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Fixed Point Theorems for Quadruple Self-Mappings Satisfying Integral Type Inequalities

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Abstract. In this paper, we study the generalization of $(\mathcal{S}, \mathcal{F})$-rational contraction pair (h, g) to almost $(\mathcal{S}, \mathcal{F}, \Gamma)$-rational contraction pair (h, g) of integral type on metric spaces. Further, we use the new generalized notion to produce some fixed point theorems via integral inequalities. This work generalizes many results in the available literature. For demonstration we give examples which show that our work generalizes many results.

1. Introduction and Preliminaries

In fixed point theory, Banach's fixed point theorem is one of the basic concepts for the solution of the equation $f(x) = x$.

Theorem 1.1. *[4] If (X, d) is a complete metric space and $f : X \to X$ satisfies that $d(f(\mu^*), f(\sigma^*)) \leq \nu d(\mu^*, \sigma^*)$, for all μ^*, $\sigma^* \in X$ and $\nu \in (0, 1)$, then f has a fixed point in X.*

Banach's fixed point theorem has been applied to various scientific problems by a number of scientists in different fields like, equilibrium problems, image processing, selection and matching problems, the study of existence and uniqueness of solutions for the integral and differential equations and many others. Banach's fixed point theorem has generalized in many directions for the new fixed point theorems and a lot of applications of the new fixed point theorems were given [1–8, 11–18]. These generalizations were carried out either by the help of the generalization of the spaces or by the generalizations of the contractions. For instance, Bhaskar [6] introduced the concept of coupled fixed point theorem which was then followed for the triple and quadruple fixed point theorem. Berinde [9], worked on the triple fixed point theorem in partially ordered metric spaces. Liu [7], has studied quadruple fixed point theorems and has given the applications in partially ordered metric spaces with the supposition of mixed g-monotone property. Aydi et al [8], studied quadruple fixed point theorems depending on another function in partially ordered metric spaces and some applications were given. Bota et al [10], studied coupled fixed point theorems and their

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applications to the existence and uniqueness of solutions of a coupled system of integral equations on the finite interval $[0, \mathcal{F}]$. Mustafa et al. [12], proved fixed point theorems on partially ordered metric spaces and generalized some fixed point theorems in the available literature for the generalized (ϕ, ψ)-contractions in partially ordered metric spaces and has given the applications of their results. Shatanawi [19], proved coupled fixed point theorems in partially ordered metric spaces for two altering distance functions.

Nadler [26] worked on the fixed point theorem for multivalued contraction mappings. Branciari [27] generalized the Banach fixed point theorem for a single valued mapping by the help of integral type of contractions. Stojakovic [25] generalized the concept of the Banach's fixed point theorem by the help of integral type contractions by following the work due to Nadler [26]. sarwar et al. [31], produced a fixed point theorem by the help of integral type contractions and provided some applications of their results in dynamic programing. Some other related results can be studied in [28–30].

Inspired from the work of the scientists [20–31], in this paper, we further extend the notion of generalized almost $(\mathcal{S}, \mathcal{F})$-rational contraction pair (h, g) to the generalized almost $(\mathcal{S}, \mathcal{F}, \Gamma)$-rational contraction pair (h, g) of integral type and prove some new fixed point theorems. Our results generalize the work in [24] and many others in the literature.

Definition 1.2. *Let (X, d) be a metric space and h, g, $\mathcal{S}$, $\mathcal{F} : X \to X$ be quadruple self-mappings. The pairs $(h, \mathcal{S})$ and $(g, \mathcal{F})$ satisfy the common limit range property with respect to mappings $\mathcal{S}$ and $\mathcal{F}$, denoted by $(CLR_{\mathcal{SF}})$ if there exist two sequences σ_n and μ_n in X such that*

$$\lim_{n\to\infty} h(\sigma_n) = \lim_{n\to\infty} \mathcal{S}(\sigma_n) = \lim_{n\to\infty} g(\mu_n) = \lim_{n\to\infty} \mathcal{F}(\mu_n) = \nu \in \mathcal{S}(X) \cap \mathcal{F}(X), \tag{1}$$

for σ_n, $\mu_n \in X$ for all $n \in \mathbb{N} \cup \{0\}$.

Definition 1.3. *[24] Generalized altering distance function is a mapping $\tau : \mathbb{R}^+ \to \mathbb{R}^+$, satisfying that:*
(i) τ is non-decreasing;
(ii) $\tau(t) = 0$ if and only if $t = 0$.
$F = \{\tau : \mathbb{R}^+ \to \mathbb{R}^+ :$ *τ satisfying (i) and (ii)*$\}$.
$\Phi = \{\phi : \mathbb{R}^+ \to \mathbb{R}^+ :$ *ϕ is right upper semi-continuous, nondecreasing, and for all $x > 0$, we have $\tau(x) > \psi(x)$ and $\tau(x)$ satisfies (i), (ii)*$\}$
$\Psi_1 = \{\psi_1 : (\mathbb{R}^+)^6 \to \mathbb{R}^+ :$ *ψ_1 satisfies $(A_1) - (A_3)$*$\}$, *where*
(A_1) ψ_1 is continuous and nondecresing in each coordinate;
(A_2) $\psi_1(x, x, x, x, x, x) \le x$ for all $x \ge 0$;
(A_3) $\psi_1 = 0$ if and only if all the components of ψ_1 are zero;
$\Psi_2 = \{\psi_2 : \mathbb{R}^{+^4} \to \mathbb{R}^+ :$ *ψ_2 is continuous and $\psi_2 = 0$ if any component of ψ_2 is zero*$\}$.

Recently, Hussein et al. [24] introduced the notion of cyclic (α, λ)-admissible pair of maps (h, g) on a metric space (X, d) and defined generalized almost $(\mathcal{S}, \mathcal{F})$-rational contraction pair (h, g), as below:

Definition 1.4. *[24] Let h, g, $\mathcal{S}$ and $\mathcal{F}$ be quadruple self-mappings of a nonempty set X and α, $\lambda : X \to \mathbb{R}^+$. Then, the pair (h, g) is called a cyclic $(\alpha, \lambda)_{(\mathcal{S},\mathcal{F})}$-admissible if*
(i) $\alpha(\mathcal{S}\sigma^*) \ge 1$ for some $\sigma^* \in X$ implies $\lambda(h\sigma^*) \ge 1$,
(ii) $\lambda(\mathcal{F}\sigma^*) \ge 1$ for some $\sigma^* \in X$ implies $\alpha(g\sigma^*) \ge 1$.

Definition 1.5. *[24] Let h, g, $\mathcal{S}$, $\mathcal{F}$ be self maps of a metric space (X, d), and (h, g) be a cyclic $(\alpha, \lambda)_{(\mathcal{S},\mathcal{F})}$-admissible pair. We say that (h, g) is a generalized almost $(\mathcal{S}, \mathcal{F})$-rational contraction pair if*

$$\alpha(\mathcal{S}(\mu^*))\lambda(\mathcal{F}(\sigma^*)) \ge 1 \text{ implies } \tau(d(h\mu^*, g\sigma^*)) \le \psi(M(\mu^*, \sigma^*)) + L\phi(N(\mu^*, \sigma^*)),$$

for all μ^*, $\sigma^* \in X$ and some $L \ge 0$, where $\tau \in F$, $\psi \in \Psi$, $\phi \in \Phi$, for

$$\begin{aligned} M(\mu^*, \sigma^*) &= \psi_1\Big(d(\mathcal{S}\mu^*, \mathcal{F}\sigma^*), d(\mathcal{S}\mu^*, h\mu^*), d(\mathcal{F}\sigma^*, g\sigma^*), \frac{d(\mathcal{S}\mu^*, g\sigma^*) + d(h\mu^*, \mathcal{F}\sigma^*)}{2}, \\ &\quad \frac{d(\mathcal{F}\sigma^*, g\sigma^*)[1 + d(\mathcal{S}\mu^*, h\mu^*)]}{1 + d(\mathcal{S}\mu^*, \mathcal{F}\sigma^*)}, \frac{d(h\mu^*, \mathcal{F}\sigma^*)[1 + d(\mathcal{S}\mu^*, g\sigma^*)]}{1 + d(\mathcal{S}\mu^*, \mathcal{F}\sigma^*)}\Big), \end{aligned}$$

$$N(\mu^*,\sigma^*) = \psi_2\Big(d(\mathcal{S}\mu^*,h\mu^*),\ d(\mathcal{S}\mu^*,g\sigma^*),d(h\mu^*,\mathcal{F}\sigma^*),\ d(\mathcal{F}\sigma^*,g\sigma^*)\Big),$$

with $\psi_1 \in \Psi_1$, $\psi_2 \in \Psi_2$.

Hussein et al [24] obtained the following Theorem for a complete metric space:

Theorem 1.6. *[24] Let* $h,\ g,\ \mathcal{S},\ \mathcal{F}$ *be self mappings of a complete metric space* (X,d) *with* $h(X) \subset \mathcal{F}(X)$, $g(X) \subset \mathcal{S}(X)$ *and* (h,g) *be a generalized almost* $(\mathcal{S},\mathcal{F})$*-rational contraction pair. Suppose that:*
(a) there exist $\mu_0^* \in X$ *such that* $\alpha(\mathcal{S}\mu_0^*) \geq 1$ *and* $\lambda(\mathcal{F}\mu_0^*) \geq 1$*;*
(b) if $\{\mu_n^*\}$ *is a sequence in X such that* $\alpha(\mu_n^*) \geq 1,\ \lambda(\mu_n^*) \geq 1$ *for all n and* $\mu_n^* \to \mu^*$ *as* $n \to \infty$*, then* $\alpha(\mu^*) \geq 1$ *and* $\lambda(\mu^*) \geq 1$.
Then the pair $\{h,\mathcal{S}\}$ *and* $\{g,\mathcal{F}\}$ *have a point of coincidence inX. Moreover, if*
(c) $\{h,\mathcal{S}\}$ *and* $\{g,\mathcal{F}\}$ *are weakly compatible,*
(d) $\alpha(\mathcal{S}u) \geq 1$ *and* $\lambda(\mathcal{F}v) \geq 1$ *whenever* $u \in C(h,S)$ *and* $v \in C(g,\mathcal{F})$*.*
Then $h,g,\mathcal{S},\mathcal{F}$ *have a common fixed point.*

In [24], Theorem 1.6 was utilized by the authors for the study of dynamic programmings.

2. Main results

In this section, we further generalize the notion of generalized almost $(\mathcal{S},\mathcal{F})$-rational contraction pair (h,g), to a generalized almost $(\mathcal{S},\mathcal{F},\Gamma)$-rational contraction of integral type, as below:

Definition 2.1. *Let* $h,\ g,\ \mathcal{S}$ *and* $\mathcal{F}$ *be quadruple of self mappings of a metric space* (X,d)*, and* (h,g) *be a cyclic* $(\alpha,\ \lambda)_{(\mathcal{S},\mathcal{F})}$*-admissible pair. We say that* (h,g) *is a generalized almost* $(\mathcal{S},\mathcal{F},\Gamma)$*-rational contraction of integral type if*

$$\alpha(\mathcal{S}\mu^*)\lambda(\mathcal{F}\sigma^*) \geq 1 \text{ implies } \tau\Big(\int_0^{d(h\mu^*,g\sigma^*)}\Gamma(t)dt\Big) \leq \psi\Big(\int_0^{\psi_1(M(\mu^*,\sigma^*))}\Gamma(t)dt\Big) + L\phi\Big(\int_0^{\psi_2(N(\mu^*,\sigma^*))}\Gamma(t)dt\Big), \tag{2}$$

for all $\mu^*,\ \sigma^* \in X$*.* $\Gamma(t)$ *is Lebesgue integrable function with finite integral such that* $\int_0^{\delta}\Gamma(t)dt > 0$*, for all* $\delta > 0$*.* $L \geq 0$, $\tau \in F$, $\psi \in \Psi$, $\phi \in \varpi$*, for*

$$\begin{aligned} \psi_1(M(\mu^*,\sigma^*)) \quad = \quad & \max\Big\{d(g(\sigma^*),\mathcal{F}(\sigma^*)),\ d(h(\mu^*),\mathcal{F}(\sigma^*)),\ d(\mathcal{F}(\sigma^*),\mathcal{S}(\mu^*)), \\ & \frac{d(h(\mu^*),\mathcal{F}(\sigma^*)) + d(h(\mu^*),\mathcal{S}(\mu^*))}{2},\ \frac{d(g(\sigma^*),\mathcal{S}(\mu^*))d(h(\mu^*),g(\sigma^*))}{1+d(g(\sigma^*),h(\mu^*))}, \\ & \frac{d(h(\mu^*),\mathcal{F}(\sigma^*))d(\mathcal{S}(\mu^*),\mathcal{F}(\sigma^*))}{1+d(g(\sigma^*),\mathcal{S}(\mu^*))}\Big\}, \end{aligned} \tag{3}$$

$$\psi_2(N(\mu^*,\sigma^*)) = \psi_2\Big(d(h(\mu^*),\mathcal{F}(\sigma^*)),\ d(g(\sigma^*),\mathcal{F}(\sigma^*)),\ d(\mathcal{F}(\sigma^*),\mathcal{S}(\mu^*)),\frac{d(h(\mu^*),\mathcal{F}(\sigma^*))d(\mathcal{S}(\mu^*),\mathcal{F}(\sigma^*))}{1+d(g(\sigma^*),\mathcal{S}(\mu^*))}\Big), \tag{4}$$

with $\psi_1 \in \Psi_1$, $\psi_2 \in \Psi_2$.

We prove the following Theorem for (X,d) with out completeness of the metric space by the use of $(CLR_{\mathcal{SF}})$ property of quadruple self mappings $h,\ g,\ \mathcal{S},\ \mathcal{F}$. This work generalize the results in [24] and many others.

Theorem 2.2. *Let* $h,\ g,\ \mathcal{S}$ *and* $\mathcal{F}$ *be quadruple of self mappings of a metric space* (X,d) *and* (h,g) *be a generalized almost* $(\mathcal{S},\mathcal{F},\Gamma)$*-rational contraction of integral type of the pair* (h,g)*. Suppose that:*
(a) there exist $\mu_0^* \in X$ *such that* $\alpha(\mathcal{S}\mu_0^*) \geq 1$ *and* $\lambda(\mathcal{F}\mu_0^*) \geq 1$*;*
(b) if $\{\mu_n^*\}$ *is a sequence in X such that* $\alpha(\mu_n^*) \geq 1,\ \lambda(\mu_n^*) \geq 1$ *for all* $n \in \mathbb{N}_0$ *and* $\mu_n^* \to \mu^*$ *as* $n \to \infty$*, then* $\alpha(\mu^*) \geq 1$ *and* $\lambda(\mu^*) \geq 1$*;*
(c) the pair (h,g) *and* $(\mathcal{S},\ \mathcal{F})$ *share* $(CLR_{\mathcal{SF}})$ *property;*
(d) $\{h,\mathcal{S}\}$ *and* $\{g,\mathcal{F}\}$ *are weakly compatible,*
then, the quadruple self mappings $h,\ g,\ \mathcal{S},\ \mathcal{F}$ *have a unique common fixed point* $\nu \in X$*.*

Proof. By the help of our assumption of the ($CLR_{\mathcal{SF}}$) property of the pairs (f,g), and $(\mathcal{S},\mathcal{F})$. We may have two sequences $\{\mu_n^*\}$ and $\{\sigma_n^*\}$ in the metric space (X,d), such that:

$$\lim_{n\to\infty} h(\mu_n^*) = \lim_{n\to\infty} \mathcal{S}(\mu_n^*) = \lim_{n\to\infty} g(\sigma_n^*) = \lim_{n\to\infty} \mathcal{F}(\sigma_n^*) = \nu \text{ for } \nu \in \mathcal{S}(X)\cap\mathcal{F}(X). \tag{5}$$

Since, for $\nu \in \mathcal{S}(X)\cap\mathcal{F}(X)$. Therefore, we may have $\nu = \mathcal{S}(u)$ for some $u \in X$. Therefore, we have

$$\lim_{n\to\infty} h(\mu_n^*) = \lim_{n\to\infty} \mathcal{S}(\mu_n^*) = \lim_{n\to\infty} g(\sigma_n^*) = \lim_{n\to\infty} \mathcal{F}(\sigma_n^*) = \nu = \mathcal{S}(u). \tag{6}$$

We show that $h(u) = \mathcal{S}(u)$. For this, we assume the contradiction, that is $h(u) \neq \mathcal{S}(u)$ and define the following sequences:

$$\sigma_{2n}^* = h(\mu_{2n}^*) = \mathcal{F}\mu_{2n+1}^* \text{ and } \sigma_{2n+1}^* = g(\mu_{2n+1}^*) = \mathcal{S}(\mu_{2n+2}^*). \tag{7}$$

By the help of (a), we have $\alpha(\mathcal{S}(\mu_0^*)) \geq 1$ and $\lambda(\mathcal{F}(\mu_0^*)) \geq 1$, where the pair (h,g) is a cyclic $(\alpha,\lambda)_{(\mathcal{S},\mathcal{F})}$-admissible. Therefore, $\alpha(\mathcal{S}(\mu_0^*)) \geq 1$ implies that $\lambda(h(\mu_0^*)) = \lambda(\mathcal{F}\mu_1^*) \geq 1$ which further implies that $\alpha(g(\mu_1^*)) = \alpha(\mathcal{S}(\mu_2^*)) \geq 1$. Continuing this procedure upto n times, we are having $\alpha(\mathcal{S}(\mu_{2n+1}^*)) \geq 1$ and $\lambda(\mathcal{F}(\mu_{2n}^*)) \geq 1$ for all $n \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}$. This means that

$$\alpha(\mathcal{S}(\mu_n^*)) \geq 1 \text{ and } \lambda(\mathcal{F}(\mu_n^*)) \geq 1, \text{ for all } n \in \mathbb{N}_0. \tag{8}$$

By the help of (8) and (b), we have $\alpha(\mathcal{S}(\mu^*))\lambda(\mathcal{F}(\mu^*)) \geq 1$ and since the pair (h, g) is a generalized almost $(\mathcal{S},\mathcal{F},\Gamma)$-rational contraction of integral type. Therefore, we have:

$$\tau\Big(\int_0^{d(hu,g(\sigma_n^*))} \Gamma(t)dt\Big) \leq \psi\Big(\int_0^{\psi_1(M(u,\sigma_n^*))} \Gamma(t)dt\Big) + L\phi\Big(\int_0^{\psi_2(N(u,\sigma_n^*))} \Gamma(t)dt\Big), \tag{9}$$

for $\mu^* = u$ and $\sigma^* = \sigma_n^*$ in the inequality (2), where

$$\begin{aligned}\psi_1(M(u,\sigma_n^*)) \;=\;& \max\Big\{d(h(u),\mathcal{F}(\sigma_n^*)),\, d(g(\sigma_n^*),\mathcal{F}(\sigma_n^*)),\, d(\mathcal{F}(\sigma_n^*),\mathcal{S}(u)),\\ & \frac{d(h(u),\mathcal{F}(\sigma_n^*)) + d(h(u),\mathcal{S}(u))}{2}, \frac{d(g(\sigma_n^*),\mathcal{S}(u))d(h(u),g(\sigma_n^*))}{1+d(g(\sigma_n^*),h(u))},\\ & \frac{d(h(u),\mathcal{F}(\sigma_n^*))d(\mathcal{S}(u),\mathcal{F}(\sigma_n^*))}{1+d(g(u),\mathcal{S}(u))}\Big\},\end{aligned} \tag{10}$$

$$\psi_2(N(u,\sigma_n^*)) = \psi_2\Big(d(h(u),\mathcal{F}(\sigma_n^*)),\, d(g(\sigma_n^*),\mathcal{F}(\sigma_n^*)),\, d(\mathcal{F}(\sigma_n^*),\mathcal{S}(u)), \frac{d(h(u),\mathcal{F}(\sigma_n^*))d(\mathcal{S}(u),\mathcal{F}(\sigma_n^*))}{1+d(g(\sigma_n^*),\mathcal{S}(u))}\Big). \tag{11}$$

Taking the $\lim_{n\to\infty}$ in (10), (11) and (9), respectively; we get:

$$\begin{aligned}\lim_{n\to\infty}\psi_1(M(u,\sigma_n^*)) \;=\;& \lim_{n\to\infty}\max\Big\{d(h(u),\mathcal{F}(\sigma_n^*)),\, d(g(\sigma_n^*),\mathcal{F}(\sigma_n^*)),\, d(\mathcal{F}(\sigma_n^*),\mathcal{S}(u)),\\ & \frac{d(h(u),\mathcal{S}(u)) + d(h(u),\mathcal{F}(\sigma_n^*))}{2}, \frac{d(g(\sigma_n^*),\mathcal{S}(u))d(h(u),g(\sigma_n^*))}{1+d(g(\sigma_n^*),h(u))},\\ & \frac{d(h(u),\mathcal{F}(\sigma_n^*))d(\mathcal{S}(u),\mathcal{F}(\sigma_n^*))}{1+d(g(\sigma_n^*),\mathcal{S}(u))}\Big\}\\ =\;& \max\Big\{d(h(u),\nu^*),\, d(\nu^*,\nu^*),\, d(\nu^*,\nu^*), \frac{d(h(u),\nu^*)+d(h(u),\nu^*)}{2},\\ & \frac{d(\nu^*,\nu^*)d(h(u),\nu^*)}{1+d(\nu^*,h(u))}, \frac{d(h(u),\nu^*)d(\nu^*,\nu^*)}{1+d(\nu^*,h(u))}\Big\}\\ =\;& \max\{d(h(u),\nu^*),0,0,d(h(u),\nu^*),0,0\} = d(h(u),\nu^*),\end{aligned} \tag{12}$$

$$\begin{aligned}
\lim_{n\to\infty}\psi_2(N(u,\sigma_n^*)) &= \lim_{n\to\infty}\psi_2\Big(d(h(u),\mathcal{F}(\sigma_n^*)),\ d(g(\sigma_n^*),\mathcal{F}(\sigma_n^*)),\ d(\mathcal{F}(\sigma_n^*),\mathcal{S}(u)),\\
&\qquad \frac{d(h(u),\mathcal{F}(\sigma_n^*))d(\mathcal{S}(\sigma_n^*),\mathcal{F}(\sigma_n^*))}{1+d(g(u),\mathcal{S}(\sigma_n^*))}\Big) \qquad (13)\\
&= \psi_2\Big(d(h(u),\nu^*),\ d(\nu^*,\nu^*),\ d(\nu^*,\ \nu^*),\ \frac{d(h(u),\ \nu^*)d(\nu^*,\ \nu^*)}{1+d(g(u),\ \nu^*)}\Big)\\
&= \psi_2\Big(d(g(u),\nu^*),\ 0,\ 0,0\Big)=0,
\end{aligned}$$

and

$$\lim_{n\to\infty}\tau\Big(\int_0^{d(hu,g(\sigma_n^*))}\Gamma(t)dt\Big)\le\lim_{n\to\infty}\psi\Big(\int_0^{\psi_1(M(u,\sigma_n^*))}\Gamma(t)dt\Big)+L\lim_{n\to\infty}\phi\Big(\int_0^{\psi_2(N(u,\sigma_n^*))}\Gamma(t)dt\Big). \qquad (14)$$

By the use of (6), (12) and (13) in (14), we have:

$$\tau\Big(\int_0^{d(hu,\nu^*)}\Gamma(t)dt\Big) \le \psi\Big(\int_0^{d(h(u),\nu^*)}\Gamma(t)dt\Big), \qquad (15)$$

which is a contradiction of $\tau(t)>\psi(t)$. This contradiction is due to our supposition that $h(u)\neq\mathcal{S}(u)$, and hence $h(u)=\mathcal{S}(u)$. Also from (5), we have $\nu^*\in\mathcal{F}(X)$. This implies $\nu^*=\mathcal{F}(v)$, for some $v\in X$. Now, we show that $\mathcal{F}(v)=g(v)$, for this we assume the contrary path, that is, $\mathcal{F}(v)\neq g(v)$. By putting $\mu^*=\mu_n^*$ and $\sigma^*=v$ in (2), and following the same lines as we followed for the proof of $\mathcal{S}(u)=h(u)=\nu^*$, we can get $\mathcal{F}(v)=g(v)=\nu^*$. Ultimately, we have

$$\mathcal{F}(v)=g(v)=\mathcal{S}(u)=h(u)=\nu^*. \qquad (16)$$

Since, (h,g) and $(\mathcal{S},\mathcal{F})$ are weakly compatible. Therefore, $\mathcal{S}(u)=h(u)$ implies $h\mathcal{S}(u)=Sh(u)$ which implies $h(\nu^*)=\mathcal{S}(\nu^*)$. Similarly, we have $\mathcal{F}(\nu^*)=g(\nu^*)$. Next, we show that ν^* is a common fixed point of h, g, $\mathcal{S}$, $\mathcal{F}$. For this let us suppose the contrary path, that is, $h(\nu^*)\neq\nu^*$. For this, putting $\mu^*=\nu^*$ and $\sigma^*=v$ in (2), we have:

$$\tau\Big(\int_0^{d(h(\nu^*),g(v))}\Gamma(t)dt\Big)\le\psi\Big(\int_0^{\psi_1(M(\nu^*,v))}\Gamma(t)dt\Big)+L\phi\Big(\int_0^{\psi_2(N(\mu^*,v))}\Gamma(t)dt\Big), \qquad (17)$$

where

$$\begin{aligned}
\psi_1(M(\nu^*,v)) &= \max\Big\{d(h(\nu^*),\mathcal{F}(v)),\ d(g(v),\mathcal{F}(v)),\ d(\mathcal{F}(v),\mathcal{S}(\nu^*)), \qquad (18)\\
&\qquad \frac{d(h(\nu^*),\mathcal{F}(v))+d(h(\nu^*),\mathcal{S}(\nu^*))}{2},\ \frac{d(g(v),\mathcal{S}(\nu^*))d(h(\nu^*),g(v))}{1+d(g(v),h(\nu^*))},\\
&\qquad \frac{d(h(\nu^*),\mathcal{F}(v))d(\mathcal{S}(\nu^*),\mathcal{F}(v))}{1+d(g(v),\mathcal{S}(\nu^*))}\Big\}\\
&= \max\Big\{d(h(\nu^*),\nu^*),\ d(\nu^*,\nu^*),\ d(\nu^*,h(\nu^*)),\ \frac{d(h(\nu^*),\nu^*)+d(h(\nu^*),h(\nu^*))}{2}, \qquad (19)\\
&\qquad \frac{d(\nu^*,h(\nu^*))d(h(\nu^*),\nu^*)}{1+d(\nu^*,h(\nu^*))},\ \frac{d(h(\nu^*),\nu^*)d(h(\nu^*),\mathcal{F}(v))}{1+d(\nu^*,\mathcal{S}(\nu^*))}\Big\},
\end{aligned}$$

and

$$\begin{aligned}
\psi_2(N(\nu^*,v)) &= \psi_2\Big(d(h(\nu^*),\mathcal{F}(v)),\ d(g(v),\mathcal{F}(v)),\ d(\mathcal{F}(v),\mathcal{S}(\nu^*)),\ \frac{d(h(\nu^*),\mathcal{F}(v))d(\mathcal{S}(\nu^*),\mathcal{F}(v))}{1+d(g(v),\mathcal{S}(\nu^*))}\Big)\\
&= \psi_2\Big(d(h(\nu^*),\nu^*),\ d(\nu^*,\nu^*),\ d(\nu^*,h(\nu^*)),\ \frac{d(h(\nu^*),\nu^*)d(h(\nu^*),\nu^*)}{1+d(\nu^*,h(\nu^*))}\Big) \qquad (20)\\
&= \psi_2\Big(d(h(\nu^*),\nu^*),\ 0,\ d(\nu^*,h(\nu^*)),\ \frac{d(h(\nu^*),\nu^*)d(h(\nu^*),\nu^*)}{1+d(\nu^*,h(\nu^*))}\Big)=0
\end{aligned}$$

By the use of (6), (12) and (13) in (14), we have:

$$\tau\Big(\int_0^{d(hz,\nu^*)}\Gamma(t)dt\Big) \leq \psi\Big(\int_0^{d(h(\nu^*),\nu^*)}\Gamma(t)dt\Big), \tag{21}$$

which is a contradiction of $\tau(t) > \psi(t)$. This contradiction is due to our supposition that $h(\nu^*) \neq \nu^*$, and hence $h(\nu^*) = \mathcal{S}(\nu^*) = \nu^*$. Similarly, we can show $g(\nu^*) = \mathcal{F}(\nu^*) = \nu^*$. Ultimately, we have $h(\nu^*) = \mathcal{S}(\nu^*) = g(\nu^*) = \mathcal{F}(\nu^*) = \nu^*$.

Finally, we prove that the common fixed point of the quadruple self-mappings h, g, $\mathcal{S}$, $\mathcal{F}$ is unique. For this, we again presume a contrary path, that is, let there exist two different fixed points, such that $h(\nu_1^*) = \mathcal{S}(\nu_1^*) = \nu_1^*$, $g(\nu_2^*) = \mathcal{F}(\nu_2^*) = \nu_2^*$, for some ν_1^*, $\nu_2^* \in X$ such that $\nu_1^* \neq \nu_2^*$. By putting $\mu^* = \nu_1^*$ and $\sigma^* = \nu_2^*$ in (2), we have:

$$\tau\Big(\int_0^{d(h\nu_1^*,g(\nu_2^*))}\Gamma(t)dt\Big) = \tau\Big(\int_0^{d(\nu_1^*,\nu_2^*)}\Gamma(t)dt\Big) \leq \psi\Big(\int_0^{\psi_1(M(\nu_1^*,\nu_2^*))}\Gamma(t)dt\Big) + L\phi\Big(\int_0^{\psi_2(N(\nu_1^*,\nu_2^*))}\Gamma(t)dt\Big). \tag{22}$$

Where,

$$\begin{aligned}
\psi_1(M(\nu_1^*,\nu_2^*)) &= \max\Big\{d(h(\nu_1^*),\mathcal{S}(\nu_1^*)),\ d(g(\nu_2^*),\mathcal{F}(\nu_2^*)),\ d(\mathcal{F}(\nu_2^*),\mathcal{S}(\nu_1^*)),\\
&\quad \frac{d(h(\nu_1^*),\mathcal{F}(\nu_2^*)) + d(h(\nu_1^*),\mathcal{S}(\nu_1^*))}{2},\ \frac{d(g(\nu_2^*),\mathcal{S}(\nu_1^*))d(h(\nu_1^*),g(\nu_2^*))}{1+d(g(\nu_2^*),h(\nu_1^*))},\\
&\quad \frac{d(h(\nu_1^*),\mathcal{F}(\nu_2^*))d(\mathcal{S}(\nu_1^*),\mathcal{F}(\nu_2^*))}{1+d(g(\nu_1^*),\mathcal{S}(\nu_1^*))}\Big\}\\
&= \max\Big\{d(\nu_1^*,\nu_1^*),\ d(\nu_2^*,\nu_2^*),\ d(\nu_2^*,\nu_1^*),\ \frac{d(\nu_1^*,\nu_2^*)+d(\nu_1^*,\nu_1^*)}{2},\ \frac{d(\nu_2^*,\nu_1^*)d(\nu_1^*,\nu_2^*)}{1+d(\nu_2^*,\nu_1^*)},\\
&\quad \frac{d(\nu_1^*,\nu_2^*)d(\nu_1^*,\nu_2^*)}{1+d(\nu_2^*,\nu_1^*)}\Big\} = d(\nu_1^*,\nu_2^*),
\end{aligned} \tag{23}$$

and

$$\begin{aligned}
\psi_2(N(\nu_1^*,\nu_2^*)) &= \psi_2\Big(d(h(\nu_1^*),\mathcal{F}(\nu_2^*)),\ d(g(\nu_2^*),\mathcal{F}(\nu_2^*)),\ d(\mathcal{F}(\nu_2^*),\mathcal{S}(\nu_1^*)),\\
&\quad \frac{d(h(\nu_1^*),\mathcal{F}(\nu_2^*))d(\mathcal{S}(\nu_1^*),\mathcal{F}(\nu_2^*))}{1+d(g(\nu_2^*),\mathcal{S}(\nu_1^*))}\Big)\\
&= \psi_2\Big(d(\nu_1^*,\nu_2^*),\ d(\nu_2^*,\nu_2^*),\ d(\nu_2^*,\nu_1^*),\ \frac{d(\nu_1^*,\nu_2^*)d(\nu_1^*,\nu_2^*)}{1+d(\nu_2^*,\nu_1^*)}\Big)\\
&= \psi_2\Big(d(\nu_1^*,\nu_2^*),\ 0,\ d(\nu_2^*,\nu_1^*),\ \frac{d(\nu_1^*,\nu_2^*)d(\nu_1^*,\nu_2^*)}{1+d(\nu_2^*,\nu_1^*)}\Big) = 0.
\end{aligned} \tag{24}$$

By the use of (23), (24) in (22), we have:

$$\tau\Big(\int_0^{d(\nu_1^*,\nu_2^*)}\Gamma(t)dt\Big) \leq \psi\Big(\int_0^{d(\nu_1^*,\nu_2^*)}\Gamma(t)dt\Big), \tag{25}$$

which is a contradiction of the fact $\tau(t) > \psi(t)$. Thus, $\nu_1^* = \nu_2^*$ and therefore, the fixed point of the quadruple self mappings h, g, $\mathcal{S}$, $\mathcal{F}$ is unique. □

Corollary 2.3. *Let h, g, $\mathcal{S}$ and $\mathcal{F}$ be quadruple of self mappings of a metric space (X,d) and the pair (h,g) be a cyclic $(\alpha,\lambda)_{(\mathcal{S},\mathcal{F})}$-admissible such that*

$$\alpha(\mathcal{S}\mu^*)\lambda(\mathcal{F}\sigma^*)\tau(\int_0^{d(h(\mu^*),g(\sigma^*))}\Gamma(t)dt) \leq \psi(\int_0^{\psi_1(M(\mu^*,\sigma^*))}\Gamma(t)dt) + L\phi(\int_0^{\psi_2(N(\mu^*,\sigma^*))}\Gamma(t)dt), \tag{26}$$

for all μ^*, $\sigma^* \in X$, and $L \geq 0$, $\tau \in F$, $\phi \in \theta$, $\psi \in \Phi$, and

$$\begin{aligned}\psi_1(M(\mu^*,\sigma^*)) \;=\; & \max\Big\{d(g(\sigma^*),\mathcal{F}(\sigma^*)),\, d(h(\mu^*),\mathcal{F}(\sigma^*)),\, d(\mathcal{F}(\sigma^*),\mathcal{S}(\mu^*)), \\ & \frac{d(h(\mu^*),\mathcal{F}(\sigma^*))+d(h(\mu^*),\mathcal{S}(\mu^*))}{2},\, \frac{d(g(\sigma^*),\mathcal{S}(\mu^*))d(h(\mu^*),g(\sigma^*))}{1+d(g(\sigma^*),h(\mu^*))}, \\ & \frac{d(h(\mu^*),\mathcal{F}(\sigma^*))d(\mathcal{S}(\mu^*),\mathcal{F}(\sigma^*))}{1+d(g(\sigma^*),\mathcal{S}(\mu^*))}\Big\},\end{aligned} \tag{27}$$

$$\psi_2(N(\mu^*,\sigma^*)) = \min\Big\{d(h(\mu^*),\mathcal{F}(\sigma^*)),\, d(g(\sigma^*),\mathcal{F}(\sigma^*)),\, d(\mathcal{F}(\sigma^*),\mathcal{S}(\mu^*)),\, \frac{d(h(\mu^*),\mathcal{F}(\sigma^*))d(\mathcal{S}(\mu^*),\mathcal{F}(\sigma^*))}{1+d(g(\sigma^*),\mathcal{S}(\mu^*))}\Big\}. \tag{28}$$

Suppose that the conditions (a)-(d) of the Theorem 2.2 are satisfied. Then, the quadruple self mappings h, g, $\mathcal{S}$, $\mathcal{F}$ have a unique common fixed point $\nu \in X$.

Assuming $\alpha(\mathcal{S}\mu^*) = 1 = \lambda(\mathcal{F}\sigma^*)$, $\tau(t) = \psi(t) = \phi(t) = t$ in the Corollary 2.3, we have the following Theorem:

Theorem 2.4. *Let h, g, $\mathcal{S}$, $\mathcal{F}$ be quadruple self-mappings of a metric space (X, d) with $L \geq 0$, such that*

$$\int_0^{d(h(\mu^*),g(\sigma^*))}\Gamma(t)dt \leq \int_0^{\psi_1(M(\mu^*,\sigma^*))}\Gamma(t)dt + L\int_0^{\psi_2(N(\mu^*,\sigma^*))}\Gamma(t)dt, \tag{29}$$

for all μ^*, $\sigma^* \in X$. $\Gamma(t)$ is Lebesgue integrable function with finite integral such that $\int_0^{\delta}\Gamma(t)dt > 0$, for all $\delta > 0$. And

$$\begin{aligned}\psi_1(M(\mu^*,\sigma^*)) \;=\; & \max\Big\{d(g(\sigma^*),\mathcal{F}(\sigma^*)),\, d(h(\mu^*),\mathcal{F}(\sigma^*)),\, d(\mathcal{F}(\sigma^*),\mathcal{S}(\mu^*)), \\ & \frac{d(h(\mu^*),\mathcal{F}(\sigma^*))+d(h(\mu^*),\mathcal{S}(\mu^*))}{2},\, \frac{d(g(\sigma^*),\mathcal{S}(\mu^*))d(h(\mu^*),g(\sigma^*))}{1+d(g(\sigma^*),h(\mu^*))}, \\ & \frac{d(h(\mu^*),\mathcal{F}(\sigma^*))d(\mathcal{S}(\mu^*),\mathcal{F}(\sigma^*))}{1+d(g(\sigma^*),\mathcal{S}(\mu^*))}\Big\},\end{aligned} \tag{30}$$

$$\psi_2(N(\mu^*,\sigma^*)) = \min\Big\{d(h(\mu^*),\mathcal{F}(\sigma^*)),\, d(g(\sigma^*),\mathcal{F}(\sigma^*)),\, d(\mathcal{F}(\sigma^*),\mathcal{S}(\mu^*)),\, \frac{d(h(\mu^*),\mathcal{F}(\sigma^*))d(\mathcal{S}(\mu^*),\mathcal{F}(\sigma^*))}{1+d(g(\sigma^*),\mathcal{S}(\mu^*))}\Big\}. \tag{31}$$

Assume that the following conditions are satisfied:
(a) The pair (h, g) and $(\mathcal{S}, \mathcal{F})$ share $(CLR_{\mathcal{S}\mathcal{F}})$ property;
(b) $\{h, \mathcal{S}\}$ and $\{g, \mathcal{F}\}$ are weakly compatible,
Then the quadruple self mappings h, g, $\mathcal{S}$, $\mathcal{F}$ have a unique common fixed point $\nu \in X$.

If we assume the value of $L = 0$, in the Corollary 2.3, we get following result:

Corollary 2.5. *Let h, g, $\mathcal{S}$ and $\mathcal{F}$ be quadruple of self mappings of a metric space (X, d) and the pair (h, g) be a cyclic $(\alpha, \lambda)_{(\mathcal{S},\mathcal{F})}$-admissible such that*

$$\alpha(\mathcal{S}\mu^*)\lambda(\mathcal{F}\sigma^*)\tau(\int_0^{d(h(\mu^*),g(\sigma^*))}\Gamma(t)dt) \leq \psi(\int_0^{\psi_1(M(\mu^*,\sigma^*))}\Gamma(t)dt) \tag{32}$$

for all μ^*, $\sigma^* \in X$. $\Gamma(t)$ is Lebesgue integrable function with finite integral such that $\int_0^{\delta}\Gamma(t)dt > 0$, for all $\delta > 0$. $\tau \in F$, $\phi \in \theta$, $\psi \in \Phi$, and

$$\begin{aligned}\psi_1(M(\mu^*,\sigma^*)) \;=\; & \max\Big\{d(g(\sigma^*),\mathcal{F}(\sigma^*)),\, d(h(\mu^*),\mathcal{F}(\sigma^*)),\, d(\mathcal{F}(\sigma^*),\mathcal{S}(\mu^*)), \\ & \frac{d(h(\mu^*),\mathcal{F}(\sigma^*))+d(h(\mu^*),\mathcal{S}(\mu^*))}{2},\, \frac{d(g(\sigma^*),\mathcal{S}(\mu^*))d(h(\mu^*),g(\sigma^*))}{1+d(g(\sigma^*),h(\mu^*))}, \\ & \frac{d(h(\mu^*),\mathcal{F}(\sigma^*))d(\mathcal{S}(\mu^*),\mathcal{F}(\sigma^*))}{1+d(g(\sigma^*),\mathcal{S}(\mu^*))}\Big\}.\end{aligned} \tag{33}$$

If the conditions (a)-(d) of the Theorem 2.2 are satisfied. Then, the quadruple of self mappings h, g, $\mathcal{S}$, $\mathcal{F}$ have a unique common fixed point $\nu \in X$.

Assuming $\psi(t) = \tau(t) - \phi(t)$ in the Corollary 2.5, we get the following result:

Corollary 2.6. *Let h, g, $\mathcal{S}$ and $\mathcal{F}$ be quadruple of self mappings of a metric space (X, d) and the pair (h, g) be a cyclic $(\alpha, \lambda)_{(\mathcal{S},\mathcal{F})}$-admissible such that*

$$\alpha(\mathcal{S}\mu^*)\lambda(\mathcal{F}\sigma^*)\tau(\int_0^{d(h(\mu^*),g(\sigma^*))}\Gamma(t)dt) \leq \tau(\int_0^{\psi_1(M(\mu^*,\sigma^*))}\Gamma(t)dt) - \phi(\int_0^{\psi_1(M(\mu^*,\sigma^*))}\Gamma(t)dt) \tag{34}$$

for all μ^*, $\sigma^* \in X$. $\Gamma(t)$ is Lebesgue integrable function with finite integral such that $\int_0^{\delta}\Gamma(t)dt > 0$, for all $\delta > 0$. $\tau \in F$, $\phi \in \theta$, $\psi \in \Phi$, and

$$\begin{aligned} \psi_1(M(\mu^*,\sigma^*)) \;=\; & \max\Big\{d(g(\sigma^*),\mathcal{F}(\sigma^*)),\, d(h(\mu^*),\mathcal{F}(\sigma^*)),\, d(\mathcal{F}(\sigma^*),\mathcal{S}(\mu^*)), \\ & \frac{d(h(\mu^*),\mathcal{F}(\sigma^*)) + d(h(\mu^*),\mathcal{S}(\mu^*))}{2},\, \frac{d(g(\sigma^*),\mathcal{S}(\mu^*))d(h(\mu^*),g(\sigma^*))}{1 + d(g(\sigma^*),h(\mu^*))}, \\ & \frac{d(h(\mu^*),\mathcal{F}(\sigma^*))d(\mathcal{S}(\mu^*),\mathcal{F}(\sigma^*))}{1 + d(g(\sigma^*),\mathcal{S}(\mu^*))}\Big\}. \end{aligned} \tag{35}$$

If the conditions (a)-(d) of the Theorem 2.2 are satisfied. Then, the quadruple self mappings h, g, $\mathcal{S}$, $\mathcal{F}$ have a unique common fixed point $\nu \in X$.

If we assume $\alpha(\mathcal{S}\mu^*) = \lambda(\mathcal{F}\sigma^*) = 1$, in the Corollary 2.6, we have the following result:

Corollary 2.7. *Let h, g, $\mathcal{S}$ and $\mathcal{F}$ be quadruple of self mappings of a metric space (X, d) and the pair (h, g) be a cyclic $(\alpha, \lambda)_{(\mathcal{S},\mathcal{F})}$-admissible such that*

$$\tau(\int_0^{d(h(\mu^*),g(\sigma^*))}\Gamma(t)dt) \leq \tau(\int_0^{\psi_1(M(\mu^*,\sigma^*))}\Gamma(t)dt) - \phi(\int_0^{\psi_1(M(\mu^*,\sigma^*))}\Gamma(t)dt) \tag{36}$$

for all μ^*, $\sigma^* \in X$. $\Gamma(t)$ is Lebesgue integrable function with finite integral such that $\int_0^{\delta}\Gamma(t)dt > 0$, for all $\delta > 0$. $\tau \in F$, $\phi \in \theta$, $\psi \in \Phi$, and

$$\begin{aligned} \psi_1(M(\mu^*,\sigma^*)) \;=\; & \max\Big\{d(g(\sigma^*),\mathcal{F}(\sigma^*)),\, d(h(\mu^*),\mathcal{F}(\sigma^*)),\, d(\mathcal{F}(\sigma^*),\mathcal{S}(\mu^*)), \\ & \frac{d(h(\mu^*),\mathcal{F}(\sigma^*)) + d(h(\mu^*),\mathcal{S}(\mu^*))}{2},\, \frac{d(g(\sigma^*),\mathcal{S}(\mu^*))d(h(\mu^*),g(\sigma^*))}{1 + d(g(\sigma^*),h(\mu^*))}, \\ & \frac{d(h(\mu^*),\mathcal{F}(\sigma^*))d(\mathcal{S}(\mu^*),\mathcal{F}(\sigma^*))}{1 + d(g(\sigma^*),\mathcal{S}(\mu^*))}\Big\}. \end{aligned} \tag{37}$$

Assume that the following conditions are satisfied:
(a) The pair (h, g) and $(\mathcal{S}, \mathcal{F})$ share $(CLR_{\mathcal{S}\mathcal{F}})$ property;
(b) $\{h, \mathcal{S}\}$ and $\{g, \mathcal{F}\}$ are weakly compatible,
Then the quadruple self mappings h, g, $\mathcal{S}$, $\mathcal{F}$ have a unique common fixed point $\nu \in X$.

3. Applications

Example 3.1. *Let $(X = [0, 1], d)$ be a metric space with $d(x, y) = |x - y|$, for x, $y \in X$. Defining h, g, $\mathcal{S}$ and $\mathcal{F}$ as under:*

$$h(x) = \begin{cases} \frac{1}{2} \text{ if } x \in [0, 0.5], \\ \frac{1}{6} \text{ if } x \in (0.5, 1], \end{cases} \qquad \mathcal{S}(x) = \begin{cases} \frac{1}{2} \text{ if } x \in [0, 0.5], \\ \frac{1}{7} \text{ if } x \in (0.5, 1], \end{cases} \tag{38}$$

$$g(x) = \begin{cases} \frac{1}{2} \text{ if } x \in [0, 0.5], \\ \frac{1}{8} \text{ if } x \in (0.5, 1], \end{cases} \qquad \mathcal{F}(x) = \begin{cases} \frac{1}{2} \text{ if } x \in [0, 0.5], \\ \frac{1}{9} \text{ if } x \in (0.5, 1]. \end{cases} \tag{39}$$

$\alpha(t) = 1 = \lambda(t)$ for all $x \in X$. One can easily check that the pair $(h, \mathcal{S})$ and $(g, \mathcal{F})$ are weakly compatible and $\alpha(t)\lambda(t) \geq 1$ for all $t \in X$. Let us consider the sequences

$$\{x_n\} = \{\frac{0.1n + 0.2}{n}\}, \qquad \{y_n\} = \{\frac{0.21n + 0.1}{0.1 + n}\}. \tag{40}$$

By the help of (38), (39) *and* (40), *we have:*

$$\lim_{n\to\infty} h(x_n) = \lim_{n\to\infty} h(\frac{0.1n + 0.2}{n}) = \frac{1}{2}, \tag{41}$$

$$\lim_{n\to\infty} \mathcal{S}(x_n) = \lim_{n\to\infty} \mathcal{S}(\frac{0.1n + 0.2}{n}) = \frac{1}{2}, \tag{42}$$

$$\lim_{n\to\infty} g(y_n) = \lim_{n\to\infty} g(\frac{0.21n + 0.1}{0.1 + n}) = \frac{1}{2}, \tag{43}$$

$$\lim_{n\to\infty} \mathcal{F}(y_n) = \lim_{n\to\infty} \mathcal{F}(\frac{0.21n + 0.1}{0.1 + n}) = \frac{1}{2}. \tag{44}$$

From (41)-(44), *it is proved that the mappings h, $\mathcal{S}$, g, $\mathcal{F}$ share $(CLR_{\mathcal{SF}})$ property. Further, we need to show that the mappings h, g, $\mathcal{S}$, $\mathcal{F}$ satisfy the inequality* (2). *For this, we study two cases, i.e., when $x \in [0, 0.5]$, $x \in (0.5, 1]$.*

Case I *For $x \in [0, 0.5]$, we have $h(x) = g(x) = \mathcal{S}(x) = \mathcal{F}(x) = 1$, which implies $d(h, g) = 0$, $\psi_1(M) = 0$ and $\psi_2(N) = 0$. And therefore, the equality exist.*

Case II *For $\sigma^* = \mu^* = x \in (0.5, 1]$, we have $h(x) = \frac{1}{6}$, $\mathcal{S}(x) = \frac{1}{7}$, $g(x) = \frac{1}{8}$, $\mathcal{F}(x) = \frac{1}{9}$, and*

$$\begin{aligned}\psi_1(M(\mu^*, \sigma^*)) &= \max\Big\{d(g(\sigma^*), \mathcal{F}(\sigma^*)),\ d(h(\mu^*), \mathcal{F}(\sigma^*)),\ d(\mathcal{F}(\sigma^*), \mathcal{S}(\mu^*)),\\ &\quad \frac{d(h(\mu^*), \mathcal{F}(\sigma^*)) + d(h(\mu^*), \mathcal{S}(\mu^*))}{2}, \frac{d(g(\sigma^*), \mathcal{S}(\mu^*))d(h(\mu^*), g(\sigma^*))}{1 + d(g(\sigma^*), h(\mu^*))},\\ &\quad \frac{d(h(\mu^*), \mathcal{F}(\sigma^*))d(\mathcal{S}(\mu^*), \mathcal{F}(\sigma^*))}{1 + d(g(\sigma^*), \mathcal{S}(\mu^*))}\Big\}\\ &= \max\Big\{d(\frac{1}{8}, \frac{1}{9}),\ d(\frac{1}{6}, \frac{1}{9}),\ d(\frac{1}{9}, \frac{1}{7}), \frac{d(\frac{1}{8}, \frac{1}{9}) + d(\frac{1}{6}, \frac{1}{9})}{2}, \frac{d(\frac{1}{8}, \frac{1}{7})d(\frac{1}{6}, \frac{1}{7})}{1 + d(\frac{1}{6}, \frac{1}{8})},\\ &\quad \frac{d(\frac{1}{6}, \frac{1}{9})d(\frac{1}{7}, \frac{1}{9})}{1 + d(\frac{1}{8}, \frac{1}{7})}\Big\} = \frac{3}{54},\end{aligned} \tag{45}$$

$$\begin{aligned}\psi_2(N(\mu^*, \sigma^*)) &= \min\Big\{d(h(\mu^*), \mathcal{F}(\sigma^*)),\ d(g(\sigma^*), \mathcal{F}(\sigma^*)),\ d(\mathcal{F}(\sigma^*), \mathcal{S}(\mu^*)),\\ &\quad \frac{d(h(\mu^*), \mathcal{F}(\sigma^*))d(\mathcal{S}(\mu^*), \mathcal{F}(\sigma^*))}{1 + d(g(\sigma^*), \mathcal{S}(\mu^*))}\Big\}\\ &= \min\Big\{d(\frac{1}{6}, \frac{1}{9}),\ d(\frac{1}{8}, \frac{1}{9}),\ d(\frac{1}{9}, \frac{1}{7}), \frac{d(\frac{1}{6}, \frac{1}{9})d(\frac{1}{7}, \frac{1}{9})}{1 + d(\frac{1}{8}, \frac{1}{7})}\Big\} = 0.00173.\end{aligned} \tag{46}$$

By the help of (2), (45) *and* (46), $L = 0.2$, $\tau(t) = 0.9t$, $\psi(t) = 0.85t$, $\phi(t) = 0.8t$, $\Gamma(t) = 2t$, *we have:*

$$\begin{aligned}0.0015625 &= \tau\Big(\int_0^{d(\frac{1}{6}, \frac{1}{8})} 2tdt\Big) \leq \psi\Big(\int_0^{\psi_1(M(\mu^*, \sigma^*))} \Gamma(t)dt\Big) + L\phi\Big(\int_0^{\psi_2(N(\mu^*, \sigma^*))} \Gamma(t)dt\Big)\\ &= \psi\Big(\int_0^{\frac{3}{54}} 2tdt\Big) + 0.2\phi\Big(\int_0^{0.00173} 2tdt\Big) = 0.002639.\end{aligned} \tag{47}$$

Thus, the inequality (2) *is also satisfied.*

Example 3.2. *Consider the following system of functional equations which is used in optimization, dynamic programing and computer [24, 31]:*

$$u_i(\mu^*) = \sup_{\sigma^* \in \mathcal{Y}} \{g(\mu^*, \sigma^*) + Q_i(\mu^*, \sigma^*, u_i(\gamma(\mu^*, \sigma^*))), \qquad \mu^* \in \mathcal{W}, \tag{48}$$

where $\gamma : \mathcal{W} \times \mathcal{Y} \to \mathcal{W}$, $g : \mathcal{W} \times \mathcal{Y} \to \mathbb{R}$ *and* $Q_i : \mathcal{W} \times \mathcal{Y} \times \mathbb{R} \to \mathbb{R}$, $i \in \{1, 2, 3, 4\}$. *Let* $\mathcal{B}(\mathcal{W})$ *denote the bounded real valued metric space of functions defined on* $\mathcal{W}$, *with the metric defined;* $d(\mu^*, \sigma^*) = |\mu^* - \sigma^*|$ *for all* μ^*, $\sigma^* \in \mathcal{B}(\mathcal{W})$. *Consider the following system for* h, g, $\mathcal{S}$, $\mathcal{F} : \mathcal{B}(\mathcal{W}) \to \mathcal{B}(\mathcal{W})$:

$$\begin{aligned} h(a(\mu^*)) &= \sup_{\sigma^* \in \mathcal{Y}} \{g(\mu^*, \sigma^*) + Q_1(\mu^*, \sigma^*, a(\gamma_1(\mu^*, \sigma^*)))\}, \\ \mathcal{S}(a(\mu^*)) &= \sup_{\sigma^* \in \mathcal{Y}} \{g(\mu^*, \sigma^*) + Q_2(\mu^*, \sigma^*, a(\gamma_2(\mu^*, \sigma^*)))\}, \\ g(a(\mu^*)) &= \sup_{\sigma^* \in \mathcal{Y}} \{g^*(\mu^*, \sigma^*) + Q_3(\mu^*, \sigma^*, a(\gamma_3(\mu^*, \sigma^*)))\}, \\ \mathcal{F}(a(\mu^*)) &= \sup_{\sigma^* \in \mathcal{Y}} \{g^*(\mu^*, \sigma^*) + Q_4(\mu^*, \sigma^*, a(\gamma_4(\mu^*, \sigma^*)))\}. \end{aligned} \tag{49}$$

Assume the following conditions:

(A) g, $g^* : \mathcal{W} \times \mathcal{Y} \to \mathbb{R}$ *and for* $i = 1, 2, 3, 4$, Q_i *are bounded;*

(B) *the pair* (h, g) *and* $(\mathcal{S}, \mathcal{F})$ *share* $(CLR_{\mathcal{SF}})$ *property;*

(C) *For all* (μ^*, σ^*, u), $(\mu^*, \sigma^*, u^*) \in \mathcal{W} \times \mathcal{Y} \times \mathbb{R}$, $i \neq j$, $i = 1, 2, 3, 4$, $j = 1, 2, 3, 4$:

$$|Q_i(\mu^*, \sigma^*, a(\gamma_4(\mu^*, \sigma^*))) - Q_j(\mu^*, \sigma^*, a(\gamma_4(\mu^*, \sigma^*)))| \leq \psi_1(M), \tag{50}$$

and

$$\begin{aligned} \tau\Big(\int_0^{|Q_i(\mu^*, \sigma^*, a(\gamma_4(\mu^*, \sigma^*))) - Q_j(\mu^*, \sigma^*, a(\gamma_4(\mu^*, \sigma^*)))|} \Gamma(t)dt\Big) &\leq \ln\Big(1 + \int_0^{\psi_1(M(\mu^*, \sigma^*))} \Gamma(t)dt\Big) \\ &+ L \ln(1 + 2\int_0^{\psi_2(N(\mu^*, \sigma^*))} \Gamma(t)dt), \end{aligned} \tag{51}$$

where

$$\begin{aligned} \psi_1(M(\mu^*, \sigma^*)) = \max\Big\{& d(g(\sigma^*), \mathcal{F}(\sigma^*)),\ d(h(\mu^*), \mathcal{F}(\sigma^*)),\ d(\mathcal{F}(\sigma^*), \mathcal{S}(\mu^*)), \\ & \frac{d(h(\mu^*), \mathcal{F}(\sigma^*)) + d(h(\mu^*), \mathcal{S}(\mu^*))}{2}, \frac{d(g(\sigma^*), \mathcal{S}(\mu^*))d(h(\mu^*), g(\sigma^*))}{1 + d(g(\sigma^*), h(\mu^*))}, \\ & \frac{d(h(\mu^*), \mathcal{F}(\sigma^*))d(\mathcal{S}(\mu^*), \mathcal{F}(\sigma^*))}{1 + d(g(\sigma^*), \mathcal{S}(\mu^*))}\Big\}, \end{aligned} \tag{52}$$

$$\begin{aligned} \psi_2(N(\mu^*, \sigma^*)) = \min\Big\{& d(h(\mu^*), \mathcal{F}(\sigma^*)),\ d(g(\sigma^*), \mathcal{F}(\sigma^*)),\ d(\mathcal{F}(\sigma^*), \mathcal{S}(\mu^*)), \\ & \frac{d(h(\mu^*), \mathcal{F}(\sigma^*))d(\mathcal{S}(\mu^*), \mathcal{F}(\sigma^*))}{1 + d(g(\sigma^*), \mathcal{S}(\mu^*))}\Big\}; \end{aligned} \tag{53}$$

(D) *for some* $a \in B(\mathcal{W})$, *such that* $\zeta_1(a) \geq 0$ *implies* $\zeta_2(h(a)) \geq 0$, *and for some* $a \in B(\mathcal{W})$, *such that* $\zeta_2(a) \geq 0$ *implies* $\zeta_1(g(a)) \geq 0$;

(E) $\{a_n\} \subset B(\mathcal{W})$ *such that* $\zeta(a_n) \geq 0$ *and* $\zeta(a_n) \geq 0$ *for all* $n \in \mathbb{N} \cup \{0\}$ *and* $a_n \to a^*$ *as* $n \to \infty$, *then* $\zeta(a^*) \geq 0$ *and* $\zeta(a^*) \geq 0$;

(F) *there exist $a_0 \in B(\mathcal{W})$, for which we have $\zeta_1(a_0) \geq 0$ and $\zeta_2(a_0) \geq 0$.*

Theorem 3.3. *For the quadruple self-mappings h, g, $\mathcal{S}$, $\mathcal{F} : B(\mathcal{W}) \to B(\mathcal{W})$, let the conditions (A)-(F) are satisfied. Then, the system* (49) *has a bounded solution in $B(\mathcal{W})$.*

Proof. Let γ^* be an orbitrary positive real number and $\mu^* \in \mathcal{W}$, $a_1(\mu^*)$, $a_2(\mu^*) \in B(\mathcal{W})$ such that $\zeta_1(a_1(\mu^*)) \geq 0$, $\zeta_2(a_2(\mu^*)) \geq 0$ and there exist y_1, $y_2 \in \mathcal{Y}$ such that

$$h(a_1(\mu^*)) - \gamma^* < g(\mu^*, y_1) + Q_1^*(\mu^*, y_1, a_1(\gamma(\mu^*, y_1))), \tag{54}$$

$$g(a_2(\mu^*)) - \gamma^* < g(\mu^*, y_2) + Q_2^*(\mu^*, y_2, a_1(\gamma(\mu^*, y_2))), \tag{55}$$

$$h(a_1(\mu^*)) \geq g(\mu^*, y_2) + Q_1^*(\mu^*, y_2, a_1(\gamma(\mu^*, y_2))), \tag{56}$$

$$g(a_2(\mu^*)) < g(\mu^*, y_1) + Q_2^*(\mu^*, y_1, a_2(\gamma(\mu^*, y_1))). \tag{57}$$

From (54) and (57), we have:

$$h(a_1(\mu^*)) - \gamma^* - g(a_2(\mu^*)) < Q_1(\mu^*, y_1, a_1(\gamma(\mu^*, y_1))) - Q_2(\mu^*, y_1, a_2(\gamma(\mu^*, y_1))), \tag{58}$$

where γ^* is an orbitrary positive real number, so we may have:

$$h(a_1(\mu^*)) - g(a_2(\mu^*)) < Q_1(\mu^*, y_1, a_1(\gamma(\mu^*, y_1))) - Q_2(\mu^*, y_1, a_2(\gamma(\mu^*, y_1))). \tag{59}$$

Similarly, by the help of (55) and (56), we have:

$$g(a_2(\mu^*)) - \gamma^* - h(a_1(\mu^*)) < Q_2(\mu^*, y_2, a_2(\gamma(\mu^*, y_2))) - Q_1(\mu^*, y_2, a_1(\gamma(\mu^*, y_2))), \tag{60}$$

for γ^* an arbitrary positive real number, we may have:

$$g(a_2(\mu^*)) - h(a_1(\mu^*)) < Q_2(\mu^*, y_2, a_2(\gamma(\mu^*, y_2))) - Q_1(\mu^*, y_2, a_1(\gamma(\mu^*, y_2))). \tag{61}$$

Define α, $\lambda : B(\mathcal{W}) \to \mathbb{R}^+$ by:

$$\alpha(\mu^*) = \begin{cases} 1 \text{ if } \zeta_1(a(\mu^*)) \geq 0, \text{ and } a(\mu^*) \in B(\mathcal{W}), \\ 0 \text{ otherwise,} \end{cases} \qquad \lambda(\mu^*) = \begin{cases} 1 \text{ if } \zeta_2(a(\mu^*)) \geq 0, \text{ and } a(\mu^*) \in B(\mathcal{W}), \\ 0 \text{ otherwise.} \end{cases} \tag{62}$$

By the help of (59) and (61), we have:

$$|h(a_1(\mu^*)) - g(a_2(\mu^*))| < \max_{\mu^*, \sigma^* \in X} \{|Q_i(\mu^*, y_1, a_1(\gamma(\mu^*, y_1))) - Q_j(\mu^*, y_1, a_2(\gamma(\mu^*, y_1)))|\} \text{ for } i \neq j. \tag{63}$$

Therefore, by the help of (51) and (63), we have:

$$\begin{aligned} \int_0^{|h(a_1(\mu^*)) - g(a_2(\mu^*))|} \Gamma(t)dt &\leq \tau\Big(\int_0^{|Q_i(\mu^*, \sigma^*, a(\gamma_4(\mu^*, \sigma^*))) - Q_j(\mu^*, \sigma^*, a(\gamma_4(\mu^*, \sigma^*)))|} \Gamma(t)dt\Big) \\ &\leq \ln\Big(1 + \int_0^{\psi_1(M(\mu^*, \sigma^*))} \Gamma(t)dt\Big) + L\ln(1 + 2\int_0^{\psi_2(N(\mu^*, \sigma^*))} \Gamma(t)dt), \end{aligned} \tag{64}$$

For $\tau(t) = t$, $\psi(t) = 1 + \ln(t)$, $\phi(t) = 1 + 2\ln(t)$, the conditions of Corollary 2.3, are satisfied. Therefore, the system (49) has a bounded solution in $B(\mathcal{W})$. □

4. Conclusion

In this paper, we extended the notion of generalized almost $(\mathcal{S}, \mathcal{F})$-rational contraction pair $(\mathcal{S}, \mathcal{F})$ to generalized almost $(\mathcal{S}, \mathcal{F}, \Gamma)$-rational contraction pair $(\mathcal{S}, \mathcal{F})$ of integral type and have produced some general fixed points theorems using integral inequalities. Our results have many special cases as their corollaries. For the applications of the results, we have given examples.

Competing Interest All the authors have equally contributed in this article and there is no competing interest.

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