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One-Sided (b, c)-Inverses in Rings

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Abstract. In this paper we introduce left and right annihilator (b, c)-inverses and we investigate some of theirs properties. Furthermore, here we study some properties of left and right (b, c)-inverses.

1. Introduction

Throughout the paper, we assume that R is a ring with identity. An involution of R is any map $* : R \to R$ satisfying

$$(a^*)^* = a,\ (ay)^* = y^* a^*,\ (a + y)^* = a^* + y^*,\ \text{for any } a, y \in R.$$

A $*$-ring R denotes the ring R with an involution $*$. Let $b, c \in R$. The concept of the (b, c)-inverse as a generalization of the Moore-Penrose inverse, the Drazin inverse, the Chipman's weighted inverse and the Bott-Duffin inverse, was for the first time introduced by M. P. Drazin in 2012 [7], in the settings of rings. Recall that an element $a \in R$ is said to be (b, c)-invertible if there exists $y \in R$ such that

$$y \in (bRy) \cap (yRc), \qquad yab = b, \qquad cay = c.$$

If such y exists, it is unique and it is called the (b, c)-inverse of a, denoted by $a^{\otimes}$. For more results on (b, c)-inverse we refer the reader to see [8, 9, 13, 22].

In [7], M. P. Drazin also introduced the hybrid and annihilator (b, c)-inverse of a. An element $y \in R$ is called the hybrid (b, c)-inverse of a if it satisfies the following equations:

$$yay = y, \quad yR = bR, \quad y^{\circ} = c^{\circ}.$$

If the condition $yR = bR$ from the above equations is replaced by ${}^{\circ}b = {}^{\circ}y$, then y is the annihilator (b, c)-inverse of a. Actually, an element $y \in R$ is called the annihilator (b, c)-inverse of a if the following hold:

$$yay = y, \quad {}^{\circ}b = {}^{\circ}y, \quad y^{\circ} = c^{\circ}.$$

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It is shown in [7] that if the hybrid (or annihilator, resp.) (b, c)-inverse of a exists, it is unique. So if a is (b, c)-invertible, it is hybrid and annihilator (b, c)-invertible, but the converse may not true.

Recently, M.P. Drazin [10] introduced left and right (b, c)-inverse. An element $a \in R$ is left (right, resp.) (b, c)-invertible if

$$Rb = Rcab \text{ (resp. } cR = cabR), \tag{1}$$

or equivalently, $a \in R$ is left (right, resp.) (b, c)-invertible if there exists $y \in R$ such that

$$Ry \subseteq Rc \quad \text{and} \quad yab = b \text{ (resp. } yR \subseteq bR \quad \text{and} \quad cay = c), \tag{2}$$

in which case any such y is called a left (right, resp.) (b, c)-inverse of a.

X. Mary in [17] introduced a new generalized inverse, called the inverse along an element. An element $a \in R$ is said to be invertible along $d \in R$ (or Mary invertible) if there exists $y \in R$ such that

$$yad = d = day, \quad yR \subseteq dR, \quad Ry \subseteq Rd.$$

If such $y \in R$ exists, it is unique and it is called the inverse along element d (or Mary inverse). This inverse unify some well-known generalized inverses, such as the group inverse, Drazin inverse and Moore-Penrose inverse. Also, the inverse along element d is a special case of the (b, c)-inverse, for $(b, c) = (d, d)$ [7, Proposition 6.1]. Several authors also have studied this new outer inverse (see [2, 3, 24, 25]).

As an extension of Mary inverse, H. H. Zhu et al. [24] recently introduced left (right, resp.) inverse along an element. Actually, an element $a \in R$ is left (right, resp.) invertible along $d \in R$ (or left (right, resp.) Mary invertible) if there exists $y \in R$ such that

$$yad = d \text{ (resp. } day = d), \quad Ry \subseteq Rd \text{ (resp. } yR \subseteq dR).$$

For the convenience of the reader, some fundamental concepts are given as follows.

An element $a \in R$ is said to be Moore-Penrose invertible if there exists $y \in R$ which satisfies the following equations:

$$(1)\ aya = a, \quad (2)\ yay = y, \quad (3)\ (ay)^* = ay, \quad (4)\ (ya)^* = ya.$$

If such y exists, it is unique and is usually denoted by $a^\dagger$. The set of all Moore-Penrose invertible elements of R will be denoted by $R^\dagger$.

An element $a \in R$ is (von Neumann) regular if it has an inner inverse y, i.e. if there exists $y \in R$ such that $aya = a$. Any inner inverse of a will be denoted by a^-. The set of all regular elements of R will be denoted by R^-. If $\delta \subseteq \{1, 2, 3, 4\}$ and y satisfies the equations (i) for all $i \in \delta$, then y is an δ-inverse of a. The set of all δ-inverse of a is denoted by $a\delta$. Clearly, $a\{1, 2, 3, 4\} = \{a^\dagger\}$.

Recall that $a \in R$ is left (right, resp.) regular [1] if there exists $x \in R$ such that $a = xa^2$ ($a = a^2x$, resp.). If a is both left and right regular, then a is strongly regular. An element a of R is said to be left (right, resp.) π-regular if there exists $x \in R$ such that $a^n = xa^{n+1}$ ($a^n = a^{n+1}x$, resp.), for some positive integer n. An element is strongly π-regular if it is both left and right π-regular. An element a is said to be left (right, resp.) $*$-regular if there exists $x \in R$ such that $a = aa^*ax$ ($a = xaa^*a$, resp.). The notions of the Drazin and group inverse can be referred to the literature [6]. It is shown in [6] that an element $a \in R$ is Drazin invertible if and only if it is strongly π-regular. Specially, a is group invertible if and only if it is strongly regular.

For an element $a \in R$, we define the following image ideals

$$aR = \{ax : x \in R\}, \quad Ra = \{xa : x \in R\},$$

and kernel ideals

$$a^\circ = \{x \in R : ax = 0\}, \quad {}^\circ a = \{x \in R : xa = 0\}.$$

Let $a, y \in R$. Then $aR = yR$ if and only if there exist $u, v \in R$ such that $a = yu$ and $y = av$. Similarly, $Ra = Ry$ if and only if there exist $s, t \in R$ such that $a = sy$ and $y = ta$.

The ring of integers is denoted by $\mathbb{Z}$, and $\mathbb{Z}_n$ stands for the factor ring of $\mathbb{Z}$ modulo n, i.e. $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$, where n is a positive integer.

The results of the paper are organized as follows. In Section 2, we define one-sided annihilator (b, c)-inverses. Also, we investigate necessary and sufficient conditions for the existence of one-sided (b, c)-inverses and one-sided annihilator (b, c)-inverses and we state some interesting special cases. In Section 3, we consider one-sided (b, c)-inverses of a product of three elements. Actually, we derive some relations of one-sided (b, c)-inverse of paq and one-sided generalized inverse of pa and aq, where $a, p, q \in R$. In Section 4, we investigate some properties of one-sided (b, c)-inverses, such as the relation between one-sided (b, c)-inverses and classical one-sided inverses.

2. Definition and existence of one-sided annihilator and one-sided (*b*, *c*)-inverses

In this section, we first introduce a class of new generalized inverses in a ring R, called a left (right) annihilator (b, c)-inverse. Then we investigate necessary and sufficient conditions for the existence of these generalized inverses.

Notice that the condition $Ry \subseteq Rc$ ($yR \subseteq bR$, resp.) from definition of left (right, resp.) (b, c)-inverse implies $c^\circ \subseteq y^\circ$ (${}^\circ b \subseteq {}^\circ y$, resp.). Thus we give the following definition of a left(right, resp.) annihilator (b, c)-inverse of a.

Definition 2.1. *Let $b, c \in R$. An element $a \in R$ is said to be left annihilator (b, c)-invertible if there exists $y \in R$ satisfying*

$$c^\circ \subseteq y^\circ \quad \text{and} \quad yab = b.$$

In this case y is called a left annihilator (b, c)-inverse of a.

Similarly, an element $a \in R$ is right annihilator (b, c)-invertible if there exists $y \in R$ which satisfies the following equations:

$${}^\circ b \subseteq {}^\circ y \quad \text{and} \quad cay = c.$$

In this case y is called a right annihilator (b, c)-inverse of a.

Remark 2.2. Let $a, b, c, y \in R$.

(i) From (2) and Definition 2.1, obviously we have the following fact: if a is left (right, resp.) (b, c)-invertible with a left (right, resp.) (b, c)-inverse y, then a is left (right, resp.) annihilator (b, c)-invertible with a left (right, resp.) annihilator (b, c)-inverse y.
However, the converse does not hold in general. For example, let $R = \mathbb{Z}$. Then 2 is left annihilator $(0, 2)$-invertible, with a left annihilator $(0, 2)$-inverse 1, but 1 is not a left $(0, 2)$-inverse of 2. Indeed, $2^\circ = 1^\circ$, but $\mathbb{Z} \nsubseteq 2\mathbb{Z}$.

(ii) In general, the condition $c^\circ \subseteq y^\circ$ (${}^\circ b \subseteq {}^\circ y$, resp.) does not imply $Ry \subseteq Rc$ ($yR \subseteq bR$, resp.). For example, let $R = \mathbb{Z}$, $y = 1$, $b = c = 2$. Obviously, $c^\circ = y^\circ = b^\circ$, but $\mathbb{Z} \nsubseteq 2\mathbb{Z}$. However, if c (b, resp.) is regular, we have $c^\circ \subseteq y^\circ$ (${}^\circ b \subseteq {}^\circ y$, resp.) if and only if $Ry \subseteq Rc$ ($yR \subseteq bR$, resp.).

From the above remark, we have the following result.

Proposition 2.3. *Let $a, b, c, y \in R$. The following statements hold.*

(i) If $c \in R^-$, then a is left (b, c)-invertible with a left (b, c)-inverse y if and only if it is left annihilator (b, c)-invertible with a left annihilator (b, c)-inverse y.

(ii) If $b \in R^-$, then a is right (b, c)-invertible with a right (b, c)-inverse y if and only if it is right annihilator (b, c)-invertible with a right annihilator (b, c)-inverse y.

It is shown in [7, Theorem 2.2] that a is (b, c)-invertible if and only if $Rb = Rcab$ and $cR = cabR$. By (1.1), a is left (right, resp.) (b, c)-invertible if and only if $Rb = Rcab$ ($cR = cabR$, resp.). Therefore, a is (b, c)-invertible if and only if it is both left and right (b, c)-invertible, that is, [10, Theorem 2.1]. According to [22] and [15], if a is (b, c)-invertible, then $b, c \in R^-$. Applying Proposition 2.3, we get the next result.

Corollary 2.4. *Let $a, b, c \in R$. Then a is (b, c)-invertible if and only if $b, c \in R^-$ and a is both left annihilator and right annihilator (b, c)-invertible.*

In the next proposition we give the relation between the left (b, c)-inverse and the right (b, c)-inverse. Since one can easily check it by using (1) and Proposition 2.3, we state it without the proof.

Proposition 2.5. *Let R be a $*$-ring and $a, b, c \in R$. Then the following statements hold.*

(i) An element a is left (b, c)-invertible if and only if a^* is right (c^*, b^*)-invertible;

(ii) Let $b, c \in R^-$. An element a is left annihilator (b, c)-invertible if and only if a^* is right annihilator (c^*, b^*)-invertible.

In the following result we present some consequences of left and right annihilator (b, c)-invertibility.

Proposition 2.6. *Let $a, b, c \in R$. Then the following is valid.*

(i) If a is left annihilator (b, c)-invertible, then $b^\circ = (cab)^\circ$;

(ii) If a is right annihilator (b, c)-invertible, then ${}^\circ c = {}^\circ(cab)$.

Proof. (i). Suppose that a is left annihilator (b, c)-invertible. By Definition 2.1, there is $y \in R$ such that $c^\circ \subseteq y^\circ$ and $yab = b$. For any $x \in (cab)^\circ$, we have $cabx = 0$. Then we obtain $abx \in c^\circ \subseteq y^\circ$, i.e. $yabx = 0$, which yields $bx = yabx = 0$. This means $(cab)^\circ \subseteq b^\circ$. Thus $b^\circ = (cab)^\circ$.

(ii). Analogously. □

In general, the converse of (i) and (ii) in Proposition 2.6 does not hold. For example, let $R = \mathbb{Z}$ and let $a = 2$, $b = c = 1$. Then we have $b^\circ = (cab)^\circ$, but there is no $y \in \mathbb{Z}$ such that $1 = b = yab = 2y$, i.e. 2 is not right annihilator $(1, 1)$-invertible in $\mathbb{Z}$.

Next we give the relation between one-sided (b, c)-inverse and the hybrid (b, c)-inverse.

Theorem 2.7. *Let $a, b, c \in R$. Then:*

(i) $y \in R$ is a right (b, c)-inverse and left annihilator (b, c)-inverse of a if and only if y is the hybrid (b, c)-inverse of a;

(ii) $y \in R$ is a left (b, c)-inverse and right annihilator (b, c)-inverse of a if and only if $yay = y$, ${}^\circ b = {}^\circ y$, $Rc = Ry$.

Proof. (i). If $y \in R$ is a right (b, c)-inverse and left annihilator (b, c)-inverse of a, by (2) and Definition 2.1 we have

$$yR \subseteq bR, \quad cay = c, \quad c^\circ \subseteq y^\circ, \quad yab = b.$$

Thus we have $b = yab \in yR$, and $c^\circ \subseteq y^\circ \subseteq (cay)^\circ = c^\circ$, i.e. $bR = yR$, $c^\circ = y^\circ$. The condition $c = cay$ implies $ay - 1 \in c^\circ = y^\circ$. Therefore, $y = yay$.

Conversely, if y is the hybrid (b, c)-inverse of a, we have $yay = y$, $yR = bR$, $y^\circ = c^\circ$. The condition $yR = bR$ gives $b = ys$, for some $s \in R$. Thus, we obtain $b = ys = yays = yab$. Moreover, since $y(ay - 1) = 0$, we have $ay - 1 \in y^\circ = c^\circ$. Hence, $c = cay$.

(ii). Similarly as in (i). □

Using [7, Definition 6.2], Theorem 2.7 and Corollary 2.4, we have the following proposition.

Proposition 2.8. *Let $a, b, c \in R$.*

(i) If $y \in R$ is the annihilator (b, c)-inverse of a, then y is a left annihilator and right annihilator (b, c)-inverse of a;

(ii) If $y \in R$ is a left (b, c)-inverse and right annihilator (b, c)-inverse of a, then y is the annihilator (b, c)-inverse of a;

(iii) If $y \in R$ is a right (b, c)-inverse and left annihilator (b, c)-inverse of a, then y is the annihilator (b, c)-inverse of a.

Moreover, if $b, c \in R^-$, then the converse of (i)-(iii) are all valid.

Proof. (i). Suppose that y is the annihilator (b,c)-inverse of a. By [7, Definition 6.2], we have

$$yay = y,\ {}^{\circ}b = {}^{\circ}y,\ y^{\circ} = c^{\circ}.$$

Since $yay = y$, we have $(ya-1)y = 0$, i.e. $ya - 1 \in {}^{\circ}y = {}^{\circ}b$. Therefore, $(ya-1)b = 0$ and $yab = b$. Similarly, $y(ay-1) = 0$ and $y^{\circ} = c^{\circ}$ imply $cay = c$. Hence, by Definition 2.1, a is left annihilator and right annihilator (b,c)-invertible.

Moreover, if $b, c \in R^{-}$, by Corollary 2.4, we get that the converse of (i) is valid.

(ii). Let $y \in R$ be a left (b,c)-inverse and right annihilator (b,c)-inverse of a. By Theorem 2.7 (ii), we have $yay = y$, ${}^{\circ}b = {}^{\circ}y$, $Rc = Ry$. Since the condition $Rc = Ry$ implies $y^{\circ} = c^{\circ}$, we have that y is the annihilator (b,c)-inverse of a.

Moreover, if $b, c \in R^{-}$, then ${}^{\circ}b = {}^{\circ}y$ and $y^{\circ} = c^{\circ}$ are equivalent to ${}^{\circ}b = {}^{\circ}y$ and $Rc = Ry$ under the condition $yay = y$. Hence, the converse of (ii) holds.

(iii). Analogously as (ii). □

Remark 2.9. Let $a, b, c, y \in R$.

(i) Note that in general the converse of (i), (ii) and (iii) from Proposition 2.8 doesn't hold. For example, let $R = \mathbb{Z}$. We have that 1 is the annihilator $(1,2)$-inverse of 1 and 1 is a right annihilator $(1,2)$-inverse of 1, but 1 is not a left $(1,2)$-inverse of 1. Indeed, $a = b = y = 1$, $c = 2$, we have ${}^{\circ}b = {}^{\circ}y = y^{\circ} = c^{\circ} = 0$, but $\mathbb{Z} \nsubseteq 2\mathbb{Z}$.

(ii) If a is both left (b,c)-invertible and left annihilator (b,c)-invertible, with a left (b,c)-inverse y and a left annihilator (b,c)-inverse z, in general y doesn't have to be equal to z. For example, let $R = \mathbb{Z}_8 = \mathbb{Z}/8\mathbb{Z}$, $a = \overline{5}, b = \overline{0}, c = \overline{2}$. Then $\overline{5}$ is left $(\overline{0},\overline{2})$-invertible and it is also left annihilator $(\overline{0},\overline{2})$-invertible. And $y = \overline{4}$ is a left $(\overline{0},\overline{2})$-inverse of $\overline{5}$, $z = \overline{6}$ is a left annihilator $(\overline{0},\overline{2})$-inverse of $\overline{5}$. But $\overline{4} \neq \overline{6}$ in $\mathbb{Z}_8$.

Next we give another existence criterion of a left (b,c)-inverse of a.

Theorem 2.10. *Let $a, b, c \in R$. Then the following are equivalent:*

(i) a is left (b,c)-invertible;

(ii) $a^{\circ} \cap bR = \{0\}$, $abR \cap c^{\circ} = \{0\}$ and $R = Rca + {}^{\circ}b$;

(iii) $a^{\circ} \cap bR = \{0\}$ and $R = Rca + {}^{\circ}b$;

(iv) $abR \cap c^{\circ} = \{0\}$ and $R = Rca + {}^{\circ}b$;

(v) $R = Rca + {}^{\circ}b$.

Proof. (i) ⇒ (ii). If a is left (b,c)-invertible, by (2), there is $y \in R$ such that $Ry \subseteq Rc$ and $yab = b$. For any $x \in a^{\circ} \cap bR$, we have $ax = 0$ and $x = bs$, for some $s \in R$. Thus $x = bs = yabs = yax = 0$, i.e. $a^{\circ} \cap bR = \{0\}$.

For any $z \in abR \cap c^{\circ}$, there is $s' \in R$ such that $z = abs'$ and $cz = 0$. Using (1), we have $Rb = Rcab$. Thus, there is $r \in R$ such that $b = rcab$. Then $z = abs' = a(rcab)s' = arcz = 0$, i.e. $abR \cap c^{\circ} = \{0\}$.

Let $u = 1 - rca$. Then $u \in {}^{\circ}b$. For any $t \in R$, we have $t = t(rca + u) = trca + tu \in Rca + {}^{\circ}b$, i.e. $R = Rca + {}^{\circ}b$.

(ii) ⇒ (iii)(or (iv), resp.). Obviously.

(iii)(or (iv), resp.) ⇒ (v). Clearly.

(v) ⇒ (i). The condition $R = Rca + {}^{\circ}b$ gives $b \in Rcab$, so $Rb = Rcab$. According to (1), a is left (b,c)-invertible. □

Dually, we have the following result for the existence of a right (b,c)-inverse of a.

Theorem 2.11. *Let $a, b, c \in R$. Then the following are equivalent:*

(i) a is right (b,c)-invertible;

(ii) ${}^{\circ}a \cap Rc = \{0\}$, $Rca \cap {}^{\circ}b = \{0\}$ and $R = abR + c^{\circ}$;

(iii) ${}^{\circ}a \cap Rc = \{0\}$ and $R = abR + c^{\circ}$;

(iv) $Rca \cap {}^{\circ}b = \{0\}$ and $R = abR + c^{\circ}$;

(v) $R = abR + c^\circ$.

In [10, Theorem 2.1], a is (b, c)-invertible if and only if it is both left and right (b, c)-invertible. Applying Theorem 2.10 and Theorem 2.11,we get the following corollary.

Corollary 2.12. *Let $a, b, c \in R$. Then the following are equivalent:*

(i) *a is (b, c)-invertible;*
(ii) $a^\circ \cap bR = \{0\}$, ${}^\circ a \cap Rc = \{0\}$, $R = abR \oplus c^\circ$ *and* $R = Rca \oplus {}^\circ b$;
(iii) $a^\circ \cap bR = \{0\}$, ${}^\circ a \cap Rc = \{0\}$, $R = abR + c^\circ$ *and* $R = Rca + {}^\circ b$;
(iv) [7, Proposition 2.7]$R = abR \oplus c^\circ$ *and* $R = Rca \oplus {}^\circ b$;
(v) [7, Proposition 2.7] $R = abR + c^\circ$ *and* $R = Rca + {}^\circ b$.

In fact, a is (b, c)-invertible if and only if one of the item of Theorem 2.10 and one of the item of Theorem 2.11 hold.

Before we present our next result, we first state the following auxiliary lemma.

Lemma 2.13. [5, 23] *Let R be a $*$-ring with 1 and let $a \in R$. Then*

(i) $a\{1, 3\} \neq \emptyset \Leftrightarrow a^*R = a^*aR \Leftrightarrow Ra = Ra^*a \Leftrightarrow R = Ra^* + {}^\circ a \Leftrightarrow R = aR + (a^*)^\circ$.
(ii) $a\{1, 4\} \neq \emptyset \Leftrightarrow aR = aa^*R \Leftrightarrow Ra^* = Raa^* \Leftrightarrow R = Ra + {}^\circ(a^*) \Leftrightarrow R = a^*R + a^\circ$.

Using (1) and Lemma 2.13, we can get the following result easily, which derives the relations between left, right (b, c)-inverse and $\{1, 3\}$, $\{1, 4\}$ and Moore-Penrose inverse.

Proposition 2.14. *Let R be a $*$-ring with 1 and let $a \in R$. Then:*

(i) *a is left $(a^*, 1)$-invertible if and only if a is $\{1, 4\}$-invertible;*
(ii) *a is right $(1, a^*)$-invertible if and only if a is $\{1, 3\}$-invertible;*
(iii) *a is Moore-Penrose invertible if and only if a is left $(a^*, 1)$-invertible and right $(1, a^*)$-invertible.*

Proof. (i). Using (1), a is left $(a^*, 1)$-invertible if and only if $Ra^* = Raa^*$. By Lemma 2.13, this is equivalent to a is $\{1, 4\}$-invertible.

(ii). Similar discuss as (i).

(iii). It is well known that $a \in R$ is Moore-Penrose invertible if and only if $a \in aa^*R \cap Ra^*a$ if and only if it is both $\{1, 3\}$ and $\{1, 4\}$-invertible. Using (i) and (ii), then (iii) holds. □

Now we consider the relations between left, right (b, c)-invertible and left, right π-regular and strongly π-regular elements.

Proposition 2.15. *Let R be a ring with 1 and let $a \in R$. Then for some $n \in \mathbb{N}$,*

(i) *a is left $(a^n, 1)$-invertible if and only if a is left π-regular;*
(ii) *a is right $(1, a^n)$-invertible if and only if a is right π-regular;*
(iii) *a is strongly π-regular (i.e. Drazin invertible) if and only if a is left $(a^n, 1)$-invertible and right $(1, a^n)$-invertible.*

Proof. (i). Using (1), a is left $(a^n, 1)$-invertible if and only if $Ra^n = Ra^{n+1}$. By the definition of left π-regular in [1], (i) holds.

(ii). Analogously as (i).

(iii). Combining (i) and (ii), we get (iii). □

Note that Proposition 2.15 provides the relations between left, right (b, c)-invertible and left, right regular and strongly regular (i.e. group invertible) elements, for $n = 1$.

In the following proposition, we give some other special cases of left and right (b, c)-invertibility.

Proposition 2.16. *Let R be a $*$-ring with 1 and let a be an element of R. Then*

(i) a is right $(a^*, 1)$-invertible if and only if $R = aa^*R$ (a is left $(1, a^*)$-invertible if and only if $R = Ra^*a$, resp.).
(ii) a is left (a, a^*)-invertible if and only if $Ra = Ra^*a^2$ (a is right (a, a^*)-invertible if and only if $a^*R = a^*a^2R$, resp.).
(iii) a is left (a^*, a)-invertible if and only if $Ra^* = Ra^2a^*$ (a is right (a^*, a)-invertible if and only if $aR = a^2a^*R$. resp.).

Remark 2.17. Let $a, b, c \in R$.

(i) According to [22, Proposition 3.3], if a is (b, c)-invertible, then $b, c \in R^-$. However, if a is left (right, resp.) (b, c)-invertible or left annihilator (right, resp.) (b, c)-invertible, the condition $b, c \in R^-$ doesn't have to hold in general. For example, let $R = \mathbb{Z}_8 = \mathbb{Z}/8\mathbb{Z}$, $a = \overline{5}$, $b = \overline{0}$, $c = \overline{2}$. Then $\overline{5}$ is left $(\overline{0}, \overline{2})$-invertible and it is also right annihilator $(\overline{0}, \overline{2})$-invertible. But $\overline{2}$ is not regular element in $\mathbb{Z}_8$.

(ii) In Proposition 2.14, 2.15 and 2.16 we have the several choices for b and c: $1, a^*, a^n$, for $n \in \mathbb{N}$. However, in each of these propositions the case when $b \neq c$ is studied. For the above mentioned choices of b and c, in the case when $b = c$, we have the following statements:
(a) a is left (right, resp.) $(1, 1)$-invertible if and only if it is left (right, resp.) invertible.
(b) a is left (right, resp.) (a, a)-invertible if and only if it is left (right, resp.) regular.
(c) a is left (right, resp.) (a^*, a^*)-invertible if and only if it is left (right, resp.) $*$-regular if and only if a is Moore-Penrose invertible.
(d) a is left (right, resp.) (a^n, a^n)-invertible if and only if it is left (right, resp.) π-regular, for some $n \in \mathbb{N}$.

(iii) If $Rc = Rb$ ($bR = cR$, resp.), then a is left (right, resp.) (b, c)-invertible if and only if it is left (right, resp.) invertible along b (c, resp.).

3. One-sided (b, c)-inverse of a product of three elements

In this section, we present several necessary and sufficient conditions for the existence of one-sided (b, c)-inverse of a product paq in a ring R, where $p, a, q \in R$.

First, we consider the relation between the left (b, c)-inverse of paq and one-sided generalized inverse of pa and aq, where $p, a, q \in R$.

Theorem 3.1. *Let $a, b, c, p, q \in R$. Then the following are equivalent:*

(i) paq is left (b, c)-invertible;
(ii) pa is left (qb, c)-invertible and aq is left (b, cp)-invertible.

Proof. (i) ⇒ (ii). Let $y \in R$ be a left (b, c)-inverse of paq. By (2), we have $Ry \subseteq Rc$ and $ypaqb = b$. Let $x = qy$ and $z = yp$. Then we have

$$Rx = Rqy \subseteq Ry \subseteq Rc \quad \text{and} \quad xpaqb = qb,$$

$$Rz = Ryp \subseteq Rcp \quad \text{and} \quad zaqb = b.$$

Therefore, pa is left (qb, c)-invertible and aq is left (b, cp)-invertible.

(ii) ⇒ (i). If (ii) holds, by (2), we have

$$Rx \subseteq Rc, \quad xpaqb = qb, \quad Rz \subseteq Rcp, \quad zaqb = b.$$

Let $y = zax$. Then $Ry = Rzax \subseteq Rx \subseteq Rc$ and $ypaqb = zaxpaqb = zaqb = b$. This means paq is left (b, c)-invertible. □

Similarly, we get the analogous result for the right (b, c)-inverse of paq.

Theorem 3.2. *Let $a, b, c, p, q \in R$. Then the following are equivalent:*

(i) paq is right (b, c)-invertible;
(ii) pa is right (qb, c)-invertible and aq is right (b, cp)-invertible.

Note that Theorem 3.1 (Theorem 3.2, resp.) provide necessary and sufficient conditions for the existence of left (right, resp.) inverse along element $d \in R$ of a product of three elements, in the case when $(b, c) = (d, d)$.

In a similar way as in the above theorems, we obtain the following result.

Theorem 3.3. *Let $a, b, c, p, q \in R$. If there exists $q' \in R$ such that $q'qb = b$, then the following are equivalent:*

(i) paq is left (b, c)-invertible;
(ii) a is left (qb, cp)-invertible.

Proof. (i) ⇒ (ii). If (i) holds, by (2), we have $Ry \subseteq Rc$ and $ypaqb = b$. Let $w = qyp$. Then we get

$$Rw = Rqyp \subseteq Ryp \subseteq Rcp \quad \text{and} \quad waqb = (qyp)aqb = q(ypaqb) = qb.$$

This implies that a is left (qb, cp)-invertible.

(ii) ⇒ (i). Assume that a has a left (qb, cp)-inverse w. By (2), we have $Rw \subseteq Rcp$ and $waqb = qb$. Then, there is $r \in R$ such that $w = rcp$. Since $q'qb = b$, multiplying by q' on the left of $waqb = qb$ gives $b = q'waqb = q'(rcp)aqb$. Let $y = q'rc$. Then $Ry = Rq'rc \subseteq Rc$ and $b = (q'rc)paqb = ypaqb$. In Consequence, by (2), paq is left (b, c)-invertible. □

Analogously, we can show the following characterization for the right (b, c)-inverse of paq.

Theorem 3.4. *Let $a, b, c, p, q \in R$. If there exists $p' \in R$ such that $cpp' = c$, then the following are equivalent:*

(i) paq is right (b, c)-invertible;
(ii) a is right (qb, cp)-invertible.

Applying Theorem 3.3, Theorem 3.4 and [10, Theorem 2.1], we have the next corollary for the (b, c)-inverse of paq.

Corollary 3.5. [14, Theorem 2.3] *Let $a, b, c, p, q \in R$. If there exist $p', q' \in R$ such that $q'qb = b$ and $cpp' = c$, then the following are equivalent:*

(i) paq is (b, c)-invertible;
(ii) a is (qb, cp)-invertible.

In this case, if $y \in R$ is the (b, c)-inverse of paq and $w \in R$ is the (qb, cp)-inverse of a, then the following relation holds:

$$w = qyp.$$

Note that Theorem 3.3 (Theorem 3.4, resp.) is also valid if we replace the condition "there exists $q' \in R$ such that $q'qb = b$" ("there exists $p' \in R$ such that $cpp' = c$", resp.) with stronger condition "q is left invertible" ("p is right invertible"). Hence, Corollary 3.5 is valid if the condition "q is left invertible and p is right invertible" holds instead of "there exist $p', q' \in R$ such that $q'qb = b$ and $cpp' = c$".

In the following result we obtain the relation between the (b, c)-inverse of paq and one-sided generalized inverses of pa and aq.

Theorem 3.6. *Let $a, b, c, p, q \in R$. If there exist $p', q' \in R$ such that $q'qc = c$ and $bpp' = b$, then the following are equivalent:*

(i) paq is (b, c)-invertible;
(ii) pa is right (qb, qc)-invertible and aq is left (bp, cp)-invertible.

Moreover, if y is the (b, c)-inverse of paq, x is a right (qb, qc)-inverse of pa and z is a left (bp, cp)-inverse of aq, then the following relation holds:

$$y = zax.$$

Proof. (i) ⇒ (ii). Suppose that paq is (b, c)-invertible, by [7, Theorem 2.2], we have $Rb = Rcpaqb$ and $cR = cpaqbR$, which imply $Rbp = Rcpaqbp$ and $qcR = qcpaqbR$. According to (1), we obtain that pa is right (qb, qc)-invertible and aq is left (bp, cp)-invertible.

(ii) ⇒ (i). If (ii) holds, using (2), we have

$$xR \subseteq qbR, \quad qcpax = qc, \quad Rz \subseteq Rcp, \quad zaqbp = bp. \tag{3}$$

Since $q'qc = c$ and $bpp' = b$, from (3) we get:

$$cpax = c, \quad zaqb = b. \tag{4}$$

The condition $xR \subseteq qbR$ and $Rz \subseteq Rcp$ implies $x = qbr_1$ and $z = r_2cp$, for some $r_1, r_2 \in R$. Let $y = zax$. Then

$$Ry = Rzax \overset{(3)}{\subseteq} Rcpax \overset{(4)}{=} Rc, \quad yR = zaxR \overset{(3)}{\subseteq} zaqbR \overset{(4)}{=} bR,$$

$$ypaqb = (zax)paqb = (r_2cp)axpaqb \overset{(4)}{=} r_2cpaqb = zaqb \overset{(4)}{=} b,$$

$$cpaqy = cpaq(zax) = cpaqza(qbr_1) \overset{(4)}{=} cpaqbr_1 = cpax \overset{(4)}{=} c.$$

Therefore, by (2), paq has a left and right (b, c)-inverse y. Hence, by Corollary 2.4, paq has a (b, c)-inverse y. □

The above theorem is also valid if the condition "p is right and q is left invertible" holds instead of "there exist $p', q' \in R$ such that $q'qc = c$ and $bpp' = b$". Note that we can get the related results for one-sided invertibility along an element, as a direct application of the above theorems.

4. Some properties of one-sided (b, c)-inverses

In this section, we investigate some properties of one-sided (b, c)-inverse in rings.

First, we need the following auxiliary lemma.

Lemma 4.1. [16, Exercise 1.6] *Let $a, b \in R$. Then*

(i) $1+ab$ *is left invertible if and only if* $1+ba$ *is left invertible. Moreover, if* $y(1+ab) = 1$*, then* $(1-bya)(1+ba) = 1$.

(ii) $1+ab$ *is right invertible if and only if* $1+ba$ *is right invertible. Moreover, if* $(1+ab)x = 1$*, then* $(1+ba)(1-bxa) = 1$.

(iii) $1 + ab$ *is invertible if and only if* $1 + ba$ *is invertible. Moreover,* $(1 + ba)^{-1} = 1 - b(1 + ab)^{-1}a$.

Note that Lemma 4.1 (iii) is known as the Jacobson's Lemma.

Now we will consider the relation between left (b, c)-inverse of $\alpha \in R$ and classical one-sided inverses of $1 + (\alpha - a)a^{\otimes}$ and $1 + a^{\otimes}(\alpha - a)$.

Theorem 4.2. *Let $a, b, c, \alpha \in R$ be such that a has a (b, c)-inverse $a^{\otimes}$. The following are equivalent:*

(i) α *is left* (b, c)*-invertible;*

(ii) α *is left annihilator* (b, c)*-invertible;*

(iii) $1 + (\alpha - a)a^{\otimes}$ *is left invertible;*

(iv) $1 + a^{\otimes}(\alpha - a)$ *is left invertible.*

Proof. (i) ⇔ (ii). Since a has a (b, c)-inverse $a^{\otimes}$, by [15], then $b, c \in R^-$. By Proposition 2.3, (i) is equivalent to (ii).

(iii) ⇔ (iv). Let $u = 1 + (\alpha - a)a^{\otimes}$ and $v = 1 + a^{\otimes}(\alpha - a)$. Applying Lemma 4.1, the left invertibility of u is equivalent to the left invertibility of v.

(i) ⇒ (iii). By Theorem 2.10, α is left (b, c)-invertible if and only if $R = Rc\alpha + {}^{\circ}b$. Since a has a (b, c)-inverse $a^{\otimes}$, by [7, Proposition 6.1], we know that $a^{\otimes}aa^{\otimes} = a^{\otimes}$, $bR = a^{\otimes}R$ and $Rc = Ra^{\otimes}$. Then $R = Rc\alpha + {}^{\circ}b =$

$Ra^{\otimes}\alpha + {}^{\circ}(a^{\otimes})$. For any $z \in R$, we have $z = z_1 a^{\otimes}\alpha + z_2$, where $z_1 \in Ra^{\otimes}a$ and $z_2 \in {}^{\circ}(a^{\otimes})$. Note that $z_1 \in Ra^{\otimes}a$ implies $z_1 = z_1 a^{\otimes} a$. Let $t = z_1 + z_2$. Then

$$t(1 + a^{\otimes}(\alpha - a)) = (z_1 + z_2)(1 + a^{\otimes}(\alpha - a)) = z_1 + z_1 a^{\otimes}(\alpha - a) + z_2 = z.$$

As $z \in R$ is arbitrary, let $z = 1$. Then $1 + a^{\otimes}(\alpha - a)$ is left invertible. By Lemma 4.1, $1 + (\alpha - a)a^{\otimes}$ is left invertible.

(iv) ⇒ (i). If $v = 1 + a^{\otimes}(\alpha - a)$ is left invertible, then there exists $t \in R$ such that $t(1 + a^{\otimes}(\alpha - a)) = 1$. Since a has a (b, c)-inverse $a^{\otimes}$, we have $a^{\otimes}aa^{\otimes} = a^{\otimes}$, $bR = a^{\otimes}R$ and $Rc = Ra^{\otimes}$. Hence the condition $bR = a^{\otimes}R$ gives ${}^{\circ}b = {}^{\circ}(a^{\otimes})$. Thus, $1 - a^{\otimes}a \in {}^{\circ}(a^{\otimes}) = {}^{\circ}b$. Therefore, for any $s \in R$,

$$s = st(1 + a^{\otimes}(\alpha - a)) = sta^{\otimes}\alpha + st(1 - a^{\otimes}a) \in Ra^{\otimes}\alpha + {}^{\circ}b = Rc\alpha + {}^{\circ}b.$$

Consequently, $R = Rc\alpha + {}^{\circ}b$. By Theorem 2.10, α is left (b, c)-invertible. □

Dually, we have the similar property for right (b, c)-inverse of $\alpha \in R$.

Theorem 4.3. *Let $a, b, c, \alpha \in R$ be such that a has a (b, c)-inverse $a^{\otimes}$. The following are equivalent:*

(i) α is right (b, c)-invertible;
(ii) α is right annihilator (b, c)-invertible;
(iii) $1 + (\alpha - a)a^{\otimes}$ is right invertible;
(iv) $1 + a^{\otimes}(\alpha - a)$ is right invertible.

Note that we can get the related results for one-sided invertibility along an element.

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