



## Viscosity Approximation Methods for Split Common Fixed Point Problems without Prior Knowledge of the Operator Norm

Pachara Jailoka<sup>a</sup>, Suthep Suantai<sup>b</sup>

<sup>a</sup>PhD Degree Program in Mathematics, Faculty of Science, Chiang Mai University, Chiang Mai 50200, Thailand

<sup>b</sup>Research Center in Mathematics and Applied Mathematics, Department of Mathematics, Faculty of Science, Chiang Mai University, Chiang Mai 50200, Thailand

**Abstract.** In this work, we study the split common fixed point problem which was first introduced by Censor and Segal [14]. We introduce an algorithm based on the viscosity approximation method without prior knowledge of the operator norm by selecting the stepsizes in the same adaptive way as López et al. [22] for solving the problem for two attracting quasi-nonexpansive operators in real Hilbert spaces. A strong convergence result of the proposed algorithm is established under some suitable conditions. We also modify our algorithm to extend to the class of demicontractive operators and the class of hemicontractive operators, and obtain strong convergence results. Moreover, we apply our main result to other split problems, that is, the split feasibility problem and the split variational inequality problem. Finally, a numerical result is also given to illustrate the convergence behavior of our algorithm.

### 1. Introduction

In 1994, the first instance of the split inverse problem (SIP) was introduced by Censor and Elfving [11] and was called the *split feasibility problem* (SFP). This split inverse problem is the problem of finding a point of a closed convex subset of a Hilbert space such that its image under a given bounded linear operator belongs to a closed convex subset of another Hilbert space. The SFP was studied by many authors (see [1, 22, 31, 34, 35]) due to its applications are desirable and can be used in real-world applications, for example, in signal processing, image recovery, modeling inverse problems, the intensity-modulated radiation therapy, etc (see [2, 5, 9, 11, 12, 22]). A fixed point of an operator is a point of the operator's domain, which is mapped to itself by the operator. In many areas, stability or equilibrium is a fundamental notions that can be explained in terms of fixed points, and it is well known that the fixed point theory is very important in nonlinear analysis and can be applied in a variety of problems. In 2009, Censor and Segal [14] introduced another problem which is a generalization of the SFP, and was called the *split common fixed point problem* (SCFP). This split problem was first considered for the class of directed operators in Euclidean spaces [14] and later has been widely studied in Hilbert spaces by many researchers, see [3, 7, 8, 15, 19–21, 25, 27, 28, 32, 36, 38], for instance. The SCFP requires to find a common fixed point of a family of

---

2010 *Mathematics Subject Classification.* Primary 47H09; Secondary 47J25, 90C25

*Keywords.* Split common fixed point problems, viscosity approximation methods, strong convergence, Hilbert spaces

Received: 13 September 2019; Accepted: 20 November 2019

Communicated by Vasile Berinde

Corresponding author: Suthep Suantai

Research supported by Chiang Mai University

*Email addresses:* pachara.j4@gmail.com (Pachara Jailoka), suthep.s@cmu.ac.th (Suthep Suantai)

operators in a Hilbert space whose image under a considered bounded linear operator is a common fixed point of another family of operators in the image space. Let us recall the SFP and the SCFP, and review some methods for solving the problems:

Let  $\mathcal{H}_1$  and  $\mathcal{H}_2$  be two real Hilbert spaces, and let  $A : \mathcal{H}_1 \rightarrow \mathcal{H}_2$  be a bounded linear operator. The SFP is formulated as finding a point

$$x^* \in C \text{ such that } Ax^* \in Q, \tag{1}$$

where  $C \subset \mathcal{H}_1$  and  $Q \subset \mathcal{H}_2$  are nonempty closed convex subsets. Byrne [1] was the first who introduced the so-called CQ algorithm which does not involve matrix inverses in finite-dimensional spaces for solving the SFP (1) as follows:

$$x_{n+1} = P_C(x_n - \gamma A^*(I - P_Q)Ax_n), \quad n \in \mathbb{N}, \tag{2}$$

where  $\gamma \in (0, \frac{2}{\|A\|^2})$ ,  $A^*$  denotes the adjoint operator of  $A$ , and  $P_C, P_Q$  are the metric projections onto  $C$  and  $Q$ , respectively.

In this work, we focus our attention on the following SCFP for two operators: Find a point

$$x^* \in F(S) \text{ such that } Ax^* \in F(T), \tag{3}$$

where  $S : \mathcal{H}_1 \rightarrow \mathcal{H}_1$  and  $T : \mathcal{H}_2 \rightarrow \mathcal{H}_2$  are two operators with nonempty fixed point sets  $F(S)$  and  $F(T)$ , respectively. We denote the solution set of the SCFP (3) by

$$\Gamma := \{x \in F(S) : Ax \in F(T)\} = F(S) \cap A^{-1}(F(T)).$$

In order to solve the SCFP (3), Censor and Segal [14] proposed an algorithm for two directed operators  $S$  and  $T$  as follows:

$$x_{n+1} = S(x_n - \gamma A^*(I - T)Ax_n), \quad n \in \mathbb{N}, \tag{4}$$

where  $\gamma \in (0, \frac{2}{\|A\|^2})$ , and proved a convergence theorem under the demiclosedness principle in finite-dimensional spaces. After that Moudafi [25] introduced the following relaxed algorithm for solving the SCFP (3):

$$\begin{cases} y_n = x_n - \gamma A^*(I - T)Ax_n, \\ x_{n+1} = (1 - \alpha_n)y_n + \alpha_n S y_n, \quad n \in \mathbb{N}, \end{cases} \tag{5}$$

where  $S$  is  $\kappa_1$ -demictractive and  $T$  is  $\kappa_2$ -demictractive,  $\alpha_n \in (0, 1)$  and  $\gamma \in (0, \frac{1-\kappa_2}{\|A\|^2})$ . He also proved a weak convergence result of this algorithm under some suitable conditions in (infinite-dimensional) real Hilbert spaces.

We see that the parameters  $\gamma$  in above mentioned algorithms ((2), (4), (5)) depend on the norm of  $A$ . In order to utilize these algorithms, we first have to calculate or estimate the operator norm  $\|A\|$ ; however, the calculation of  $\|A\|$  is not an easy work in general practice.

Question: How do we construct an algorithm which is independent of  $\|A\|$  for solving the SFP or the SCFP?

López et al. [22] presented one of the ways to select the stepsize  $\gamma_n$  for replacing the parameter  $\gamma$  in Algorithm (2) for solving the SFP (1) as follows:

$$\gamma_n := \frac{\lambda_n \|(I - T)Ax_n\|^2}{\|A^*(I - T)Ax_n\|^2}, \tag{6}$$

where  $\lambda_n \in (0, 2)$  and  $T := P_Q$ . It can be seen that the choice of the stepsize  $\gamma_n$  in (6) does not depend on  $\|A\|$  (indeed, it depends on  $x_n$ ). For the SCFP, Maingé [28] introduced an algorithm based on the viscosity

method by choosing the stepsize in the similar way to (6) in the cases of quasi-nonexpansive or directed operators. Cui and Wang [15], and Boikanyo [3] also introduced algorithms by choosing the stepsizes in the same way as (6) with  $\lambda_n = \frac{1-\kappa_2}{2}$  for solving the SCFP (3) where  $S$  and  $T$  are demicontractive operators with coefficients  $\kappa_1$  and  $\kappa_2$ , respectively. Cegielski [8] studied some properties of an extrapolation of the Landweber-type operator where its stepsize is defined by (6) with  $\lambda_n = 1$ , and applied to the SCFP.

In this paper, inspired and motivated by these works, we are interested to study the SCFP for two operators in real Hilbert spaces. Our main objective is to construct some efficient algorithms based on the viscosity approximation method [24] without prior knowledge of the operator norm for solving the SCFP. In Section 3, we first propose a viscosity-type algorithm by selecting the stepsize in the same way as (6) for two attracting quasi-nonexpansive operators, and prove a strong convergence result under some suitable conditions of the proposed algorithm. In Section 4, we modify our algorithm to extend the class of operators to the class of demicontractive operators and the class of hemicontractive operators, and also obtain strong convergence results. Moreover, we apply our main result to other split problems, that is, the split feasibility problem and the split variational inequality problem as seen in Section 5. Finally, in Section 6, we give a numerical example to demonstrate the convergence of our algorithm and also compare the convergence behavior of our algorithm with a viscosity-type algorithm depending on the operator norm.

## 2. Preliminaries

Throughout this paper, we adopt the following notations:

- $\mathbb{N}$  : the set of positive integers,
- $\mathbb{R}$  : the set of real numbers,
- $I$  : the identity operator on a Hilbert space,
- $x_n \rightarrow x$  :  $\{x_n\}$  converges strongly to  $x$ ,
- $x_n \rightharpoonup x$  :  $\{x_n\}$  converges weakly to  $x$ ,

and also assume that  $\mathcal{H}$ ,  $\mathcal{H}_1$  and  $\mathcal{H}_2$  are real Hilbert spaces with inner products  $\langle \cdot, \cdot \rangle$  and induced norms  $\| \cdot \|$ . Let  $x, y \in \mathcal{H}$ , and let  $\mu \in \mathbb{R}$ . Then the following identities hold on  $\mathcal{H}$ :

$$\| \mu x + (1 - \mu)y \|^2 = \mu \|x\|^2 + (1 - \mu) \|y\|^2 - \mu(1 - \mu) \|x - y\|^2; \tag{7}$$

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle. \tag{8}$$

Let  $D$  be a nonempty closed convex subset of  $\mathcal{H}$ . Recall that the (metric) projection  $P_D$  from  $\mathcal{H}$  onto  $D$  is defined, for each  $x \in \mathcal{H}$ ,  $P_D x$  is the unique point in  $D$  such that

$$\|x - P_D x\| = d(x, D) := \inf \{ \|x - y\| : y \in D \}.$$

It is well known that  $P_D x \in D$  is characterized by the property:

$$\langle x - P_D x, y - P_D x \rangle \leq 0, \quad \forall y \in D.$$

An operator  $f : \mathcal{H} \rightarrow \mathcal{H}$  is called a  $\tau$ -contraction with respect to  $D$ , where  $\tau \in [0, 1)$  if  $\|f(x) - f(y)\| \leq \tau \|x - y\|$  for all  $x \in \mathcal{H}$  and  $y \in D$ . It is easy to check that if  $f$  is a  $\tau$ -contraction with respect to  $D$ , then  $P_D f$  is also a  $\tau$ -contraction with respect to  $D$ .

Let  $T : \mathcal{H} \rightarrow \mathcal{H}$  be an operator. Denote by  $F(T)$  the set of all fixed points of  $T$ , i.e.,  $F(T) := \{u \in \mathcal{H} : u = Tu\}$ . The operator  $T_\lambda : \mathcal{H} \rightarrow \mathcal{H}$  with  $\lambda \in [0, 2]$  defined by

$$T_\lambda := I + \lambda(T - I)$$

is called a  $\lambda$ -relaxation of  $T$ . It is clear that  $F(T) = F(T_\lambda)$  for  $\lambda \neq 0$ .

Now, let us recall the definitions of some operators occurring in our study.

**Definition 2.1.** An operator  $T : \mathcal{H} \rightarrow \mathcal{H}$  having a fixed point is said to be

(i) *quasi-nonexpansive* if

$$\|Tx - u\| \leq \|x - u\|, \quad \forall x \in \mathcal{H}, \forall u \in F(T),$$

(ii) *demicontractive* [17, 23] if there exists  $\kappa \in [0, 1)$  such that

$$\|Tx - u\|^2 \leq \|x - u\|^2 + \kappa\|x - Tx\|^2, \quad \forall x \in \mathcal{H}, \forall u \in F(T),$$

(also called  $\kappa$ -*demicontractive*),

(iii) *hemiccontractive* [30] if

$$\|Tx - u\|^2 \leq \|x - u\|^2 + \|x - Tx\|^2, \quad \forall x \in \mathcal{H}, \forall u \in F(T).$$

**Remark 2.2.** It can be easily observed from Definition 2.1 that

$$T\text{-quasi-nonexpansive} \Rightarrow T\text{-demicontractive} \Rightarrow T\text{-hemiccontractive}.$$

We also need the following classes of operators included in the class of quasi-nonexpansive operators.

**Definition 2.3.** An operator  $T : \mathcal{H} \rightarrow \mathcal{H}$  having a fixed point is said to be  $\rho$ -*attracting quasi-nonexpansive* ( $\rho$ -AQNE) [37] where  $\rho \geq 0$  if

$$\|Tx - u\|^2 \leq \|x - u\|^2 - \rho\|x - Tx\|^2, \quad \forall x \in \mathcal{H}, \forall u \in F(T). \tag{9}$$

If  $T$  satisfies (9) with  $\rho > 0$ , then we call  $T$  *attracting quasi-nonexpansive* (AQNE). In particular, if  $\rho = 1$ , then  $T$  is called a *directed operator* [4, 14] (also named a *firmly quasi-nonexpansive operator* [37], or a *cutter* [10]).

**Remark 2.4.** We have the following implications:

$$T\text{-directed} \Rightarrow T\text{-AQNE} \Rightarrow T\text{-quasi-nonexpansive}.$$

A characterization of  $\rho$ -attracting quasi-nonexpansive operator is shown below.

**Proposition 2.5.** ([6]) An operator  $T : \mathcal{H} \rightarrow \mathcal{H}$  is  $\rho$ -AQNE, where  $\rho \geq 0$  if and only if

$$\|x - Tx\|^2 \leq \frac{2}{\rho + 1} \langle x - Tx, x - u \rangle$$

for all  $x \in \mathcal{H}$  and  $u \in F(T)$ .

We recall the notion of the so-called demiclosedness principle.

**Definition 2.6.** Given an operator  $T : \mathcal{H} \rightarrow \mathcal{H}$ , we say that  $I - T$  is *demiclosed at 0* if for any sequence  $\{x_n\} \subset \mathcal{H}$ , it holds that

$$(x_n \rightharpoonup u \text{ and } x_n - Tx_n \rightarrow 0) \Rightarrow u \in F(T).$$

It is well known that a nonexpansive operator  $T : \mathcal{H} \rightarrow \mathcal{H}$  (i.e.,  $\|Tx - Ty\| \leq \|x - y\|$  for all  $x, y \in \mathcal{H}$ ) can guarantee the demiclosedness of  $I - T$  at 0 (see [29, Lemma 2]).

We next give some significant tools for proving our main result.

**Lemma 2.7.** ([33]) Suppose that  $\{a_n\}$  is a sequence of nonnegative real numbers such that

$$a_{n+1} \leq (1 - \mu_n)a_n + \mu_n\sigma_n + \tau_n, \quad n \in \mathbb{N},$$

where  $\{\mu_n\}$ ,  $\{\sigma_n\}$  and  $\{\tau_n\}$  satisfy the following conditions:

- (i)  $\{\mu_n\} \subset [0, 1], \sum_{n=1}^{\infty} \mu_n = \infty;$
- (ii)  $\limsup_{n \rightarrow \infty} \sigma_n \leq 0$  or  $\sum_{n=1}^{\infty} |\mu_n \sigma_n| < \infty;$
- (iii)  $\tau_n \geq 0$  for all  $n \in \mathbb{N}, \sum_{n=1}^{\infty} \tau_n < \infty.$

Then  $\lim_{n \rightarrow \infty} a_n = 0.$

**Lemma 2.8.** ([26]) *Let  $\{a_n\}$  be a sequence of real numbers such that there exists a subsequence  $\{n_i\}$  of  $\{n\}$  which satisfies  $a_{n_i} < a_{n_i+1}$  for all  $i \in \mathbb{N}$ . Define a sequence of positive integers  $\{\omega(n)\}$  by*

$$\omega(n) := \max\{m \leq n : a_m < a_{m+1}\}$$

for all  $n \geq n_0$  (for some  $n_0$  large enough). Then  $\{\omega(n)\}$  is a nondecreasing sequence such that  $\omega(n) \rightarrow \infty$  as  $n \rightarrow \infty$ , and it holds that

$$a_{\omega(n)} \leq a_{\omega(n)+1} \text{ and } a_n \leq a_{\omega(n)+1}.$$

### 3. A Viscosity-Type Algorithm for the Split Common Fixed Point Problem

In this section, we present an iterative method whose stepsize does not depend on the operator norms for solving the SCFP (3), and also prove a strong convergence theorem for two attracting quasi-nonexpansive operators.

We first give the following useful lemma for proving our main result.

**Lemma 3.1.** *Let  $A : \mathcal{H}_1 \rightarrow \mathcal{H}_2$  be a bounded linear operator, and let  $T : \mathcal{H}_2 \rightarrow \mathcal{H}_2$  be a  $\rho$ -AQNE operator ( $\rho \geq 0$ ) with  $A^{-1}(F(T)) \neq \emptyset$ . If  $x \in \mathcal{H}_1$  with  $Ax \notin F(T)$  and  $u \in A^{-1}(F(T))$ , then*

$$\|x - \gamma A^*(I - T)Ax - u\|^2 \leq \|x - u\|^2 - (\rho + 1 - \lambda)\lambda \frac{\|(I - T)Ax\|^4}{\|A^*(I - T)Ax\|^2}, \tag{10}$$

where

$$\gamma := \frac{\lambda \|(I - T)Ax\|^2}{\|A^*(I - T)Ax\|^2}$$

and  $\lambda \in (0, \rho + 1).$

*Proof.* Let  $x \in \mathcal{H}_1$  with  $Ax \notin F(T)$  and  $u \in A^{-1}(F(T)).$  If  $A^*(I - T)Ax = 0,$  then by Proposition 2.5, we have

$$\|(I - T)Ax\|^2 \leq \frac{2}{\rho + 1} \langle (I - T)Ax, Ax - Au \rangle = \frac{2}{\rho + 1} \langle A^*(I - T)Ax, x - u \rangle = 0,$$

which is a contradiction. Thus,  $A^*(I - T)Ax \neq 0$  and hence  $\gamma$  is well defined. By using Proposition 2.5, we have

$$\begin{aligned} \|x - \gamma A^*(I - T)Ax - u\|^2 &= \|x - u\|^2 - 2\gamma \langle A^*(I - T)Ax, x - u \rangle + \gamma^2 \|A^*(I - T)Ax\|^2 \\ &= \|x - u\|^2 - 2\gamma \langle (I - T)Ax, Ax - Au \rangle + \gamma^2 \|A^*(I - T)Ax\|^2 \\ &\leq \|x - u\|^2 - (\rho + 1)\gamma \|(I - T)Ax\|^2 + \gamma^2 \|A^*(I - T)Ax\|^2 \\ &= \|x - u\|^2 - (\rho + 1 - \lambda)\lambda \frac{\|(I - T)Ax\|^4}{\|A^*(I - T)Ax\|^2}. \end{aligned}$$

□

We now prove our main result.

**Theorem 3.2.** Let  $A : \mathcal{H}_1 \rightarrow \mathcal{H}_2$  be a bounded linear operator. Let  $S : \mathcal{H}_1 \rightarrow \mathcal{H}_1$  be  $\rho_1$ -AQNE operator ( $\rho_1 > 0$ ) and  $T : \mathcal{H}_2 \rightarrow \mathcal{H}_2$  a  $\rho_2$ -AQNE operator ( $\rho_2 \geq 0$ ) such that both  $I - S$  and  $I - T$  are demiclosed at 0. Assume that  $\Gamma \neq \emptyset$ . Let  $f : \mathcal{H}_1 \rightarrow \mathcal{H}_1$  be a  $\tau$ -contraction with respect to  $\Gamma$ . Let  $\{x_n\} \subset \mathcal{H}_1$  be a sequence generated iteratively by  $x_1 \in \mathcal{H}_1$  and

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)S(x_n - \gamma_n A^*(I - T)Ax_n), \quad n \in \mathbb{N}, \tag{11}$$

where the stepsize  $\gamma_n$  is selected in such a way:

$$\gamma_n := \begin{cases} \frac{\lambda_n \|(I-T)Ax_n\|^2}{\|A^*(I-T)Ax_n\|^2}, & \text{if } Ax_n \notin F(T), \\ 0, & \text{otherwise,} \end{cases} \tag{12}$$

and the sequences  $\{\lambda_n\}$  and  $\{\alpha_n\}$  satisfy the following conditions:

(C1)  $0 < a \leq \lambda_n \leq b < \rho_2 + 1$ ;

(C2)  $\alpha_n \in (0, 1)$  such that  $\lim_{n \rightarrow \infty} \alpha_n = 0$  and  $\sum_{n=1}^{\infty} \alpha_n = \infty$ .

Then the sequence  $\{x_n\}$  converges strongly to a point  $x^* \in \Gamma$ , where  $x^* = P_{\Gamma}f(x^*)$ .

*Proof.* One can show that  $P_{\Gamma}f$  is a contraction on  $\Gamma$ . By Banach fixed point theorem, there exists  $x^* \in \Gamma$  such that  $x^* = P_{\Gamma}f(x^*)$ . Thus, by characterization of  $P_{\Gamma}$ , we have

$$\langle f(x^*) - x^*, u - x^* \rangle \leq 0, \quad \forall u \in \Gamma. \tag{13}$$

Since  $x^* \in \Gamma$ ,  $x^* \in F(S)$  and  $Ax^* \in F(T)$ . We first show that  $\{x_n\}$  is bounded. Let  $y_n = x_n - \gamma_n A^*(I - T)Ax_n$ . If  $Ax_n \notin F(T)$ , then it follows from Lemma 3.1 that

$$\|y_n - x^*\|^2 \leq \|x_n - x^*\|^2 - (\rho_2 + 1 - \lambda_n)\lambda_n \frac{\|(I - T)Ax_n\|^4}{\|A^*(I - T)Ax_n\|^2} \tag{14}$$

$$\leq \|x_n - x^*\|^2 - \frac{(\rho_2 + 1 - \lambda_n)\lambda_n}{\|A\|^2} \|(I - T)Ax_n\|^2. \tag{15}$$

In the case of  $Ax_n \in F(T)$ , (15) still holds. Since  $S$  is  $\rho_1$ -AQNE and by using (15), we have

$$\begin{aligned} \|Sy_n - x^*\|^2 &\leq \|y_n - x^*\|^2 - \rho_1 \|y_n - Sy_n\|^2 \\ &\leq \|x_n - x^*\|^2 - \frac{(\rho_2 + 1 - \lambda_n)\lambda_n}{\|A\|^2} \|(I - T)Ax_n\|^2 - \rho_1 \|y_n - Sy_n\|^2. \end{aligned} \tag{16}$$

It follows that  $\|Sy_n - x^*\| \leq \|x_n - x^*\|$ . Thus, we have

$$\begin{aligned} \|x_{n+1} - x^*\| &= \|\alpha_n(f(x_n) - x^*) + (1 - \alpha_n)(Sy_n - x^*)\| \\ &\leq \alpha_n \|f(x_n) - x^*\| + (1 - \alpha_n) \|Sy_n - x^*\| \\ &\leq \alpha_n (\|f(x_n) - f(x^*)\| + \|f(x^*) - x^*\|) + (1 - \alpha_n) \|x_n - x^*\| \\ &\leq \alpha_n (\tau \|x_n - x^*\| + \|f(x^*) - x^*\|) + (1 - \alpha_n) \|x_n - x^*\| \\ &= (1 - \alpha_n(1 - \tau)) \|x_n - x^*\| + \alpha_n(1 - \tau) \frac{\|f(x^*) - x^*\|}{1 - \tau} \\ &\leq \max \left\{ \|x_n - x^*\|, \frac{\|f(x^*) - x^*\|}{1 - \tau} \right\}. \end{aligned}$$

By mathematical induction, we obtain

$$\|x_n - x^*\| \leq \max \left\{ \|x_1 - x^*\|, \frac{\|f(x^*) - x^*\|}{1 - \tau} \right\}$$

for all  $n \in \mathbb{N}$ . Therefore,  $\{x_n\}$  is bounded. This implies that  $\{f(x_n)\}$  is also bounded. Now, from (7) and (16), we have

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &= \|\alpha_n(f(x_n) - x^*) + (1 - \alpha_n)(Sy_n - x^*)\|^2 \\ &\leq \alpha_n\|f(x_n) - x^*\|^2 + (1 - \alpha_n)\|Sy_n - x^*\|^2 \\ &\leq \alpha_n\|f(x_n) - x^*\|^2 + \|x_n - x^*\|^2 - \frac{(\rho_2 + 1 - \lambda_n)\lambda_n}{\|A\|^2}\|(I - T)Ax_n\|^2 - \rho_1\|y_n - Sy_n\|^2. \end{aligned}$$

Thus, above inequality leads to the following two inequalities:

$$\frac{(\rho_2 + 1 - \lambda_n)\lambda_n}{\|A\|^2}\|(I - T)Ax_n\|^2 \leq \alpha_n\|f(x_n) - x^*\|^2 + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \tag{17}$$

and

$$\rho_1\|y_n - Sy_n\|^2 \leq \alpha_n\|f(x_n) - x^*\|^2 + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2. \tag{18}$$

Here we divide the rest of the proof into two cases.

*Case 1.* Suppose that there exists  $n_0 \in \mathbb{N}$  such that  $\{\|x_n - x^*\|\}_{n \geq n_0}$  is either nonincreasing or nondecreasing. Since  $\{\|x_n - x^*\|\}$  is bounded, then it converges, and hence  $\|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \rightarrow 0$  as  $n \rightarrow \infty$ . Taking the limit as  $n \rightarrow \infty$  into (17) and (18) yields

$$\lim_{n \rightarrow \infty} \|(I - T)Ax_n\| = 0 \tag{19}$$

and

$$\lim_{n \rightarrow \infty} \|y_n - Sy_n\| = 0. \tag{20}$$

We show that  $\|y_n - x_n\| \rightarrow 0$  as  $n \rightarrow \infty$ . If  $Ax_n \in F(T)$ , then  $\|y_n - x_n\| = 0$ . Thus, we assume that  $Ax_n \notin F(T)$ . By the quasi-nonexpansivity of  $S$  and using (14), we have

$$\|Sy_n - x^*\|^2 \leq \|y_n - x^*\|^2 \leq \|x_n - x^*\|^2 - (\rho_2 + 1 - \lambda_n)\lambda_n \frac{\|(I - T)Ax_n\|^4}{\|A^*(I - T)Ax_n\|^2},$$

which implies that

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &\leq \alpha_n\|f(x_n) - x^*\|^2 + (1 - \alpha_n)\|Sy_n - x^*\|^2 \\ &\leq \alpha_n\|f(x_n) - x^*\|^2 + \|x_n - x^*\|^2 - (\rho_2 + 1 - \lambda_n)\lambda_n \frac{\|(I - T)Ax_n\|^4}{\|A^*(I - T)Ax_n\|^2} \end{aligned}$$

or

$$(\rho_2 + 1 - \lambda_n)\lambda_n \frac{\|(I - T)Ax_n\|^4}{\|A^*(I - T)Ax_n\|^2} \leq \alpha_n\|f(x_n) - x^*\|^2 + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2.$$

Taking the limit as  $n \rightarrow \infty$  into above inequality yields  $\frac{\|(I - T)Ax_n\|^4}{\|A^*(I - T)Ax_n\|^2} \rightarrow 0$  as  $n \rightarrow \infty$ . Since  $\|y_n - x_n\|^2 = \lambda_n^2 \frac{\|(I - T)Ax_n\|^4}{\|A^*(I - T)Ax_n\|^2}$ , we have

$$\lim_{n \rightarrow \infty} \|y_n - x_n\| = 0. \tag{21}$$

We next show that

$$\limsup_{n \rightarrow \infty} \langle f(x^*) - x^*, x_n - x^* \rangle \leq 0.$$

To show this, let  $\{x_{n_j}\}$  be a subsequence of  $\{x_n\}$  such that

$$\lim_{j \rightarrow \infty} \langle f(x^*) - x^*, x_{n_j} - x^* \rangle = \limsup_{n \rightarrow \infty} \langle f(x^*) - x^*, x_n - x^* \rangle.$$

Since  $\{x_{n_j}\}$  is bounded, there exists a subsequence  $\{x_{n_{j_k}}\}$  of  $\{x_{n_j}\}$  and  $u \in \mathcal{H}_1$  such that  $x_{n_{j_k}} \rightarrow u$ . Without loss of generality, we assume that  $x_{n_j} \rightarrow u$ . Since  $A$  is a bounded linear operator,  $\langle y, Ax_{n_j} - Au \rangle = \langle A^*y, x_{n_j} - u \rangle \rightarrow 0$  as  $j \rightarrow \infty$ , for all  $y \in \mathcal{H}_2$ , that is,  $Ax_{n_j} \rightarrow Au$ . From (19) and by the demiclosedness of  $I - T$  at 0, we have  $Au \in F(T)$ . Since  $x_{n_j} \rightarrow u$ , it follows from (21) that  $y_{n_j} \rightarrow u$ . From (20) and by the demiclosedness of  $I - S$  at 0, we get  $u \in F(S)$ . Therefore,  $u \in \Gamma$ . Since  $x^*$  solves the variational inequality (13), we have

$$\limsup_{n \rightarrow \infty} \langle f(x^*) - x^*, x_n - x^* \rangle = \lim_{j \rightarrow \infty} \langle f(x^*) - x^*, x_{n_j} - x^* \rangle = \langle f(x^*) - x^*, u - x^* \rangle \leq 0.$$

From (8), we have

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &= \|(1 - \alpha_n)(Sy_n - x^*) + \alpha_n(f(x_n) - x^*)\|^2 \\ &\leq (1 - \alpha_n)^2 \|Sy_n - x^*\|^2 + 2\alpha_n \langle f(x_n) - x^*, x_{n+1} - x^* \rangle \\ &= (1 - \alpha_n)^2 \|Sy_n - x^*\|^2 + 2\alpha_n \langle f(x_n) - f(x^*), x_{n+1} - x^* \rangle + 2\alpha_n \langle f(x^*) - x^*, x_{n+1} - x^* \rangle \\ &\leq (1 - \alpha_n)^2 \|x_n - x^*\|^2 + 2\alpha_n \tau \|x_n - x^*\| \|x_{n+1} - x^*\| + 2\alpha_n \langle f(x^*) - x^*, x_{n+1} - x^* \rangle \\ &\leq (1 - \alpha_n)^2 \|x_n - x^*\|^2 + \alpha_n \tau (\|x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2) + 2\alpha_n \langle f(x^*) - x^*, x_{n+1} - x^* \rangle. \end{aligned}$$

It follows that

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &\leq \frac{(1 - \alpha_n)^2 + \alpha_n \tau}{1 - \alpha_n \tau} \|x_n - x^*\|^2 + \frac{2\alpha_n}{1 - \alpha_n \tau} \langle f(x^*) - x^*, x_{n+1} - x^* \rangle \\ &= \left(1 - \frac{(1 - \tau)\alpha_n}{1 - \alpha_n \tau}\right) \|x_n - x^*\|^2 + \frac{(\alpha_n - (1 - \tau)\alpha_n)}{1 - \alpha_n \tau} \|x_n - x^*\|^2 + \frac{2\alpha_n}{1 - \alpha_n \tau} \langle f(x^*) - x^*, x_{n+1} - x^* \rangle \\ &\leq \left(1 - \frac{(1 - \tau)\alpha_n}{1 - \alpha_n \tau}\right) \|x_n - x^*\|^2 + \frac{(1 - \tau)\alpha_n}{1 - \alpha_n \tau} \left\{ \left(\frac{\alpha_n}{1 - \tau} - 1\right) M + \frac{2}{1 - \tau} \langle f(x^*) - x^*, x_{n+1} - x^* \rangle \right\} \\ &= (1 - \mu_n) \|x_n - x^*\|^2 + \mu_n \sigma_n, \end{aligned} \tag{22}$$

where  $M = \sup\{\|x_n - x^*\|^2 : n \in \mathbb{N}\}$ ,  $\mu_n = \frac{(1 - \tau)\alpha_n}{1 - \alpha_n \tau}$ , and  $\sigma_n = \left(\frac{\alpha_n}{1 - \tau} - 1\right) M + \frac{2}{1 - \tau} \langle f(x^*) - x^*, x_{n+1} - x^* \rangle$ . Obviously,  $\{\mu_n\} \subset [0, 1]$ ,  $\sum_{n=1}^{\infty} \mu_n = \infty$  and  $\limsup_{n \rightarrow \infty} \sigma_n \leq 0$ . By applying Lemma 2.7 to (22), we conclude that  $x_n \rightarrow x^*$  as  $n \rightarrow \infty$ .

*Case 2.* Suppose that  $\{\|x_n - x^*\|\}$  is not a monotone sequence. Thus, there exists a subsequence  $\{n_i\}$  of  $\{n\}$  such that  $\|x_{n_i} - x^*\| < \|x_{n_{i+1}} - x^*\|$  for all  $i \in \mathbb{N}$ . Define a positive integer sequence  $\{\omega(n)\}$  by

$$\omega(n) := \max\{m \leq n : \|x_m - x^*\| < \|x_{m+1} - x^*\|\}$$

for all  $n \geq n_0$  (for some  $n_0$  large enough). It follows from Lemma 2.8 that  $\{\omega(n)\}$  is a nondecreasing sequence such that  $\omega(n) \rightarrow \infty$  as  $n \rightarrow \infty$  and

$$\|x_{\omega(n)} - x^*\|^2 - \|x_{\omega(n)+1} - x^*\|^2 \leq 0$$

for all  $n \geq n_0$ . From (17), we have

$$\lim_{n \rightarrow \infty} \|(I - T)Ax_{\omega(n)}\| = 0. \tag{23}$$

From (18), we get

$$\lim_{n \rightarrow \infty} \|y_{\omega(n)} - Sy_{\omega(n)}\| = 0. \tag{24}$$



By (23), (24) and by the same proof as in Case 1, we deduce that

$$\limsup_{n \rightarrow \infty} \langle f(x^*) - x^*, x_{\omega(n)} - x^* \rangle \leq 0.$$

By the same computation as in Case 1, we also have

$$\|x_{\omega(n)+1} - x^*\|^2 \leq (1 - \mu_{\omega(n)})\|x_{\omega(n)} - x^*\|^2 + \mu_{\omega(n)}\sigma_{\omega(n)}, \tag{25}$$

where  $\mu_{\omega(n)} = \frac{(1-\tau)\alpha_{\omega(n)}}{1-\alpha_{\omega(n)\tau}}$ ,  $\sigma_{\omega(n)} = \left(\frac{\alpha_{\omega(n)}}{1-\tau} - 1\right)M + \frac{2}{1-\tau}\langle f(x^*) - x^*, x_{\omega(n)+1} - x^* \rangle$  and  $M = \sup\{\|x_{\omega(n)} - x^*\|^2 : n \in \mathbb{N}\}$ . Clearly,  $\limsup_{n \rightarrow \infty} \sigma_{\omega(n)} \leq 0$ . Since  $\|x_{\omega(n)} - x^*\|^2 \leq \|x_{\omega(n)+1} - x^*\|^2$ , it follows from (25) that  $\|x_{\omega(n)} - x^*\|^2 \leq \sigma_{\omega(n)}$ . This implies that  $\|x_{\omega(n)} - x^*\| \rightarrow 0$  as  $n \rightarrow \infty$ . It follows from Lemma 2.8 and (25) that

$$0 \leq \|x_n - x^*\| \leq \|x_{\omega(n)+1} - x^*\| \rightarrow 0$$

as  $n \rightarrow \infty$ . Therefore,  $\{x_n\}$  converges strongly to  $x^*$ . This completes the proof.  $\square$

**Remark 3.3.** In Theorem 3.2, if  $f$  is a constant function, i.e.,  $f(x) = u_0$  for some  $u_0 \in \mathcal{H}_1$ , then Algorithm (11) becomes the Halpern-type algorithm [16]. In particular, if  $u_0 = 0$ , then  $x^*$  is the unique minimum norm solution in  $\Gamma$ .

Taking  $\rho_1 = 1 = \rho_2$  in Theorem 3.2, we obtain a convergence result for solving the SCFP (3) for two directed operators as follows.

**Corollary 3.4.** Let  $A : \mathcal{H}_1 \rightarrow \mathcal{H}_2$  be a bounded linear operator. Let  $S : \mathcal{H}_1 \rightarrow \mathcal{H}_1$  and  $T : \mathcal{H}_2 \rightarrow \mathcal{H}_2$  be directed operators such that both  $I - S$  and  $I - T$  are demiclosed at 0. Assume that  $\Gamma \neq \emptyset$ . Let  $f : \mathcal{H}_1 \rightarrow \mathcal{H}_1$  be a  $\tau$ -contraction with respect to  $\Gamma$ . Then the sequence  $\{x_n\}$  generated by Algorithm (11) converges strongly to a point  $x^* \in \Gamma$ , provided that  $0 < a \leq \lambda_n \leq b < 2$ , and  $\alpha_n \in (0, 1)$  such that  $\lim_{n \rightarrow \infty} \alpha_n = 0$  and  $\sum_{n=1}^{\infty} \alpha_n = \infty$ .

**Remark 3.5.** In the case of a directed operator  $S$ , the algorithms in [3, 28] are invented by using a relaxation of  $S$  for solving the SCFP (3); however, in this case, Algorithm (11) is constructed without the relaxation of operators.

#### 4. Extending Classes of Operators for the Split Common Fixed Point Problem

In this section, we slightly modify Algorithm 11 to extend to the class of demicontractive operators and the class of hemicontractive operators, respectively, for solving the SCFP (3). Furthermore, strong convergence results are also obtained.

##### 4.1. An Algorithm for Demicontractive Operators

We first give the following lemma showing a relationship between a demicontractive operator and its relaxation (see [19, Lemma 3.4]).

**Lemma 4.1.** ([19]) Let  $T : \mathcal{H} \rightarrow \mathcal{H}$  be an operator having a fixed point and let  $\kappa \in [0, 1)$ ,  $\lambda \in (0, 1 - \kappa)$ . Then  $T$  is  $\kappa$ -demicontractive if and only if  $T_\lambda$  is  $\left(\frac{1-\kappa-\lambda}{\lambda}\right)$ -AQNE.

**Theorem 4.2.** Let  $A : \mathcal{H}_1 \rightarrow \mathcal{H}_2$  be a bounded linear operator. Let  $S : \mathcal{H}_1 \rightarrow \mathcal{H}_1$  be a  $\kappa_1$ -demicontractive operator and  $T : \mathcal{H}_2 \rightarrow \mathcal{H}_2$  a  $\kappa_2$ -demicontractive operator such that both  $I - S$  and  $I - T$  are demiclosed at 0. Assume that  $\Gamma \neq \emptyset$ . Let  $f : \mathcal{H}_1 \rightarrow \mathcal{H}_1$  be a  $\tau$ -contraction with respect to  $\Gamma$ . Let  $\{x_n\} \subset \mathcal{H}_1$  be a sequence generated iteratively by  $x_1 \in \mathcal{H}_1$  and

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)S_{\lambda_1}(x_n - \gamma_n A^*(I - T)Ax_n), \quad n \in \mathbb{N}, \tag{26}$$

where the stepsize  $\gamma_n$  is selected in such a way:

$$\gamma_n := \begin{cases} \frac{\lambda_2 \|(I-T)Ax_n\|^2}{\|A^*(I-T)Ax_n\|^2}, & \text{if } Ax_n \notin F(T), \\ 0, & \text{otherwise,} \end{cases} \tag{27}$$

and the parameters  $\lambda_1, \lambda_2$  and the sequence  $\{\alpha_n\}$  satisfy the following conditions:

- (C1)  $\lambda_1 \in (0, 1 - \kappa_1)$  and  $\lambda_2 \in (0, 1 - \kappa_2)$ ;
- (C2)  $\alpha_n \in (0, 1)$  such that  $\lim_{n \rightarrow \infty} \alpha_n = 0$  and  $\sum_{n=1}^{\infty} \alpha_n = \infty$ .

Then the sequence  $\{x_n\}$  converges strongly to a point  $x^* \in \Gamma$ , where  $x^* = P_{\Gamma}f(x^*)$ .

*Proof.* Set  $U := S_{\lambda_1}$  and  $G := T_{\lambda_2}$ . Thus,  $F(U) = F(S)$  and  $F(G) = F(T)$ . Now Algorithm (26) can be rewritten in the form:

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)U(x_n - \gamma'_n A^*(I - G)Ax_n),$$

where

$$\gamma'_n := \begin{cases} \frac{\|(I-G)Ax_n\|^2}{\|A^*(I-G)Ax_n\|^2}, & \text{if } Ax_n \notin F(G), \\ 0, & \text{otherwise.} \end{cases}$$

By Lemma 4.1,  $U$  is  $\left(\frac{1-\kappa_1-\lambda_1}{\lambda_1}\right)$ -AQNE and  $G$  is  $\left(\frac{1-\kappa_2-\lambda_2}{\lambda_2}\right)$ -AQNE, where  $\frac{1-\kappa_1-\lambda_1}{\lambda_1}, \frac{1-\kappa_2-\lambda_2}{\lambda_2} > 0$ . Clearly,  $I - U = \lambda_1(I - S)$  and  $I - G = \lambda_2(I - T)$  are demiclosed at 0. Therefore, it follows directly from Theorem 3.2 that  $x_n \rightarrow x^* \in \Gamma$ , where  $x^* = P_{\Gamma}f(x^*)$ .  $\square$

Taking  $\kappa_1 = 0 = \kappa_2$  in Theorem 4.2, we obtain a convergence result for solving the SCFP (3) for two quasi-nonexpansive operators as follows.

**Corollary 4.3.** *Let  $A : \mathcal{H}_1 \rightarrow \mathcal{H}_2$  be a bounded linear operator. Let  $S : \mathcal{H}_1 \rightarrow \mathcal{H}_1$  and  $T : \mathcal{H}_2 \rightarrow \mathcal{H}_2$  be quasi-nonexpansive operators such that both  $I - S$  and  $I - T$  are demiclosed at 0. Assume that  $\Gamma \neq \emptyset$ . Let  $f : \mathcal{H}_1 \rightarrow \mathcal{H}_1$  be a  $\tau$ -contraction with respect to  $\Gamma$ . Then the sequence  $\{x_n\}$  generated by Algorithm (26) converges strongly to a point  $x^* \in \Gamma$ , provided that  $\lambda_1, \lambda_2 \in (0, 1)$ , and  $\alpha_n \in (0, 1)$  such that  $\lim_{n \rightarrow \infty} \alpha_n = 0$  and  $\sum_{n=1}^{\infty} \alpha_n = \infty$ .*

**Remark 4.4.** The results in [3] are consequences of our results as follows:

- (i) Taking  $f(x) = x_0$  and  $\lambda_2 = \frac{1-\kappa_2}{2}$  in Theorem 4.2, we obtain a result in [3, Theorem 4.1].
- (ii) Taking  $f(x) = x_0$  and  $\lambda_2 = \frac{1}{2}$  in Corollary 4.3, we obtain a result in [3, Theorem 5.1].

#### 4.2. An Algorithm for Hemicontractive Operators

Recall that an operator  $T : \mathcal{H} \rightarrow \mathcal{H}$  is said to be Lipschitzian if there exists  $L > 0$  such that

$$\|Tx - Ty\| \leq L\|x - y\|, \quad \forall x, y \in \mathcal{H}.$$

We first give some properties of the operator  $TT_{\delta}$ , where  $T$  is Lipschitzian.

**Lemma 4.5.** ([36, 38]) *Let  $T : \mathcal{H} \rightarrow \mathcal{H}$  be a Lipschitzian operator with a constant  $L > 1$ , and let  $\delta \in \left(0, \frac{1}{L}\right)$ . Then the following hold:*

- (i)  $F(TT_{\delta}) = F(T)$ ;
- (ii) If  $I - T$  is demiclosed at 0, then  $I - TT_{\delta}$  is also demiclosed at 0.

In [36, 38], they proved that the operator  $(TT_\delta)_\lambda$  is quasi-nonexpansive when  $T$  is Lipschitzian hemicontractive. The following lemma gives more desirable result under the same conditions.

**Lemma 4.6.** *Let  $T : \mathcal{H} \rightarrow \mathcal{H}$  be a Lipschitzian hemicontractive operator with a constant  $L > 1$ . Then the operator  $(TT_\delta)_\lambda$  is  $\left(\frac{\delta-\lambda}{\lambda}\right)$ -AQNE, where  $0 < \lambda < \delta < \frac{1}{\sqrt{1+L^2}+1}$ .*

*Proof.* We will prove this result by applying Lemma 4.1, so it is sufficient to show that the operator  $TT_\delta$  is  $(1 - \delta)$ -demicontractive. By Lemma 4.5 (i),  $F(TT_\delta) = F(T)$ . Let  $x \in \mathcal{H}$  and  $u \in F(T)$ . By the hemicontractivity of  $T$ , we get

$$\|TT_\delta x - u\|^2 \leq \|T_\delta x - u\|^2 + \|T_\delta x - TT_\delta x\|^2. \tag{28}$$

Since the equality (7) holds and  $T$  is hemicontractive, we have

$$\begin{aligned} \|T_\delta x - u\|^2 &= \|(1 - \delta)(x - u) + \delta(Tx - u)\|^2 \\ &= (1 - \delta)\|x - u\|^2 + \delta\|Tx - u\|^2 - \delta(1 - \delta)\|x - Tx\|^2 \\ &\leq (1 - \delta)\|x - u\|^2 + \delta(\|x - u\|^2 + \|x - Tx\|^2) - \delta(1 - \delta)\|x - Tx\|^2 \\ &= \|x - u\|^2 + \delta^2\|x - Tx\|^2. \end{aligned} \tag{29}$$

Since  $T$  is Lipschitzian with the coefficient  $L$ ,

$$\|Tx - TT_\delta x\| \leq L\|x - T_\delta x\| = \delta L\|x - Tx\|. \tag{30}$$

From (7) and (30), we have

$$\begin{aligned} \|T_\delta x - TT_\delta x\|^2 &= \|(1 - \delta)(x - TT_\delta x) + \delta(Tx - TT_\delta x)\|^2 \\ &= (1 - \delta)\|x - TT_\delta x\|^2 + \delta\|Tx - TT_\delta x\|^2 - \delta(1 - \delta)\|x - Tx\|^2 \\ &\leq (1 - \delta)\|x - TT_\delta x\|^2 + \delta^3 L^2\|x - Tx\|^2 - \delta(1 - \delta)\|x - Tx\|^2 \\ &= (1 - \delta)\|x - TT_\delta x\|^2 - \delta(1 - \delta - \delta^2 L^2)\|x - Tx\|^2. \end{aligned} \tag{31}$$

By substituting (29) and (31) into (28) and by simplifying it, we obtain

$$\|TT_\delta x - u\|^2 \leq \|x - u\|^2 + (1 - \delta)\|x - TT_\delta x\|^2 - \delta(1 - 2\delta - \delta^2 L^2)\|x - Tx\|^2.$$

This together with the condition of  $\delta$  implies

$$\|TT_\delta x - u\|^2 \leq \|x - u\|^2 + (1 - \delta)\|x - TT_\delta x\|^2, \tag{32}$$

i.e.,  $TT_\delta$  is  $(1 - \delta)$ -demicontractive. Therefore, it follows directly from Lemma 4.1 that the operator  $(TT_\delta)_\lambda$  is  $\left(\frac{\delta-\lambda}{\lambda}\right)$ -AQNE.  $\square$

**Theorem 4.7.** *Let  $A : \mathcal{H}_1 \rightarrow \mathcal{H}_2$  be a bounded linear operator. Let  $S : \mathcal{H}_1 \rightarrow \mathcal{H}_1$  and  $T : \mathcal{H}_2 \rightarrow \mathcal{H}_2$  be Lipschitzian hemicontractive operators with coefficients  $L_1, L_2 > 1$ , respectively, such that both  $I - S$  and  $I - T$  are demiclosed at 0. Assume that  $\Gamma \neq \emptyset$ . Let  $f : \mathcal{H}_1 \rightarrow \mathcal{H}_1$  be a  $\tau$ -contraction with respect to  $\Gamma$ . Let  $\{x_n\} \subset \mathcal{H}_1$  be a sequence generated iteratively by  $x_1 \in \mathcal{H}_1$  and*

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)(SS_{\delta_1})_{\lambda_1}(x_n - \gamma_n A^*(I - TT_{\delta_2})Ax_n), \quad n \in \mathbb{N}, \tag{33}$$

where the stepsize  $\gamma_n$  is selected in such a way:

$$\gamma_n := \begin{cases} \frac{\lambda_2 \|(I - TT_{\delta_2})Ax_n\|^2}{\|A^*(I - TT_{\delta_2})Ax_n\|^2}, & \text{if } Ax_n \notin F(T), \\ 0, & \text{otherwise,} \end{cases} \tag{34}$$

and the parameters  $\delta_1, \delta_2, \lambda_1, \lambda_2$  and the sequence  $\{\alpha_n\}$  satisfy the following conditions:

(C1)  $0 < \lambda_1 < \delta_1 < \frac{1}{\sqrt{1+L_1^2+1}}$  and  $0 < \lambda_2 < \delta_2 < \frac{1}{\sqrt{1+L_2^2+1}}$ ;

(C2)  $\alpha_n \in (0, 1)$  such that  $\lim_{n \rightarrow \infty} \alpha_n = 0$  and  $\sum_{n=1}^{\infty} \alpha_n = \infty$ .

Then the sequence  $\{x_n\}$  converges strongly to a point  $x^* \in \Gamma$ , where  $x^* = P_{\Gamma}f(x^*)$ .

*Proof.* Set  $U := (SS_{\delta_1})_{\lambda_1}$  and  $G := (TT_{\delta_2})_{\lambda_2}$ . By Lemma 4.5 (i),  $F(U) = F(SS_{\delta_1}) = F(S)$  and  $F(G) = F(TT_{\delta_2}) = F(T)$ . We can rewrite Algorithm (33) in the form:

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)U(x_n - \gamma'_n A^*(I - G)Ax_n),$$

where

$$\gamma'_n := \begin{cases} \frac{\|(I-G)Ax_n\|^2}{\|A^*(I-G)Ax_n\|^2}, & \text{if } Ax_n \notin F(G), \\ 0, & \text{otherwise.} \end{cases}$$

By Lemma 4.6,  $U$  is  $(\frac{\delta_1 - \lambda_1}{\lambda_1})$ -AQNE and  $G$  is  $(\frac{\delta_2 - \lambda_2}{\lambda_2})$ -AQNE, where  $\frac{\delta_1 - \lambda_1}{\lambda_1}, \frac{\delta_2 - \lambda_2}{\lambda_2} > 0$ . We also have from Lemma 4.5 (ii) that  $I - U = \lambda_1(I - SS_{\delta_1})$  and  $I - G = \lambda_2(I - TT_{\delta_2})$  are demiclosed at  $\bar{0}$ . Therefore, the result is obtained directly by Theorem 3.2.  $\square$

### 5. Applications

In this section, we apply our main result to the split feasibility problem and the split variational inequality problem, respectively.

#### 5.1. Split Feasibility Problems

Let  $C$  and  $Q$  be nonempty closed convex subsets of  $\mathcal{H}_1$  and  $\mathcal{H}_2$ , respectively, and let  $A : \mathcal{H}_1 \rightarrow \mathcal{H}_2$  be a bounded linear operator. Recall the *split feasibility problem* (SFP) [11] is the problem of finding a point

$$x^* \in C \text{ such that } Ax^* \in Q.$$

Applying Theorem 3.2, we obtain a strongly convergent algorithm which is independent of the operator norms for solving the SFP as shown below.

**Theorem 5.1.** *Let  $A : \mathcal{H}_1 \rightarrow \mathcal{H}_2$  be a bounded linear operator. Let  $C$  and  $Q$  be nonempty closed convex subsets of  $\mathcal{H}_1$  and  $\mathcal{H}_2$ , respectively. Assume that  $\Omega := \{x \in C : Ax \in Q\} \neq \emptyset$ . Let  $f : \mathcal{H}_1 \rightarrow \mathcal{H}_1$  be a contraction with respect to  $\Omega$ . Let  $\{x_n\} \subset \mathcal{H}_1$  be a sequence generated iteratively by  $x_1 \in \mathcal{H}_1$  and*

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)P_C(x_n - \gamma_n A^*(I - P_Q)Ax_n), \quad n \in \mathbb{N}, \tag{35}$$

where the stepsize  $\gamma_n$  is selected in such a way:

$$\gamma_n := \begin{cases} \frac{\lambda_n \|(I - P_Q)Ax_n\|^2}{\|A^*(I - P_Q)Ax_n\|^2}, & \text{if } Ax_n \notin Q, \\ 0, & \text{otherwise,} \end{cases} \tag{36}$$

and the sequences  $\{\lambda_n\}$  and  $\{\alpha_n\}$  satisfy the following conditions:

(C1)  $0 < a \leq \lambda_n \leq b < 2$ ;

(C2)  $\alpha_n \in (0, 1)$  such that  $\lim_{n \rightarrow \infty} \alpha_n = 0$  and  $\sum_{n=1}^{\infty} \alpha_n = \infty$ .

Then the sequence  $\{x_n\}$  converges strongly to a point  $x^* \in \Omega$ , where  $x^* = P_{\Omega}f(x^*)$ .

*Proof.* Take  $S := P_C$  and  $T := P_Q$ . Thus,  $F(S) = C$  and  $F(T) = Q$ . By the firm nonexpansivity of the metric projection, we have  $S$  and  $T$  are 1-AQNE, and  $I - S, I - T$  are demiclosed at  $\bar{0}$ . So, the result is obtained directly by Theorem 3.2.  $\square$

5.2. Split Variational Inequality Problems

Let  $C$  be a nonempty closed convex subset of  $\mathcal{H}$ , and let  $g : \mathcal{H} \rightarrow \mathcal{H}$  be an operator. The *variational inequality problem* is to find a point  $x \in C$  such that

$$\langle g(x), y - x \rangle \geq 0, \quad \forall y \in C. \tag{37}$$

The solution set of (37) is denoted by  $VIP(C, g)$ . It is known that  $VIP(C, g) = \text{Fix}(P_C(I - \mu g))$  for  $\mu > 0$ . We also know that if  $g$  is  $\delta$ -inverse strongly monotone, where  $\delta > 0$ , i.e.,

$$\langle x - y, g(x) - g(y) \rangle \geq \delta \|g(x) - g(y)\|^2, \quad \forall x, y \in \mathcal{H},$$

then  $P_C(I - \mu g)$  is nonexpansive, where  $\mu \in (0, 2\delta)$ , see [18].

Let  $A : \mathcal{H}_1 \rightarrow \mathcal{H}_2$  be a bounded linear operator. Let  $C \subset \mathcal{H}_1$  and  $Q \subset \mathcal{H}_2$  be nonempty closed convex subsets, and let  $g : \mathcal{H}_1 \rightarrow \mathcal{H}_1$  and  $h : \mathcal{H}_2 \rightarrow \mathcal{H}_2$  be operators. Then, the *split variational inequality problem* (SVIP) [13] is to find a point

$$x^* \in VIP(C, g) \text{ such that } Ax^* \in VIP(Q, h). \tag{38}$$

Applying Theorem 3.2, we get a strongly convergent algorithm which is independent of the operator norms for solving the SVIP as follows.

**Theorem 5.2.** *Let  $A : \mathcal{H}_1 \rightarrow \mathcal{H}_2$  a bounded linear operator. Let  $C$  and  $Q$  be nonempty closed convex subsets of  $\mathcal{H}_1$  and  $\mathcal{H}_2$ , respectively. Let  $g : \mathcal{H}_1 \rightarrow \mathcal{H}_1$  and  $h : \mathcal{H}_2 \rightarrow \mathcal{H}_2$  be inverse strongly monotone operators with coefficients  $\delta_1$  and  $\delta_2$ , respectively. Assume that  $\Omega := \{x \in VIP(C, g) : Ax \in VIP(Q, h)\} \neq \emptyset$ . Let  $f : \mathcal{H}_1 \rightarrow \mathcal{H}_1$  be a contraction with respect to  $\Omega$ . Let  $\{x_n\} \subset \mathcal{H}_1$  be a sequence generated iteratively by  $x_1 \in \mathcal{H}_1$  and*

$$\begin{cases} y_n = x_n - \gamma_n A^*(I - P_Q(I - \mu_2 h))Ax_n, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)P_C(I - \mu_1 g)y_n, \end{cases} \quad n \in \mathbb{N}, \tag{39}$$

where the stepsize  $\gamma_n$  is selected in such a way:

$$\gamma_n := \begin{cases} \frac{\lambda_n \|(I - P_Q(I - \mu_2 h))Ax_n\|^2}{\|A^*(I - P_Q(I - \mu_2 h))Ax_n\|^2}, & \text{if } Ax_n \notin VIP(Q, h), \\ 0, & \text{otherwise,} \end{cases} \tag{40}$$

the parameters  $\mu_1, \mu_2$  and the sequences  $\{\lambda_n\}, \{\alpha_n\}$  satisfy the following conditions:

- (C1)  $\mu_1 \in (0, 2\delta_1)$  and  $\mu_2 \in (0, 2\delta_2)$ ;
- (C2)  $0 < a \leq \lambda_n \leq b < 1$ ;
- (C3)  $\alpha_n \in (0, 1)$  such that  $\lim_{n \rightarrow \infty} \alpha_n = 0$  and  $\sum_{n=1}^{\infty} \alpha_n = \infty$ .

Then the sequence  $\{x_n\}$  converges strongly to a point  $x^* \in \Omega$ , where  $x^* = P_\Omega f(x^*)$ .

*Proof.* Take  $S := P_C(I - \mu_1 g)$  and  $T := P_Q(I - \mu_2 h)$ . Then,  $F(S) = VIP(C, g)$  and  $F(T) = VIP(Q, h)$ . Since  $S$  and  $T$  are nonexpansive, we have  $S$  and  $T$  are 0-AQNE, and  $I - S, I - T$  are demiclosed at 0. Therefore, the result is obtained directly by Theorem 3.2.  $\square$

6. A Numerical Example

In this section, we provide a numerical result to illustrate the convergence of Algorithm 11 in Theorem 3.2 and also compare the convergence behavior of our algorithm with a viscosity-type algorithm depending on the operator norm.

**Example 6.1.** Let  $\mathcal{H}_1 = \mathbb{R}^3 = \mathcal{H}_2$  with the usual norm. Define two operators  $S, T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  by

$$S \begin{bmatrix} a \\ b \\ c \end{bmatrix} := \frac{1}{2} \begin{bmatrix} a \\ b \\ 2c \end{bmatrix} \quad \text{and} \quad T \begin{bmatrix} a \\ b \\ c \end{bmatrix} := \left( \frac{1}{3} B^\top B + I \right)^{-1} \begin{bmatrix} a \\ b \\ c \end{bmatrix},$$

where  $B = \begin{bmatrix} 1 & 2 & -3 \\ -5 & 0 & 1 \end{bmatrix}$ . It is easy to verify that  $S$  and  $T$  are 1-AQNE (i.e., directed), and  $I - S, I - T$  are also demiclosed at 0. Given a bounded linear operator

$$A := \begin{bmatrix} 1 & 2 & -3 \\ 2 & 7 & 0 \\ -1 & -3 & 0 \end{bmatrix},$$

then we see that  $0 \in \Gamma := \{x \in F(S) : Ax \in F(T)\}$ . Put  $\alpha_n = \frac{1}{n+3}$ , and let a contraction  $f : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  be such that  $f(x) = \frac{1}{2}x$ . Then the viscosity iterative method for the SCFP (3) can be written in the form:

*Initialization:* Let  $x_1 \in \mathbb{R}^3$  be arbitrary.

*Iterative step:* For  $n \in \mathbb{N}$ , let

$$x_{n+1} = \frac{1}{2n+6}x_n + \left(1 - \frac{1}{n+3}\right)S(x_n - \gamma_n A^\top(I - T)Ax_n). \tag{41}$$

We now consider two algorithms defined by (41) with different stepsizes  $\gamma_n$  as follows:

**Algorithm 1.** (Algorithm 11, Theorem 3.2). Take

$$\gamma_n := \begin{cases} \frac{\lambda \| (I-T)Ax_n \|^2}{\| A^\top(I-T)Ax_n \|^2}, & \text{if } Ax_n \notin F(T), \\ 0, & \text{otherwise,} \end{cases}$$

where  $\lambda \in (0, 2)$ .

**Algorithm 2.** Take  $\gamma_n := \gamma \in \left(0, \frac{2}{\|A\|^2}\right)$ .

We start with the initial point  $x_1 = (3, 6, -5)$  and the stopping criterion for our testing process is set as  $\|x_n - x_{n-1}\| < 10^{-6}$  where  $x_n = (a_n, b_n, c_n)$ . Now, we show the convergence behaviour of Algorithms 1 and 2 by Table 1, and plot the number of iterations  $n$  against  $\|x_n - x_{n-1}\|$  as seen in Figure 1. In Table 2, the iteration numbers of Algorithms 1 and 2 are shown while their stepsizes are chosen differently.

|                                         | $n$        | $a_n$      | $b_n$      | $c_n$      | $\ x_n - x_{n-1}\ $ |
|-----------------------------------------|------------|------------|------------|------------|---------------------|
| Algorithm 1<br>( $\lambda = 1$ )        | 2          | 0.7155288  | 0.7009828  | -3.1105405 | 6.0719395           |
|                                         | 3          | 0.0942971  | -0.3336636 | -1.9846520 | 1.6504687           |
|                                         | 4          | -0.0320429 | -0.1128823 | -0.4231261 | 1.5821091           |
|                                         | 5          | -0.0024916 | 0.0331737  | -0.1931336 | 0.2740478           |
|                                         | ⋮          | ⋮          | ⋮          | ⋮          | ⋮                   |
|                                         | 23         | -0.0000007 | 0.0000003  | -0.0000006 | 0.0000012           |
| 24                                      | -0.0000004 | 0.0000000  | -0.0000004 | 0.0000005  |                     |
| Algorithm 2<br>( $\gamma = 1/\ A\ ^2$ ) | 2          | 0.9738642  | 1.4580758  | -3.5269417 | 5.1869261           |
|                                         | 3          | 0.3096212  | 0.2439270  | -2.7353280 | 1.5943739           |
|                                         | 4          | 0.0862869  | -0.0393379 | -2.2216787 | 0.6276566           |
|                                         | 5          | 0.0076107  | -0.0875128 | -1.8447723 | 0.3880325           |
|                                         | ⋮          | ⋮          | ⋮          | ⋮          | ⋮                   |
|                                         | 24         | -0.0040449 | -0.0059952 | -0.1162249 | 0.0165296           |
| ⋮                                       | ⋮          | ⋮          | ⋮          | ⋮          |                     |
| 101                                     | -0.0000003 | -0.0000004 | -0.0000086 | 0.0000011  |                     |
| 102                                     | -0.0000003 | -0.0000004 | -0.0000076 | 0.0000097  |                     |

Table 1: Numerical experiment of Algorithms 1 and 2

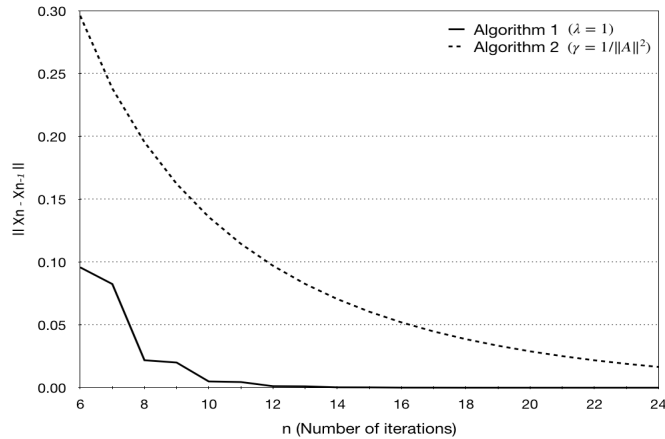


Figure 1: A graph of error of Algorithms 1 and 2

| Algorithm 1          |                   | Algorithm 2         |                   |
|----------------------|-------------------|---------------------|-------------------|
| Choices of $\lambda$ | No. of iterations | Choices of $\gamma$ | No. of iterations |
| 0.01                 | 1259              | $0.01/\ A\ ^2$      | 4488              |
| 0.1                  | 164               | $0.1/\ A\ ^2$       | 729               |
| 0.5                  | 32                | $0.5/\ A\ ^2$       | 187               |
| 1                    | 24                | $1/\ A\ ^2$         | 102               |
| 1.5                  | 27                | $1.5/\ A\ ^2$       | 71                |
| 1.9                  | 31                | $1.9/\ A\ ^2$       | 58                |
| 1.99                 | 31                | $1.99/\ A\ ^2$      | 55                |
| 1.9999               | 31                | $1.9999/\ A\ ^2$    | 55                |

Table 2: The number of iterations of Algorithms 1 and 2 by choosing different stepsizes

**Remark 6.2.** By testing the convergence behavior of Algorithms 1 and 2 in Example 6.1, we observe that

- (i) Both algorithms converge to  $0 \in \Gamma$ ;
- (ii) The number of iterations of Algorithm 1 is smaller than that of Algorithm 2.

### 7. Concluding Remarks

In this work, we study the split common fixed point problem (SCFP) for two operators in real Hilbert spaces. This split problem is the problem of finding a fixed point of an operator in a real Hilbert space such that its image under a given bounded linear operator is a fixed point of another operator in the image space. Various algorithms were introduced for solving the problem and most of them depend on the norm of the bounded linear operators; however, the calculation of the operator norms is not an easy work in general practice. We first present a viscosity-type algorithm whose stepsize does not depend on the operator norms for solving the SCFP for two attracting quasi-nonexpansive operators, and also obtain some sufficient conditions for the strong convergence of the proposed algorithm. After that we modify our algorithm to extend to the class of demicontractive operators and the class of hemicontractive operators, and also obtain strong convergence results. Moreover, strong convergence theorems for solving the split feasibility problem and the split variational inequality problem are consequences of our main result. We finally give the numerical example to illustrate the convergence behavior of our algorithm and it is observed that our algorithm requires the smaller number of iterations than a viscosity-type algorithm depending on the operator norm.

Our algorithms improve the many algorithms such as in [19, 21, 32, 36, 38] for solving the SCFP (3) in the sense that our algorithms do not depend on the norms of the operators as follows:

- In [21, Theorem 3.2], they introduced an algorithm in the similar way to Algorithm 11 (in the case of a constant function  $f$ ) with the stepsize  $\gamma_n := \gamma \in \left(0, \frac{1}{\|A\|^2}\right)$ .
- In [19, Theorem 4.4] and [32, Corollary 3.5], they introduced an algorithm in the similar way to Algorithm 26 with the stepsize  $\gamma_n := \gamma \in \left(0, \frac{1-\kappa_2}{\|A\|^2}\right)$ , where  $S$  is  $\kappa_1$ -demicontractive and  $T$  is  $\kappa_2$ -demicontractive.
- The algorithms in [36, 38] were introduced for hemicontractive operators; however, their algorithms still depend on the operator norms.

## Acknowledgments

The authors would like to thank the referees for valuable comments and suggestions for improving this work and Chiang Mai University for the financial support.

## References

- [1] C. Byrne, Iterative oblique projection onto convex sets and the split feasibility problem, *Inverse Problems* 18 (2002) 441–453.
- [2] C. Byrne, A unified treatment of some iterative algorithms in signal processing and image reconstruction, *Inverse Problems* 20 (2004) 103–120.
- [3] O. A. Boikanyo, A strongly convergent algorithm for the split common fixed point problem, *Appl. Math. Comput.* 265 (2015) 844–853.
- [4] H. H. Bauschke, P. L. Combettes, A weak-to-strong convergence principle for Fejér-monotone methods in Hilbert spaces, *Math. Oper. Res.* 26 (2001) 248–264.
- [5] P. L. Combettes, The convex feasibility problem in image recovery, In: P. Hawkes (Editor), *Advances in Imaging and Electron Physics*, Academic Press, New York 95 (1996) 155–270.
- [6] A. Cegielski, *Iterative Methods for Fixed Point Problems in Hilbert Spaces*, Lecture Notes in Mathematics 2057, Springer, Heidelberg, 2012.
- [7] A. Cegielski, General method for solving the split common fixed point problem, *J. Optim. Theory Appl.* 165 (2015) 385–404.
- [8] A. Cegielski, Landweber-type operator and its properties, *Contemp. Math.* 658 (2016) 139–148.
- [9] Y. Censor, T. Bortfeld, B. Martin, A. Trofimov, A unified approach for inversion problems in intensity-modulated radiation therapy, *Phys. Med. Biol.* 51 (2006) 2353–2365.
- [10] A. Cegielski, Y. Censor, Opial-type theorems and the common fixed point problem, in: H. H. Bauschke, R. S. Burachik, P. L. Combettes, V. Elser, D. R. Luke, H. Wolkowicz (Editors), *Fixed-Point Algorithms for Inverse Problems in Science and Engineering*, Springer-Verlag, New York (2011) 55–183.
- [11] Y. Censor, T. Elfving, A multiprojection algorithm using Bregman projections in a product space, *Numer. Algorithms* 8 (1994) 221–239.
- [12] Y. Censor, T. Elfving, N. Kopf, T. Bortfeld, The multiple-sets split feasibility problem and its applications for inverse problems, *Inverse Problems* 21 (2005) 2071–2084.
- [13] Y. Censor, A. Gibali, S. Reich, Algorithms for the split variational inequality problem, *Numer. Algorithms* 59 (2012) 301–323.
- [14] Y. Censor, A. Segal, The split common fixed point problem for directed operators, *J. Convex Anal.* 16 (2009) 587–600.
- [15] H. Cui, F. Wang, Iterative methods for the split common fixed point problem in Hilbert spaces, *Fixed Point Theory Appl.* 78 (2014) <https://doi.org/10.1186/1687-1812-2014-78>.
- [16] B. Halpern, Fixed points of nonexpanding maps, *Bull. Amer. Math. Soc.* 73 (1967) 957–961.
- [17] T. L. Hicks, J. D. Kubicek, On the Mann iteration process in a Hilbert space, *J. Math. Anal. Appl.* 59 (1977) 498–504.
- [18] H. Iiduka, W. Takahashi, Strong convergence theorems for nonexpansive mappings and inverse-strongly monotone mappings, *Nonlinear Anal.* 61 (2005) 341–350.
- [19] P. Jailoka, S. Suantai, Split common fixed point and null point problems for demicontractive operators in Hilbert spaces, *Optim. Methods Softw.* 34 (2019) 248–263.
- [20] P. Jailoka, S. Suantai, The split common fixed point problem for multivalued demicontractive mappings and its applications, *Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Math. RACSAM* 113 (2019) 689–706.
- [21] R. Kraikaew, S. Saejung, On split common fixed point problems, *J. Math. Anal. Appl.* 415 (2014) 513–524.
- [22] G. López, V. Martín-Márquez, F. Wang, H. K. Xu, Solving the split feasibility problem without prior knowledge of matrix norms, *Inverse Problems* 28 (2012) doi:10.1088/0266-5611/28/8/085004.
- [23] Ş. Mârşter, The solution by iteration of nonlinear equations in Hilbert spaces, *Proc. Amer. Math. Soc.* 63 (1977) 69–73.
- [24] A. Moudafi, Viscosity approximation methods for fixed-points problems, *J. Math. Anal. Appl.* 241 (2000) 46–55.
- [25] A. Moudafi, The split common fixed-point problem for demicontractive mappings, *Inverse Problems* 26 (2010) 587–600.
- [26] P. E. Maingé, Strong convergence of projected subgradient methods for nonsmooth and nonstrictly convex minimization, *Set-Valued Anal.* 16 (2008) 899–912.
- [27] A. Moudafi, A note on the split common fixed-point problem for quasi-nonexpansive operators, *Nonlinear Anal.* 74 (2011) 4083–4087.



- [28] P. E. Maingé, A viscosity method with no spectral radius requirements for the split common fixed point problem, *European J. Oper. Res.* 235 (2014) 17–27.
- [29] Z. Opial, Weak convergence of the sequence of successive approximations for nonexpansive mappings, *Bull. Am. Math. Soc.* 73 (1967) 591–597.
- [30] L. Qihou, On Naimpally and Singh’s Open Questions, *J. Math. Anal. Appl.* 124 (1987) 157–164.
- [31] B. Qu, N. Xiu, A note on the CQ algorithm for the split feasibility problem, *Inverse Problems* 21 (2005) 1655–1665.
- [32] D. V. Thong, Viscosity approximation methods for solving fixed-point problems and split common fixed-point problems, *J. Fixed Point Theory Appl.* 19 (2017) 1481–1499.
- [33] H. K. Xu, Iterative algorithms for nonlinear operators, *J. Lond. Math. Soc.* 66 (2002) 240–256.
- [34] H. K. Xu, Iterative methods for the split feasibility problem in infinite-dimensional Hilbert spaces, *Inverse Problems* 26 (2010) doi:10.1088/0266-5611/26/10/105018.
- [35] Q. Yang, The relaxed CQ algorithm for solving the split feasibility problem, *Inverse Problems* 20 (2004) 1261–1266.
- [36] Y. Yao, Y. C. Liou, J. C. Yao, Split common fixed point problem for two quasi-pseudo-contractive operators and its algorithm construction, *Fixed Point Theory Appl.* 127 (2015) <https://doi.org/10.1186/s13663-015-0376-4>.
- [37] I. Yamada, N. Ogura, Hybrid steepest descent method for variational inequality problem over the fixed point set of certain quasi-nonexpansive mappings, *Numer. Funct. Anal. Optim.* 25 (2004) 619–655.
- [38] L. J. Zhu, Y. C. Liou, J. C. Yao, Y. Yao, New algorithms designed for the split common fixed point problem of quasi-pseudocontractions, *J. Inequal. Appl.* 304 (2014) <https://doi.org/10.1186/1029-242X-2014-304>.