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Bounds for the Zeros of Polynomials from Compression Matrix Inequalities

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Abstract. In this paper, some compression matrix inequalities are applied to the Frobenius companion matrices of monic polynomials in order to obtain new upper bounds for the zeros of such polynomials.

1. Introduction

Locating the zeros of polynomials is a classical problem, which has attracted the attention of many mathematicians beginning with Cauchy. This problem, which is still a fascinating topic to both complex and numerical analysts, has many applications in diverse fields of mathematics. The Frobenius companion matrix plays an important link between matrix analysis and the geometry of polynomials. It has been used for the location of the zeros of polynomials by matrix methods (see, e.g., [2], [4], [6], [9], [10], [17]-[23], and references therein). In Section 2, we employ several matrix inequalities involving the spectral norm, the spectral radius, and the numerical radius to derive new bounds for the zeros of polynomials.

Suppose that $p(z) = z^n + a_n z^{n-1} + ... + a_2 z + a_1$ is a complex monic polynomial with $n \geq 2$ and $a_1 \neq 0$. Let $z_1, z_2, z_3, ..., z_n$ be the zeros of p arranged in such a way that $|z_1| \geq |z_1| \geq ... \geq |z_n|$. The Frobenius companion matrix C_p of p is defined as

$$C_p = \begin{bmatrix} -a_n & -a_{n-1} & ... & -a_2 & -a_1 \\ 1 & 0 & ... & 0 & 0 \\ 0 & 1 & ... & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}.$$

It is well-known that the characteristic polynomial of C_p is p itself. Thus, the zeros of p are exactly the

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eigenvalues of C_p (see, e.g., [14, p. 316]). Note that

$$C_p^2 = \begin{bmatrix} b_n & b_{n-1} & \dots & b_3 & b_2 & b_1 \\ -a_n & -a_{n-1} & \dots & -a_3 & -a_2 & -a_1 \\ 1 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 & 0 \end{bmatrix},$$

where $b_j = a_n a_j - a_{j-1}$ for $j = 1, 2, \dots, n$, with $a_0 = 0$. Let $p_1(z) = (z - a_n)p(z) = z^{n+1} - b_n z^{n-1} - b_{n-1} z^{n-2} - \dots - b_2 z - b_1$. Then $z_1, z_2, z_3, \dots, z_n$ and a_n are the zeros of p_1. The corresponding Frobenius companion matrix C_{p_1} of p_1 is given by

$$C_{p_1} = \begin{bmatrix} 0 & b_n & b_{n-1} & \dots & b_2 & b_1 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \end{bmatrix}.$$

We have

$$C_p^3 = \begin{bmatrix} c_n & c_{n-1} & c_{n-2} & \dots & c_4 & c_3 & c_2 & c_1 \\ b_n & b_{n-1} & b_{n-2} & \dots & b_4 & b_3 & b_2 & b_1 \\ -a_n & -a_{n-1} & -a_{n-2} & \dots & -a_4 & -a_3 & -a_2 & -a_1 \\ 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 & 0 & 0 \end{bmatrix},$$

where $b_j = a_n a_j - a_{j-1}$ and $c_j = -a_n b_j + a_{n-1} a_j - a_{j-2}$ for $j = 1, 2, \dots, n$, with $a_0 = a_{-1} = 0$. Let $p_2(z) = (z^2 - a_n z + a_n^2 - a_{n-1})p(z) = z^{n+2} - c_n z^{n-1} - \dots - c_2 z - c_1$. The corresponding Frobenius companion matrix C_{p_2} of p_2 is given by

$$C_{p_2} = \begin{bmatrix} 0 & 0 & c_n & c_{n-1} & c_{n-2} & \dots & c_2 & c_1 \\ 1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & 1 & 0 \end{bmatrix}.$$

We have

$$C_p^4 = \begin{bmatrix} d_n & d_{n-1} & \dots & d_5 & d_4 & d_3 & d_2 & d_1 \\ c_n & c_{n-1} & \dots & c_5 & c_4 & c_3 & c_2 & c_1 \\ b_n & b_{n-1} & \dots & b_5 & b_4 & b_3 & b_2 & b_1 \\ -a_n & -a_{n-1} & \dots & -a_5 & -a_4 & -a_3 & -a_2 & -a_1 \\ 1 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 & 0 & 0 & 0 \end{bmatrix},$$

where $b_j = a_n a_j - a_{j-1}, c_j = -a_n b_j + a_{n-1} a_j - a_{j-2}$, and $d_j = -a_n c_j - a_{n-1} b_{j-1} + a_{n-2} a_j - a_{j-3}$ for $j = 1, 2, ..., n$, with $a_0 = a_{-1} = a_{-2} = 0$. Let $p_3(z) = \left(z^3 - a_n z^2 + \left(a_n^2 - a_{n-1}\right) z - a_n^3 + 2a_n a_{n-1} - a_{n-2}\right) p(z) = z^{n+3} - d_n z^{n-1} - ... - d_2 z - d_1$. The corresponding Frobenius companion matrix C_{p_3} of p_3 is given by

$$C_{p_3} = \begin{bmatrix} 0 & 0 & 0 & d_n & d_{n-1} & ... & d_2 & d_1 \\ 1 & 0 & 0 & 0 & 0 & ... & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & ... & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & ... & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & ... & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & ... & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & ... & 1 & 0 \end{bmatrix}.$$

In fact, for $k < n$, the entries of the first row of C_p^k are the negative of the coefficients of the polynomial obtained by multiplying p by a polynomial of degree $k - 1$. We leave the details to the interested reader. It should be mentioned here that the zeros of p are contained in the zeros of p_1, p_2, and p_3. So, any upper bound for the zeros of p_1, p_2, or p_3 can be considered as an upper bound for the zeros of p.

Let $\mathbb{M}_n(\mathbb{C})$ denote the algebra of all $n \times n$ complex matrices. The eigenvalues of A are denoted by $\lambda_1(A), \lambda_2(A), ..., \lambda_n(A)$, and are arranged so that $|\lambda_1(A)| \geq |\lambda_2(A)| \geq ... \geq |\lambda_n(A)|$. The singular values of A (i.e., the eigenvalues of $|A| = (A^*A)^{\frac{1}{2}}$) are denoted by $s_1(A), s_2(A), ..., s_n(A)$, and arranged so that $s_1(A) \geq s_2(A) \geq ... \geq s_n(A)$. Recall that $s_j^2(A) = \lambda_j(A^*A) = \lambda_j(AA^*)$ for $j = 1, 2, ..., n$. For $A \in \mathbb{M}_n(\mathbb{C})$, let $r(A)$, $w(A)$, and $\|A\|$ denote the spectral radius, the numerical radius, and the spectral norm of A, respectively. Recall that $w(A) = \max_{\|x\|=1} |\langle Ax, x\rangle|$. It is known that

$$\left|\lambda_j(A)\right| \leq r(A) \leq w(A) \leq \|A\| = s_1(A) \tag{1}$$

(see, e.g., [7] or [15]). Let $A \in \mathbb{M}_n(\mathbb{C})$, and let $A = U|A|$ be the polar decomposition of A. The generalized Aluthge transform of A is defined as

$$A(t) = |A|^t U |A|^{1-t}$$

for $0 < t < 1$ (see, e.g., [3]). This transform is well-defined, as it is independent of the choice of the partial isometry U in the polar decomposition of A.

A compression of a partitioned block matrix $A = \left[A_{ij}\right]$ with respect to a certain real-valued function f is a matrix obtained from A by replacing each of its blocks by $f\left(A_{ij}\right)$. Inequalities relating $f(A) = f\left(\left[A_{ij}\right]\right)$ to its compression matrix $\left[f\left(A_{ij}\right)\right]$ are called compression inequalities.

In this paper, we employ spectral norm, spectral radius, and numerical radius compression inequalities to obtain new bounds for the zeros of polynomials.

2. Main results

Matrix analysis methods have been successfully utilized to derive several bounds for the zeros of polynomials. We employ various matrix inequalities to the companion matrices C_p, C_{p_1}, C_{p_2}, and C_{p_3} to derive new bounds for the zeros of p. This will be accomplished by applying certain lemmas.

The following lemma can be found in [16].

Lemma 2.1. *Let* $A \in \mathbb{M}_n(\mathbb{C})$ *be partitioned as*

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

where A_{ij} is an $n_i \times n_j$ matrix for $i, j = 1, 2$ with $n_1 + n_2 = n$. If

$$\tilde{A} = \begin{bmatrix} \|A_{11}\| & \|A_{12}\| \\ \|A_{21}\| & \|A_{22}\| \end{bmatrix},$$

then

$$r(A) \le r\left(\tilde{A}\right), \tag{2}$$

$$w(A) \le w\left(\tilde{A}\right), \tag{3}$$

and

$$\|A\| \le \left\|\tilde{A}\right\|.$$

Theorem 2.2. *If z is any zero of p_1, then*

$$|z| \le \frac{1}{2}\left(\max\{1, |b_n|\} + 1 + \sqrt{(\max\{1, |b_n|\} - 1)^2 + 4\sqrt{\sum_{j=1}^{n-1} \left|b_j\right|^2}}\right).$$

Proof. Partition C_{p_1} as

$$C_{p_1} = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

where $A = \begin{bmatrix} 0 & b_n \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} b_{n-1} & \dots & b_2 & b_1 \\ 0 & \dots & 0 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix}$, and $D = \begin{bmatrix} 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix}$. Using the inequality (2) in Lemma 2.1, we have

$$\begin{aligned} r\left(C_{p_1}\right) &\le r\left(\begin{bmatrix} \|A\| & \|B\| \\ \|C\| & \|D\| \end{bmatrix}\right) \\ &= \frac{1}{2}\left(\|A\| + \|D\| + \sqrt{(\|A\| - \|D\|)^2 + 4\|B\|\|C\|}\right). \end{aligned}$$

Since $\|A\| = \max\{1, |b_n|\}$, $\|B\| = \sqrt{\sum_{j=1}^{n-1} \left|b_j\right|^2}$, and $\|C\| = \|D\| = 1$, it follows that

$$r\left(C_{p_1}\right) \le \frac{1}{2}\left(\max\{1, |b_n|\} + 1 + \sqrt{(\max\{1, |b_n|\} - 1)^2 + 4\sqrt{\sum_{j=1}^{n-1} \left|b_j\right|^2}}\right),$$

from which the result follows by the inequalities (1). $\square$

The following two theorems, however, can be proved similarly as in Theorem 2.2.

Theorem 2.3. *If z is any zero of p_2, then*

$$|z| \le 1 + \left(\sum_{j=1}^{n} \left|c_j\right|^2\right)^{\frac{1}{4}}.$$

Theorem 2.4. *If z is any zero of p_3, then*

$$|z| \leq 1 + \left(\sum_{j=1}^{n} \left| d_j \right|^2 \right)^{\frac{1}{4}}.$$

The following lemma can be found in [11, pp. 8-9].

Lemma 2.5. *Let L_n be the $n \times n$ matrix given by*

$$L_n = \begin{bmatrix} 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix}.$$

Then

$$w(L_n) = \cos \frac{\pi}{n+1}.$$

The following two lemmas are well-known and they can be found in [24] and [5], respectively. The first lemma gives a useful formulation of the numerical radius, and the second one is an improvement of the inequality (3) in Lemma 2.1. Here, $\mathbb{M}_{r \times s}(\mathbb{C})$ denotes the space of all $r \times s$ complex matrices.

Lemma 2.6. *Let $A \in \mathbb{M}_n(\mathbb{C})$. Then*

$$w(A) = \max_{\theta \in \mathbb{R}} \left\| \mathrm{Re}\left(e^{i\theta} A \right) \right\|.$$

Lemma 2.7. *Let $A \in \mathbb{M}_k(\mathbb{C})$, $B \in \mathbb{M}_{k \times m}(\mathbb{C})$, $C \in \mathbb{M}_{m \times k}(\mathbb{C})$ and $D \in \mathbb{M}_m(\mathbb{C})$, and let $T = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$. Then*

$$\begin{aligned} w(T) &\leq w\left(\begin{bmatrix} w(A) & w(T_o) \\ w(T_o) & w(D) \end{bmatrix} \right) \\ &= \frac{1}{2}\left(w(A) + w(D) + \sqrt{(w(A) - w(D))^2 + 4w^2(T_o)} \right), \end{aligned}$$

where $T_o = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}$.

The following lemma [1] gives a bound for the spectral radii of sums of two matrices involving the Aluthge transforms of these matrices.

Lemma 2.8. *Let $A, B \in \mathbb{M}_n(\mathbb{C})$. Then*

$$r(A+B) \leq r\left(\begin{bmatrix} w(A(t)) & \|AB\|^{\frac{t}{2}} \|BA\|^{\frac{1-t}{2}} \\ \|AB\|^{\frac{t}{2}} \|BA\|^{\frac{1-t}{2}} & w(B^*(t)) \end{bmatrix} \right)$$

for $0 < t < 1$.

Theorem 2.9. *If z is any zero of p_1, then*

$$|z| \leq \frac{1}{2}\left(\cos \frac{\pi}{n+1} + \sqrt{\cos^2 \frac{\pi}{n+1} + 4b} \right),$$

where $b = \left(\sum_{j=1}^{n} \left| b_j \right|^2 \right)^{\frac{1}{2}}$.

Proof. Write C_{p_1} as $C_{p_1} = A + B$, where $A = L_{n+1}$, and

$$B = \begin{bmatrix} 0 & b_n & b_{n-1} & \dots & b_2 & b_1 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}.$$

Note that $|A| = I_n \oplus 0$ and $|B^*| = b \oplus 0_n$ in which $b = \left(\sum_{j=1}^{n} \left|b_j\right|^2\right)^{\frac{1}{2}}$, where $I_n, 0_n$ are, respectively, the identity and the zero matrices in $\mathbb{M}_n(\mathbb{C})$. Let $U = A$ and $V = \frac{1}{b}B$. Then U and V are partial isometries. Moreover, A and B can be written, respectively, as $A = U|A|$ and $B = V|B|$ in a polar decomposition. Now, it is easy to see that $A(t) = L_n \oplus 0$ and $B^*(t) = 0_{n+1}$ for $0 < t < 1$. Consequently, $w(B^*(t)) = 0$, and by considering Lemma 2.5, we obtain $w(A(t)) = \cos\frac{\pi}{n+1}$. Furthermore, using the inequality $|z| \le r\left(C_{p_1}\right)$ and Lemma 2.8, we have

$$\begin{aligned} |z| &\le \inf_{0<t<1} r\left(\begin{bmatrix} \cos\frac{\pi}{n+1} & \|AB\|^{\frac{t}{2}}\|BA\|^{\frac{1-t}{2}} \\ \|AB\|^{\frac{t}{2}}\|BA\|^{\frac{1-t}{2}} & 0 \end{bmatrix}\right) \\ &= \frac{1}{2}\left(\cos\frac{\pi}{n+1} + \sqrt{\cos^2\frac{\pi}{n+1} + 4\inf_{0<t<1}\|AB\|^{\frac{t}{2}}\|BA\|^{\frac{1-t}{2}}}\right). \end{aligned}$$

Since $\|AB\| = \|BA\| = b$, we have $\inf_{0<t<1}\|AB\|^t\|BA\|^{1-t} = b$, from which our result follows. □

Now, the following two theorems can be proved in a similar manner to Theorem 2.9.

Theorem 2.10. *If z is any zero of p_2, then*

$$|z| \le \frac{1}{2}\left(\cos\frac{\pi}{n+2} + \sqrt{\cos^2\frac{\pi}{n+2} + 4c}\right),$$

where $c = \left(\sum_{j=1}^{n}\left|c_j\right|^2\right)^{\frac{1}{2}}$.

Theorem 2.11. *If z is any zero of p_3, then*

$$|z| \le \frac{1}{2}\left(\cos\frac{\pi}{n+3} + \sqrt{\cos^2\frac{\pi}{n+3} + 4d}\right),$$

where $d = \left(\sum_{j=1}^{n}\left|d_j\right|^2\right)^{\frac{1}{2}}$.

The following two lemmas can be found in [15, p. 44] and [25, p. 133], respectively.

Lemma 2.12. *Let $A = \left[a_{ij}\right] \in \mathbb{M}_n(\mathbb{C})$. Then*

$$w(A) \le w\left(\left[\left|a_{ij}\right|\right]\right) = \frac{1}{2}r\left(\left[\left|a_{ij}\right| + \left|a_{j_i}\right|\right]\right). \tag{4}$$

Moreover, if $a_{ij} \ge 0$, then

$$w(A) = r\left(\left[\frac{a_{ij} + a_{j_i}}{2}\right]\right). \tag{5}$$

Lemma 2.13. *Let T_n be the $n \times n$ tridiagonal matrix given by*

$$T_n = \begin{bmatrix} 0 & \frac{1}{2} & 0 & \ldots & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & \ldots & 0 \\ 0 & \frac{1}{2} & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \frac{1}{2} \\ 0 & 0 & \ldots & \frac{1}{2} & 0 \end{bmatrix}.$$

Then the eigenvalues of T_n are

$$\lambda_j = \cos \frac{j\pi}{n+1} \text{ for } j = 1, 2, \ldots, n.$$

Based on Lemma 2.12 and Lemma 2.13, we have the following bounds for the zeros of polynomials. Related results can be found in [18].

Theorem 2.14. *If z is any zero of p_1, then*

$$|z| \leq \frac{1}{2}\left(\cos \frac{\pi}{n+1} + \sqrt{\cos^2 \frac{\pi}{n+1} + (1 + |b_n|)^2 + \sum_{j=1}^{n-1} \left|b_j\right|^2}\right).$$

Proof. By applying the inequality (4) in Lemma 2.12 to C_{p_1}, we obtain

$$w\left(C_{p_1}\right) = w\left(\begin{bmatrix} 0 & b_n & b_{n-1} & \ldots & b_2 & b_1 \\ 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & 0 \end{bmatrix}\right) \leq r\left(\begin{bmatrix} 0 & u^t \\ u & T_n \end{bmatrix}\right), \tag{6}$$

where $u = \left[\frac{1}{2}(1 + |b_n|), \frac{1}{2}|b_{n-1}|, \ldots, \frac{1}{2}|b_1|\right]^t$, and T_n is the $n \times n$ tridiagonal matrix given in Lemma 2.13. Since T_n is Hermitian, we have $\|T_n\| = r(T_n) = \cos \frac{\pi}{n+1}$. By using the inequality (6) and the inequality (2) in Lemma 2.1, we obtain

$$\begin{aligned} w\left(C_{p_1}\right) &\leq r\left(\begin{bmatrix} 0 & \|u\| \\ \|u\| & \cos \frac{\pi}{n+1} \end{bmatrix}\right) \\ &= \frac{1}{2}\left(\cos \frac{\pi}{n+1} + \sqrt{\cos^2 \frac{\pi}{n+1} + 4\|u\|^2}\right) \\ &= \frac{1}{2}\left(\cos \frac{\pi}{n+1} + \sqrt{\cos^2 \frac{\pi}{n+1} + (1 + |b_n|)^2 + \sum_{j=1}^{n-1} \left|b_j\right|^2}\right). \end{aligned}$$

Now, the desired result follows from the inequalities (1). □

Similarly, we can prove the following two theorems.

Theorem 2.15. *If z is any zero of p_2, then*

$$|z| \leq \frac{1}{2}\left(\cos \frac{\pi}{n+2} + \sqrt{\cos^2 \frac{\pi}{n+2} + 1 + \sum_{j=1}^{n} \left|c_j\right|^2}\right).$$

Theorem 2.16. *If z is any zero of p_3, then*

$$|z| \leq \frac{1}{2}\left(\cos\frac{\pi}{n+3} + \sqrt{\cos^2\frac{\pi}{n+3} + 1 + \sum_{j=1}^{n}\left|d_j\right|^2}\right).$$

Example 2.17. *Consider the polynomial $p(z) = z^3 + z^2 + \frac{1}{2}z + 1$. Then the upper bounds for the zeros of this polynomial $p(z)$ estimated by different mathematicians are as shown in the following table*

Bound	***Value***
Montel [9]	3.5
Fujii and Kubo [10]	1.9571
Cauchy [14]	2
Kittaneh [18]	2.0574
Linden [21]	1.9492

But if z is a zero of the polynomial $p(z) = z^3 + z^2 + \frac{1}{2}z + 1$, then Theorem 2.9 gives $|z| \leq 1.5153$, Theorem 2.10 gives $|z| \leq 1.6333$, Theorem 2.14 gives $|z| \leq 1.3536$, and Theorem 2.15 gives $|z| \leq 1.3355$, which are better than all the estimates mentioned above.

Now, we are in a position to derive a new bound for the zeros of p_1.

Theorem 2.18. *If z is any zero of p_1, then*

$$|z| \leq \frac{\sqrt{\sum_{j=1}^{n}\left|b_j\right|^2}}{2} + \cos\frac{\pi}{n+1}.$$

Proof. Let $u = \left[\frac{1}{2}|b_n|, \frac{1}{2}|b_{n-1}|, ..., \frac{1}{2}|b_1|\right]^t$. By (5), (6), the triangle inequality for the spectral norm, and the inequality (3) in Lemma 2.1, we have

$$\begin{aligned}
w\left(C_{p_1}\right) &\leq w\left(\begin{bmatrix} 0 & u^t \\ u & 0 \end{bmatrix} + T_n\right) \\
&\leq w\left(\begin{bmatrix} 0 & u^t \\ u & 0 \end{bmatrix}\right) + w\left(T_n\right) \\
&= r\left(\begin{bmatrix} 0 & u^t \\ u & 0 \end{bmatrix}\right) + \cos\frac{\pi}{n+1} \\
&\leq r\left(\begin{bmatrix} 0 & \|u\| \\ \|u\| & 0 \end{bmatrix}\right) + \cos\frac{\pi}{n+1} \\
&= \frac{\sqrt{\sum_{j=1}^{n}\left|b_j\right|^2}}{2} + \cos\frac{\pi}{n+1}.
\end{aligned}$$

Now, the desired bound follows from the fact that $|z| \leq w\left(C_{p_1}\right)$. □

In a similar manner to Theorem 2.18, we can prove the following two theorems.

Theorem 2.19. *If z is any zero of p_2, then*

$$|z| \leq \frac{\sqrt{\sum_{j=1}^{n}\left|c_j\right|^2}}{2} + \cos\frac{\pi}{n+2}.$$

Theorem 2.20. *If z is any zero of p_3, then*

$$|z| \leq \frac{\sqrt{\sum_{j=1}^{n} \left|d_j\right|^2}}{2} + \cos\frac{\pi}{n+3}.$$

The following two lemmas can be found in [10] and [13], respectively. The second lemma is an immediate consequence of Lemma 2.6.

Lemma 2.21. *Let*

$$R = \begin{bmatrix} a_1 & a_2 & \dots & a_n \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}.$$

Then

$$w(R) = \frac{\left(\sum_{j=1}^{n}\left|a_j\right|^2\right)^{\frac{1}{2}} + |a_1|}{2}.$$

Lemma 2.22. *Let $A \in \mathbb{M}_{k\times l}(\mathbb{C})$, $B \in \mathbb{M}_{l\times k}(\mathbb{C})$. Then* $w\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) = \frac{1}{2}\max_{\theta\in\mathbb{R}}\left\|e^{i\theta}A + e^{-i\theta}B^*\right\|$.

Theorem 2.23. *If z is any zero of p with $n > 4$, then*

$$|z| \leq \frac{\left(\sum_{j=n-2}^{n}\left|a_j\right|^2\right)^{\frac{1}{2}} + |a_n|}{2} + \frac{1}{2}\left[\frac{1}{\sqrt{2}} + \cos\frac{\pi}{n-2} + \sqrt{\left(\frac{1}{\sqrt{2}} - \cos\frac{\pi}{n-2}\right)^2 + 4\alpha^2}\right],$$

where

$$\alpha = \frac{1}{2}\left(\frac{1 + \sum_{j=1}^{n-3}\left|a_j\right|^2 + \sqrt{\left(1 + \sum_{j=1}^{n-3}\left|a_j\right|^2\right)^2 - 4\sum_{j=1}^{n-4}\left|a_j\right|^2}}{2}\right)^{\frac{1}{2}}.$$

Proof. Let $L = \begin{bmatrix} -a_n & -a_{n-1} & -a_{n-2} & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 \end{bmatrix}$ and $N = \begin{bmatrix} 0 & 0 & 0 & -a_{n-3} & \dots & -a_2 & -a_1 \\ 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & 0 \end{bmatrix}$. Then $C_p = L + N$. So, by the triangle inequality, we have $w\left(C_p\right) \leq w(L) + w(N)$. By using Lemma 2.21, we have

$$w(L) = \frac{\left(\sum_{j=n-2}^{n}\left|a_j\right|^2\right)^{\frac{1}{2}} + |a_n|}{2}.$$

By applying Lemma 2.7 to N, partitioned as

$$N = \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix},$$

where

$$N_{11} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},\ N_{12} = \begin{bmatrix} -a_{n-3} & -a_{n-4} & \ldots & -a_1 \\ 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \end{bmatrix},$$

$$N_{21} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix},\ \text{and } N_{22} = \begin{bmatrix} 0 & 0 & \ldots & 0 & 0 \\ 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \end{bmatrix},$$

we have

$$\begin{aligned} w(N) &\leq w\left(\begin{bmatrix} w(N_{11}) & w(T_o) \\ w(T_o) & w(N_{22}) \end{bmatrix}\right) \\ &= \frac{1}{2}\left(w(N_{11}) + w(N_{22}) + \sqrt{(w(N_{11}) - w(N_{22}))^2 + 4w^2(T_o)}\right). \end{aligned}$$

According to Lemma 2.22, we have that

$$w(T_o) = w\left(\begin{bmatrix} 0 & N_{12} \\ N_{21} & 0 \end{bmatrix}\right) = \alpha.$$

By using Lemma 2.5, we have $w(N_{11}) = \frac{1}{\sqrt{2}}$ and $w(N_{22}) = \cos\frac{\pi}{n-2}$, and so

$$w(N) \leq \frac{1}{2}\left[\frac{1}{\sqrt{2}} + \cos\frac{\pi}{n-2} + \sqrt{\left(\frac{1}{\sqrt{2}} - \cos\frac{\pi}{n-2}\right)^2 + 4\alpha^2}\right].$$

Hence,

$$w\left(C_p\right) \leq \frac{\left(\sum_{j=n-2}^{n} \left|a_j\right|^2\right)^{\frac{1}{2}} + |a_n|}{2} + \frac{1}{2}\left[\frac{1}{\sqrt{2}} + \cos\frac{\pi}{n-2} + \sqrt{\left(\frac{1}{\sqrt{2}} - \cos\frac{\pi}{n-2}\right)^2 + 4\alpha^2}\right].$$

Recalling that $|z| \leq w\left(C_p\right)$, the result follows. □

Theorem 2.24. *If z is any zero of p_3, then*

$$|z| \leq \frac{1}{2}\left[\frac{1}{\sqrt{2}} + \cos\frac{\pi}{n+1} + \sqrt{\left(\frac{1}{\sqrt{2}} - \cos\frac{\pi}{n+1}\right)^2 + 4\beta^2}\right],$$

where

$$\beta = \frac{1}{2}\left(\frac{1 + \sum_{j=1}^{n} \left|d_j\right|^2 + \sqrt{\left(1 + \sum_{j=1}^{n} \left|d_j\right|^2\right)^2 - 4\sum_{j=1}^{n-1} \left|d_j\right|^2}}{2}\right)^{\frac{1}{2}}.$$

Proof. Partition C_{p_3} as

$$C_{p_3} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix},$$

where

$$C_{11} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \; C_{12} = \begin{bmatrix} d_n & d_{n-1} & \ldots & d_1 \\ 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \end{bmatrix},$$

$$C_{21} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix}, \text{ and } C_{22} = \begin{bmatrix} 0 & 0 & \ldots & 0 & 0 \\ 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \end{bmatrix}.$$

Then by Lemma 2.7, we have

$$\begin{aligned} w\left(C_{p_3}\right) &\leq w\left(\begin{bmatrix} w(C_{11}) & w(T_o) \\ w(T_o) & w(C_{22}) \end{bmatrix}\right) \\ &= \frac{1}{2}\left(w(C_{11}) + w(C_{22}) + \sqrt{(w(C_{11}) - w(C_{22}))^2 + 4w^2(T_o)}\right). \end{aligned}$$

According to Lemma 2.22, we have that

$$w(T_o) = w\left(\begin{bmatrix} 0 & C_{12} \\ C_{21} & 0 \end{bmatrix}\right) = \beta.$$

By using simple computations, we have $w(C_{11}) = \frac{1}{\sqrt{2}}$ and $w(C_{22}) = \cos\frac{\pi}{n+1}$, and so

$$w\left(C_{p_3}\right) \leq \frac{1}{2}\left[\frac{1}{\sqrt{2}} + \cos\frac{\pi}{n+1} + \sqrt{\left(\frac{1}{\sqrt{2}} - \cos\frac{\pi}{n+1}\right)^2 + 4\beta^2}\right].$$

Recalling that $|z| \leq w\left(C_{p_3}\right)$, the result follows. □

The following lemma can be found in [8].

Lemma 2.25. *Let* $T = \begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}$ *with* $X \in \mathbb{M}_{k\times m}(\mathbb{C})$ *and* $Y \in \mathbb{M}_{m\times k}(\mathbb{C})$. *Then*

$$w^4(T) \leq \frac{1}{16}\|P\|^2 + \frac{1}{4}w^2(XY) + \frac{1}{8}w(XYP + PXY),$$

where $P = |X^*|^2 + |Y|^2$.

Theorem 2.26. *If z is any zero of* p_3, *then*

$$|z| \leq \cos\frac{\pi}{n+1} + \frac{1}{2}\left(\alpha^2 + |d_n|^2 + (1+\alpha)|d_n|\right)^{\frac{1}{4}},$$

where $\alpha = \sum_{j=1}^{n} \left|d_j\right|^2$.

Proof. Let

$$C_{11} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \; C_{12} = \begin{bmatrix} d_n & d_{n-1} & \ldots & d_1 \\ 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \end{bmatrix},$$

$$C_{21} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix}, \text{ and } C_{22} = \begin{bmatrix} 0 & 0 & \ldots & 0 & 0 \\ 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \end{bmatrix}.$$

Then

$$C_{p_3} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}.$$

So, by the triangle inequality, we have

$$w\left(C_{p_3}\right) \leq w\left(\begin{bmatrix} C_{11} & 0 \\ 0 & C_{22} \end{bmatrix}\right) + w\left(\begin{bmatrix} 0 & C_{12} \\ C_{21} & 0 \end{bmatrix}\right).$$

Since $w\left(\begin{bmatrix} C_{11} & 0 \\ 0 & C_{22} \end{bmatrix}\right) = \max\{w(C_{11}), w(C_{22})\} = \cos\frac{\pi}{n+1}$, by applying Lemma 2.25, we get

$$w^4\left(\begin{bmatrix} 0 & C_{12} \\ C_{21} & 0 \end{bmatrix}\right) \leq \frac{1}{16}\|P\|^2 + \frac{1}{4}w^2(C_{12}C_{21}) + \frac{1}{8}w(C_{12}C_{21}P + PC_{12}C_{21}),$$

where $P = C_{12}C_{12}^* + C_{21}^*C_{21}$. Since $\|P\| = \left\|C_{12}C_{12}^* + C_{21}^*C_{21}\right\| = \alpha$, and since by Lemma 2.21, we have $w^2(C_{12}C_{21}) = \frac{|d_n|^2}{4}$ and $w(C_{12}C_{21}P + PC_{12}C_{21}) = \frac{(1+\alpha)|d_n|}{2}$, it follows that

$$w\left(C_{p_3}\right) \leq \cos\frac{\pi}{n+1} + \frac{1}{2}\left(\alpha^2 + |d_n|^2 + (1+\alpha)|d_n|\right)^{\frac{1}{4}}.$$

Recalling that $|z| \leq w\left(C_{p_3}\right)$, the result follows. □

The following lemma can be found in [4].

Lemma 2.27. *Let A and $B \in \mathbb{M}_n(\mathbb{C})$. Then*

$$w(A+B) \leq \sqrt{w^2(A) + w^2(B) + \|A\|\|B\| + w(B^*A)}.$$

Theorem 2.28. *If z is any zero of p, then*

$$|z| \leq \frac{\left(\sum_{j=1}^{n}\left|a_j - a_{j-1}\right|^2\right)^{\frac{1}{2}} + |a_n - a_{n-1}|}{2} + \sqrt{\delta^2 + \cos^2\frac{\pi}{n+1} + \sqrt{\sum_{j=1}^{n-1}\left|a_j\right|^2}},$$

where $\delta = \frac{\left(\sum_{j=1}^{n-1}\left|a_j\right|^2\right)^{\frac{1}{2}} + |a_{n-1}|}{2}$.

Proof. Let $A = \begin{bmatrix} a_n & a_{n-1} & \dots & a_1 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix}$. Then $C_p = B - A$. So, by the triangle inequality, we have $w\left(C_p\right) \leq w(A - AB) + w(B - AB)$. By using Lemma 2.21, we have

$$w(A - AB) = \frac{\left(\sum_{j=1}^{n}\left|a_j - a_{j-1}\right|^2\right)^{\frac{1}{2}} + |a_n - a_{n-1}|}{2}.$$

Let $B - AB = L + M$, where $L = \begin{bmatrix} -a_{n-1} & -a_{n-2} & \dots & -a_1 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 \end{bmatrix}$ and $M = \begin{bmatrix} 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix}$. Now, using Lemma 2.27, we get

$$w(B - AB) = w(L + M) \leq \sqrt{w^2(L) + w^2(M) + \|L\|\,\|M\| + w(M^*L)},$$

where $w(L) = \delta$, $w(M) = \cos\frac{\pi}{n+1}$, $\|L\| = \sqrt{\sum_{j=1}^{n-1}\left|a_j\right|^2}$, $\|M\| = 1$, and $w(M^*L) = 0$. Consequently,

$$w\left(C_p\right) \leq \frac{\left(\sum_{j=1}^{n}\left|a_j - a_{j-1}\right|^2\right)^{\frac{1}{2}} + |a_n - a_{n-1}|}{2} + \sqrt{\delta^2 + \cos^2\frac{\pi}{n+1} + \sqrt{\sum_{j=1}^{n-1}\left|a_j\right|^2}},$$

which yields the desired inequality. □

Proposition 2.29. *Let $A, B \in \mathbb{M}_n(\mathbb{C})$. Then*

$$r(A+B) \leq \frac{1}{2}\left(w(A) + w(B) + \sqrt{(w(A) - w(B))^2 + 4w^2\left(\begin{bmatrix} 0 & I \\ BA & 0 \end{bmatrix}\right)}\right).$$

Proof. We have

$$\begin{aligned} r(A+B) &= r\left(\begin{bmatrix} A+B & 0 \\ 0 & 0 \end{bmatrix}\right) \\ &= r\left(\begin{bmatrix} A & I \\ 0 & 0 \end{bmatrix}\begin{bmatrix} I & 0 \\ B & 0 \end{bmatrix}\right) \\ &= r\left(\begin{bmatrix} I & 0 \\ B & 0 \end{bmatrix}\begin{bmatrix} A & I \\ 0 & 0 \end{bmatrix}\right) \\ &= r\left(\begin{bmatrix} A & I \\ BA & B \end{bmatrix}\right) \\ &\leq w\left(\begin{bmatrix} A & I \\ BA & B \end{bmatrix}\right) \\ &= \frac{1}{2}\left(w(A) + w(B) + \sqrt{(w(A) - w(B))^2 + 4w^2\left(\begin{bmatrix} 0 & I \\ BA & 0 \end{bmatrix}\right)}\right) \text{(by Lemma 2.7)}, \end{aligned}$$

as required. □

Theorem 2.30. *If z is any zero of p, then*

$$|z| \leq \frac{1}{2}\left(\cos\frac{\pi}{n+1} + \xi + \sqrt{\left(\cos\frac{\pi}{n+1} - \xi\right)^2 + 4w^2\left(\begin{bmatrix} 0 & I \\ S & 0 \end{bmatrix}\right)}\right),$$

where $\xi = \frac{\left(\sum_{j=1}^{n}|a_j|^2\right)^{\frac{1}{2}} + |a_n|}{2}$, and $S = \begin{bmatrix} -a_{n-1} & -a_{n-2} & \dots & -a_1 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 \end{bmatrix}$.

Proof. Let $M = \begin{bmatrix} 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix}$ and $N = \begin{bmatrix} -a_n & -a_{n-1} & \dots & -a_2 & -a_1 \\ 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 \end{bmatrix}$. Then $C_p = M + N$. By using Proposition 2.29, we have

$$r\left(C_p\right) \le \frac{1}{2}\left(w(M) + w(N) + \sqrt{(w(M) - w(N))^2 + 4w^2\left(\begin{bmatrix} 0 & I \\ NM & 0 \end{bmatrix}\right)}\right).$$

By using Lemma 2.5, we have $w(M) = \cos\frac{\pi}{n+1}$, and applying Lemma 2.21, we have

$$w(N) = \xi,$$

and so

$$r\left(C_p\right) \le \frac{1}{2}\left(\cos\frac{\pi}{n+1} + \xi + \sqrt{\left(\cos\frac{\pi}{n+1} - \xi\right)^2 + 4w^2\left(\begin{bmatrix} 0 & I \\ NM & 0 \end{bmatrix}\right)}\right),$$

where $NM = S$. Consequently,

$$|z| \le \frac{1}{2}\left(\cos\frac{\pi}{n+1} + \xi + \sqrt{\left(\cos\frac{\pi}{n+1} - \xi\right)^2 + 4w^2\left(\begin{bmatrix} 0 & I \\ S & 0 \end{bmatrix}\right)}\right).$$

Recalling that $|z| \le r\left(C_p\right)$, the result follows. □

Remark 2.31. *It is well-known that if $T \in \mathbb{M}_n(\mathbb{C})$ is nilpotent of index 2, i.e., if $T^2 = 0$, then*

$$w(T) = \frac{1}{2}\|T\|.$$

An estimate for the numerical radius of a nilpotent matrix has been given by Haagerup and de. la Harpe [12]. This says that if $T \in \mathbb{M}_n(\mathbb{C})$ such that $T^k = 0$ for some $k \ge 1$, then

$$w(T) \le \|T\|\cos\frac{\pi}{k+1}.$$

Using this result of Haagerup and de. la Harpe, we have the following estimates for the numerical radii of 2×2 off diagonal block matrices with certain conditions.

Proposition 2.32. *Let $A \in \mathbb{M}_{k\times m}(\mathbb{C})$, $B \in \mathbb{M}_{m\times k}(\mathbb{C})$ such that $AB = 0$. If $T = \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}$, then*

$$w(T) \le \frac{\max\{\|A\|, \|B\|\}}{\sqrt{2}}.$$

Proof. Since $T^3 = \begin{bmatrix} 0 & ABA \\ BAB & 0 \end{bmatrix} = 0$, it follows that

$$\begin{aligned} w(T) &\le \|T\|\cos\frac{\pi}{4} \\ &= \frac{\max\{\|A\|, \|B\|\}}{\sqrt{2}}, \end{aligned}$$

as required. □

Proposition 2.33. *Let $A \in \mathbb{M}_{k\times m}(\mathbb{C})$, $B \in \mathbb{M}_{m\times k}(\mathbb{C})$ such that $AB = BA = 0$. If $T = \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}$, then*

$$w(T) = \frac{1}{2}\max\{\|A\|, \|B\|\}.$$

Proof. We have $T^2 = \begin{bmatrix} AB & 0 \\ 0 & BA \end{bmatrix} = 0$. Then $w(T) = \frac{1}{2}\|T\| = \frac{1}{2}\max\{\|A\|, \|B\|\}$, as required. □

Theorem 2.34. *If z is any zero of p_3, then*

$$|z| \le \frac{1}{2}\left[\max\left\{\sqrt{\sum_{j=1}^{n-1}\left|d_j\right|^2}, 1\right\} + \frac{1}{\sqrt{2}} + \cos\frac{\pi}{n+1} + \sqrt{\left(\frac{1}{\sqrt{2}} - \cos\frac{\pi}{n+1}\right)^2 + |d_n|^2}\right].$$

Proof. Let $L = \begin{bmatrix} 0 & 0 & 0 & 0 & d_{n-1} & d_{n-2} & \dots & d_2 & d_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}$ and $N = \begin{bmatrix} 0 & 0 & 0 & d_n & 0 & \dots & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & 1 & 0 \end{bmatrix}$. Then $C_{p_3} = L + N$. So, by the triangle inequality, we have $w\left(C_{p_3}\right) \le w(L) + w(N)$. By applying Proposition 2.33 to L, partitioned as

$$L = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix},$$

where

$$L_{11} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \; L_{12} = \begin{bmatrix} 0 & d_{n-1} & \dots & d_1 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \end{bmatrix},$$

$$L_{21} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix}, \text{ and } L_{22} = \begin{bmatrix} 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 \end{bmatrix},$$

we have

$$\begin{aligned} w(L) &= w\left(\begin{bmatrix} 0 & L_{12} \\ L_{21} & 0 \end{bmatrix}\right) \\ &= \frac{1}{2}\max\{\|L_{12}\|, \|L_{21}\|\}. \\ &= \frac{1}{2}\max\left\{\sqrt{\sum_{j=1}^{n-1}\left|d_j\right|^2}, 1\right\}. \end{aligned}$$

Also, by applying Lemma 2.7 to N, partitioned as

$$N = \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix},$$

where

$$N_{11} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, N_{12} = \begin{bmatrix} d_n & 0 & ... & 0 \\ 0 & 0 & ... & 0 \\ 0 & 0 & ... & 0 \end{bmatrix},$$

$$N_{21} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix}, \text{ and } N_{22} = \begin{bmatrix} 0 & 0 & ... & 0 & 0 \\ 1 & 0 & ... & 0 & 0 \\ 0 & 1 & ... & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & ... & 1 & 0 \end{bmatrix},$$

we have

$$\begin{aligned} w(N) &\leq w\left(\begin{bmatrix} w(N_{11}) & w(T_o) \\ w(T_o) & w(N_{22}) \end{bmatrix}\right) \\ &= \frac{1}{2}\left(w(N_{11}) + w(N_{22}) + \sqrt{(w(N_{11}) - w(N_{22}))^2 + 4w^2(T_o)}\right). \end{aligned}$$

By using Lemma 2.5, we have $w(N_{11}) = \frac{1}{\sqrt{2}}$ and $w(N_{22}) = \cos\frac{\pi}{n+1}$. Since $w^2(T_o) = w^2\left(\begin{bmatrix} 0 & N_{12} \\ 0 & 0 \end{bmatrix}\right) = \frac{1}{4}\|N_{12}\|^2 = \frac{1}{4}|d_n|^2$, it follows that

$$w(N) \leq \frac{1}{2}\left[\frac{1}{\sqrt{2}} + \cos\frac{\pi}{n+1} + \sqrt{\left(\frac{1}{\sqrt{2}} - \cos\frac{\pi}{n+1}\right)^2 + |d_n|^2}\right].$$

Consequently,

$$w\left(C_{p_3}\right) \leq \frac{1}{2}\left[\max\left\{\sqrt{\sum_{j=1}^{n-1}\left|d_j\right|^2}, 1\right\} + \frac{1}{\sqrt{2}} + \cos\frac{\pi}{n+1} + \sqrt{\left(\frac{1}{\sqrt{2}} - \cos\frac{\pi}{n+1}\right)^2 + |d_n|^2}\right].$$

Now, the desired bound follows from the fact $|z| \leq w\left(C_{p_3}\right)$. □

Finally, we remark that lower bound counterparts of the upper bounds obtained in this paper can be derived by considering the polynomial $\frac{z^n}{a_1}p\left(\frac{1}{z}\right)$ whose zeros are the reciprocals of those of p. This enables us to describe annuli in the complex plane containing all the zeros of p. Moreover, for $k < n$, compression matrix inequalities may be applied to C_p^k in order to obtain further bounds for the zeros of p. Thus, by the spectral mapping theorem and the inequalities (1), if z is any zero of p, then $|z| \leq \left(w\left(C_p^k\right)\right)^{\frac{1}{k}} \leq \left\|C_p^k\right\|^{\frac{1}{k}}$.

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