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Some Results About Concircular Vector Fields on Riemannian Manifolds

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Abstract. In this article, we show that the presence of a concircular vector field on a Riemannian manifold can be used to obtain rigidity results for Riemannian and Kaehler manifolds. More precisely, we find new geometrical characterizations of spheres, Euclidean spaces as well as of complex Euclidean spaces using non-trivial concircular vector fields.

1. Introduction

Throughout this article, we assume that manifolds are connected and differentiable. The notion of concircular vector fields was first introduced by A. Fialkow in [14]. By a *concircular vector field* on a Riemannian manifold (M, g), we mean a smooth vector field ξ defined on M satisfying

$$\nabla_X \xi = \rho X, \qquad X \in \mathfrak{X}(M), \tag{1}$$

where ∇ denotes the covariant derivative operator with respect to the Riemannian connection of (M, g), $\rho : M \to \mathbf{R}$ is a smooth function, and $\mathfrak{X}(M)$ is the Lie algebra of smooth vector fields on M. The function ρ in equation (1) is called the potential function of ξ.

A concircular vector field ξ on M is called *non-trivial* if the zero set $Z(\rho) = \{p \in M : \rho(p) = 0\}$ of its potential function ρ is of measure zero in M. Further, a concircular vector field ξ is called a *concurrent vector field* if its potential function ρ in (1) is a non-zero constant. (Note that non-trivial concircular vector fields defined in this article is different from the one defined in [6]).

It is well-known that concircular vector fields play important roles in differential geometry as well as in physics. For example, concircular vector fields appeared in the study of concircular mappings, i.e., conformal mappings preserving geodesic circles [6, 22]. Such vector fields play important roles in the theories of projective and conformal transformations as well. Further, concircular vector fields have interesting applications in general relativity, e.g., trajectories of time-like concircular fields in the de Sitter space-time model determine the world lines of receding or colliding galaxies satisfying the Weyl hypothesis

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(see [21]). Moreover, it was proved by one of the authors in [3] that a Lorentzian manifold is a generalized Robertson-Walker space-time if and only if it admits a time-like concircular vector field (for a nice survey on generalized Robertson-Walker spacetimes, see [16]).

In differential geometry, one important research topic is to discover how the existence of a special vector field on a Riemannian manifold affects the geometry and topology of the Riemannian manifold. Some important special vector fields on Riemannian manifolds include geodesic vector fields, concurrent vector fields, concircular vector fields, Killing vector fields, and conformal vector fields. For such special vector fields, see e.g., [3–11, 13, 15, 16, 17].

The main purpose of this article is to prove that the presence of a concircular vector field on a Riemannian manifold can be used to obtain rigidity results for some Riemannian manifolds. More precisely, in this article by applying non-trivial concircular vector fields, we obtain new geometrical characterizations of spheres, Euclidean spaces, and complex Euclidean spaces.

2. Preliminaries

Let ξ be a concircular vector field on a Riemannian manifold (M, g) with the potential function ρ. We denote by α the smooth 1-form dual to the concircular vector field ξ. Then it follows from (1) that the 1-form α is closed.

The curvature tensor field R and the Ricci tensor Ric of (M, g) are given respectively by (cf. e.g., [1, 2])

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \tag{2}$$

and

$$Ric(X,Y) = \sum_{i=1}^{n} g\left(R(e_i, X)Y, e_i\right), \tag{3}$$

where $n = \dim M$ and $\{e_1, .., e_n\}$ is a local orthonormal frame on M.

The Ricci operator Q of M is a symmetric operator defined by

$$g(QX,Y) = Ric(X,Y)$$

for $X, Y \in \mathfrak{X}(M)$. The scalar curvature S of M is the trace of the Ricci operator, i.e., $S = \operatorname{Tr} Q$. The gradient ∇S of the scalar curvature satisfies (cf. [6])

$$\frac{1}{2}\nabla S = \sum_{i=1}^{n} (\nabla Q)(e_i, e_i), \tag{4}$$

where the covariant derivative ∇Q of Q is defined by

$$(\nabla Q)(X,Y) = \nabla_X QY - Q\nabla_X Y.$$

By choosing $Z = \xi$ in equation (2) and by using equations (1) and (4), we have

$$R(X,Y)\xi = (X\rho)Y - (Y\rho)X \tag{5}$$

for $X, Y \in \mathfrak{X}(M)$, which gives

$$Ric(Y,\xi) = -(n-1)(Y\rho).$$

Hence, we have

$$Q(\xi) = -(n-1)\nabla\rho, \tag{6}$$

where $\nabla\rho$ denotes the gradient of the potential function ρ.

Associated with the concircular vector field ξ on M, we define a smooth function $h : M \to \mathbf{R}$ by

$$h = \frac{1}{2}\,\|\xi\|^2 . \tag{7}$$

Hence, after applying equation (1), we derive the following simple expression of the gradient ∇h of h:

$$\nabla h = \rho\xi. \tag{8}$$

Note that for a smooth function $f : M \to \mathbf{R}$ on a Riemannian manifold M, the Hessian operator $\mathcal{H}_f$ and the Laplacian Δf of f are defined respectively by

$$\mathcal{H}_f X = \nabla_X \nabla f, \quad \Delta f = \operatorname{div}(\nabla \mathrm{f}), \tag{9}$$

where

$$\operatorname{div}(X) = \sum_{i=1}^{n} g\left(\nabla_{e_i} X, e_i\right).$$

The Hessian of f, Hess_f, is given by

$$\mathrm{Hess}_f(X, Y) = g\left(\mathcal{H}_f X, Y\right), \quad X, Y \in \mathfrak{X}(M). \tag{10}$$

Recall that if f is a non-constant function on a compact Riemannian manifold M satisfying

$$\int_M f dv = 0,$$

then a minimum principle yields

$$\int_M \|\nabla f\|^2 dv \geq \lambda_1 \int_M f^2 dv, \tag{11}$$

where dv is the volume element of M and λ_1 denotes the first nonzero eigenvalue of the Laplace operator Δ acting on smooth functions.

3. A geometric characterization of spheres

In this section, we apply non-trivial concircular vector fields to derive a geometric characterization of spheres.

Let $S^n(c)$ denote the hypersphere with radius $\frac{1}{\sqrt{c}}$ centered at the origin o in a Euclidean $(n+1)$-space $\mathbb{E}^{n+1}$ and let N be a unit normal vector field of $S^n(c)$ in $\mathbb{E}^{n+1}$. Then the Weingarten map $A = A_N$ of $S^n(c)$ satisfies $A = -\sqrt{c}I$ (cf. [2]). From Gauss' formula, we have

$$D_X Y = \nabla_X Y - \sqrt{c} g(X, Y) N \tag{12}$$

for vector fields $X, Y \in \mathfrak{X}(S^n(c))$, where g denotes the induced metric on $S^n(c)$, and D and ∇ are the Riemannian connections of $\mathbb{E}^{n+1}$, and of $S^n(c)$, respectively.

For a given constant non-zero vector w in $\mathbb{E}^{n+1}$, let us consider the function $f = \langle w, N\rangle$ defined on $S^n(c)$, where $\langle\ ,\ \rangle$ is the inner product on $\mathbb{E}^{n+1}$. Then for any vector field X tangent to $S^n(c)$, we have

$$Xf = \left\langle w, \sqrt{c}X\right\rangle = \sqrt{c} g(w^T, X), \tag{13}$$

where w^T is the component of w tangential to $S^n(c)$. From (13), we get

$$\nabla f = \sqrt{c}w^T. \tag{14}$$

Since w is a constant vector field in $\mathbb{E}^{n+1}$, we have $D_X w = 0$. Thus, it follows from formula (12) of Gauss and equation (14) that

$$\begin{aligned} 0 &= \langle D_X w, Y\rangle = X\langle w, Y\rangle - \left\langle w, \nabla_X Y - \sqrt{c}g(X,Y)N\right\rangle \\ &= Xg(w^T, Y) - g(w^T, \nabla_X Y) + \sqrt{c}fg(X,Y) \\ &= g\left(\nabla_X w^T, Y\right) + \sqrt{c}fg(X,Y). \end{aligned}$$

Thus

$$\nabla_X w^T = -\sqrt{c}fX, \tag{15}$$

which shows that w^T is a concircular vector field on $S^n(c)$. Also, it is to verify that the potential function $\rho = -\sqrt{c}f$ of w^T vanishes exactly on the great hypersphere $S^{n-1}(c) \subset S^n(c)$ obtained by the intersection $S^n(c) \cap E$, where E is the hyperplane of $\mathbb{E}^{n+1}$ containing the origin o with N as its hyperplane normal. Since $S^{n-1}(c)$ is of measure zero in $S^n(c)$, w^T is a non-trivial concircular vector field on $S^n(c)$.

Further, by using equations (14) and (15), we have

$$\Delta\rho = -\sqrt{c}\Delta f = -c\,(\operatorname{div} w^T) = c\sqrt{c}nf = -nc\rho.$$

Hence

$$(\Delta\rho)^2 = n^2c^2\rho^2. \tag{16}$$

Note that the first non-zero eigenvalue λ_1 of the Laplace operator Δ on $S^n(c)$ is $\lambda_1 = nc$. Therefore, we obtain

$$nc\,(2\lambda_1 - nc)\,\rho^2 = n^2c^2\rho^2. \tag{17}$$

Thus, it follows from equations (16) and (17) that the potential function of ρ of the non-trivial concircular vector field w^T on $S^n(c)$ satisfies the following equation:

$$(\Delta\rho)^2 = nc\,(2\lambda_1 - nc)\,\rho^2.$$

Now, we prove the following geometrical characterization of spheres.

Theorem 3.1. *Let M be an n-dimensional compact Riemannian manifold with positive Ricci curvature. Then M admits a non-trivial concircular vector field whose potential function ρ satisfies*

$$(\Delta\rho)^2 \le nc\,(2\lambda_1 - nc)\,\rho^2$$

for a constant c if and only if M is isometric to the n-sphere.

Proof. Assume that (M, g) is an n-dimensional compact Riemannian manifold of positive Ricci curvature which admits a non-trivial concircular vector field ξ whose potential function ρ satisfying

$$(\Delta\rho)^2 \le nc\,(2\lambda_1 - nc)\,\rho^2, \tag{18}$$

for some constant c. Then equation (1) yields $\operatorname{div}(\xi) = n\rho$. Thus we have

$$\int_M \rho dv = 0. \tag{19}$$

We observe that the constant c appearing in inequality (18) is non-zero. Since otherwise, if $c = 0$, then inequality (18) implies $\Delta\rho = 0$. Thus, by compactness of M, ρ is a constant and then equation (19) gives $\rho = 0$. This contradicts to ξ being a non-trivial concircular vector field.

Now, using equation (1), we find that $\mathrm{div}(\rho\xi) = \xi\rho + n\rho^2$. So, after integrating this equation over M, we obtain

$$\int_M (\xi\rho)dv = -n\int_M \rho^2 dv. \tag{20}$$

On the other hand, by applying equation (6), we get

$$Ric(\xi,\xi) = -(n-1)\xi(\rho) \tag{21}$$

and

$$Ric(\nabla\rho,\xi) = -(n-1)\left\|\nabla\rho\right\|^2. \tag{22}$$

From the Bochner formula for the potential function ρ, we have

$$\int_M \left\{Ric(\nabla\rho,\nabla\rho) + \left\|\mathcal{H}_\rho\right\|^2 - (\Delta\rho)^2\right\} dv = 0. \tag{23}$$

Now, we may use the non-zero constant c to compute

$$Ric\,(\nabla\rho + c\xi, \nabla\rho + c\xi) = Ric(\nabla\rho,\nabla\rho) + 2cRic\,(\nabla f,\xi) + c^2 Ric(\xi,\xi),$$

which on using equations (21) and (22) gives

$$Ric\,(\nabla\rho + c\xi, \nabla\rho + c\xi) = Ric(\nabla\rho,\nabla\rho) - 2(n-1)c\left\|\nabla\rho\right\|^2 - (n-1)c^2\xi(\rho).$$

Thus, integrating the above equation and applying equations (20) and (23), we may conclude that

$$\begin{aligned}\int_M & Ric\,(\nabla\rho + c\xi, \nabla\rho + c\xi)\,dv \\ &= \int_M \left(-\left\|\mathcal{H}_\rho\right\|^2 + (\Delta\rho)^2 - 2(n-1)c\left\|\nabla\rho\right\|^2 + n(n-1)c^2\rho^2\right) dv.\end{aligned}$$

Now, using inequality (11) (which holds in view of equation (25)), we obtain

$$\int_M Ric\,(\nabla\rho + c\xi, \nabla\rho + c\xi)\,dv \leq \int_M \left\{-\left\|\mathcal{H}_\rho\right\|^2 + (\Delta\rho)^2 - 2(n-1)c\lambda_1\rho^2 + n(n-1)c^2\rho^2\right\} dv,$$

which can be rearranged as

$$\int_M Ric(\nabla\rho + c\xi, \nabla\rho + c\xi)dv \leq \int_M \left\{-\left(\left\|\mathcal{H}_\rho\right\|^2 - \frac{1}{n}(\Delta\rho)^2\right) - \frac{(n-1)}{n}\left\{nc\,(2\lambda_1 - nc)\,\rho^2 - (\Delta\rho)^2\right\}\right\} dv.$$

Next, by applying Schwartz's inequality $n\left\|\mathcal{H}_\rho\right\|^2 \geq (\Delta\rho)^2$, the inequality (18), and the fact that the Ricci curvature of M is positive in above inequality, we conclude $\nabla\rho + c\xi = 0$, or

$$\nabla\rho = -c\xi. \tag{24}$$

The above equation implies $\Delta\rho = -nc\rho$. Therefore, the non-constant function ρ is an eigenfunction of Δ with eigenvalue $-nc$. Hence, as M being compact, it implies that $c > 0$. Now, by taking the covariant derivative of equation (24) with respect to any $X \in \mathfrak{X}(M)$ and using equation (1), we find

$$\nabla_X\nabla\rho = -c\rho X, \tag{25}$$

Therefore, the compact Riemannian manifold M admits a non-constant function ρ which satisfies the Obata differential equation (25) (see [18]). Consequently, the Riemannian manifold M is isometric to a sphere of constant curvature c according to Obata's theorem (cf. [18]).

The converse was already proved before the statement of the theorem. □

4. A simple characterization of Euclidean spaces

In this section, we use concircular vector fields on complete Riemannian manifolds to derive a simple geometrical characterization of Euclidean spaces. In order to do so, we need the following.

Lemma 4.1. *A complete Riemannian manifold admits a concurrent vector field if and only if it is isometric to a Euclidean space.*

Proof. If M is a Euclidean n-space $\mathbb{E}^n$, it is well-known that the position vector field of $\mathbb{E}^n$ is a concurrent vector field.

Conversely, assume that M is a complete Riemannian manifold which admits a concurrent vector field η such that

$$\nabla_X \eta = cX, \quad X \in \mathfrak{X}(M), \tag{26}$$

for a non-zero constant c.

Let us put $f = \frac{1}{2}\|\eta\|^2$. Then it follows from (26) that the gradient of f satisfies

$$\nabla f = c\eta. \tag{27}$$

Now, by taking the covariant derivative of equation (27) with respect to a tangent vector field X, we have the following expression of the Hessian operator $\mathcal{H}_f$ of f:

$$\mathcal{H}_f(X) = c^2 X.$$

Thus we find

$$\mathrm{Hess}_f(\,\cdot\,,\cdot\,) = c^2 g(\,\cdot\,,\cdot\,).$$

Therefore, M is isometric to a Euclidean space according to Theorem 1 of [20]. □

Theorem 4.2. *A complete Riemannian manifold admits a non-trivial concircular vector field ξ whose potential function ρ is constant along each integral curve of ξ if and only if it is isometric to a Euclidean space.*

Proof. Suppose that ξ is a non-trivial concircular vector field on an n-dimensional complete Riemannian manifold M whose potential function ρ satisfies $\xi\rho = 0$. As before, let h be the function given by $h = \frac{1}{2}\|\xi\|^2$. Then we know from Section 2 that the gradient of h satisfies equation (8), i.e., $\nabla h = \rho\xi$.

Now, we observe that the function h is non-constant, due to ξ is non-trivial. This can be seen as follows. First, we have $\nabla_\xi \xi = \rho\xi$ from equation (1). So, after taking the inner product of this equation with ξ, we get $\xi h = 2\rho h$. Hence

$$2\rho = \xi\,(\ln h) \tag{28}$$

on any open subset $U \subset M$ on which $\xi \neq 0$. If h is a constant function, equation (28) implies $\rho = 0$ on U, which contradicts to the assumption that the concircular vector field ξ is non-trivial. This shows the observation.

Next, by taking covariant derivative of equation (8) with respect to a tangent vector field X, we obtain the following expression for the Hessian operator $\mathcal{H}_h$:

$$\mathcal{H}_h(X) = (X\rho)\xi + \rho^2 X,$$

which implies

$$\mathrm{Hess}_h(X, Y) = (X\rho)\eta(Y) + \rho^2 g(X, Y), \tag{29}$$

where η denotes the 1-form dual to ξ. So, by using symmetry of the Hessian in equation (29), we conclude

$$(X\rho)\eta(Y) = (Y\rho)\eta(X)$$

or equivalently, $(X\rho)\xi = \eta(X)\nabla\rho$. Hence, using $\xi\rho = 0$ in the above equation, we obtain

$$\eta(\xi)\nabla\rho = \|\xi\|^2\,\nabla\rho = 0. \tag{30}$$

Since the concircular vector field ξ vanishes only on a measure zero subset of M, by continuity we get $\nabla\rho = 0$ from (30). Therefore, ρ is a constant c. Moreover, since ξ is a non-trivial concircular vector field, we must have $c \neq 0$. Hence, ξ is a concurrent vector field. Consequently, by applying Lemma 4.1, we conclude that M is isometric to a Euclidean space.

The converse is trivial since the position vector field of $\mathbb{E}^n$ is concurrent. □

5. Concircular vector fields on Kaehler manifolds

A Riemannian metric g on a complex manifold $(\tilde{M}, J)$ is called Hermitian if the metric g and the complex structure J on $\tilde{M}$ are compatible, i.e.,

$$g(JX, JY) = g(X, Y), \quad X, Y \in \mathfrak{X}(\tilde{M}). \tag{31}$$

A Hermitian manifold $(\tilde{M}, g, J)$ is called a Kaehler manifold if its complex structure J is parallel with respect to its Riemannian connection ∇, i.e., $\nabla J = 0$.

It is well-known that the Riemann curvature tensor R of a Kaehler manifold $(\tilde{M}, g, J)$ satisfies the following relations (cf., e.g. [2]):

$$R(X, Y) = -R(Y, X), \tag{32}$$
$$R(X, Y)JZ = J(R(X, Y)Z), \tag{33}$$
$$R(JX, JY)Z = R(X, Y)Z, \tag{34}$$
$$g(R(X, Y)Z, W) = g(R(Z, W)X, Y), \tag{35}$$

for $X, Y, Z, W \in \mathfrak{X}(\tilde{M})$.

For concircular vector fields on Kaehler manifolds, we have the following.

Theorem 5.1. *If a complete Kaehler n-manifold $(\tilde{M}, J, g)$ with $n \geq 2$ admits a non-trivial concircular vector field, then it is holomorphically isometric to a complex Euclidean n-space $\mathbb{C}^n$.*

Proof. Let $(\tilde{M}, J, g)$ be a complete Kaehler n-manifold with $n = \dim_{\mathbf{C}} \tilde{M} \geq 2$. Assume that $(\tilde{M}, J, g)$ admits a non-trivial concircular vector field ξ such that

$$\nabla_X \xi = \rho X, \quad X \in \mathfrak{X}(\tilde{M}). \tag{36}$$

Then it follows from (2) and (36) that the curvature tensor R satisfies

$$R(X, \xi)\xi = (X\rho)\xi - (\xi\rho)X \tag{37}$$

for any tangent vector field X. By taking the inner product of (37) with ξ we get

$$(X\rho)\|\xi\|^2 = (\xi\rho)g(X, \xi), \tag{38}$$

which implies $X\rho = 0$ whenever $\xi \neq 0$ and $g(X, \xi) = 0$. Since ξ is a non-trivial concircular vector field, equation (1) shows that the zero set $Z(\xi)$ of ξ is of measure zero. Hence, by continuity, we obtain

$$X\rho = 0, \quad \text{whenever} \quad g(X, \xi) = 0. \tag{39}$$

Also, by taking the inner product of (37) with any vector field X perpendicular to ξ, we have

$$g(R(X,\xi)\xi, X) = -(\xi\rho)\|X\|^2. \tag{40}$$

Similarly, by applying (2), (33) and (36), we find

$$R(Y, J\xi)J\xi = -(Y\rho)\xi - ((J\xi)\rho)JY \tag{41}$$

for any tangent vector field Y. Thus, after combining (38) with (41), we obtain

$$g(R(Y, J\xi)J\xi, Y) = 0 \tag{42}$$

for any tangent vector field Y satisfying $g(Y, \xi) = 0$.

Next, by applying (32), (34), (35) and (42) we have

$$\begin{aligned} 0 &= -g(R(Y, J\xi)J\xi, Y) = g(R(JY, \xi)J\xi, Y) \\ &= g(R(J\xi, Y)JY, \xi) = -g(R(\xi, JY)JY, \xi) \\ &= g(R(JY, \xi)\xi, JY) \end{aligned} \tag{43}$$

for any tangent vector field Y satisfying $g(Y, \xi) = 0$. Now, by combining (40) and (43) we get

$$(\xi\rho)\|X\|^2 = 0 \tag{44}$$

for any tangent vector field X satisfying $g(X, \xi) = g(JX, \xi) = 0$. Since $\dim_{\mathbf{C}} \tilde{M} \geq 2$ by hypothesis, there exists a non-zero vector field X which is perpendicular to ξ and $J\xi$. Hence, $\xi\rho = 0$ by (44). Thus, after combining this with (39), we conclude that ρ is a non-zero constant function. Hence, ξ is a concurrent vector field (cf. [6]). Consequently, $\tilde{M}$ is isometric to a Euclidean $2n$-space $\mathbb{E}^{2n}$ according to Lemma 4.1. Hence, the Kaehler n-manifold $(\tilde{M}, J, g)$ is a complete, simply-connected, flat Kaehler manifold. Consequently, $(\tilde{M}, J, g)$ is holomorphically isometric to a complex Euclidean n-space $\mathbb{C}^n$. □

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