), \frac{p(x,Ty)+p(y,Tx)}{2}\Big\},$$
$$N_s^T(x,y) = \max\Big\{p(x,y), p(x,Tx), p(y,Ty)\Big\}.$$

Then T has a unique fixed point in X.

Proof. Proof follows on the similar lines as done in the Theorem 2.3. □

Remark 2.13. *By taking $M_s^T(x,y) = p(x,y)$ and $\psi(N_s^T(x,y)) = \tau$, Corollary 2.12 reduces to Theorem 3.1 of Ðukic et al.[8] in the sense of F-contraction along with Suzuki type contraction.*

Remark 2.14. *By taking $\psi(N_s^T(x,y)) = \tau$, Corollary 2.12 reduces to Corollary 1 of Dinarvand [7] in the sense of F-contraction along with Suzuki type contraction.*

3. Fixed point results for (α, β)-Suzuki-Geraghty type generalized (F, ψ)-contractions:

This section is devoted to establish the results based on generalized contraction invoking (α, β)-admissible mappings.

Definition 3.1. *Let (X, p_b) be a partial b-metric space and $T : X \to X$ be a mapping. Then T is said to be a (α, β)-Suzuki-Geraghty type generalized (F, ψ)-contraction on X, if there exists $F \in \Delta_F$, $\theta \in \Theta$ and $\psi \in \Psi$ such that for all $x, y \in X$ and $s > 1$,*

$$\begin{aligned} &\frac{1}{2s} p_b(x, Tx) < p_b(x, y) \\ &\Rightarrow \alpha(x, y)\beta(x, y)F(s^{\epsilon} p_b(Tx, Ty)) \leq \theta\Big(M_s^T(x, y)\Big)F\Big(M_s^T(x, y)\Big) - \psi\Big(M_s^T(x, y)\Big), \end{aligned} \tag{34}$$

where

$$M_s^T(x, y) = \max\Big\{p_b(x, y), p_b(x, Tx), p_b(y, Ty), \frac{p_b(x, Ty) + p_b(y, Tx)}{2s}\Big\},$$

and $\epsilon > 1$ is a constant.

Theorem 3.2. *Let (X, p_b) be a complete partial b-metric space. Let T be a self mapping on X satisfying the following conditions:*

1. *T is (α, β)-admissible;*
2. *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\beta(x_0, Tx_0) \geq 1$;*
3. *T is continuous;*
4. *T is an (α, β)-Suzuki-Geraghty type generalized (F, ψ)-contraction on (X, p_b).*

Then T has a unique fixed point $u \in X$.

Proof. Let x_0 be an arbitrary point in X such that $\alpha(x_0, Tx_0) \geq 1$ and $\beta(x_0, Tx_0) \geq 1$. Set $Tx_0 = x_1$ and $Tx_1 = x_2$. Continuing this process, we define a sequence $\{x_n\}$ in X such that

$$x_{n+1} = Tx_n \text{ for all } n \in \mathbb{N}.$$

If there exist $n_0 \in \mathbb{N}$ such that $x_{n_0+1} = x_{n_0}$ for any $n_0 \in \mathbb{N}$, then x_{n_0} is a fixed point of T and we are done .
Consequently, we suppose that $x_{n+1} \neq x_n$ for all $n \in \mathbb{N}$.
Since, T is an (α, β)-admissible mapping, it follows from (2) that $\alpha(x_0, Tx_0) = \alpha(x_0, x_1) \geq 1$, $\alpha(Tx_0, Tx_1) = \alpha(x_1, x_2) \geq 1$. By induction, we get
$\alpha(x_n, x_{n+1}) \geq 1$ for all $n \geq 0$.
Similarly, $\beta(x_n, x_{n+1}) \geq 1$ for all $n \geq 0$.
Since $p_b(x_n, Tx_n) > 0$ for all $n \in \mathbb{N}$, we have

$$\frac{1}{2s} p_b(x_n, Tx_n) < p_b(x_n, Tx_n), \quad \text{for all } n \in \mathbb{N}.$$

To avoid the repetition of similar treatment of the Theorem 2.3, for the sake of brevity , we omit the remaining part of proof. □

Example 3.3. *Let $X = [0, 1.2]$ be equipped with partial metric $p_b : X \times X \to [0, \infty)$ defined by*

$$p_b(x, y) = [\max\{x, y\}]^2,$$

for all $x, y \in X$. It is obvious that, (X, p_b) is a complete partial b-metric space with $s = 2$.
Let the mapping $T : X \to X$ is defined by

$$Tx = \log\left(\sqrt{2 + x}\right)^{x^2}.$$

Also define $\alpha, \beta : X \times X \to [0, \infty)$*as :*

$$\alpha(x,y) = \beta(x,y) = \begin{cases} 1, & x, y \in [0,1], \\ 0, & \text{otherwise}, \end{cases}$$

Let $\alpha(x,y) \geq 1$ *and* $\beta(x,y) \geq 1$ *for all* $x, y \in X$*, then* $x, y \in [0,1]$*. On the other side,* $Tx \leq 1$ *for all* $x \in [0,1]$*. It follows that* $\alpha(Tx, Ty) \geq 1$ *and* $\beta(Tx, Ty) \geq 1$*. Also, there exists* $x_0 = 0 \in X$ *such that* $\alpha(0, T0) = \alpha(0,0) \geq 1$ *and* $\beta(0, T0) = \beta(0,0) \geq 1$*. Define* $\theta : [0, \infty) \to [0, 1)$ *by* $\theta(k) = \frac{76}{76.243 + \log(10^k)}$*. Also let* $\psi : [0, \infty) \to [0, \infty)$ *be given by* $\psi(k) = \log(1 + k)$*,* $F(k) = k + \log(k)$ *for all* $p \in \mathbb{R}^+$*.*
Without loss of generality we take $x, y \in X$ *with* $x > y$*. It is evident that* $\frac{1}{2s} p_b(x, Tx) < p_b(x, y)$ *for all* $x, y \in X$ *with* $x > y$*.*
Case(i): *If* $x, y \in [0,1]$*, then* $\alpha(x,y) = \beta(x,y) = 1$*.*
Now consider the L.H.S. of (34)

$$\begin{aligned} \alpha(x,y)\beta(x,y)F(s^{\epsilon} p_b(Tx, Ty)) &= F\Big(s^{\epsilon} \max\{Tx, Ty\}^2\Big). \\ &= s^{\epsilon}\Big(\log\Big(\sqrt{2+x}\Big)^{x^2}\Big)^2 + \log\Big\{s^{\epsilon}\Big(\log\Big(\sqrt{2+x}\Big)^{x^2}\Big)^2\Big\}. \end{aligned} \tag{35}$$

For R.H.S. of 34, utilizing the definition of $M_s^T(x,y)$ *and* $N_s^T(x,y)$*, we deduce* $M_s^T(x,y) = x^2$ *and* $N_s^T(x,y) = x^2$*. Further,*

$$\theta\big(x^2\big)F\big(x^2\big) - \psi\big(x^2\big) = \frac{76(x^2 + \log x^2)}{76.243 + \log(10^{x^2})} - \log(1 + x^2). \tag{36}$$

Now, validity of condition 34 for $\epsilon = 3.02$ *is shown by Figure*(3).

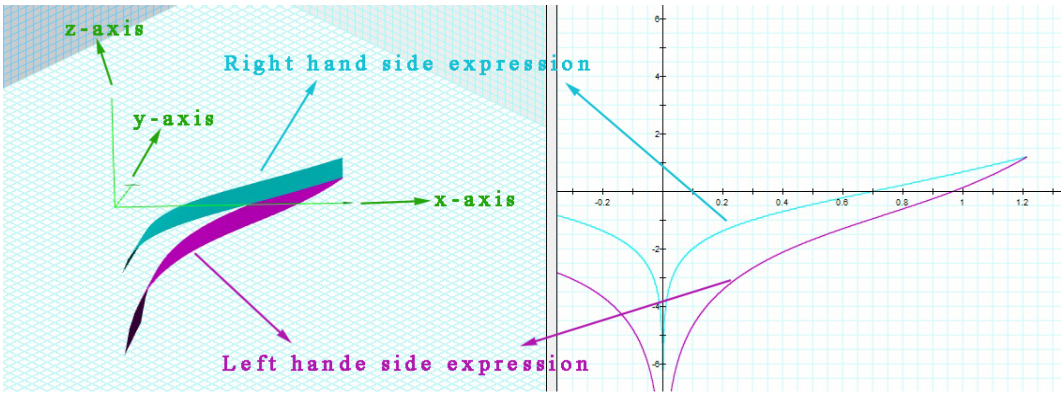

Figure 3: Plot of inequality with $\epsilon = 3.02$, 3D view and 2D view.

Case(ii): *If* $x, y \in (1, 1.2]$*, then* $\alpha(x,y) = \beta(x,y) = 0$*, we have*

$$\alpha(x,y)\beta(x,y)F(s^{\epsilon} p_b(Tx, Ty)) = 0.$$

For R.H.S. of (34), one can verify that $M_s^T(x,y) = N_s^T(x,y) = x^2$*. Subsequently*

$$\frac{76(x^2 + \log x^2)}{76.243 + \log(10^{x^2})} - \log(1 + x^2).$$

Domination of R.H.S. over L.H.S. of 34 is authenticated by Figure(4).

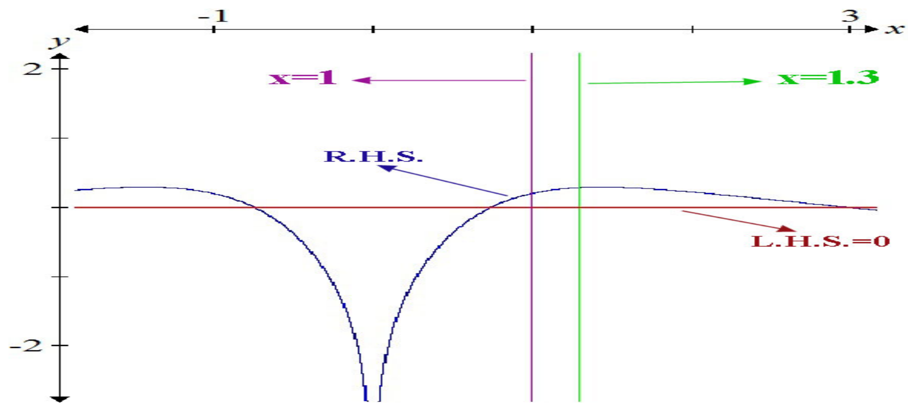

Figure 4: Plot of inequality 2D view.

Case(iii): *If $x \in (1, 1.2]$ and $y \in [0, 1]$ then, $\alpha(x, y)\beta(x, y) = 0$ and $M_s^T(x, y) = N_s^T(x, y) = x^2$. By repeating the same technique as mentioned in Case(ii), one can conclude that the condition 34 is satisfied for all $x \in (1, 1.3]$ and $y \in [0, 1]$. Thus all the conditions of Theorem 3.2 are fulfilled. Hence T is (α, β)-Suzuki-Geraghty type generalized (F, ψ)-contraction and has a unique fixed point $0 \in X$.*

Remark 3.4. *If we take $\epsilon \in (1, 3.04165]$, the inequality 34 is still valid.*

Remark 3.5. *By introducing Theorem 3.2 we generalized Theorem 3.5 of Rosa and Vetro [17] and obtained the F-contraction version of the said theorem in the setting of partial b-metric spaces.*

For $s = 1$, one can prove the following theorem as an independent result in the setting of partial metric space.

Theorem 3.6. *Let (X, p) be a complete partial metric space. Let T be a self mapping on X satisfying the following conditions:*

1. *T is (α, β)-admissible;*
2. *T is continuous;*
3. *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\beta(x_0, Tx_0) \geq 1$;*
4. *if there exists $F \in \Delta_F$, $\theta \in \Theta$ and $\psi \in \Psi$ such that for all $x, y \in X$,*

$$\frac{1}{2}p(x, Tx) < p(x, y) \Rightarrow \alpha(x, y)\beta(x, y)F(p_b(Tx, Ty)) \leq \theta\Big(M^T(x, y)\Big)F\Big(M^T(x, y)\Big) - \psi\Big(M^T(x, y)\Big), \tag{37}$$

where

$$M^T(x, y) = \max\Big\{p(x, y), p(x, Tx), p(y, Ty), \frac{p(x, Ty) + p(y, Tx)}{2}\Big\}.$$

Then T has a unique fixed point $u \in X$.

4. Applications

4.1. Application to graph theory

Jachymski [13], introduced a different approach in metric fixed point theory by replacing the order structure with a graph structure on a metric space. Let (X, p) be a partial metric space with diagonal of the cartesian product $X \times X$ denoted by $\Delta = \{(z, z) : z \in X\}$. Consider, a directed graph $G = \big(V(G), E(G)\big)$ with the set $V(G)$ of its vertices coinciding with X and the set $E(G)$ of its edges as a superset of Δ. Assume that G has no parallel edges, i.e.; $(x, y),\ (y, x) \in E(G) \Longrightarrow x = y$. Also, G is directed if the edges have a direction associated with them. If x and y are vertices in a graph G, then a path in G from x to y of length m is a sequence $\{x_n\}_{n=0}^m$ of $(m + 1)$ vertices such that $x_0 = x$, $x_m = y$ and $(x_{n-1}, x_n) \in E(G)$ for $n = 1, \ldots m$. Moreover, a graph is called connected if there is a path between any two vertices.

Definition 4.1. *[13] A mapping $T : X \times X$ is called G-continuous, if we have a given $x \in X$ and a sequence $\{x_n\}$ such that $x_n \to x$, as $n \to \infty$, $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N} \Longrightarrow Tx_n \to Tx$.*

Definition 4.2. *[13] A mapping $T : X \to X$ is a Banach G-contraction or simply G-contraction if T preserves edges of G, that is*
for all $x, y \in X$, $\quad (x, y) \in E(G) \Longrightarrow \ (Tx, Ty) \in E(G)$
and T decreases weights of edges of G in the following way:
there exists $k \in (0, 1)$ such that for all $x, y \in X$, $\quad (x, y) \in E(G) \Longrightarrow d(Tx, Ty) \leq kd(x, y)$.

Above definitions motivate us to form the following.

Definition 4.3. *Let (X, p) be a partial metric space endowed with a graph G and $T : X \to X$ be a mapping. Then T is said to be Suzuki-Geraghthy type generalized graphic (F, ψ)-contraction on a partial metric space (X, p), if there exists $F \in \Delta_F$, $\theta \in \Theta$ and $\psi \in \Psi$ such that for all $x, y \in X$ and $s > 1$,*

$$\frac{1}{2}p(x, Tx) < p(x, y) \Rightarrow F(p(Tx, Ty)) \leq \theta\Big(M^T(x, y)\Big)F\Big(M^T(x, y)\Big) - \psi\Big(M^T(x, y)\Big), \tag{38}$$

where ϵ and $M^T(x, y)$ are as in the Theorem 3.6.

Theorem 4.4. *Let (X, p) be a complete partial metric space endowed with a graph G. $T : X \to X$ is self mapping on X satisfying the following conditions:*

1. *T is Suzuki-Geraghthy type generalized graphic (F, ψ)-contraction on partial metric space (X, p);*
2. *T is G-continuous on (X, p);*
3. *T preserves edges of G;*
4. *there exists $x_0 \in X$ such that $(x_0, Tx_0) \in E(G)$.*

Then T has a fixed point $u \in X$.

Proof. Define $\alpha, \beta : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \beta(x, y) = \begin{cases} 1, & if \ (x, y) \in E(G); \\ 0, & otherwise \end{cases}$$

Firstly, we prove that T is an (α, β)-admissible mapping. If $\alpha(x, y) \geq 1$ for any $x, y \in X$, then by the definition of α, we obtain that $(x, y) \in E(G)$. Owing to (iii), we have2 $(Tx, Ty) \in E(G)$. Again, from the definition of α, we have $\alpha(Tx, Ty) \geq 1$.
Similarly as above, we can prove that $\beta(x, y) \geq 1 \Rightarrow \beta(Tx, Ty) \geq 1$ so that T is an (α, β)-admissible mapping. Now, choose $x_0 \in X$ such that $(x_0, Tx_0) \in E(G)$, i.e., $\alpha(x_0, Tx_0) \geq 1$ and $\beta(x_0, Tx_0) \geq 1$.
From (i) and $\alpha(x, y) \geq 1$, $\beta(x, y) \geq 1$, implies

$$\alpha(x, y)\beta(x, y)F(p(Tx, Ty)) \leq \theta\Big(M^T(x, y)\Big)F\Big(M^T(x, y)\Big) - \psi\Big(M^T(x, y)\Big).$$

Hence, all the conditions of Theorem 3.6 are satisfied. Therefore, T has a fixed point. □

Next, an example is presented which substantiates the hypothesis of Theorem 4.4 involving a directed graph.

Example 4.5. *Let $X = \{1, 2, 3, 4, 5\}$ be endowed with with the partial metric $p : X \times X \to [0, \infty)$ defined by*

$$p(x, y) = \max\{x, y\}.$$

Then (X, p) is a complete partial metric space.
We define $T : X \to X$ as follows

$$Tx = \begin{cases} 2 & \text{if } x \in \{3, 4\}, \\ 1 & \text{if } x \in \{1, 2, 5\}. \end{cases}$$

Define $\theta : [0, \infty) \to [0, 1)$ by $\theta(k) = \frac{19}{19.135+k}$. And let $\psi : [0, \infty) \to [0, \infty)$ be given by $\psi(k) = \frac{k}{1000}$. Let $F(k) = \log k$ for all $k \in \mathbb{R}^+$.
Consider the directed graph $G = \big(V(G), E(G)\big)$ defined by $V(G) = X$ and $E(G) = \{(x, y) : x, y \in \{1, 2, 3, 4, 5\}\} \cup \Delta$.
It is easy to deduce that T preserves edges in G and T is G-continuous. Also, there exists $x_0 = 1 \in X$ such that $(1, T1) = (1, 1) \in E(G)$. Without loss of generality we take $x, y \in X$ such that $x \neq y$. To prove that the contractive condition (38) of Theorem 4.4 holds with $x, y \in E(G)$, we distinguish the following cases:
***Case I:** $x \in \{1, 2, 5\}$ and $y \in \{3, 4\}$.*
***Case II:** $x \neq y$ and $x, y \in \{1, 2, 5\}$.*
***Case III:** $x, y \in \{3, 4\}$.*
Evidently, for each of the aforementioned cases, inequality $\frac{1}{2}p(x, Tx) < p(x, y)$ holds. Following Table demonstrates that the condition (38) is satisfied for each of the above cases.

Cases	x	y	$F(p(Tx, Ty))$	$\theta\big(M^T(x, y)\big)F\big(M^T(x, y)\big) - \psi\big(M^T(x, y)\big)$
Case I	1	3	0.30103	0.406546
	1	4	0.30103	0.490452
	2	3	0.30103	0.406546
	2	4	0.30103	0.490452
	5	3	0.30103	0.594974
	5	4	0.30103	0.594974
Case II	1	2	0	0.268621
	1	5	0	0.545256
	2	5	0	0.545256
Case III	3	4	0.30103	0.490452

Thus all the hypothesis of Theorem 4.5 are fulfilled and consequently T has a fixed point, which is $x = 1$. Figure(5) represents the graph with all the discussed cases.

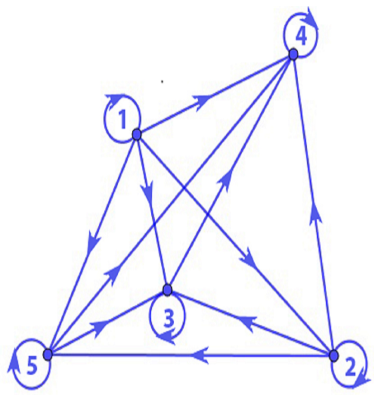

Figure 5: Directed graph G defined in Example 4.5

4.2. Application to solution of some integral equations

In this section we will focus on the applicability of the acquired results.
We present the application of the existence of fixed point for Suzuki-Geraghty type generalized (F,ψ)-contraction to the following equation of integral equation for an unknown function u:

$$u(t) = g(t) + \int_a^b K(t,z)f(z,u(z))dz, \qquad t \in [a,b], \tag{39}$$

where $f : [a,b] \times \mathbb{R} \to \mathbb{R}$, $K : [a,b] \times [a,b] \to [0,\infty)$, $g : [a,b] \to \mathbb{R}$ are given continuous functions.
Let X be the set $C[a,b]$ of real continuous functions defined on $[a,b]$ and let $p_b : X \times X \to [0,\infty)$ be equipped with the metric defined by

$$p_b(u,v) = \max_{a \le t \le b} |u(t) - v(t)|^2. \tag{40}$$

One can easily verify that (X,p_b) is a complete partial b-metric space. Let the self map $T : X \to X$ is defined by

$$Tu(t) = g(t) + \int_a^b K(t,z)f(z,u(z))dz, \qquad t \in [a,b], \tag{41}$$

then u is a fixed point of T if and only it is a solution of (39. Now, we formulate the following subsequent theorem to show the existence of solution of integral equation.

Theorem 4.6. *Assume that the following assumptions hold:*

(1)

$$\max_{a \le t \le b} \int_a^b |K(t,z)|^2 dz \le \frac{1}{b-a};$$

(2) Suppose that for all $x, y \in \mathbb{R}$,

$$\frac{1}{2s} p_b(x,Tx) < p_b(x,y) \Longrightarrow |f(z,x) - f(z,y)|^2 \le \frac{1}{2s^\epsilon} |x(t) - y(t)|^2 e^{-\tau}.$$

Then the integral equation (39) has a solution.

Proof. Employing the conditions (1) − (2) along with inequality (39), we have

$$\begin{aligned}
p_b(Tu_1,Tu_2) &= \max_{a \le t \le b} |Tu_1(t) - Tu_2(t)|^2 \\
&= \max_{a \le t \le b} \left| g(t) + \int_a^b K(t,z)f(z,u_1(z))dz - \Big(g(t) + \int_a^b K(t,z)f(z,u_2(z))dz\Big) \right|^2 \\
&= \max_{a \le t \le b} \left\{ \left| \int_a^b \Big(K(t,z)f(z,u_1(z)) - K(t,z)f(z,u_2(z))\Big)dz \right|^2 \right\} \\
&\le \max_{a \le t \le b} \left\{ \int_a^b |K(t,z)|^2 dz. \int_a^b \Big| f(z,u_1(z)) - f(z,u_2(z)) \Big|^2 dz \right\}
\end{aligned}$$

$$
\begin{aligned}
&= \Big\{\max_{a\le t\le b}\int_a^b |K(t,z)|^2 dz\Big\}.\Big\{\int_a^b \Big|f(z,u_1(z)) - f(z,u_2(z))\Big|^2 dz\Big\}\\
&= \Big\{\max_{a\le t\le b}\int_a^b |K(t,z)|^2 dz\Big\}.\Big\{\int_a^b \Big|f(z,u_1(z)) - f(z,u_2(z))\Big|^2 dz\Big\}\\
&\le \Big\{\frac{1}{b-a}\Big\}.\Big\{\frac{1}{2s^{\epsilon}}\int_a^b |u_1(z) - u_2(z)|^2 e^{-\tau} dz\Big\}\\
&\le \frac{1}{2s^{\epsilon}(b-a)}\int_a^b \max_{a\le t\le b}|u_1(t) - u_2(t)|^2 e^{-\tau} dz\\
&= \left(\frac{1}{2s^{\epsilon}}\right)\max_{a\le t\le b}|u_1(t) - u_2(t)|^2 e^{-\tau}
\end{aligned}
$$

$$
\begin{aligned}
\textit{i.e.,}\quad p_b(Tu_1,Tu_2) &= \left(\frac{1}{2s^{\epsilon}}\right)\Big(p_b(u_1,u_2)\Big)e^{-\tau}\\
&\le \left(\frac{1}{s^{\epsilon}}\right)\frac{M_s^T(u_1,u_2)}{2}e^{-\tau}
\end{aligned}
$$

Which amounts to say that

$$s^{\epsilon}p_b(Tu_1,Tu_2) \le \frac{M_s^T(u_1,u_2)}{2}e^{-\tau},$$

where

$$M_s^T(u_1,u_2) = \max\Big\{p_b(u_1,u_2), p_b(u_1,Tu_1), p_b(u_2,Tu_2), \frac{p_b(u_1,Tu_2)+p_b(u_2,Tu_1)}{2s}\Big\}.$$

Taking $\theta(M_s^T(u_1,u_2)) = \frac{1}{2}$, above inequality turns into

$$s^{\epsilon}p_b(Tu_1,Tu_2) \le \theta\Big(M_s^T(u_1,u_2)\Big)M_s^T(u_1,u_2)e^{-\tau}$$

Consequently, by applying to logarithms, we obtain

$$\ln\Big(s^{\epsilon}p_b(Tu_1,Tu_2)\Big) \le \ln\Bigg(\theta\Big(M_s^T(u_1,u_2)\Big)M_s^T(u_1,u_2)\Bigg) - \tau.$$

For $\ln(p) = p$, $p > 0$, above inequality turns into

$$F\Big(s^{\epsilon}p_b(Tu_1,Tu_2)\Big) \le F\Bigg(\theta\Big(M_s^T(u_1,u_2)\Big)M_s^T(u_1,u_2)\Bigg) - \tau.$$

Hence $\frac{1}{2s}p_b(u_1,Tu_1) < p_b(u_1,u_2) \Rightarrow$

$$F\Big(s^{\epsilon}p_b(Tu_1,Tu_2)\Big) \le F\Bigg(\theta\Big(M_s^T(u_1,u_2)\Big)M_s^T(u_1,u_2)\Bigg) - \tau.$$

Thus, all the hypothesis of Corollary 2.11 are satisfied, we conclude that T has a unique fixed point x^* in X. Which amounts to say that the integral equation (39) has a unique solution which belongs to $X = C[a,b]$. □

Following example furnishes the validity of Theorem 4.6.

Example 4.7. *Consider the following integral equation in $X = C([0,1],\mathbb{R})$.*

$$u(t) = \frac{t^2}{2+t} + \frac{1}{3}\int_0^1 \frac{s^2}{(2+t)}\frac{1}{(2+u(s))}ds; \quad t \in [0,1]. \tag{42}$$

In order to find the solution of (42), we will prove that $u(t)$ is a fixed point of the mapping $Tu(t)$, that is , $u(t) = Tu(t)$, where

$$Tu(t) = \frac{t^2}{2+t} + \frac{1}{3}\int_0^1 \frac{s^2}{(2+t)}\frac{1}{(2+u(s))}ds; \quad t \in [0,1]. \tag{43}$$

One can observe that integral equation (42) is a special case of (39), in which
$f(s,t) = \frac{1}{3(2+u(s))}$;
$K(t,s) = \frac{s^2}{(2+t)}$;
$g(t) = \frac{t^2}{2+t}$.
Indeed, functions $f(s,t)$, $g(t)$ and $K(t,s)$ are continuous. Thus the assumptions with respect to aforesaid functions are satisfied.
Further, for all $u, v \in \mathbb{R}$, we have

$$\begin{aligned} 0 \leq |f(s,u) - f(s,v)|^2 &\leq \left|\frac{1}{3(2+u)} - \frac{1}{3(2+v)}\right|^2 \\ &\leq \frac{1}{9}|v-u|^2 \\ &\leq \frac{1}{2(2^{1.2})}|v-u|^2 e^{-0.1} \end{aligned}$$

i.e.,

$$|f(s,u) - f(s,v)|^2 \leq \frac{1}{2s^{\epsilon}}|v-u|^2 e^{-\tau}$$

for $\tau = 0.1, \epsilon = 1.2$ and $s = 2$. Hence, condition (2) of Theorem 4.6 is verified. For condition (1), we have

$$\max_{a\leq t\leq b}\int_0^1 |K(t,z)|^2 ds = \max_{a\leq t\leq b}\int_0^1 \left(\frac{s^2}{2+t}\right)^2 = \max_{a\leq t\leq b}\frac{1}{5(2+t)^2} \leq \frac{1}{(b-a)}.$$

Hence condition (1) is also proved for all, $t \in [0,1]$. Consequently, all the conditions of Theorem 4.6 are fulfilled and hence the integral equation (42) has a solution in $X = C([0,1],\mathbb{R})$. Furthermore, the approximate solution of the integral equation (42) is

$$u(t) = \frac{0.15878 + 3t^2}{3(2+t)}, \tag{44}$$

which is demonstrated geometrically by Figure(6).

Making use of the obtained approximate solution and (43), we acquire

$$Tu(t) = \frac{t^2}{2+t} + \frac{1}{3(2+t)}\int_0^1 \frac{3s^2(2+s)}{2s^2 + 6s + 12.15878}ds; \quad t \in [0,1]. \tag{45}$$

Figure(7) represents the plot of the integral equation 45.
From Figure(6) and Figure(7), it is evident that the plot of approximate solution of intrgral equation (42) almost coincide with the plot of integral equation (45). This authenticates that the approximate solution mentioned in equation (44) is a fixed point of (42) and hence is a solution of integral equation (42).
Moreover, the error between the value of $u(t)$ and the approximate solution is visualized in Figure(8).

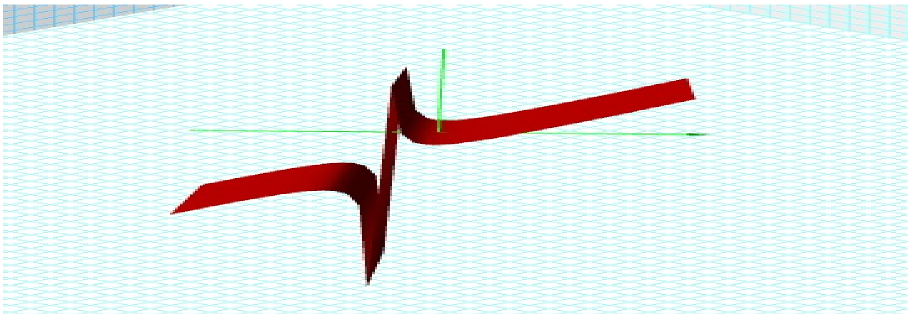

Figure 6: Geometrical representation of approximate solution of (42), 3D view.

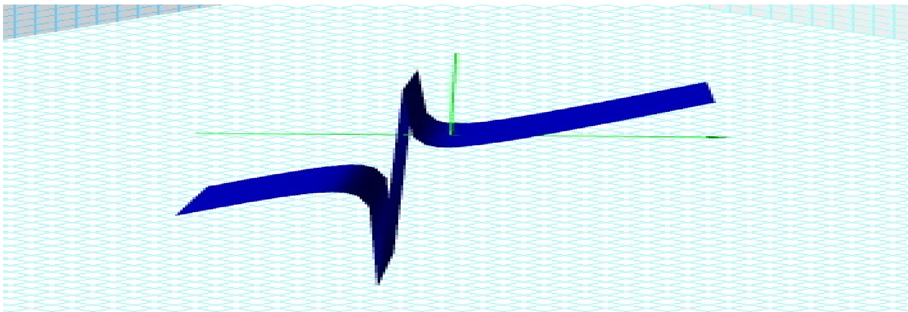

Figure 7: Geometrical representation of integral equation (45), 3D view.

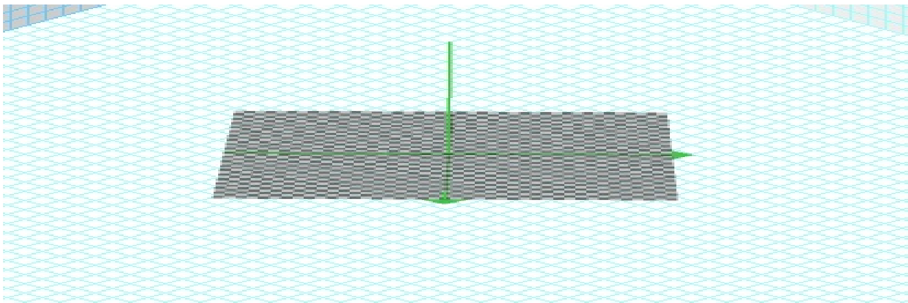

Figure 8: Error between approximate solution and integral equation.

Open Problem: For future reading, as an application an open open problem is suggested as follows:
A discretized population balance for continuous systems at steady state can be modeled by the following integral equation

$$f(t) = \frac{a}{2(1+2a)} \int_0^t f(t-x)f(x)dx + e^{-t}.$$

Whether the existence of solution of the above integral equation can be derived from results established in this note?

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