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Functional Equation and Its Modular Stability With and Without Δ_p−Condition

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Abstract. Mixed type is a further step of development in functional equations. In this paper, the authors made an attempt to introduce such equation of the following form with its general solution

$$h(py+z)+h(py-z)+h(y+pz)+h(y-pz)=(p+p^2)[h(y+z)+h(y-z)]+2h(py)-2(p^2+p-1)h(y)$$

for all $y, z \in \mathbb{R}, p \neq 0, \pm 1$. Also, without Fatou property authors investigate its various stabilities related to Ulam problem in modular space by considering with and without Δ_p−condition.

1. Introduction

For the detailed study on Ulam problem and its recent developments called generalized Hyers-Ulam-Rassias stability, one can refer [1, 8, 11]. In 1950, Nakano [7] established the modular linear spaces and further developed by many authors, one can refer [5, 6, 9]. The definitions related to our main theorem related to modular space can be referred in [3, 4].

In 2015, Abasalt Bodaghi et al.[1] investigated the stabilities of following mixed type equation

$$h(3y+z)-5h(2y+z)+h(2y-z)+10h(y+z)-5h(y-z)=10h(z)+4h(2y)-8h(y)$$

for all $y, z \in \mathbb{R}$.

In 2016, Pasupathi Narasimman et al.[8] introduced the equations quintic and sextic, respectively of the form

$$\begin{aligned} &p[h(py-z)+h(py+z)]+h(y-pz)+h(y+pz)\\ &\quad =(p^4+p^2)[h(y-z)+h(y+z)]+2(p^6-p^4-p^2+1)h(y),\\ &h(py-z)+h(py+z)+h(y-pz)+h(y+pz)\\ &\quad =(p^4+p^2)[h(y-z)+h(y+z)]+2(p^6-p^4-p^2+1)[h(y)+h(z)] \end{aligned}$$

with $p \in \mathbb{R} - \{0, \pm 1\}$ also discussed their various stabilities related to Ulam problem.

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In 2017, authors Hark-Mahn Kim and Young Soon Hong [2] investigated the alternative stability theorem in a modular space using Δ_3-condition of a modified quadratic equation.

In 2019, authors John Michael Rassias, Hemen Dutta and Narasimman Pasupathi [10] investigated Ulam stability problem in non-Archimedean intuitionistic fuzzy normed spaces of the generalized quartic equation

$$h(py-z)+h(py+z)+h(y-pz)+h(y+pz)=2p^2\{h(y-z)+h(y+z)\}+2(p^2-1)^2\{h(y)+h(z)\}$$

where $p\neq 0,\pm 1$. Motivation from the above literature, the authors made an attempt to introduce a new mixed type equation satisfied by $h(x)=x+x^3$ of the form

$$\begin{aligned} &h(py-z)+h(py+z)+h(y-pz)+h(y+pz) \\ &\quad =(p+p^2)[h(y-z)+h(y+z)]+2h(py)-2(p^2+p-1)h(y) \end{aligned} \tag{1}$$

for all $y,z\in\mathbb{R}, p\neq 0,\pm 1$. Mainly, authors investigate various stabilities concerning Ulam problem in modular spaces and its general solution.

In Section-2 and Section-3, authors obtain the solution of (1) in additive case and cubic case, respectively. Authors provide the various stabilities of equation (1) in modular space in Sections-4 for additive case and in Section-5 for cubic case, and we given the conclusion in Section-6.

2. General Solution of (1): Additive Case

Lemma 2.1. *Let X and Y are linear spaces, a mapping $h:X\longrightarrow Y$ is additive and odd if h satisfies*

$$h(py-z)+h(py+z)+h(y-pz)+h(y+pz)=(p+p^2)[h(y-z)+h(y+z)]-2(p^2-1)h(y) \tag{2}$$

for all $y,z\in X$.

Proof. Consider h satisfies (2). Replacing (y,z) by $(0,0)$ and $(y,0)$ in (2), we get $h(0)=0$ and

$$h(py)=ph(y) \tag{3}$$

respectively, for all $y\in X$. Therefore, h is additive function. Let $(y,z)=(0,y)$ in (2) and by (3), we reached

$$h(-y)=-h(y);\quad y\in X. \tag{4}$$

Thus h is an odd function. □

Theorem 2.2. *A function $h:X\to Y$ is a solution of (2) iff $A(y)$ is the diagonal of the additive symmetric map $A_1:X\to Y$ such that h is of the form $h(y)=A(y)$ for all $y\in X$.*

Proof. Let h satisfies (2) when h is additive. We can rewrite (2) as follows

$$\begin{aligned} &h(y)+\frac{1}{2(p^2-1)}h(py+z)+\frac{1}{2(p^2-1)}h(py-z)+\frac{1}{2(p^2-1)}h(y+pz) \\ &\quad +\frac{1}{2(p^2-1)}h(y-pz)-\frac{p+p^2}{2(p^2-1)}h(y+z)-\frac{p+p^2}{2(p^2-1)}h(y-z)=0 \end{aligned} \tag{5}$$

for all $y,z\in X$. Theorems 3.5 and 3.6 in [12] implies that h is of the form

$$h(y)=A^1(y)+A^0(y) \tag{6}$$

for all $y\in X$, $A^0(y)=A^0$ and for $i=1$, $A^i(y)$ is the diagonal of the i-additive symmetric map $A_i:X^i\to Y$. We get $A^0(y)=A^0=0$ and h is odd, by $h(0)=0$ and $h(-y)=-h(y)$, respectively. It follows that $h(y)=A^1(y)$.

Conversely, $A^1(y)$ is the diagonal of the additive symmetric map $A_1:X^1\to Y$ such that $h(y)=A^1(y)$ for all $y\in X$. From

$$A^1(y+z)=A^1(y)+A^1(z),\quad A^1(ry)=r^1A^1(y);\quad y,z\in X, r\in Q,$$

we see that h satisfies (2) and this completes the proof of Theorem 2.2. □

3. General Solution of (1): Cubic Case

Lemma 3.1. *Let X and Y are linear spaces, a mapping $h : X \longrightarrow Y$ is cubic and odd if h satisfies*

$$h(py+z)+h(py-z)+h(y+pz)+h(y-pz)=(p+p^2)[h(y+z)+h(y-z)]+2(p^3-p^2-p+1)h(y) \tag{7}$$

for all $y, z \in X$.

Proof. Consider h satisfies (7). Replacing (y, z) by $(0, 0)$ and $(y, 0)$ in (7), we get $h(0) = 0$ and

$$h(py) = p^3 h(y) \tag{8}$$

respectively, for all $y \in X$. Therefore, h is cubic function. Let (y, z) by $(0, y)$ in (7) and using (8), we obtain

$$h(-y) = -h(y); \quad y \in X. \tag{9}$$

Thus h is an odd function. □

Theorem 3.2. *A function $h : X \to Y$ is a solution of (7) iff $C^3(y)$ is the diagonal of the 3-additive symmetric map $C_3 : X^3 \to Y$ such that h is of the form $h(y) = C^3(y)$ for all $y \in X$.*

Proof. Let h satisfies (7) when h is cubic. We can rewrite (7) as follows

$$\begin{aligned} h(y) &+ \frac{1}{2(p^2-1)}h(py+z) + \frac{1}{2(p^2-1)}h(py-z) + \frac{1}{2(p^2-1)}h(y+pz) \\ &+ \frac{1}{2(p^2-1)}h(y-pz) - \frac{p+p^2}{2(p^2-1)}h(y+z) - \frac{p+p^2}{2(p^2-1)}h(y-z) = 0 \end{aligned} \tag{10}$$

for all $y, z \in X$. Theorems 3.5 and 3.6 in [12] implies that h is of the form

$$h(y) = C^3(y) + C^2(y) + C^1(y) + C^0(y) \tag{11}$$

for all $y \in X$, where $C^0(y) = C^0$ and $i = 1, 2, 3$, $C^i(y)$ is the diagonal of the i-additive symmetric map $C_i : X^i \to Y$. We get $C^0(y) = C^0 = 0$ and h is odd, by $h(0) = 0$ and $h(-y) = -h(y)$, respectively. Therefore $C^2(y) = 0$. It follows that $h(y) = C^3(y) + C^1(y)$. By (8) and $C^n(ry) = r^n C^n(y)$ for all $y \in X$ and $r \in Q$, we obtain $n^1 C^1(y) = n^3 C^1(y)$. Hence, $C^1(x) = 0$ for all $y \in X$. Therefore $h(y) = C^3(y)$.

Conversely, $C^3(y)$ is the diagonal of the 3−additive symmetric map $C_3 : X^3 \to Y$ such that $h(y) = C^3(y)$ for all $y \in X$. From

$$\begin{aligned} &C^3(y+z) = C^3(y) + 3C^{2,1}(y,z) + 3C^{1,2}(y,z) + C^3(z),\ C^3(ry) = r^3C^3(y), \\ &C^{2,1}(y,rz) = r^1C^{2,1}(y,z),\ C^{2,1}(ry,z) = r^2C^{2,1}(y,z),\ C^{1,2}(y,rz) = r^2C^{1,2}(y,z),\ C^{1,2}(ry,z) = r^1C^{1,2}(y,z) \end{aligned}$$

for all $y, z \in X, r \in Q$, we see that h satisfies (7) and this completes the proof of Theorem 3.2. □

4. Stability of Functional Equation (1): Additive Case

Assume that the linear space X, μ−complete convex modular space $\mathbb{X}_\mu$ in the following theorems and corollaries. Now, we obtain the stability of (1) called generalized Hyers-Ulam-Rassias in modular spaces without Δ_p−condition and the Fatou property. Here after, we use the following notation

$$D_A h(y,z) = h(py+z) + h(py-z) + h(y+pz) + h(y-pz) - (p+p^2)[h(y+z)+h(y-z)] + 2(p^2-1)h(y)$$

for all $y, z \in X$.

Theorem 4.1. *Let a mapping* $h : X \to \mathbb{X}_\mu$ *satisfies*

$$\mu(D_A h(y,z)) \leq \nu(y,z) \tag{12}$$

and a mapping $\nu : X^2 \to [0,\infty)$ *such that*

$$\zeta(y,z) = \sum_{j=0}^{\infty} \frac{\nu(p^j y, p^j z)}{p^j} < \infty, \quad y,z \in X. \tag{13}$$

Then there exists $A_1 : X \to \mathbb{X}_\mu$ *a unique additive mapping defined by* $A_1(y) = lim_{n\to\infty} \frac{h(p^n y)}{p^n}, y \in X$, *which satisfies (2) and*

$$\mu(h(y) - A_1(y)) \leq \frac{1}{2p}\zeta(y,0), \quad \forall y \in X. \tag{14}$$

Proof. Substituting $z = 0$ in (12), we obtain

$$\mu(h(py) - ph(y)) \leq \frac{1}{2}\nu(y,0) \tag{15}$$

and so

$$\mu\left(h(y) - \frac{h(py)}{p}\right) \leq \frac{1}{2p}\nu(y,0), \quad \forall y \in X. \tag{16}$$

By induction on n, we arrive

$$\mu\left(h(y) - \frac{h(p^n y)}{p^n}\right) \leq \frac{1}{2}\sum_{j=0}^{n-1} \frac{\nu(p^j y, 0)}{p^{j+1}}, \quad \forall y \in X. \tag{17}$$

Substituting y by $p^m y$ in (17), we obtain

$$\mu\left(\frac{h(p^m y)}{p^m} - \frac{h(p^{n+m} y)}{p^{n+m}}\right) \leq \frac{1}{2p}\sum_{j=m}^{n+m-1} \frac{\nu(p^j y, 0)}{p^j} \tag{18}$$

by assumption (13) it converges to zero as $m \to \infty$. Hence, by inequality (18) the sequence $\left\{\frac{h(p^n y)}{p^n}\right\}, \ \forall y \in X$ is $\mu-$Cauchy and hence it is convergent in $\mathbb{X}_\mu$ since $\mathbb{X}_\mu$ is $\mu-$complete. Thus, a mapping $A_1 : X \to \mathbb{X}_\mu$ is defined by

$$A_1(y) = \mu - \lim_{n\to\infty} \left\{\frac{h(p^n y)}{p^n}\right\}$$

for all $y \in X$, which implies

$$\lim_{n\to\infty} \mu\left(\frac{h(p^n y)}{p^n} - A_1(y)\right) = 0, \quad \forall y \in X.$$

Next, we claim the mapping A_1 satisfies (2). Setting $(y,z) = (p^n y, p^n z)$ in (12), and dividing the resultant by p^n, we arrive

$$\frac{\mu(D_A h(p^n y, p^n z))}{p^n} \leq \frac{\nu(p^n y, p^n z)}{p^n}, \quad \forall y,z \in X.$$

Hence, by property $\mu(\alpha u) \leq \alpha\mu(u), 0 < \alpha \leq 1, u \in \mathbb{X}_\mu$, we get

$$\begin{aligned}
&\mu\left(\frac{1}{4p^2+2p+3}DA_1(y,z)\right) \\
&\quad\leq \mu\left(\frac{1}{4p^2+2p+3}DA_1(y,z) - \frac{Dh(p^n y, p^n z)}{(4p^2+2p+3)p^n} + \frac{Dh(p^n y, p^n z)}{(4p^2+2p+3)p^n}\right) \\
&\quad\leq \frac{1}{4p^2+2p+3}\mu\left(A_1(py+z) - \frac{h(p^n(py+z))}{p^n}\right) + \frac{1}{4p^2+2p+3}\mu\left(A_1(py-z) - \frac{h(p^n(py-z))}{p^n}\right) \\
&\qquad + \frac{1}{4p^2+2p+3}\mu\left(A_1(y+pz) - \frac{h(p^n(y+pz))}{p^n}\right) + \frac{1}{4p^2+2p+3}\mu\left(A_1(y-pz) - \frac{h(p^n(y-pz))}{p^n}\right) \\
&\qquad + \frac{p+p^2}{4p^2+2p+3}\mu\left(A_1(y+z) - \frac{h(p^n(y+z))}{p^n}\right) + \frac{p+p^2}{4p^2+2p+3}\mu\left(A_1(y-z) - \frac{h(p^n(y-z))}{p^n}\right) \\
&\qquad + \frac{2(p^2-1)}{4p^2+2p+3}\mu\left(A_1(y) - \frac{h(p^n y)}{p^n}\right) + \frac{1}{4p^2+2p+3}\mu\left(\frac{Dh(p^n y, p^n z)}{p^n}\right)
\end{aligned}$$

for all $y, z \in X$ and n is positive integers. We obtain $\mu\left(\frac{1}{4p^2+2p+3}DA_1(y,z)\right) = 0$, if $n \to \infty$. Hence $DA_1(y,z) = 0$ for all $y, z \in X$. Thus A_1 satisfies (2) and hence it is additive. Since $\sum_{i=0}^{n} \frac{1}{p^{i+1}} + \frac{1}{p} \leq 1$ for all $n \in \mathbb{N}$, by the convexity of modular μ and (15), we arrive

$$\begin{aligned}
\mu(h(y) - A_1(y)) &= \mu\left(h(y) - \frac{h(p^n y)}{p^n}\right) + \mu\left(\frac{h(p^n y)}{p^n} - A_1(y)\right) \\
&\leq \frac{1}{2}\sum_{i=0}^{n-1} \frac{1}{p^{i+1}}\nu(p^i y, 0) + \mu\left(\frac{h(p^n y)}{p^n} - A_1(y)\right) \leq \frac{1}{2}\sum_{i=0}^{\infty} \frac{1}{p^{i+1}}\nu(p^i y, 0) = \frac{1}{2p}\zeta(y, 0)
\end{aligned}$$

for all $y \in X$. Now, to prove the uniqueness of A_1, we consider that there exists a additive mapping $D_1 : X \to \mathbb{X}_\mu$ satisfying

$$\mu(h(y) - D_1(y)) \leq \frac{1}{2}\sum_{j=0}^{\infty} \frac{1}{p^{j+1}}\nu(p^j y, 0), \quad \forall y \in X.$$

But, if $A_1(y_0) \neq D_1(y_0)$ for some $y_0 \in X$. Then there exists a constant $\epsilon > 0$ which is positive such that $\epsilon < \mu(A_1(y_0) - D_1(y_0))$. By (13), there is a positive integer $n_0 \in \mathbb{N}$ such that $\sum_{j=n_0}^{\infty} \frac{1}{p^{j+1}}\nu(p^j y, 0) < \frac{\epsilon}{2}$. Since A_1 and D_1 are additive mappings, by $A_1(p^{n_0} y_0) = p^{n_0} A_1(y_0)$ and $D_1(p^{n_0} y_0) = p^{n_0} D_1(y_0)$, we arrive

$$\begin{aligned}
&\epsilon < \mu(A_1(y_0) - D_1(y_0)) \\
&\quad= \mu\left(\frac{A_1(p^{n_0} y_0) - h(p^{n_0} y_0)}{p^{n_0}} + \frac{h(p^{n_0} y_0) - D_1(p^{n_0} y_0)}{p^{n_0}}\right) \\
&\quad\leq \frac{1}{p^{n_0}}\mu(A_1(p^{n_0} y_0) - h(p^{n_0} y_0)) + \frac{1}{p^{n_0}}\mu(h(p^{n_0} y_0) - D_1(p^{n_0} y_0)) \\
&\quad\leq \frac{1}{p^{n_0}}\sum_{j=0}^{\infty} \frac{\nu(p^{j+n_0} y_0, 0)}{p^{j+1}} \leq \sum_{j=n_0}^{\infty} \frac{\nu(p^j y_0, 0)}{p^{j+1}} < \epsilon,
\end{aligned}$$

which implies a contradiction. Therefore the mapping A_1 is a unique additive mapping near h satisfying (14) in $\mathbb{X}_\mu$. □

Letting $\nu(y,z) = \epsilon$ and $\nu(y,z) = \epsilon(||y||^m + ||z||^m)$ in Theorem 4.1, we obtain Hyers-Ulam and generalized Hyers-Ulam stabilities, respectively in the following corollaries.

Corollary 4.2. *Let a mapping $h : X \to \mathbb{X}_\mu$ satisfying*

$$\mu(D_A h(y,z)) \leq \epsilon, \quad \forall y,z \in X$$

for some $\epsilon > 0$. Then there exists $A_1 : X \to \mathbb{X}_\mu$, a unique additive mapping satisfies (2) and

$$\mu(h(y) - A_1(y)) \leq \frac{\epsilon}{2(p-1)} \tag{19}$$

for all $y \in X$ and $p \neq 1$.

Corollary 4.3. *If $h : X \to \mathbb{X}_\rho$ a mapping satisfies*

$$\mu(D_A h(y,z)) \leq \epsilon\left(\|y\|^m + \|z\|^m\right), \quad \forall y,z \in X,\ m < 1$$

a real numbers $\epsilon > 0$, then there exists $A_1 : X \to \mathbb{X}_\rho$ a unique additive mapping satisfying

$$\mu(h(y) - A_1(y)) \leq \frac{\epsilon}{2(p-p^m)}\|y\|^m, \quad \forall y \in X \tag{20}$$

where $y \neq 0$ and $p^m < p$.

Assuming μ satisfies the Δ_p−condition and if there exists $\beta > 0$ defined by $\mu(py) \leq \beta\mu(y)$ for all $y \in \mathbb{X}_\mu$.

Theorem 4.4. *Letting $h : X \to \mathbb{X}_\mu$ and $\nu : X^2 \to [0,\infty)$ be the mappings satisfies*

$$\mu(D_A h(y,z)) \leq \nu(y,z) \tag{21}$$

and

$$\Psi(y,z) = \sum_{j=1}^{\infty} \frac{\beta^{2j}}{p^j}\nu\left(\frac{y}{p^j}, \frac{z}{p^j}\right) < \infty, \quad \forall y,z \in X. \tag{22}$$

Then there exists $A_2 : X \to \mathbb{X}_\mu$ a unique additive mapping such that $A_2(y) = lim_{n\to\infty} p^n h\left(\frac{y}{p^n}\right)$ which satisfies(2) and

$$\mu(h(y) - A_2(y)) \leq \frac{1}{2p}\Psi(y,0), \quad \forall y \in X. \tag{23}$$

Proof. The equation (15), implies that

$$\mu\left(h(y) - ph\left(\frac{y}{p}\right)\right) \leq \frac{1}{2}\nu\left(\frac{y}{p}, 0\right), \quad y \in X. \tag{24}$$

Hence, by the convexity μ, we have

$$\mu\left(h(y) - p^2 h\left(\frac{y}{p^2}\right)\right) \\ \leq \frac{1}{p}\mu\left(ph(y) - p^2 h\left(\frac{y}{p}\right)\right) + \frac{1}{p}\mu\left(p^2 h\left(\frac{y}{p}\right) - p^3 h\left(\frac{y}{p^2}\right)\right) \leq \frac{\beta}{2p}\nu\left(\frac{y}{p}, 0\right) + \frac{\beta^2}{2p}\nu\left(\frac{y}{p^2}, 0\right), \forall y \in X.$$

Then by induction on $n > 1$, we have

$$\mu\left(h(y) - p^n h\left(\frac{y}{p^n}\right)\right) \leq \frac{1}{2}\sum_{j=1}^{n-1} \frac{\beta^{2j-1}}{p^j}\nu\left(\frac{y}{p^j}, 0\right) + \frac{1}{2}\frac{\beta^{2(n-1)}}{p^{n-1}}\nu\left(\frac{y}{p^n}, 0\right) \tag{25}$$

for all $y \in X$. Considering (25) holds true for n and we deduce the following by using the convexity of μ,

$$\begin{aligned}
&\mu\left(h(y) - p^{n+1}h\left(\frac{y}{p^{n+1}}\right)\right) \qquad (26)\\
&\quad = \frac{1}{p}\mu\left(ph(y) - p^2h\left(\frac{y}{p}\right)\right) + \frac{1}{p}\mu\left(p^2h\left(\frac{y}{p}\right) - p^{n+2}h\left(\frac{y}{p^{n+1}}\right)\right)\\
&\quad \le \frac{\beta}{p}\mu\left(h(y) - ph\left(\frac{y}{p}\right)\right) + \frac{\beta^2}{p}\mu\left(h\left(\frac{y}{p}\right) - p^nh\left(\frac{y}{p^{n+1}}\right)\right)\\
&\quad \le \frac{\beta}{2p}\nu\left(\frac{y}{p},0\right) + \frac{\beta^2}{2p}\sum_{j=1}^{n-1}\frac{\beta^{2j-1}}{p^j}\nu\left(\frac{y}{p^{j+1}},0\right) + \frac{\beta^2}{2p}\frac{\beta^{2(n-1)}}{p^{n-1}}\nu\left(\frac{y}{p^{n+1}},0\right)\\
&\quad = \frac{1}{2}\sum_{j=1}^{n}\frac{\beta^{2j-1}}{p^j}\nu\left(\frac{y}{p^j},0\right) + \frac{1}{2}\frac{\beta^{2n}}{p^n}\nu\left(\frac{y}{p^{n+1}},0\right).
\end{aligned}$$

The above inequality proves (25) for $n+1$. Substituting y by $\frac{y}{p^m}$ in (25), we arrive

$$\begin{aligned}
&\mu\left(p^mh\left(\frac{y}{p^m}\right) - p^{n+m}h\left(\frac{y}{p^{n+m}}\right)\right)\\
&\quad \le \beta^m\mu\left(h\left(\frac{y}{p^m}\right) - p^nh\left(\frac{y}{p^{n+m}}\right)\right)\\
&\quad \le \beta^m\frac{1}{2}\sum_{j=1}^{n-1}\frac{\beta^{2j-1}}{p^j}\nu\left(\frac{y}{p^{j+m}},0\right) + \beta^m\frac{1}{2}\frac{\beta^{2(n-1)}}{p^{n-1}}\nu\left(\frac{y}{p^{n+m}},0\right)\\
&\quad \le \frac{p^m}{2\beta^m}\sum_{j=m+1}^{n+m-1}\frac{\beta^{2j-1}}{p^j}\nu\left(\frac{y}{p^j},0\right) + \frac{p^m}{2\beta^m}\frac{\beta^{2(n+m-1)}}{p^{n+m-1}}\nu\left(\frac{y}{p^{n+m}},0\right)
\end{aligned}$$

by (22) it converges to zero as $m \to \infty$. Hence, $\{p^nh\left(\frac{y}{p^n}\right)\}$ is $\mu-$Cauchy for all $y \in X$ and hence it is $\mu-$convergent in $\mathbb{X}_\mu$ since $\mathbb{X}_\mu$ is $\mu-$complete. Hence, we have

$$A_2(y) = \mu - \lim_{n\to\infty} p^nh\left(\frac{y}{p^n}\right) \qquad (27)$$

for all $y \in X$, which implies

$$\lim_{n\to\infty}\mu\left(p^nh\left(\frac{y}{p^n}\right) - A_2(y)\right) = 0, \quad \forall y \in X.$$

Hence by the Δ_p-condition, we arrive the following by taking $n \to \infty$.

$$\begin{aligned}
&\mu\left(h(y) - A_2(y)\right)\\
&\quad \le \frac{1}{p}\mu\left(ph(y) - p^{n+1}h\left(\frac{y}{p^n}\right)\right) + \frac{1}{p}\mu\left(p^{n+1}h\left(\frac{y}{p^n}\right) - pA_2(y)\right)\\
&\quad \le \frac{\beta}{p}\mu\left(h(y) - p^nh\left(\frac{y}{p^n}\right)\right) + \frac{\beta}{p}\mu\left(p^nh\left(\frac{y}{p^n}\right) - A_2(y)\right)\\
&\quad \le \frac{\beta}{2p}\sum_{j=1}^{n-1}\frac{\beta^{2j-1}}{p^j}\nu\left(\frac{y}{p^j},0\right) + \frac{\beta}{2p}\frac{\beta^{2(n-1)}}{p^{n-1}}\nu\left(\frac{y}{p^n},0\right) + \frac{\beta}{p}\mu\left(p^nh\left(\frac{y}{p^n}\right) - A_2(y)\right)\\
&\quad \le \frac{1}{2p}\sum_{j=1}^{\infty}\frac{\beta^{2j}}{p^j}\nu\left(\frac{y}{p^j},0\right) \le \frac{1}{2p}\Psi(y,0).
\end{aligned}$$

Next, we prove A_2 satisfies (2). Assuming $(y,z) = \left(\frac{y}{p^n}, \frac{z}{p^n}\right)$ in (21), and multiplying the resultant by p^n, we obtain

$$\mu\left(p^n D_A h\left(\frac{y}{p^n}, \frac{z}{p^n}\right)\right) \le \beta^n \nu\left(\frac{y}{p^n}, \frac{z}{p^n}\right) \le \frac{\beta^{2n}}{p^n}\nu\left(\frac{y}{p^n}, \frac{z}{p^n}\right)$$

as $n \to \infty$, which tends to zero. Hence, the property $\mu(\gamma u) \le \gamma\mu(u), 0 < \gamma \le 1, u \in \mathbb{X}_\mu$ implies that

$$\begin{aligned}
&\mu\left(\frac{1}{4p^2+2p+3}D_A A_2(y,z)\right)\\
&\le \mu\left(\frac{1}{4p^2+2p+3}D_A A_2(y,z) - p^n\frac{D_A h(\frac{y}{p^n},\frac{z}{p^n})}{(4p^2+2p+3)} + p^n\frac{D_A h(\frac{y}{p^n},\frac{z}{p^n})}{(4p^2+2p+3)}\right)\\
&\le \frac{1}{4p^2+2p+3}\mu\left(A_2(py+z) - p^n h\left(\frac{py+z}{p^n}\right)\right) + \frac{1}{4p^2+2p+3}\mu\left(A_2(py-z) - p^n h\left(\frac{py-z}{p^n}\right)\right)\\
&\quad + \frac{1}{4p^2+2p+3}\mu\left(A_2(y+pz) - p^n h\left(\frac{y+pz}{p^n}\right)\right) + \frac{1}{4p^2+2p+3}\mu\left(A_2(y-pz) - p^n h\left(\frac{y-pz}{p^n}\right)\right)\\
&\quad + \frac{p+p^2}{4p^2+2p+3}\mu\left(A_2(y+z) - p^n h\left(\frac{y+z}{p^n}\right)\right) + \frac{p+p^2}{4p^2+2p+3}\mu\left(A_2(y-z) - p^n h\left(\frac{y-z}{p^n}\right)\right)\\
&\quad + \frac{2(p^2-1)}{4p^2+2p+3}\mu\left(A_2(y) - p^n h\left(\frac{y}{p^n}\right)\right) + \frac{1}{4p^2+2p+3}\mu\left(p^n D_A h\left(\frac{y}{p^n}, \frac{z}{p^n}\right)\right), \quad \forall y,z \in X.
\end{aligned}$$

As the limit $n \to \infty$, we obtain

$$\mu\left(\frac{1}{4p^2+2p+3}D_A A_2(y,z)\right) = 0$$

for all $y,z \in X$. Hence, $D_A A_2(y,z) = 0$ and A_2 satisfies (2). Hence, it is additive. To prove the uniqueness of A_2, assume that $D_2 : X \to \mathbb{X}_\rho$, a additive mapping satisfies

$$\mu(h(y) - D_2(y)) \le \frac{1}{2p}\sum_{j=1}^{\infty}\frac{\beta^{2j}}{p^j}\nu\left(\frac{y}{p^j}, 0\right), \quad \forall y \in X.$$

Since A_2 and D_2 are additive mappings and $p^n A_2\left(\frac{x}{p^n}\right) = A_2(x), p^n D_2\left(\frac{x}{p^n}\right) = D_2(x)$ implies that

$$\begin{aligned}
&\mu(D_2(y) - A_2(y))\\
&= \mu\left(\frac{p^{n+1}}{p}\left(D_2\left(\frac{y}{p^n}\right) - h\left(\frac{y}{p^n}\right)\right) + \frac{p^{n+1}}{p}\left(h\left(\frac{y}{p^n}\right) - A_2\left(\frac{y}{p^n}\right)\right)\right)\\
&\le \frac{\beta^{n+1}}{p}\mu\left(D_2\left(\frac{y}{p^n}\right) - h\left(\frac{y}{p^n}\right)\right) + \frac{\beta^{n+1}}{p}\mu\left(h\left(\frac{y}{p^n}\right) - A_2\left(\frac{y}{p^n}\right)\right)\\
&\le \frac{\beta^{n+1}}{p}\frac{1}{2p}\sum_{j=1}^{\infty}\frac{\beta^{2j}}{p^j}\nu\left(\frac{y}{p^{j+n}}, 0\right) + \frac{\beta^{n+1}}{p}\frac{1}{2p}\sum_{j=1}^{\infty}\frac{\beta^{2j}}{p^j}\nu\left(\frac{y}{p^{j+n}}, 0\right)\\
&\le \frac{\beta^{n+1}}{p^2}\sum_{j=1}^{\infty}\frac{\beta^{2j}}{p^j}\nu\left(\frac{y}{p^{j+n}}, 0\right) \le \frac{\beta p^n}{p^2\beta^n}\sum_{j=1}^{\infty}\frac{\beta^{2(j+n)}}{p^{j+n}}\nu\left(\frac{y}{p^{j+n}}, 0\right)
\end{aligned}$$

for all $y \in X$ and as $n \to \infty$ it tends to zero. Therefore, A_2 satisfying (23) and is a unique additive mapping. □

Considering $\nu(y,z) = \epsilon$ and $\nu(y,z) = \epsilon\left(\|y\|^m + \|z\|^m\right)$ in Theorem 4.4, we obtain the following Hyers-Ulam and Hyers-Ulam-Rassias stabilities, respectively.

Corollary 4.5. *Let a mapping $h : X \to \mathbb{X}_\mu$ satisfying*

$$\mu(D_A h(y,z)) \leq \epsilon$$

for all $y, z \in X$, $\epsilon > 0$. Hence there exists a unique additive mapping $A_2 : X \to \mathbb{X}_\mu$ which satisfies (2) and

$$\mu(h(y) - A_2(y)) \leq \frac{\epsilon\beta^2}{2p(p-\beta^2)} \tag{28}$$

for all $y \in X$ and for some $\beta^2 < p$.

Corollary 4.6. *If $h : X \to \mathbb{X}_\mu$ a mapping satisfies*

$$\mu(D_A h(y,z)) \leq \epsilon\left(\|y\|^m + \|z\|^m\right)$$

for all $y, z \in X$. Then there exists $A_2 : X \to \mathbb{X}_\mu$ a unique additive mapping such that

$$\mu(h(y) - A_2(y)) \leq \frac{\epsilon\beta^2}{2p(p^{m+1}-\beta^2)}\|y\|^m \tag{29}$$

for all $y \in X, y \neq 0$, for given real numbers $\beta^2 < p^{m+1}$ and $\epsilon > 0$.

5. Stability of Functional Equation (1): Cubic Case

We obtain generalized Hyers-Ulam-Rassias stability of (1) in modular spaces without Δ_p−condition and the Fatou property. Here after, we use the following notation

$$D_C h(y,z) = h(py+z) + h(py-z) + h(y+pz) + h(y-pz) - (p+p^2)[h(y+z) + h(y-z)] - 2(p^3-p^2-p+1)h(y)$$

for all $y, z \in X$.

Theorem 5.1. *Considering $h : X \to \mathbb{X}_\mu$ a mapping satisfies*

$$\mu(D_C h(y,z)) \leq \nu(y,z) \tag{30}$$

and a mapping $\nu : X^2 \to [0,\infty)$ satisfies

$$\zeta(y,z) = \sum_{j=0}^{\infty} \frac{\nu(p^j y, p^j z)}{p^{3j}} < \infty, \quad \forall y, z \in X. \tag{31}$$

Then there exists $C_1 : X \to \mathbb{X}_\mu$ a unique cubic mapping defined by $C_1(y) = lim_{n\to\infty} \frac{h(p^n y)}{p^{3n}}$, $y \in X$ which satisfies the equation (7) and

$$\mu(h(y) - C_1(y)) \leq \frac{1}{2p^3}\zeta(y,0), \quad \forall y \in X \tag{32}$$

Proof. Assuming $y = 0$ in (30), we obtain

$$\mu(h(py) - p^3 h(y)) \leq \frac{1}{2}\nu(y,0) \tag{33}$$

and hence

$$\mu\left(h(y)-\frac{h(py)}{p^3}\right)\leq\frac{1}{2p^3}\nu(y,0),\quad\forall y\in X. \tag{34}$$

Generalizing, we arrive

$$\mu\left(h(y)-\frac{h(p^ny)}{p^{3n}}\right)\leq\frac{1}{2}\sum_{j=0}^{n-1}\frac{\nu(p^jy,0)}{p^{3(j+1)}},\quad\forall y\in X. \tag{35}$$

Substituting y by p^my in (35), we obtain

$$\mu\left(\frac{h(p^my)}{p^{3m}}-\frac{h(p^{n+m}y)}{p^{3(n+m)}}\right)\leq\frac{1}{2p^3}\sum_{j=m}^{n+m-1}\frac{\nu(p^jy,0)}{p^{3j}} \tag{36}$$

by the assumption (31) it converges to zero as $m\to\infty$. Hence (36) implies that the sequence $\left\{\frac{h(p^ny)}{p^{3n}}\right\}$ is $\mu-$Cauchy and therefore it is convergent in $\mathbb{X}_\mu$ since the $\mathbb{X}_\mu$ is $\mu-$complete. Hence we define $C_1 : X\to\mathbb{X}_\rho$ as

$$C_1(y)=\mu-\lim_{n\to\infty}\left\{\frac{h(p^ny)}{p^{3n}}\right\},\quad\forall y\in X,$$

which implies

$$\lim_{n\to\infty}\mu\left(\frac{h(p^ny)}{p^{3n}}-C_1(y)\right)=0,\quad\forall y\in X.$$

Here after we complete this proof by similar way of Theorem 4.1. □

Assuming $\nu(y,z)=\epsilon$ and $\nu(y,z)=\epsilon\left(\|y\|^m+\|z\|^m\right)$ in Theorem 5.1, we obtain the following stabilities called Hyers-Ulam and Hyers-Ulam-Rassias respectively.

Corollary 5.2. *Let a mapping $h : X\to\mathbb{X}_\mu$ satisfying*

$$\mu(D_Ch(y,z))\leq\epsilon$$

for all $y,z\in X$. Then there exists $C_1 : X\to\mathbb{X}_\mu$ a unique cubic mapping which satisfies (7) and

$$\mu(h(y)-C_1(y))\leq\frac{\epsilon}{2(p^3-1)} \tag{37}$$

for all $y\in X$, for some $\epsilon>0$ and $p^3>1$.

Corollary 5.3. *If $h : X\to\mathbb{X}_\mu$ a mapping satisfies*

$$\mu(D_Ch(y,z))\leq\epsilon\left(\|y\|^m+\|z\|^m\right),\quad\forall y,z\in X,$$

then there exists a unique cubic mapping $C_1 : X\to\mathbb{X}_\mu$ such that

$$\mu(h(y)-C_1(y))\leq\frac{\epsilon}{2(p^3-p^m)}\|y\|^m \tag{38}$$

for all $y\in X, y\neq 0$, for given real numbers $m<3$ and $\epsilon>0$.

Assuming a nontrivial convex modular μ satisfies the Δ_p-condition if there exists $\beta>0$ such that $\mu(py)\leq\beta\mu(y)$ for all $y\in\mathbb{X}_\mu$, where $\beta\geq p$ and hence $\mu(p^3y)\leq M\mu(y)$

Theorem 5.4. *If a mapping $h : X \to \mathbb{X}_\mu$ satisfies*

$$\mu(D_C h(y,z)) \leq \nu(y,z) \tag{39}$$

and $\nu : X^2 \to [0,\infty)$ is a mapping such that

$$\Psi(y,z) = \sum_{j=1}^{\infty} \frac{M^{2j}}{p^{3j}} \nu\left(\frac{y}{p^j}, \frac{z}{p^j}\right) < \infty, \ \ \forall y,z \in X. \tag{40}$$

Then a unique cubic mapping $C_2 : X \to \mathbb{X}_\rho$ exists and defined by $C_2(y) = lim_{n\to\infty} p^{3n} h\left(\frac{y}{p^n}\right), y \in X$, which satisfies (7) and

$$\mu(h(y) - C_2(y)) \leq \frac{1}{2p}\Psi(y,0), \forall y \in X. \tag{41}$$

Proof. Equation (33) implies that

$$\mu\left(h(y) - p^3 h\left(\frac{y}{p}\right)\right) \leq \frac{1}{2}\nu\left(\frac{y}{p}, 0\right), \ \ \forall y \in X. \tag{42}$$

Hence by the convexity μ, we arrive

$$\begin{aligned}
&\mu\left(h(y) - (p^3)^2 h\left(\frac{y}{p^2}\right)\right) \\
&\quad \leq \frac{1}{p^3}\mu\left(p^3 h(y) - (p^3)^2 h\left(\frac{y}{p}\right)\right) + \frac{1}{p^3}\mu\left((p^3)^2 h\left(\frac{y}{p}\right) - (p^3)^3 h\left(\frac{y}{p^2}\right)\right) \\
&\quad \leq \frac{M}{2p^3}\nu\left(\frac{y}{p}, 0\right) + \frac{M^2}{2p^3}\nu\left(\frac{y}{p^2}, 0\right), \ \ \forall y \in X.
\end{aligned}$$

Generalizing, we obtain

$$\mu\left(h(y) - (p^3)^n h\left(\frac{y}{p^n}\right)\right) \leq \frac{1}{2}\sum_{j=1}^{n-1} \frac{M^{2j-1}}{p^{3j}}\nu\left(\frac{y}{p^j}, 0\right) + \frac{1}{2}\frac{M^{2(n-1)}}{p^{3(n-1)}}\nu\left(\frac{y}{p^n}, 0\right) \tag{43}$$

for all $y \in X$. The rest of proof is similar to that of Theorem 4.4. □

Assuming $\nu(y,z) = \epsilon$ and $\nu(y,z) = \epsilon\left(||y||^m + ||z||^m\right)$ in Theorem 5.4, we obtain the following stabilities called Hyers-Ulam and Hyers-Ulam-Rassias respectively.

Corollary 5.5. *If a mapping $h : X \to \mathbb{X}_\mu$ satisfying*

$$\mu(D_C h(y,z)) \leq \epsilon, \ \ \forall y,z \in X,$$

then there exists $C_2 : X \to \mathbb{X}_\mu$ a unique cubic mapping which satisfies (7) and

$$\mu(h(y) - C_2(y)) \leq \frac{\epsilon M^2}{2p(p^3 - M^2)}, \ \ \forall y \in X, \tag{44}$$

for some $\epsilon > 0$ and $M^2 < p^3$.

Corollary 5.6. *If $h : X \to \mathbb{X}_\rho$ a mapping satisfies*

$$\mu(D_C h(y,z)) \leq \epsilon\left(||y||^m + ||z||^m\right), \ \ \forall y,z \in X,$$

then a unique cubic mapping $C_2 : X \to \mathbb{X}_\mu$ exists such that

$$\mu(h(y) - C_2(y)) \leq \frac{\epsilon M^2}{2p(p^{m+3} - M^2)}||y||^m, \ \ \forall y \in X, \tag{45}$$

where $y \neq 0$, for given real numbers $M^2 < p^{m+3}$ and $\epsilon > 0$.

6. Conclusion

We introduced a generalized mixed type of additive and cubic functional equation with its general solution and various stabilities concerning Ulam problem in modular spaces by considering with and without Δ_p-condition.

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