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On a Ricci Quarter-Symmetric Metric Recurrent Connection and a Projective Ricci Quarter-Symmetric Metric Recurrent Connection in a Riemannian Manifold

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Abstract. Two new types of connections, Ricci quarter-symmetric metric recurrent connection and projective Ricci quarter-symmetric metric recurrent connection, were introduced and some interesting geometrical and physical characteristics were achieved.

1. Introduction

The concept of the semi-symmetric connection was introduced by Friedman and Schouten in [6] for the first time, Hayden in [11] introduced the metric connection with torsion, and Yano in [21] defined a semi-symmetric metric connection and studied its geometric properties. N. Agache and M. Chafle [1] investigated the semi-symmetric non-metric connection. Recently, De, Han and Zhao in [2] studied the semi-symmetric non-metric connection. On the other hand, the Schur's theorem of a semi-symmetric non-metric connection is well known ([12, 13]) based only on the second Bianchi identity. A semi-symmetric metric connection that is a geometrical model for scalar-tensor theories of gravitation was studied ([3]) and a conjugate symmetry condition of the Amari-Chentsov connection with metric recurrent was also studied. Recently in [9] the similar topics were further studied in sub-Riemannian manifolds. A quarter-symmetric connection in [8] was defined and studied. Afterwards, several types of a quarter-symmetric metric connection were studied ([4, 10, 19, 22]). In [7, 14, 20, 23, 24], the geometric and physic properties of conformal and projective the semi-symmetric metric recurrent connections were studied. And in [17, 18] a projective conformal quarter-symmetric metric connection and a generalized quarter-symmetric metric recurrent connection were studied. In [5] a curvature copy problem of the symmetric connection was studied. And in [18] the mutual connection of a semi-symmetric connection was studied.

Motivated by the previous researches we define newly in this note the Ricci quarter-symmetric metric recurrent connection and the projective Ricci quarter-symmetric metric recurrent connection and study

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their properties. And the Schur's theorem of the Ricci quarter-symmetric metric recurrent connection and the projective Ricci quarter-symmetric metric recurrent connection and several types of these connections with constant curvature are discovered.

2. A Ricci Quarter Symmetric Metric Recurrent Connection

Let (M, g) be a Riemannian manifold ($dimM \geq 2$), g be the Riemannian metric on M, and $\widetilde{\nabla}$ be the Levi-Civita connection with respect to g. Let $\mathcal{X}(M)$ denote the collection of all vector fields on M.

Definition 2.1. *A connection ∇ is called a Ricci quarter-symmetric metric recurrent connection, if it satisfies*

$$\nabla_Z g(X, Y) = 2\omega(Z)g(X, Y), T(X, Y) = \pi(Y)UX - \pi(X)UY \tag{1}$$

where U is a Ricci operator, ω and π are 1-form respectively. If $U(X) = X$, then ∇ is a semi-symmetric metric recurrent connection studied in [24].

Let (x^i) be the local coordinate, then $g, \widetilde{\nabla}, \nabla, \omega, \pi, U$ and T have the local expressions $g_{ji}, \{^k_{ji}\}, \Gamma^k_{ji}, \omega_i, \pi_i, U^j_i$ and T^k_{ji} respectively. At the same time the expression (1) can be rewritten as

$$\nabla_k g_{ji} = 2\omega_k g_{ji}, T^k_{ji} = \pi_i U^k_j - \pi_j U^k_i \tag{2}$$

The coefficient of ∇ is given as

$$\Gamma^k_{ij} = \{^k_{ij}\} - \omega_i \delta^k_j - \omega_j \delta^k_i + g_{ij}\omega^k + \pi_j U^k_i - U_{ij}\pi^k \tag{3}$$

where U_{ij} is a Ricci tensor of the Levi-Civita connection $\widetilde{\nabla}$. From (3), the curvature tensor of ∇, by a direct computation, is

$$\begin{aligned} R^l_{ijk} &= K^l_{ijk} + \delta^l_i a_{jk} - \delta^l_j a_{ik} + g_{jk}a^l_i - g_{ik}a^l_j + U^l_j b_{ik} - U^l_i b_{jk} \\ &+ U_{ik}b^l_j - U_{jk}b^l_i + c^l_{ij}\pi_k - c^l_{ji}\pi_k - c_{ijk}\pi^l + c_{jik}\pi^l - \delta^l_k(\omega_{ij} - \omega_{ji}) \end{aligned} \tag{4}$$

where K^l_{ijk} is the curvature tensor of the Levi-Civita connection $\widetilde{\nabla}$ and other notations are given as

$$\begin{aligned} a_{ik} &= \widetilde{\nabla}_i\omega_k + \omega_i\omega_k + U_{ik}\omega_p\pi^p - U^p_i\omega_p\pi_k - \frac{1}{2}g_{ik}\omega_p\omega^p \\ b_{ik} &= \widetilde{\nabla}_i\pi_k + \pi_i\omega_k - U^p_i\pi_p\pi_k - \frac{1}{2}U_{ik}\pi_p\pi^p \\ c_{ijk} &= \widetilde{\nabla}_i U_{jk} \\ \omega_{ij} &= \widetilde{\nabla}_i\omega_j \end{aligned}$$

Let

$$A^l_{ijk} = \delta^l_i a_{jk} + a^l_i g_{jk} - U^l_i b_{jk} - b^l_i U_{jk} + c^l_{ij}\pi_k - c_{ijk}\pi^l - \delta^l_k\omega_{ij}$$

Then, we get

$$R^l_{ijk} = K^l_{ijk} + A^l_{ijk} - A^l_{jik} \tag{5}$$

So there exists the following.

Theorem 2.2. *When $A^l_{ijk} = A^l_{jik}$, then the curvature tensor will keep unchanged under the connection transformation $\widetilde{\nabla} \to \nabla$.*

From (3), the coefficient of dual connection $\widehat{\nabla}$ of the Ricci quarter-symmetric metric recurrent connection ∇ is

$$\widehat{\Gamma}^k_{ij} = \{^k_{ij}\} + \omega_i\delta^k_j - \omega_j\delta^k_i + g_{ij}\omega^k + \pi_j U^k_i - U_{ij}\pi^k \tag{6}$$

By using the expression (6), the curvature tensor of dual connection $\widehat{\nabla}$ is

$$\begin{aligned}\widehat{R}^l_{ijk} &= K^l_{ijk} + \delta^l_i a_{jk} - \delta^l_j a_{ik} + g_{jk}a^l_i - g_{ik}a^l_j + U^l_j b_{ik} - U^l_i b_{jk} \\ &+ U_{ik}b^l_j - U_{jk}b^l_i + c^l_{ij}\pi_k - c^l_{ji}\pi_k - c_{ijk}\pi^l + c_{jik}\pi^l + \delta^l_k(\omega_{ij} - \omega_{ji})\end{aligned} \tag{7}$$

In the Riemannian manifold (M, g) if $R^l_{ijk} = \widehat{R}^l_{ijk}$, then the connection ∇ is called a conjugate symmetry and if $R_{jk} = \widehat{R}_{jk}$, then the connection ∇ is called a conjugate Ricci symmetry, and if $P_{ij} = \widehat{P}_{ij}$, then the connection ∇ is called a conjugate quasi-Ricci (or Volume) symmetry, where $P_{ji} = g^{hl}R_{jihl}$.

Theorem 2.3. *In a Riemannian manifold (M, g) with a Ricci quarter-symmetric metric recurrent connection ∇ if a 1-form ω is a closed form, then the Riemannian manifold (M, g, ∇) is a quasi-Ricci flat and the Ricci quarter-symmetric metric recurrent connection is a conjugate symmetric.*

Proof. By using the contraction of the indices k and l in the (4) we have

$$P_{ji} = \widehat{P}_{ji} - n(\omega_{ji} - \omega_{ij})$$

where $\widetilde{P}_{ij} = K_{ijk}{}^k = 0$. If a 1-form ω is a closed form, then $\omega_{ij} = \omega_{ji}$. Hence $P_{ji} = 0$. Consequently the Riemannian manifold (M, g, ∇) is a quasi-Ricci flat. On the other hand, from the expressions (4) and (7), we obtain

$$\widehat{R}^l_{ijk} = R_{ijk}{}^l + 2\delta^l_k(\omega_{ij} - \omega_{ji}) \tag{8}$$

If a 1-form ω is a closed form, then $\omega_{ij} = \omega_{ji}$. Hence from the expression (8), we have $\widehat{R}^l_{ijk} = R_{ijk}{}^l$. Consequently, the Ricci quarter-symmetric metric recurrent connection ∇ is a conjugate symmetry. □

Theorem 2.4. *The Ricci quarter-symmetric metric recurrent connection ∇ on a Riemannian manifold (M, g) is a conjugate symmetry if and only if It is a conjugate Ricci symmetry or a conjugate volume symmetry.*

Proof. By using the contraction of the indices i and l in (8) we have

$$\widehat{R}_{jk} = R_{jk} - 2(\omega_{jk} - \omega_{kj}).$$

From this expression, we arrive at

$$\omega_{jk} - \omega_{kj} = \frac{1}{2}(R_{jk} - \widehat{R}_{jk}).$$

Substituting this expression into (8), we have

$$\widehat{R}^l_{ijk} + \delta^l_k\widehat{R}_{ij} = R_{ijk}{}^l + \delta^l_k R_{ij} \tag{9}$$

From the equation (9) it is easy to show that $R_{ijk}{}^l = \widehat{R}^l_{ijk}$ if and only if $R_{jk} = \widehat{R}_{jk}$. On the other hand, by using the contraction of the indices k and l in (8), we have

$$\widehat{P}_{ij} = P_{ij} + 2n(\omega_{ij} - \omega_{ji})$$

From this expression, we arrive at

$$\omega_{jk} - \omega_{kj} = \frac{1}{2n}(R_{jk} - \widehat{R}_{jk}).$$

Substituting this expression into (8) we have

$$\widehat{R}^{l}_{ijk} - \frac{1}{n}\delta^{l}_{k}\widehat{P}_{ij} = R_{ijk}{}^{l} - \frac{1}{n}\delta^{l}_{k}P_{ij} \tag{10}$$

From the equation (10), it is easy to show that $R_{ijk}{}^{l} = \widehat{R}^{l}_{ijk}$ if and only if $P_{ij} = \widehat{P}_{ij}$. □

It is well known that a sectional curvature at a point p in a Riemannian manifold is independent of Π (a 2-dimensional subspace of $T_p(M)$), the curvature tensor is

$$R_{ijk}{}^{l} = k(p)(\delta^{l}_{i}g_{jk} - \delta^{l}_{j}g_{ik}) \tag{11}$$

In this case, if $k(p)$ =const, then the Riemannian manifold is a constant curvature manifold.

Theorem 2.5. *Suppose that $(M, g)(dimM \geq 3)$ is a connected Riemannian manifold associated with an isotropic Ricci quarter-symmetric metric recurrent connection ∇. If there holds*

$$\omega_h = -s_h \tag{12}$$

then (M, g, ∇) is a constant curvature manifold, where $s_h = \frac{1}{n-1}T^{p}_{hp}$(Schur's theorem for the Ricci quarter-symmetric metric recurrent connection)

Proof. Substituting the expression (11) and using the expression (2) into the second Bianchi identity of the curvature tensor of the Ricci quarter-symmetric metric recurrent connection ∇, we get

$$\nabla_h R_{ijk}{}^{l} + \nabla_i R_{jhk}{}^{l} + \nabla_j R_{hik}{}^{l} = T^{m}_{hi}R_{jmk}{}^{l} + T^{m}_{ij}R_{hmk}{}^{l} + T^{m}_{jh}R_{imk}{}^{l}$$

then we have

$$\begin{aligned}
&(\nabla_h k(p) + 2\omega_h k(p))(\delta^{l}_{i}g_{jk} - \delta^{l}_{j}g_{ik}) + (\nabla_i k(p) + 2\omega_i k(p))(\delta^{l}_{j}g_{hk} - \delta^{l}_{h}g_{jk})\\
&\quad +(\nabla_j k(p) + 2\omega_j k(p))(\delta^{l}_{h}g_{ik} - \delta^{l}_{i}g_{hk})\\
&= k(p)[\pi_h(\delta^{l}_{i}U_{jk} - \delta^{l}_{j}U_{ik} + U^{l}_{i}g_{jk} - U^{l}_{j}g_{ik}) + \pi_i(\delta^{l}_{j}U_{hk} - \delta^{l}_{h}U_{jk} + U^{l}_{j}g_{hk} - U^{l}_{h}g_{jk})\\
&\quad +\pi_j(\delta^{l}_{h}U_{ik} - \delta^{l}_{i}U_{hk} + U^{l}_{h}g_{ik} - U^{l}_{i}g_{hk})]
\end{aligned}$$

Contracting the indices i and l, then we obtain

$$\begin{aligned}
&(n-2)(\nabla_h k(p) + 2\omega_h k(p))g_{jk} - (n-2)(\nabla_j k(p) + 2\omega_j k(p))g_{hk}\\
&= k(p)[(n-3)(\pi_h U_{jk} - \pi_j U_{hk}) + (\pi_h U^{i}_{i} - U^{i}_{h}\pi_i)g_{jk} - (\pi_j U^{i}_{i} - \pi_i U^{i}_{j})g_{hk}]
\end{aligned}$$

Multiplying both sides of this expression by g^{jk}, then we have

$$(n-1)(n-2)(\nabla_h k(p) + 2\omega_h k(p)) = 2(n-2)k(p)(\pi_h U^{p}_{p} - U^{p}_{h}\pi_p)$$

From this equation above we obtain

$$\nabla_h k(p) = -2(\omega_h + s_h)k(p).$$

Consequently, from that we know $k(p)$ =const if and only if $\omega_h = -s_h$. □

By Theorem 2.5, the expression (2) for the Ricci quarter-symmetric metric connection with a constant curvature satisfies

$$\nabla_k g_{ji} = -2s_k g_{ji}, T^{k}_{ij} = \pi_j U^{k}_{i} - \pi_i U^{k}_{j} \tag{13}$$

Similarly, the formula (3) shows

$$\Gamma^{k}_{ij} = \{^{k}_{ij}\} + s_i\delta^{k}_{j} + s_j\delta^{k}_{i} - g_{ij}s^{k} + \pi_j U^{k}_{i} - U_{ij}\pi^{k} \tag{14}$$

If the Riemannian manifold is an Einstein manifold, then we obtain

$$U_{jk} = \frac{k}{n} g_{jk} \tag{15}$$

From the expression (15), we have

$$s_h = -\frac{k}{n}\pi_h.$$

Hence, for an Einstein manifold, the expression (13) shows

$$\nabla_k g_{ij} = \frac{2k}{n}\pi_k g_{ij}, T^k_{ij} = \frac{k}{n}(\pi_j \delta^k_i - \pi_i \delta^k_j) \tag{16}$$

Similarly, the formula (14) shows

$$\Gamma^k_{ij} = \{^k_{ij}\} - \frac{k}{n}\pi_i \delta^k_j \tag{17}$$

This connection was studied in [3].

From the expression (3), the coefficient of mutual connection $\overset{m}{\nabla}$ of the Ricci quarter-symmetric metric recurrent connection ∇ is

$$\overset{m}{\Gamma}{}^k_{ij} = \{^k_{ij}\} - \omega_i \delta^k_j - \omega_j \delta^k_i + g_{ij}\omega^k + \pi_i U^k_j - U_{ij}\pi^k \tag{18}$$

This connection satisfies the relation

$$\overset{m}{\nabla}_k g_{ij} = 2\omega_k g_{ij} - 2\pi_k U_{ij} + U_{ki}\pi_j + U_{kj}\pi_i, \overset{m}{T}{}^k_{ij} = \pi_i U^k_j - \pi_j U^k_i. \tag{19}$$

From the expressions (18) and (19), the coefficient of dual connection $\widehat{\overset{m}{\nabla}}$ of the mutual connection $\overset{m}{\nabla}$ is

$$\widehat{\overset{m}{\Gamma}}{}^k_{ij} = \{^k_{ij}\} + \omega_i \delta^k_j - \omega_j \delta^k_i + g_{ij}\omega^k - \pi_i U^k_j + \pi_j U^k_i. \tag{20}$$

On the other hand, in a Riemannian manifold the Weyl connection $\overset{w}{\nabla}$ satisfies the relation

$$\overset{w}{\nabla}_k g_{ij} = 2\omega_k g_{ij}, \overset{w}{T}{}^k_{ij} = 0. \tag{21}$$

and the coefficient of $\overset{w}{\nabla}$ is

$$\overset{m}{\Gamma}{}^k_{ij} = \{^k_{ij}\} - \omega_i \delta^k_j - \omega_j \delta^k_i + g_{ij}\omega^k. \tag{22}$$

From the expressions (21) and (22), the coefficient of a dual connection $\widehat{\overset{w}{\nabla}}$ of the Weyl connection $\overset{w}{\nabla}$ is

$$\widehat{\overset{m}{\Gamma}}{}^k_{ij} = \{^k_{ij}\} + \omega_i \delta^k_j - \omega_j \delta^k_i + g_{ij}\omega^k. \tag{23}$$

Theorem 2.6. *In a Riemannian manifold (M, g) the dual connection $\widehat{\overset{m}{\nabla}}$ of the mutual connection $\overset{m}{\nabla}$ of a Ricci quarter-symmetric metric recurrent connection ∇ is projective equivalent to dual connection $\widehat{\overset{w}{\nabla}}$ of the Weyl connection $\overset{w}{\nabla}$.*

Proof. From the expressions (20) and (23), we have

$$\widehat{\overset{m}{\Gamma}}{}^k_{(ij)} = \widehat{\overset{w}{\Gamma}}{}^k_{(ij)},$$

where (ij) expresses the symmetry of the indices. Hence the connection $\widehat{\overset{m}{\nabla}}$ has the same geodesic as $\widehat{\overset{w}{\nabla}}$. Thus the connection $\widehat{\overset{m}{\nabla}}$ is projective equivalent to the connection $\widehat{\overset{w}{\nabla}}$. □

3. A Projective Ricci Quarter-Symmetric Metric Recurrent Connection

Definition 3.1. *In a Riemannian manifold (M, g), a connection $\overset{p}{\nabla}$ is called a projective Ricci quarter-symmetric metric recurrent connection, if the $\overset{p}{\nabla}$ is projective equivalent to a Ricci quarter-symmetric metric recurrent connection ∇.*

In a Riemannian manifold (M, g), a projective Ricci quarter-symmetric metric recurrent connection $\overset{p}{\nabla}$ satisfies the relation

$$\begin{aligned} \overset{p}{\nabla}_Z g(X,Y) &= -2[\Psi(Z)-\omega(Z)]g(X,Y) - \Psi(X)g(Y,Z) - \Psi(Y)g(X,Z), \\ \overset{p}{T}(X,Y) &= \pi(Y)UY - \pi(X)UY. \end{aligned}$$

The local expression of this relation is

$$\begin{cases} \overset{p}{\nabla}_k g_{ij} &= -2(\Psi_k-\omega_k)g_{ij} - \Psi_j g_{ik} - \Psi_i g_{jk}, \\ \overset{p}{T}{}^k_{ij} &= \pi_j U^k_i - \pi_i U^k_j \end{cases} \tag{24}$$

and the coefficient of $\overset{p}{\nabla}$ is

$$\overset{p}{\Gamma}{}^k_{ij} = \{^k_{ij}\} + (\Psi_i-\omega_i)\delta^k_j + (\Psi_j-\omega_j)\delta^k_i + g_{ij}\omega^k + \pi_j U^k_i - U_{ij}\pi^k. \tag{25}$$

where Ψ_i is a projective component.

From (25), we find that the curvature tensor of $\overset{p}{\nabla}$ is

$$\begin{aligned} \overset{p}{R}{}_{ijk}{}^l &= K_{ijk}{}^l + \delta^l_j \overset{p}{a}_{ik} - \delta^l_i \overset{p}{a}_{jk} + g_{jk}\overset{p}{b}{}^l_i - g_{ik}\overset{p}{b}{}^l_j + U^l_j \overset{p}{c}_{ik} - U^l_i \overset{p}{c}_{jk} \\ &+ U_{ik}\overset{p}{d}{}^l_j - U_{jk}\overset{p}{d}{}^l_i + (\overset{p}{e}{}^l_{ij} - \overset{p}{e}{}^l_{ji})\pi_k - (\overset{p}{e}_{ijk} - \overset{p}{e}_{jik})\pi^l \\ &- \delta^l_k(\omega_{ij}-\omega_{ji}) + \delta^l_k(\Psi_{ij}-\Psi_{ji}) \end{aligned} \tag{26}$$

where $K_{ijk}{}^l$ is the curvature tensor of the Levi-Civita connection $\widetilde{\nabla}$, and the other notations are given as

$$\begin{cases} \overset{p}{a}_{ik} &= \widetilde{\nabla}_i(\Psi_k-\omega_k) - (\Psi_i-\omega_k)(\Psi_k-\omega_k) \\ &\quad + U_{ik}(\Psi_p-\omega_p)\pi^p - U^p_i(\Psi_p-\omega_p)\pi_k - g_{ik}(\Psi_p-\omega_p)\omega^p \\ \overset{p}{b}_{ik} &= \widetilde{\nabla}_i\omega_k + \omega_i\omega_k + U_{ik}\omega_p\pi^p - U^p_i\omega_p\pi_k \\ \overset{p}{c}_{ik} &= \widetilde{\nabla}_i\pi_k - \pi_i(\Psi_k-\omega_k) - U^p_i\pi_p\pi_k + \frac{1}{2}U_{ik}\pi_p\pi^p \\ \overset{p}{d}_{ik} &= \widetilde{\nabla}_i\pi_k + \pi_i\omega_k - U_{ip}\pi^p\pi_k + \frac{1}{2}U_{ik}\pi_p\pi^p \\ \overset{p}{e}_{ijk} &= \widetilde{\nabla}_i U_{jk} \\ \Psi_{ij} &= \widetilde{\nabla}_i\Psi_j \end{cases} \tag{27}$$

Let

$$B_{ijk}{}^l = \delta^l_i \overset{p}{a}_{jk} + g_{jk}\overset{p}{b}{}^l_i - U^l_i \overset{p}{c}_{jk} - U_{jk}\overset{p}{d}{}^l_i + \overset{p}{e}{}^l_{ij}\pi_k - \overset{p}{e}_{ijk}\pi^l + \delta^l_k\Psi_{ij} - \delta^l_k\omega_{ij}$$

Then we get

$$\overset{p}{R}{}_{ijk}{}^l = K_{ijk}{}^l + B_{jik}{}^l - B_{ijk}{}^l.$$

So there exists the following.

Theorem 3.2. *When $B_{jik}{}^{l} = B_{ijk}{}^{l}$, then the curvature tensor will keep unchanged under the connection transformation $\widetilde{\nabla} \to \overset{p}{\nabla}$.*

From (25) and (26), the coefficient of dual connection $\widehat{\overset{p}{\nabla}}$ of the projective Ricci quarter-symmetric metric recurrent connection $\overset{p}{\nabla}$ is

$$\widehat{\overset{p}{\Gamma}}{}^{k}_{ij} = \{^{k}_{ij}\} - (\Psi_i - \omega_i)\delta^k_j - (\Psi^k - \omega^k)g_{ij} - \omega_j\delta^k_i + \pi_j U^k_i - U_{ij}\pi^k. \tag{28}$$

By using the expression (28), the curvature tensor of dual connection $\widehat{\overset{p}{\nabla}}$ is

$$\begin{aligned}\widehat{\overset{p}{R}}_{ijk}{}^{l} &= K_{ijk}{}^{l} + \delta^l_i \overset{p}{b}_{jk} - \delta^l_j \overset{p}{b}_{ik} + g_{ik}\overset{p}{a}{}^{l}_{j} - g_{jk}\overset{p}{a}{}^{l}_{i} + U^l_j \overset{p}{d}_{ik} - U^l_i \overset{p}{d}_{jk} \\ &+ U_{ik}\overset{pl}{c}_j - U_{jk}\overset{pl}{c}_i + (\overset{pl}{e}_{ij} - \overset{pl}{e}_{ji})\pi_k - (\overset{p}{e}_{ijk} - \overset{p}{e}_{jik})\pi^l \\ &+ \delta^l_k(\omega_{ij} - \omega_{ji}) + \delta^l_k(\Psi_{ij} - \Psi_{ji})\end{aligned} \tag{29}$$

From the expressions (26) and (29), we have

$$\begin{aligned}\widehat{\overset{p}{R}}_{ijk}{}^{l} &= \overset{p}{R}_{ijk}{}^{l} + \delta^l_i(\overset{p}{a}_{jk} + \overset{p}{b}_{jk}) - \delta^l_j(\overset{p}{a}_{ik} + \overset{p}{b}_{ik}) + g_{ik}(\overset{pl}{a}_j + \overset{pl}{b}_j) - g_{jk}(\overset{pl}{a}_i + \overset{pl}{b}_i) \\ &+ U^l_j(\overset{p}{c}_{ik} + \overset{p}{d}_{ik}) - U^l_i(\overset{p}{c}_{jk} + \overset{p}{d}_{jk}) + U_{ik}(\overset{pl}{c}_j + \overset{pl}{d}_j) - U_{jk}(\overset{pl}{c}_i + \overset{pl}{d}_i) \\ &+ 2\delta^l_k(\Psi_{ij} - \Psi_{ji}) + 2\delta^l_k(\omega_{ij} - \omega_{ji})\end{aligned} \tag{30}$$

Let

$$D_{ijk}{}^{l} = \delta^l_i(\overset{p}{a}_{jk} + \overset{p}{b}_{jk}) + g_{ik}(\overset{pl}{a}_j + \overset{pl}{b}_j) + U^l_j(\overset{p}{c}_{ik} + \overset{p}{d}_{ik}) + U_{ik}(\overset{pl}{c}_j + \overset{pl}{d}_j) + 2\delta^l_k(\Psi_{ij} + \omega_{ij})$$

Then we get

$$\widehat{\overset{p}{R}}_{ijk}{}^{l} = \overset{p}{R}_{ijk}{}^{l} + D_{jik}{}^{l} - D_{ijk}{}^{l}. \tag{31}$$

So there exists the following.

Theorem 3.3. *In the Riemannian manifold $(M, g, \overset{p}{\nabla})$, if 1-form Ψ and ω are of closed forms, then the Riemannian manifold is a quasi-Ricci(or volume) flat and if $D_{jik}{}^{l} = D_{ijk}{}^{l}$, then the projective Ricci quarter-symmetric recurrent connection $\overset{p}{\nabla}$ is a conjugate symmetry.*

Proof. By using the contraction of the indices k and l in the expression (26) we have

$$\begin{aligned}\overset{p}{P}_{ij} &= \widetilde{P}_{ij} + \overset{p}{a}_{ij} - \overset{p}{a}_{ji} + \overset{p}{b}_{ij} - \overset{p}{b}_{ji} + U^k_j\overset{p}{c}_{ik} - U^k_i\overset{p}{c}_{jk} + U_{ik}\overset{p}{d}{}^{k}_{j} - U_{jk}\overset{p}{d}{}^{k}_{i} \\ &+ (\overset{pk}{e}_{ij} - \overset{pk}{e}_{ji})\pi_k + (e_{jik} - e_{ijk})\pi^k + n(\Psi_{ij} - \Psi_{ji}) - n(\omega_{ij} - \omega_{ji})\end{aligned} \tag{32}$$

where $P_{ij} = R_{ijkl}g^{kl}, \widetilde{P}_{ij} = K_{ijkl}g^{kl} = 0$, and $e^k_{ij}\pi_k = e_{ijk}\pi^k$.

Using the expression (28), there holds the following

$$\begin{aligned}
\overset{p}{a}_{ij} - \overset{p}{a}_{ji} &= (\Psi_{ij} - \Psi_{ji}) - (\omega_{ij} - \omega_{ji}) - U_i^p(\Psi_p - \omega_p)\pi_j + U_j^p(\Psi_p - \omega_p)\pi_i,\\
\overset{p}{b}_{ij} - \overset{p}{b}_{ji} &= \omega_{ij} - \omega_{ji} - U_i^p\omega_p\pi_j + U_j^p\omega_p\pi_i,\\
U_j^k\overset{p}{c}_{ik} - U_i^k\overset{p}{c}_{jk} &= U_j^k\widetilde{\nabla}_i\pi_k - U_i^k\widetilde{\nabla}_j\pi_k - U_j^k(\Psi_k - \omega_k)\pi_i + U_i^k(\Psi_k - \omega_k)\pi_j,\\
U_{ik}\overset{p}{d}_j{}^k - U_{jk}\overset{p}{d}_i{}^k &= U_{ik}\widetilde{\nabla}_j\pi^k - U_{jk}\widetilde{\nabla}_i\pi^k + U_{ik}\omega^k\pi_j - U_{jk}\omega^k\pi_i,\\
e_{ij}^k\pi_k - e_{ji}^k\pi_k &= 0,\\
e_{ijk}\pi^k - e_{jik}\pi^k &= 0.
\end{aligned}$$

Substituting these expressions into the expression (32) and using 1-form Ψ and ω are of closed 1-forms, then $\overset{p}{P}_{ij} = 0$. Hence the Riemannian manifold $(M, g, \overset{p}{\nabla})$ is a qusai-Ricci(or volume) flat.

On the other hand, from the expression (31) if $D_{jik}{}^l = D_{ijk}{}^l$, then $\widehat{\overset{p}{R}}_{ijk}{}^l = \overset{p}{R}_{ijk}{}^l$. Hence the projective Ricci quarter-symmetric recurrent connection is of conjugate symmetry. □

Theorem 3.4. *Suppose that $(M, g)(dimM \geq 3)$ is a connected Riemannian manifold associated with an isotropic Ricci quarter-symmetric metric recurrent projective connection. If there holds*

$$\Psi_h = 2(\omega_h + s_h) \tag{33}$$

then $(M, g, \overset{p}{\nabla})$ is a constant curvature manifold, where $s_h = \frac{1}{n-1}T_{hp}^p$ (the Schur's theorem for the Ricci quarter-symmetric metric recurrent projective connection)

Proof. Substituting the expression (11) into the second Bianchi identity of the curvature tensor of the projective Ricci quarter-symmetric metric recurrent connection, we get

$$\overset{p}{\nabla}_h\overset{p}{R}_{ijk}{}^l + \overset{p}{\nabla}_i\overset{p}{R}_{jhk}{}^l + \overset{p}{\nabla}_j\overset{p}{R}_{hik}{}^l = \overset{p}{T}_{hi}{}^s\overset{p}{R}_{jsk}{}^l + \overset{p}{T}_{ij}{}^s\overset{p}{R}_{hsk}{}^l + \overset{p}{T}_{jh}{}^s\overset{p}{R}_{isk}{}^l$$

then by using the expression (24) we have

$$\begin{aligned}
&[\overset{p}{\nabla}_h k(p) + (2\omega_h - \Psi_h)k(p)](\delta_i^l g_{jk} - \delta_j^l g_{ik}) + [\overset{p}{\nabla}_i k(p) + (2\omega_i - \Psi_i)k(p)](\delta_j^l g_{hk} - \delta_h^l g_{jk})\\
&\quad + [\overset{p}{\nabla}_j k(p) + (2\omega_j - \Psi_j)k(p)](\delta_h^l g_{ik} - \delta_i^l g_{hk})\\
&= k(p)\Big[\pi_h(\delta_i^l U_{jk} - \delta_j^l U_{ik} + U_i^l g_{jk} - U_j^l g_{ik}) + \pi_i(\delta_j^l U_{hk} - \delta_h^l U_{jk} + U_j^l g_{hk} - U_h^l g_{jk})\\
&\quad + \pi_j(\delta_h^l U_{ik} - \delta_i^l U_{hk} + U_h^l g_{ik} - U_i^l g_{hk})\Big]
\end{aligned}$$

Contracting the indices i and l, we obtain

$$\begin{aligned}
&(n-1)[\overset{p}{\nabla}_h k(p) + (2\omega_h - \Psi_h)k(p)]g_{jk} - (n-1)[\overset{p}{\nabla}_j k(p) + (2\omega_j - \Psi_j)k(p)]g_{hk}\\
&\quad + [\overset{p}{\nabla}_j k(p) + (2\omega_j - \Psi_j)]g_{hk} - [\overset{p}{\nabla}_h k(p) + (2\omega_h - \Psi_h)]g_{jk}\\
&= k(p)\Big\{\pi_h[(n-2)U_{jk} + g_{jk}U_s^s] - \pi_j[(n-2)U_{hk} + g_{hk}U_s^s] + \pi_j U_{hk} - \pi_h U_{jk}\\
&\quad + g_{hk}U_j^s\pi_s - g_{jk}U_h^s\pi_s\Big\}
\end{aligned}$$

Multiplying both sides of this expression by g^{jk}, then we have

$$(n-1)(n-2)\Big[\overset{p}{\nabla}_h k(p) + (2\omega_h - \Psi_h)k(p)\Big] = 2(n-2)k(p)(\pi_h U_s^s - \pi_s U_h^s)$$

From this equation above we obtain

$$\overset{p}{\nabla}_h k(p) = [\Psi_h - 2(\omega_h + s_h)]k(p)$$

Consequently from that we know $k(p)$ =const if and only if $\Psi_h = 2(\omega_h + s_h)$. □

Theorem 3.5. *If an Einstein manifold $(M, g)(dimM \geq 3)$ associated with a projective Ricci quarter-symmetric metric recurrent connection $\overset{p}{\nabla}$ has a constant curvature, then the Riemannian manifold $(M, g, \overset{p}{\nabla})$ is conformal flat.*

Proof. Adding the expressions (26) and (29), we obtain

$$\begin{aligned}\widetilde{\overset{p}{R}}_{ijk}{}^{l} + \overset{p}{R}_{ijk}{}^{l} &= 2K_{ijk}{}^{l} + \delta^l_j(\overset{p}{a}_{ik} - \overset{p}{b}_{ik}) - \delta^l_i(\overset{p}{a}_{jk} - \overset{p}{b}_{jk}) + g_{ik}(\overset{p}{a}_j{}^{l} - \overset{p}{b}_j{}^{l}) \\ &- g_{jk}(\overset{p}{a}_i{}^{l} - \overset{p}{b}_i{}^{l}) + U^l_j(\overset{p}{c}_{ik} + \overset{p}{d}_{ik}) - U^l_i(\overset{p}{c}_{jk} + \overset{p}{d}_{jk}) + U_{ik}(\overset{p}{c}_j{}^{l} + \overset{p}{d}_j{}^{l}) \\ &- U_{jk}(\overset{p}{c}_i{}^{l} + \overset{p}{d}_i{}^{l}) + 2(\overset{p}{e}_{ij}{}^{l} - \overset{p}{e}_{ji}{}^{l})\pi_k - 2(\overset{p}{e}_{ijk} - \overset{p}{e}_{jik})\pi^l\end{aligned} \tag{34}$$

From the assumption that a Riemannian manifold is an Einstein manifold, we have

$$U_{jk} = \frac{k}{n} g_{jk}.$$

Using this expression, from (27) we obtain

$$\overset{p}{e}_{ijk} = 0. \tag{35}$$

Using these expressions, from the expression (34), we have

$$\widetilde{\overset{p}{R}}_{ijk}{}^{l} + \overset{p}{R}_{ijk}{}^{l} = 2K_{ijk}{}^{l} + \delta^l_j\alpha_{ik} - \delta^l_i\alpha_{jk} + g_{ik}\alpha^l_j - g_{jk}\alpha^l_i \tag{36}$$

where $\alpha_{ik} = a_{ik} - b_{ik} + \frac{k}{n}(c_{ik} + d_{ik})$. Contracting the indices i and l of (36), we get

$$\overset{p}{R}_{jk} + \widetilde{\overset{p}{R}}_{jk} = 2K_{jk} - (n-2)\alpha_{jk} - g_{jk}\alpha^i_i \tag{37}$$

Multiplying both sides of (37) by g^{jk}, then we arrive at

$$\overset{p}{R} + \widetilde{\overset{p}{R}} = 2K - 2(n-1)\alpha^i_i.$$

From this expression above we have

$$\alpha^i_i = \frac{1}{2(n-1)}[2K - (\overset{p}{R} + \widetilde{\overset{p}{R}})]$$

Using the expression from (37), we have

$$\alpha_{jk} = \frac{1}{n-2}\Big\{2K_{jk} - (\overset{p}{R}_{jk} + \widetilde{\overset{p}{R}}_{jk} - \frac{1}{2(n-1)} g_{jk}[2K - (\overset{p}{R} + \widetilde{\overset{p}{R}})])\Big\}$$

Substituting this expression into (36) and putting

$$\overset{p}{C}_{ijk}{}^{l} = \overset{p}{R}_{ijk}{}^{l} - \frac{1}{n-2}\Big(\delta^l_i\overset{p}{R}_{jk} - \delta^l_j\overset{p}{R}_{ik} + g_{jk}\overset{p}{R}{}^l_i - g_{ik}\overset{p}{R}{}^l_j\Big) + \frac{\overset{p}{R}}{(n-1)(n-2)}(\delta^l_i g_{jk} - \delta^l_j g_{ik})$$

$$\widehat{\overset{p}{C}}{}^{l}_{ijk} = \widehat{\overset{p}{R}}{}^{l}_{ijk} - \frac{1}{n-2}\Big(\delta^l_i \widehat{\overset{p}{R}}_{jk} - \delta^l_j \widehat{\overset{p}{R}}_{ik} + g_{jk}\widehat{\overset{p}{R}}{}^l_i - g_{ik}\widehat{\overset{p}{R}}{}^l_j\Big) + \frac{\widehat{\overset{p}{R}}}{(n-1)(n-2)}(\delta^l_i g_{jk} - \delta^l_j g_{ik})$$

$$\widetilde{C}^l_{ijk} = K_{ijk}{}^l - \frac{1}{n-2}\Big(\delta^l_i K_{jk} - \delta^l_j K_{ik} + g_{jk}K^l_i - g_{ik}K^l_j\Big) + \frac{K}{(n-1)(n-2)}(\delta^l_i g_{jk} - \delta^l_j g_{ik})$$

then by a direct computation, we obtain

$$\overset{p}{C}_{ijk}{}^{l} + \widehat{\overset{p}{C}}{}^{l}_{ijk} = 2\widetilde{C}^l_{ijk} \tag{38}$$

By using the fact that $\overset{p}{\nabla}$ has a constant curvature, thus we have $\overset{p}{C}_{ijk}{}^{l} = \widehat{\overset{p}{C}}{}^{l}_{ijk} = 0$. Hence, one gets

$$\widetilde{C}^l_{ijk} = 0.$$

This means that the Riemannian manifold $(M, g, \widetilde{\nabla})$ is of conformal flat. □

Theorem 3.6. *The projective Ricci quarter-symmetric metric recurrent connection* $\overset{p}{\nabla}$ *on an Einstein manifold* $(M, g)(dimM \geq 3)$ *is a conjugate symmetry if and only if it is a conjugate Ricci symmetry and conjugate volume symmetry.*

Proof. From (26) and (29), we get

$$\widehat{\overset{p}{R}}{}^{l}_{ijk} = \overset{p}{R}_{ijk}{}^{l} + \delta^l_i\beta_{jk} - \delta^l_j\beta_{ik} + g_{ik}\beta^l_j - g_{jk}\beta^l_i + 2\delta^l_k\gamma_{ij} \tag{39}$$

where $\beta_{jk} = \overset{p}{a}_{jk} + \overset{p}{b}_{jk} + \frac{K}{n}(\overset{p}{c}_{jk} + \overset{p}{d}_{jk})$, $\gamma_{ij} = (\omega_{ij} - \omega_{ji}) - (\Psi_{ij} - \Psi_{ji})$. By using contraction of indices i and l of (39), we obtain

$$\widehat{\overset{p}{R}}_{jk} = \overset{p}{R}_{jk} + n\beta_{jk} - g_{jk}\beta^i_i - 2\gamma_{jk}. \tag{40}$$

Alternating the indices k and j of this expression, we obtain

$$\widehat{\overset{p}{R}}_{jk} - \widehat{\overset{p}{R}}_{kj} = \overset{p}{R}_{jk} - \overset{p}{R}_{kj} + n(\beta_{jk} - \beta_{kj}) - 4\gamma_{jk}$$

On one hand, contracting the indices k and l of (39) and changing index i for j, index j for k, we get

$$\widehat{\overset{p}{P}}_{jk} = \overset{p}{P}_{jk} + 2(\beta_{jk} - \beta_{kj}) - 2n\gamma_{jk}$$

From these expressions above we have

$$\gamma_{jk} = \frac{1}{2(n^2-4)}\Big\{2\Big[(\widehat{\overset{p}{P}}_{jk} - \widehat{\overset{p}{P}}_{kj}) - (\overset{p}{R}_{jk} - \overset{p}{R}_{kj})\Big] + n(\widehat{\overset{p}{R}}_{jk} - \overset{p}{R}_{jk})\Big\}$$

Using this expression, from (40) we have

$$\beta_{jk} = \frac{1}{n}\Big(\widehat{\overset{p}{R}}_{jk} - \overset{p}{R}_{jk} + g_{jk}\beta^i_i + \frac{1}{n^2-4}\Big\{2\Big[(\widehat{\overset{p}{P}}_{jk} - \widehat{\overset{p}{P}}_{kj}) - (\overset{p}{R}_{jk} - \overset{p}{R}_{kj})\Big] + n(\widehat{\overset{p}{R}}_{jk} - \overset{p}{R}_{jk})\Big\}\Big)$$

Substituting the above two expressions into (39), we obtain

$$\begin{aligned}
&\overset{p}{R}_{ijk}{}^{l} - \frac{1}{n}(\delta^l_i \overset{p}{R}_{jk} - \delta^l_j \overset{p}{R}_{ik} + g_{ik}\overset{p}{R}{}^{l}_{j} - g_{jk}\overset{p}{R}{}^{l}_{i}) - \frac{2}{n(n^2-4)}[\delta^l_i(\overset{p}{R}_{jk} - \overset{p}{R}_{kj}) \\
&-\delta^l_j(\overset{p}{R}_{ik} - \overset{p}{R}_{ki}) + g_{ik}(\overset{p}{R}_j{}^l - \overset{p}{R}_{\cdot j}{}^l) - g_{jk}(\overset{p}{R}_i{}^l - \overset{p}{R}_{\cdot i}{}^l) + n\delta^l_k(\overset{p}{R}_{ij} - \overset{p}{R}_{ji})] \\
&-\frac{1}{n^2-4}(\delta^l_i \overset{p}{P}_{jk} - \delta^l_j \overset{p}{P}_{ik} + g_{ik}\overset{p}{P}_j{}^l - g_{jk}\overset{p}{P}_i{}^l + n\delta^l_k \overset{p}{P}_{ij}) \\
= \;& \widehat{\overset{p}{R}}_{ijk}{}^{l} - \frac{1}{n}(\delta^l_i \widehat{\overset{p}{R}}_{jk} - \delta^l_j \widehat{\overset{p}{R}}_{ik} + g_{ik}\widehat{\overset{p}{R}}{}^{l}_{j} - g_{jk}\widehat{\overset{p}{R}}{}^{l}_{i}) - \frac{2}{n(n^2-4)}[\delta^l_i(\widehat{\overset{p}{R}}_{jk} - \widehat{\overset{p}{R}}_{kj}) \\
&-\delta^l_j(\widehat{\overset{p}{R}}_{ik} - \widehat{\overset{p}{R}}_{ki}) + g_{ik}(\widehat{\overset{p}{R}}_j{}^l - \widehat{\overset{p}{R}}_{\cdot j}{}^l) - g_{jk}(\widehat{\overset{p}{R}}_i{}^l - \widehat{\overset{p}{R}}_{\cdot i}{}^l) + n\delta^l_k(\widehat{\overset{p}{R}}_{ij} - \widehat{\overset{p}{R}}_{ji})] \\
&-\frac{1}{n^2-4}(\delta^l_i \widehat{\overset{p}{P}}_{jk} - \delta^l_j \widehat{\overset{p}{P}}_{ik} + g_{ik}\widehat{\overset{p}{P}}_j{}^l - g_{jk}\widehat{\overset{p}{P}}_i{}^l + n\delta^l_k \widehat{\overset{p}{P}}_{ij})
\end{aligned}$$

From this expression we arrive at $R_{ijk}{}^l = \widehat{R}_{ijk}{}^l$ if and only if $R_{jk} = \widehat{R}_{jk}, P_{jk} = \widehat{P}_{jk}$. Where $\overset{p}{R}_j{}^l = \overset{p}{R}_{js}g^{sl}, \overset{p}{R}_{\cdot j}{}^l = \overset{p}{R}_{sj}g^{sl}$. This ends the proof of Theorem 3.6. □

From the expression (25), the coefficient of mutual connection $\widehat{\overset{p}{\nabla}}$ of the projective Ricci quarter-symmetric metric recurrent connection $\overset{p}{\nabla}$ is

$$\overset{pm}{\Gamma}_{ij}{}^k = \{^k_{ij}\} - (\Psi_i - \omega_i)\delta^k_j + (\Psi^j - \omega^j)\delta^k_i + g_{ij}\omega^k + \pi_i U^k_j - U_{ij}\pi^k. \tag{41}$$

This connection satisfies the relation

$$\overset{pm}{\nabla}_k g_{ij} = -2(\Psi_k - \omega_k)g_{ij} - \Psi_i g_{jk} - \Psi_j g_{ik} - 2\pi_k U_{ij} + U_{ik}\pi_j + U_{jk}\pi_i \tag{42}$$

$$\overset{mk}{T}_{ij} = \pi_i U^k_j - \pi_j U^k_i \tag{43}$$

From the expressions (41) and (42), the coefficient of dual connection $\widehat{\overset{pm}{\nabla}}$ of the mutual connection $\overset{pm}{\nabla}$ is

$$\widehat{\overset{pm}{\Gamma}}_{ij}{}^k = \{^k_{ij}\} - (\Psi_i - \omega_i)\delta^k_j - (\Psi^k - \omega^k)g_{ij} - \omega_j\delta^k_i - \pi_i U^k_j + U_{ij}\pi^k. \tag{44}$$

On the other hand, the coefficient of a dual connection $\widehat{\overset{pw}{\nabla}}$ of the Weyl projective connection $\overset{pw}{\nabla}$ is given as

$$\widehat{\overset{pw}{\Gamma}}_{ij}{}^k = \{^k_{ij}\} - (\Psi_i - \omega_i)\delta^k_j - (\Psi^k - \omega^k)g_{ij} - \omega_j\delta^k_i. \tag{45}$$

Theorem 3.7. *In a Riemannian manifold the dual connection $\widehat{\overset{pm}{\nabla}}$ of the mutual connection $\overset{pm}{\nabla}$ of the projective Ricci quarter-symmetric metric recurrent connection $\overset{p}{\nabla}$ is projective equivalent to dual connection $\widehat{\overset{pw}{\nabla}}$ of the Weyl projective connection $\overset{pw}{\nabla}$.*

Proof. From the expressions (44) and (45), we have

$$\widehat{\overset{pm}{\Gamma}}{}^k_{(ij)} = \widehat{\overset{pw}{\Gamma}}{}^k_{(ij)}$$

Hence, the connection $\widehat{\overset{pm}{\nabla}}$ has the same geodesic as $\widehat{\overset{pw}{\nabla}}$. Thus the connection $\widehat{\overset{pm}{\nabla}}$ is projective equivalent to the connection $\widehat{\overset{pw}{\nabla}}$. □

4. Ackonowedement

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