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Polar Decomposition and Characterization of Binormal Operators

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Abstract. We illustrate the matrix representation of the closed range operator that enables us to determine the polar decomposition with respect to the orthogonal complemented submodules. This result proves that the reverse order law for the Moore–Penrose inverse of operators holds. Also, it is given some new characterizations of the binormal operators via the generalized Aluthge transformation. New characterizations of the binormal operators enable us to obtain equivalent conditions when the inner product of the binormal operator with its generalized Aluthge transformation is positive in the general setting of adjointable operators on Hilbert C^*-modules.

1. Introduction and preliminaries

Hilbert C^*-modules are generalizations of both C^*-algebras and Hilbert spaces, which possess some new phenomena compared with that of Hilbert spaces. For instance, a closed submodule of a Hilbert C^*-module may fail to be orthogonally complemented [15, p. 7] and an adjointable operator from one Hilbert C^*-module to another may have no polar decomposition [24, Theorem 15.3.7].

A significant progress has been made in the study of the polar decomposition and its applications both for Hilbert space operators [7, 9–11, 23], and for adjointable operators on Hilbert C^*-modules [5, 8, 17, 20, 24].

One application of the polar decomposition is the study of the closed range operator, which its useful transformation and application were introduced, for example, in [14].

Another application of the polar decomposition is the study of the binormal operator, which was defined by Campbell in [3] and some properties of binormal operators were shown in [4, 12].

The study of Moore–Penrose inverse and Aluthge transformation of binormal operators has a long history.

Aluthge [1] defined a transformation $T_{\frac{1}{2},\frac{1}{2}} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$, which was later called the *Aluthge transformation*. The Aluthge transformation is very useful, and many authors have obtained results by using it. The Aluthge transformation of binormal operators is an attractive and important problem, which appears in the operator theory. Hence, authors are mainly focused on binormal operators, for example, [12, 13, 17]. Yamazaki [26] introduced the notion of the $*$-Aluthge transform $T^{(*)}_{\frac{1}{2},\frac{1}{2}}$ of T by setting $T^{(*)}_{\frac{1}{2},\frac{1}{2}} = |T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}}$. These concepts were also generalized as follows: For every $\alpha, \beta > 0$, the generalized Aluthge transformation $T_{\alpha,\beta}$ is defined by $T_{\alpha,\beta} = |T|^{\alpha}U|T|^{\beta}$ and the generalized $*$-Aluthge transformation $T^{(*)}_{\alpha,\beta}$ is defined by $T^{(*)}_{\alpha,\beta} = |T^*|^{\alpha}U|T^*|^{\beta}$; see [6, 22].

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In this paper, the block operator matrix is illustrated that enables us to determine a part of the polar decomposition of closed range operators concerning the orthogonal complemented submodules. These results prove that the reverse order law for the Moore–Penrose inverse of operators holds. Also, by applying the matrix representation in binormal operators with closed range, we obtain new results.

Furthermore, we provide conditions under which the polar decomposition is guaranteed in Hilbert C^*-modules. Using this fact and related results lead to some new characterizations of binormal operators via generalized Aluthge transformation, $*$-generalized Aluthge transformation and satisfy the triangle equality. New characterizations of the binormal operators are provided to get equivalent conditions for the inner product of binormal operators when its generalized Aluthge transformation is positive.

The paper is organized as follows. We establish several properties concerning commuting pieces and real powers of certain operators constructed from other adjointable operators on Hilbert C^*-modules. The basic idea is the decomposition of an operator T into a 2×2 operator matrix as in Section 2. In Section 3, this is applied to MP-invertible operators. Section 4 concerns binormal MP-invertible operators. Section 5 contains a sufficient condition for the existence of the polar decomposition and again some results on commuting pieces and real powers. In Section 6, several results are established under the binormality assumption for the Aluthge transformation and its generalizations.
Let us fix our notation and terminology.

For an inner-product module over a C^*-algebra, $\mathfrak{A}$ is a right $\mathfrak{A}$-module equipped with an $\mathfrak{A}$-valued inner product $\langle \cdot, \cdot \rangle : \mathcal{X} \times \mathcal{X} \to \mathfrak{A}$. If $\mathcal{X}$ is complete with respect to the induced norm defined by $\|x\| = \|\langle x, x\rangle\|^{\frac{1}{2}}$ $(x \in \mathcal{X})$, then $\mathcal{X}$ is called a *Hilbert $\mathfrak{A}$-module*.

More precisely, inner product C^*-modules are generalizations of inner product spaces by allowing inner products to take values in some C^*-algebras instead of the field of complex numbers. Every C^* -algebra can be regarded as a Hilbert C^* -module over itself, where the inner product is defined by $\langle a, b\rangle := a^*b$. Furthermore, if x is an element of a Hilbert $\mathcal{A}$ -module $\mathcal{X}$, then $|x| \in \mathcal{A}$ denotes the unique positive square root of $\langle x, x\rangle$. In the case of a C^* -algebra, we get the usual $|a| = (a^*a)^{\frac{1}{2}}$.

Throughout the rest of this paper, $\mathfrak{A}$ denotes a C^*-algebra and $\mathcal{X}$ and $\mathcal{Y}$ are Hilbert $\mathfrak{A}$-modules. Let $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ be the set of operators $T : \mathcal{X} \to \mathcal{Y}$ for which there is an operator $T^* : \mathcal{Y} \to \mathcal{X}$ such that $\langle Tx, y\rangle = \langle x, T^*y\rangle$ for any $x \in \mathcal{X}$ and $y \in \mathcal{Y}$. It is known that any element $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ must be bounded and $\mathfrak{A}$-linear. We call $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ the set of adjointable operators from $\mathcal{X}$ to $\mathcal{Y}$. For any $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, the range and the null spaces of T are represented by $\mathcal{R}(T)$ and $\mathcal{N}(T)$, respectively. In the case $\mathcal{X} = \mathcal{Y}$, the space $\mathcal{L}(\mathcal{X}, \mathcal{X})$, which is abbreviated to $\mathcal{L}(\mathcal{X})$, is a C^*-algebra.

A closed submodule M of $\mathcal{X}$ is said to be *orthogonally complemented* if $\mathcal{X} = M \oplus M^\perp$, where $M^\perp = \Big\{x \in \mathcal{X} : \langle x, y\rangle = 0 \text{ for any } y \in M\Big\}$. In this case, the projection from $\mathcal{X}$ onto M is denoted by P_M. If $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ does not have a closed range, then neither $\mathcal{N}(T)$ nor $\overline{\mathcal{R}(T)}$ needs to be orthogonally complemented. In addition, if $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ and $\overline{\mathcal{R}(T^*)}$ is not orthogonally complemented, then it may happen that $\mathcal{N}(T)^\perp \neq \overline{\mathcal{R}(T^*)}$; see [15, 19]. The above facts show that the theory of Hilbert C^*-modules is much different and more complicated than that of Hilbert spaces.

The polar decomposition of $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ can be characterized as

$$T = U|T| \text{ and } U^*U = P_{\overline{\mathcal{R}(T^*)}}$$

where $U \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ is a partial isometry [16, Definition 3.10]. Also, orthogonality complemented of submodules $\overline{\mathcal{R}(T)}$ and $\overline{\mathcal{R}(T^*)}$ insure uniqueness of the polar decomposition. In this case, $T^* = U^*|T^*|$ is the polar decomposition of T^*; see [16, Remark 3.11].

The Moore–Penrose inverse of $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ is denoted by $T^\dagger$, which is the unique element $X \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$ satisfying

$$TXT = T,\ XTX = X,\ (TX)^* = TX, \text{ and } (XT)^* = XT.$$

If such a $T^\dagger$ exists, then T is said to be Moore–Penrose invertible (In brief, MP-invertible). Xu and Sheng [25, Theorem 2.2] showed that, for any $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, T is MP-invertible if and only if $\mathcal{R}(T)$ is closed.

The term orthogonal projection will be reserved for T, which is self-adjoint and idempotent. From the definition of MP-invertible, it can be proved that the MP-invertible of an operator (if it exists) is unique and that $T^\dagger T$ and $TT^\dagger$ are orthogonal projections into $\mathcal{R}(T^*)$ and $\mathcal{R}(T)$, respectively. Clearly, T is MP-invertible if and only if T^* is MP-invertible [15, Theorem 3.2], and in this case $(T^*)^\dagger = (T^\dagger)^*$, $(TT^*)^\dagger = (T^*)^\dagger T^\dagger$, $T^* = T^*TT^\dagger$, and $T^\dagger = T^*(TT^*)^\dagger$.

An operator T is said to be binormal or weakly centered [21], if $|T|$ and $|T^*|$ are commutative.

Throughout of this paper, parameters $\alpha, \beta, \alpha', \beta'$, r, and s are positive real numbers.

2. Matrix representation of parts of polar decomposition

The useful block operator matrix is illustrated that enables us to determinate the part of polar decomposition of MP-invertible operator with respect to the orthogonal sums of submodules.

Let us begin with auxiliary lemmas.

Lemma 2.1. *[20, Theorem 2.1] Let $T \in \mathcal{L}(\mathcal{X})$ be MP-invertible and let $T = U|T|$ be a polar decomposition of T. Then $U = T|T|^\dagger$.*

Lemma 2.2. *If $S = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} \in \mathcal{L}(\mathcal{X} \oplus \mathcal{X})$ and $W = AA^* + BB^* \in \mathcal{L}(\mathcal{X})$ is invertible, then*

$$(S^*S)^{\frac{1}{2^n}} = \left(\begin{bmatrix} A^*A & A^*B \\ B^*A & B^*B \end{bmatrix} \right)^{\frac{1}{2^n}} = \begin{bmatrix} A^*W^{-\frac{1}{2^n}}A & A^*W^{-\frac{1}{2^n}}B \\ B^*W^{-\frac{1}{2^n}}A & B^*W^{-\frac{1}{2^n}}B \end{bmatrix}. \tag{1}$$

Proof. Letting $n = 1$, we have

$$\begin{aligned} &\begin{bmatrix} A^*W^{-\frac{1}{2}}A & A^*W^{-\frac{1}{2}}B \\ B^*W^{-\frac{1}{2}}A & B^*W^{-\frac{1}{2}}B \end{bmatrix} \begin{bmatrix} A^*W^{-\frac{1}{2}}A & A^*W^{-\frac{1}{2}}B \\ B^*W^{-\frac{1}{2}}A & B^*W^{-\frac{1}{2}}B \end{bmatrix} \\ &= \begin{bmatrix} A^*W^{-\frac{1}{2}}(AA^*+BB^*)W^{-\frac{1}{2}}A & A^*W^{-\frac{1}{2}}(AA^*+BB^*)W^{-\frac{1}{2}}B \\ B^*W^{-\frac{1}{2}}(AA^*+BB^*)W^{-\frac{1}{2}}A & B^*W^{-\frac{1}{2}}(AA^*+BB^*)W^{-\frac{1}{2}}B \end{bmatrix} \\ &= \begin{bmatrix} A^*A & A^*B \\ B^*A & B^*B \end{bmatrix}. \end{aligned}$$

Since S^*S is positive, then it has the square root. Hence $|S| = \left(\begin{bmatrix} A^*A & A^*B \\ B^*A & B^*B \end{bmatrix} \right)^{\frac{1}{2}} = \begin{bmatrix} A^*W^{-\frac{1}{2}}A & A^*W^{-\frac{1}{2}}B \\ B^*W^{-\frac{1}{2}}A & B^*W^{-\frac{1}{2}}B \end{bmatrix}$. By induction, the desired result follows. □

Lemma 2.3. *If $S = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} \in \mathcal{L}(\mathcal{X} \oplus \mathcal{X})$ and $W = AA^* + BB^* \in \mathcal{L}(\mathcal{X})$ is invertible, then S is MP-invertible and*

$$(S^*S)^\dagger = \begin{bmatrix} A^*A & A^*B \\ B^*A & B^*B \end{bmatrix}^\dagger = \begin{bmatrix} A^*W^{-2}A & A^*W^{-2}B \\ B^*W^{-2}A & B^*W^{-2}B \end{bmatrix}.$$

Proof. Letting $S^\dagger = \begin{bmatrix} A^*W^{-1} & 0 \\ B^*W^{-1} & 0 \end{bmatrix}$ ensures that S has a closed range. Since the reverse order law holds for the product of operators S and S^*. Hence, we conclude that

$$(S^*S)^\dagger = S^\dagger(S^*)^\dagger = \begin{bmatrix} A^*W^{-1} & 0 \\ B^*W^{-1} & 0 \end{bmatrix} \begin{bmatrix} W^{-1}A & W^{-1}B \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} A^*W^{-2}A & A^*W^{-2}B \\ B^*W^{-2}A & B^*W^{-2}B \end{bmatrix}.$$

□

Theorem 2.4. *Suppose that $\mathcal{X}$ is a Hilbert $\mathcal{A}$-module, that $T \in \mathcal{L}(\mathcal{X})$ is MP-invertible, and that $T = U|T|$ is the polar decomposition of T. Then the polar decompositions of T can be decomposed as follows with respect to the orthogonal complemented submodules:*

(i) *If* $T = \begin{bmatrix} T_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(T^*) \\ \mathcal{N}(T) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(T) \\ \mathcal{N}(T^*) \end{bmatrix}$, *then*

$$T = U|T| = \begin{bmatrix} T_1 D^{-\frac{1}{2}} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} D^{\frac{1}{2}} & 0 \\ 0 & 0 \end{bmatrix},$$

where $D = T_1^* T_1$ *is positive and invertible. The first matrix of the last equation coincides with U and the second one coincides with* $|T|$.

(ii) *If* $T = \begin{bmatrix} T_1 & T_2 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(T) \\ \mathcal{N}(T^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(T) \\ \mathcal{N}(T^*) \end{bmatrix}$, *then*

$$T = U|T| = \begin{bmatrix} D^{-\frac{1}{2}} T_1 & D^{-\frac{1}{2}} T_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T_1^* D^{-\frac{1}{2}} T_1 & T_1^* D^{-\frac{1}{2}} T_2 \\ T_2^* D^{-\frac{1}{2}} T_1 & T_2^* D^{-\frac{1}{2}} T_2 \end{bmatrix},$$

where $D = T_1 T_1^* + T_2 T_2^*$ *is positive and invertible. The first matrix of the last equation coincides with U and the second one coincides with* $|T|$.

(iii) *If* $T = \begin{bmatrix} T_1 & 0 \\ T_2 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(T^*) \\ \mathcal{N}(T) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(T^*) \\ \mathcal{N}(T) \end{bmatrix}$, *then*

$$T = U|T| = \begin{bmatrix} T_1 D^{-\frac{1}{2}} & 0 \\ T_2 D^{-\frac{1}{2}} & 0 \end{bmatrix} \begin{bmatrix} D^{\frac{1}{2}} & 0 \\ 0 & 0 \end{bmatrix},$$

where $D = T_1^* T_1 + T_2^* T_2$ *is positive and invertible. The first matrix of the last equation coincides with U and the second one coincides with* $|T|$.

Proof. (i) Since $|T| = \begin{bmatrix} T_1^* T_1 & 0 \\ 0 & 0 \end{bmatrix}^{\frac{1}{2}} = \begin{bmatrix} D^{\frac{1}{2}} & 0 \\ 0 & 0 \end{bmatrix}$, Lemma 2.1 implies that $U = T|T|^{\dagger} = \begin{bmatrix} T_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} D^{\frac{1}{2}} & 0 \\ 0 & 0 \end{bmatrix}^{\dagger} = \begin{bmatrix} T_1 D^{-\frac{1}{2}} & 0 \\ 0 & 0 \end{bmatrix}$.

(ii) By virtue of Lemma 2.2, we conclude that

$$|T| = \begin{bmatrix} T_1^* T_1 & T_1^* T_2 \\ T_2^* T_1 & T_2^* T_2 \end{bmatrix}^{\frac{1}{2}} = \begin{bmatrix} T_1^* D^{-\frac{1}{2}} T_1 & T_1^* D^{-\frac{1}{2}} T_2 \\ T_2^* D^{-\frac{1}{2}} T_1 & T_2^* D^{-\frac{1}{2}} T_2 \end{bmatrix},$$

where $D = T_1 T_1^* + T_2 T_2^* \in \mathcal{L}(\mathcal{R}(T))$ is positive and invertible and then Lemmas 2.1 and 2.3 immediately lead to

$$\begin{aligned} U &= T|T|^{\dagger} = \begin{bmatrix} T_1 & T_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T_1^* D^{-\frac{1}{2}} T_1 & T_1^* D^{-\frac{1}{2}} T_2 \\ T_2^* D^{-\frac{1}{2}} T_1 & T_2^* D^{-\frac{1}{2}} T_2 \end{bmatrix}^{\dagger} \\ &= \begin{bmatrix} T_1 & T_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T_1^* D^{-\frac{3}{2}} T_1 & T_1^* D^{-\frac{3}{2}} T_2 \\ T_2^* D^{-\frac{3}{2}} T_1 & T_2^* D^{-\frac{3}{2}} T_2 \end{bmatrix} \\ &= \begin{bmatrix} D^{-\frac{1}{2}} T_1 & D^{-\frac{1}{2}} T_2 \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

(iii) Since $D = T_1^* T_1 + T_2^* T_2 \in \mathcal{L}(\mathcal{R}(T^*))$ is positive and invertible, then $|T| = \begin{bmatrix} D^{\frac{1}{2}} & 0 \\ 0 & 0 \end{bmatrix}$ and $|T|^{\dagger} = \begin{bmatrix} D^{-\frac{1}{2}} & 0 \\ 0 & 0 \end{bmatrix}$. Indeed, by applying Lemma 2.1, we have

$$U = T|T|^{\dagger} = \begin{bmatrix} T_1 D^{-\frac{1}{2}} & 0 \\ T_2 D^{-\frac{1}{2}} & 0 \end{bmatrix}.$$

□

3. Application of Theorem 2.4 to the reverse order law

Lemma 3.1. *Let $T \in \mathcal{L}(\mathcal{X})$ be MP-invertible . Then*

(i) $(|T|^\dagger)^\alpha = (|T|^\alpha)^\dagger = |(T^\dagger)^*|^\alpha$;
(ii) $(|T^*|^\dagger)^\alpha = (|T^*|^\alpha)^\dagger = |T^\dagger|^\alpha$.

Proof. (i) If $T = \begin{bmatrix} T_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(T^*) \\ \mathcal{N}(T) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(T) \\ \mathcal{N}(T^*) \end{bmatrix}$, then the invertibility of T_1 implies that

$$\begin{aligned}
(|T|^\dagger)^\alpha &= \left(\begin{bmatrix} \left(T_1^*T_1\right)^{\frac{1}{2}} & 0 \\ 0 & 0 \end{bmatrix}^\dagger\right)^\alpha = \begin{bmatrix} \left(T_1^*T_1\right)^{-\frac{1}{2}} & 0 \\ 0 & 0 \end{bmatrix}^\alpha \\
&= \begin{bmatrix} \left(T_1^*T_1\right)^{\frac{\alpha}{2}} & 0 \\ 0 & 0 \end{bmatrix}^\dagger = \begin{bmatrix} \left(T_1^{-1}(T_1^*)^{-1}\right)^{\frac{1}{2}} & 0 \\ 0 & 0 \end{bmatrix}^\alpha.
\end{aligned}$$

The last matrix of equation coincides with $|(T^\dagger)^*|^\alpha$ and the previous one coincides with $(|T|^\alpha)^\dagger$. Then the desired result holds.

(ii) It is evident. □

Theorem 3.2. *Let $T \in \mathcal{L}(\mathcal{X})$ be MP-invertible and let $T = U|T|$ be the polar decomposition of T. Then the following properties hold:*

(i) $(T|T|^\alpha)^\dagger = (|T|^\alpha)^\dagger T^\dagger$;
(ii) $(|T^*|^\alpha T)^\dagger = T^\dagger(|T^*|^\alpha)^\dagger$;
(iii) $(|T^*|^\beta T|T|^\alpha)^\dagger = (|T|^\alpha)^\dagger T^\dagger(|T^*|^\beta)^\dagger$;
(iv) $(|T|^\alpha T^*)^\dagger = (T^*)^\dagger(|T|^\alpha)^\dagger$;
(v) $(T|T|^\alpha T^*)^\dagger = (T^*)^\dagger(|T|^\alpha)^\dagger T^\dagger$;
(vi) $(U|T|^\alpha)^\dagger = (|T|^\alpha)^\dagger U^*$;
(vii) $(|T^*|^\alpha U)^\dagger = U^*(|T^*|^\alpha)^\dagger$;
(viii) $(|T^*|^\beta U|T|^\alpha)^\dagger = (|T|^\alpha)^\dagger U^*(|T^*|^\beta)^\dagger$;

Proof. Applying the matrix decomposition of T in the implication (i) of Theorem 2.4 and using Lemma 3.1 conclude these results. □

4. Moore–Penrose inverse of binormal operators

In the section, we present the results related to the reverse order law for the MP inverse of the MP-invertible binormal operators.
We recall that MP-invertible T is called EP, if T commutes with $T^\dagger$.

Let us begin with auxiliary propositions.

Proposition 4.1. *Let $\mathfrak{A}$ be a C^*-algebra and let $a, b \in \mathfrak{A}$ be such that a and b are commutative self adjoint elements. Then for all $\alpha, \beta \in \mathbb{R}^+$, $f(a^\alpha)$ and $g(b^\beta)$ commute, whenever f and g are continuous complex-valued functions on the interval $[-\|a\|, \|a\|]$ and $[-\|b\|, \|b\|]$, respectively.*

Proof. Since a is a self adjoint element of a C^*-algebra $\mathfrak{A}$, then for all $\alpha \in \mathbb{R}^+$, a^α is the norm limit of $p_n(a)$, where $p_n(a)$ is a sequence of polynomials without constant terms. Choose any sequence $\{q_n\}_{n=1}^\infty$ of polynomials such that $q_n(t) \to f(t)$ uniformly on $sp(a)$ to a continuous function f on an interval containing $sp(a)$ and 0, so that $f(0) = 0$. Then for all $\alpha \in \mathbb{R}$, we have $\|q_n(p_n(a)) - f(a^\alpha)\| \to 0$ as $n \to \infty$; hence

$$f(a^\alpha)b = \lim_{n\to\infty} q_n(p_n(a))b = \lim_{n\to\infty} b\, q_n(p_n(a)) = bf(a^\alpha).$$

Since $f(a^\alpha)$ commutes with b, whenever f is a continuous complex-valued function on the interval $[-\|a\|, \|a\|]$. Then, by the previous argument, $f(a^\alpha)$ and $g(b^\beta)$ commute, whenever g is a continuous complex-valued function on the interval $[-\|b\|, \|b\|]$. □

Proposition 4.2. *Let $T, S \in \mathcal{L}(\mathcal{X})$ be self adjoint such that T and S are MP-invertible operators and $TS = ST$. Then*

(i) $(TS)^\dagger = S^\dagger T^\dagger = T^\dagger S^\dagger$;
(ii) $TS^\dagger = S^\dagger T$ *and* $T^\dagger S = S^\dagger T$.

Proof. The block matrix forms of T and S immediately imply them. □

Remark 4.3. *As an application of Proposition 4.1, T is a binormal operator if and only if for all $\alpha, \beta > 0$, $|T|^\alpha$ and $|T^*|^\beta$ commute, if and only if there exist $\alpha > 0$ and $\beta > 0$ such that $|T|^\alpha$ and $|T^*|^\beta$ commute.*

The following theorem proves some results related with binormal operator.

Theorem 4.4. *Suppose that $T \in \mathcal{L}(\mathcal{X})$ is a binormal MP-invertible operator. Then the following statements are valid:*

(i) *T is a binormal MP-invertible operator iff $T^\dagger$ is a binormal operator;*
(ii) *$|T|$ is an EP operator;*
(iii) $(|T|^\alpha |T^*|^\beta)^\dagger = (|T^*|^\beta |T|^\alpha)^\dagger = (|T^*|^\beta)^\dagger (|T|^\alpha)^\dagger = (|T|^\alpha)^\dagger (|T^*|^\beta)^\dagger$;
(iv) $(|T|^\alpha (|T^*|^\dagger)^\beta)^\dagger = ((|T^*|^\dagger)^\beta)^\dagger |T|^\alpha)^\dagger = |T^*|^\beta (|T|^\alpha)^\dagger = (|T|^\alpha)^\dagger |T^*|^\beta$;
(v) $(|T|^\alpha |T^\dagger|^\beta)^\dagger = (|T^\dagger|^\beta)^\dagger (|T|^\alpha)^\dagger = (|T|^\alpha)^\dagger (|T^\dagger|^\beta)^\dagger$;
(vi) $T|T|^\alpha = |T^*|^\alpha T$;
(vii) $T = |T^*|^\alpha T(|T|^\alpha)^\dagger$;
(viii) $T = (|T^*|^\alpha)^\dagger T|T|^\alpha$.

Proof. Since T is a binormal operator with closed range. Then decomposition of operator T in Theorem 2.4 part (ii) implies that

$$\begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}^{\frac{1}{2}} \begin{bmatrix} T_1^*T_1 & T_1^*T_2 \\ T_2^*T_1 & T_2^*T_2 \end{bmatrix}^{\frac{1}{2}} = \begin{bmatrix} T_1^*T_1 & T_1^*T_2 \\ T_2^*T_1 & T_2^*T_2 \end{bmatrix}^{\frac{1}{2}} \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}^{\frac{1}{2}}. \tag{2}$$

Lemma 2.2 concludes that

$$\begin{bmatrix} D^{\frac{1}{2}} T_1^* D^{-\frac{1}{2}} T_1 & D^{\frac{1}{2}} T_1^* D^{-\frac{1}{2}} T_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} T_1^* D^{\frac{-1}{2}} T_1 D^{\frac{1}{2}} & 0 \\ T_2^* D^{\frac{-1}{2}} T_1 D^{\frac{1}{2}} & 0 \end{bmatrix}.$$

The invertibility of D implies that $T_1^* D^{-\frac{1}{2}} T_2 = 0$. Taking the $*$-operation, we get $T_2^* D^{-\frac{1}{2}} T_1 = 0$. Therefore,

$$\begin{aligned} T &= U|T| \\ &= \begin{bmatrix} T_1 & T_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} D^{-\frac{1}{2}} T_1 & D^{-\frac{1}{2}} T_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T_1^* D^{-\frac{1}{2}} T_1 & 0 \\ 0 & T_2^* D^{-\frac{1}{2}} T_2 \end{bmatrix} \\ &= \begin{bmatrix} D^{-\frac{1}{2}} T_1 T_1^* D^{\frac{-1}{2}} T_1 & D^{-\frac{1}{2}} T_2 T_2^* D^{-\frac{1}{2}} T_2 \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

Then

$$T_1 = D^{-\frac{1}{2}} T_1 T_1^* D^{-\frac{1}{2}} T_1 \tag{3}$$

$$T_2 = D^{-\frac{1}{2}} T_2 T_2^* D^{-\frac{1}{2}} T_2. \tag{4}$$

By multiplying T_2^* on the left of equation (3), we have

$$T_2^* T_1 = T_2^* D^{-\frac{1}{2}} T_1 T_1^* D^{-\frac{1}{2}} T_1 = 0. \tag{5}$$

By (5), equation (2) get into

$$\begin{bmatrix} D^{\frac{1}{2}} (T_1^* T_1)^{\frac{1}{2}} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} (T_1^* T_1)^{\frac{1}{2}} D^{\frac{1}{2}} & 0 \\ 0 & 0 \end{bmatrix}. \tag{6}$$

(i) Equation (6) implies that $|T_1^*|$ and $|T_1|$ commute. Applying Lemma 4.2 concludes that

$$\begin{aligned}(|T_1^*||T_1|)^\dagger &= (|T_1|)^\dagger|T_1^*|)^\dagger = (|T_1^*|)^\dagger(|T_1|)^\dagger &(7)\\ &= |(T_1^\dagger)^*|\,|T_1^\dagger| = |T_1^\dagger|\,|(T_1^\dagger)^*|. \qquad \text{By Lemma 3.1} &(8)\end{aligned}$$

Then

$$\begin{bmatrix} ((T_1^*T_1)^{\frac{1}{2}})^\dagger D^{-\frac{1}{2}} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} D^{-\frac{1}{2}}((T_1^*T_1)^{\frac{1}{2}})^\dagger & 0 \\ 0 & 0 \end{bmatrix}. \tag{9}$$

Therefore $(|T||T^*|)^\dagger = (|T^*||T|)^\dagger = |T^\dagger||(T^\dagger)^*| = |(T^\dagger)^*||T^\dagger|$. Two terms of the last equation lead to that $T^\dagger$ is a binormal operator.

Conversely, since $(T^\dagger)^\dagger = T$ replaces T with $T^\dagger$, then the desired result follows.

(ii) Equation (6) implies that

$$\begin{bmatrix} (T_1^*T_1)^{\frac{1}{2}} D^{-\frac{1}{2}} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} D^{-\frac{1}{2}}(T_1^*T_1)^{\frac{1}{2}} & 0 \\ 0 & 0 \end{bmatrix}, \tag{10}$$

which is equivalently $|T||(T^*)^\dagger| = |(T^*)^\dagger||T|$. Lemma 3.1 yields that $|T||T|^\dagger = |T|^\dagger|T|$. That is, $|T|$ is an EP operator.

(iii) Equation (6) implies that $|T_1^*|$ and $|T_1|$ commute. It is enough in Proposition 4.1 we let $f(x) = g(x) = x$. It concludes that $(|T|^\alpha|T^*|^\beta)^\dagger = (|T^*|^\beta|T|^\alpha)^\dagger = (|T^*|^\beta)^\dagger(|T|^\alpha)^\dagger = (|T|^\alpha)^\dagger(|T^*|^\beta)^\dagger$.

(iv) Equation (6) implies that $|T_1|\,|T_1^*| = |T_1^*|\,|T_1|$. Proposition 4.1 implies that $|T_1|^\alpha\,|T_1^*|^\beta = |T_1^*|^\beta\,|T_1|^\alpha$. By Proposition 4.2, the implication (ii) concludes that $|T_1|^\alpha\,(|T_1^*|^\beta)^\dagger = (|T_1^*|^\beta)^\dagger\,|T_1|^\alpha$.

Now, Proposition 4.2 implies that the desired result yields, that is, $(|T|^\alpha(|T^*|^\dagger)^\beta)^\dagger = ((|T^*|^\dagger)^\beta)^\dagger|T|^\alpha)^\dagger = |T^*|^\beta(|T|^\alpha)^\dagger = (|T|^\alpha)^\dagger|T^*|^\beta$.

(v) By the implication (iv) and Lemma 3.1, we infer it.

(vi) Equation (5) and Lemma 5.2 part (i) imply that

$$\begin{bmatrix} D^{-\frac{1}{2}}T_1 & D^{-\frac{1}{2}}T_2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \left(T_1^*T_1\right)^{\frac{\alpha}{2}} & 0 \\ 0 & \left(T_2^*T_2\right)^{\frac{\alpha}{2}} \end{bmatrix} = \begin{bmatrix} D^{\frac{\alpha}{2}} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} D^{-\frac{1}{2}}T_1 & D^{-\frac{1}{2}}T_2 \\ 0 & 0 \end{bmatrix}.$$

Then, $D^{-\frac{1}{2}}T_1\left(T_1^*T_1\right)^{\frac{\alpha}{2}} = D^{\frac{-1}{2}}D^{\frac{\alpha}{2}}T_1$ and $D^{-\frac{1}{2}}T_2\left(T_2^*T_2\right)^{\frac{\alpha}{2}} = D^{-\frac{1}{2}}D^{\frac{\alpha}{2}}T_2$. That is, $T_1\left(T_1^*T_1\right)^{\frac{\alpha}{2}} = D^{\frac{\alpha}{2}}T_1$ and $T_2\left(T_2^*T_2\right)^{\frac{\alpha}{2}} = D^{\frac{\alpha}{2}}T_2$, or equivalently

$$\begin{bmatrix} T_1 & T_2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \left(T_1^*T_1\right)^{\frac{\alpha}{2}} & 0 \\ 0 & \left(T_2^*T_2\right)^{\frac{\alpha}{2}} \end{bmatrix} = \begin{bmatrix} D^{\frac{\alpha}{2}} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} T_1 & T_2 \\ 0 & 0 \end{bmatrix}.$$

This means that $T|T|^\alpha = |T^*|^\alpha T$.

(vii) Since $\mathcal{R}(|T|^\alpha) = \mathcal{R}(T^*T) = \mathcal{R}(T^*)$, therefore $|T|^\alpha(|T|^\alpha)^\dagger = T^\dagger T$. Now, by the implication (vi), we have $T|T|^\alpha = |T^*|^\alpha T$. Post multiplying $(|T|^\alpha)^\dagger$ concludes $T|T|^\alpha(|T|^\alpha)^\dagger = |T^*|^\alpha T(|T|^\alpha)^\dagger$, and then $T = |T^*|^\alpha T(|T|^\alpha)^\dagger$.

(viii) We have $\mathcal{R}(|T^*|^\alpha) = \mathcal{R}(TT^*) = \mathcal{R}(T)$; therefore $|T^*|^\alpha(|T^*|^\alpha)^\dagger = TT^\dagger$.

Now, we know that $T|T|^\alpha = |T^*|^\alpha T$. Taking $*$-operation, we have $|T|^\alpha T^* = T^*|T^*|^\alpha$. By multiplying $(|T^*|^\alpha)^\dagger$ on the right-side of this equation, we have

$$\begin{aligned}|T|^\alpha T^*(|T^*|^\alpha)^\dagger &= T^*|T^*|^\alpha(|T^*|^\alpha)^\dagger\\ &= T^*TT^\dagger = T^*.\end{aligned}$$

Then $T = (|T^*|^\alpha)^\dagger T|T|^\alpha$. □

5. The polar decomposition for adjointable operators

An adjointable operator between Hilbert C^*-modules may have no polar decomposition unless some additional conditions are satisfied. Hence, we state the following theorem. Furthermore, we obtain useful corollaries.

Theorem 5.1. *Suppose that $T, S, H \in \mathcal{L}(\mathcal{X})$ and that $\overline{\mathcal{R}(T)}$ and $\overline{\mathcal{R}(T^*)}$ are orthogonally complemented. Then T has the polar decomposition $T = U|T|$, where $U \in \mathcal{L}(\mathcal{X})$ is a partial isometry and $|SHT^*|^\alpha = U\big||S|H|T|\big|^\alpha U^*$ for every $\alpha > 0$.*

Proof. By [16, Lemma 3.3.], we have $\overline{\mathcal{R}(T^*T)} = \overline{\mathcal{R}(T^*)}$. Since T^*T is positive, then [16, Proposition 2.9.] implies that $\overline{\mathcal{R}(|T|)} = \overline{\mathcal{R}(T^*T)}$. Therefore, $\overline{\mathcal{R}(|T|)} = \overline{\mathcal{R}(T^*)}$. Hence, $\overline{\mathcal{R}(T)}$ and $\overline{\mathcal{R}(|T|)}$ are orthogonally complemented. Equivalently, parts (*ii*) and (*iii*) of [16, Lemma 3.5.] conclude that T has the polar decomposition $T = U|T|$, where $U \in \mathcal{L}(\mathcal{X})$ is a partial isometry such that $\mathcal{R}(U^*) = \overline{\mathcal{R}(|T|)}$.

Now, let $A = \big||S|H|T|\big|^2$. Since U^*U is the orthogonal projection onto $\overline{\mathcal{R}(|T|)}$ and $\mathcal{R}(A) \subseteq \mathcal{R}(|T|)$, then $(UAU^*)^m = UA^mU^*$ for every $m \geq 1$. Hence, for every continuous function f defined in $[0, \infty)$ such that $f(0) = 0$, we choose any sequence $\{P_m\}_{m=1}^{\infty}$ of polynomials such that $P_m(0) = 0$, $m \in \mathbb{N}$, and $P_m(t) \to f(t)$ uniformly on the interval $[0, \|\,|A|\,\|]$. Then

$$Uf(A)U^* = \lim_{m\to\infty} UP_m(A)U^* = \lim_{m\to\infty} P_m(UAU^*) = f(UAU^*).$$

On the other hand, we have

$$\begin{aligned} |SHT^*|^2 &= (SHT^*)^*SHT^* = TH^*S^*SHT^* = U|T|H^*|S|^2H|T|U^* \\ &= U\big||S|H|T|\big|^2U^*. \end{aligned}$$

By applying $f(t) = t^{\frac{\alpha}{2}}$, the desired result follows. □

Implications (i)–(iii) in the below corollary are done in [16, Lemma 3.12.].

Corollary 5.2. *Suppose that $T \in \mathcal{L}(\mathcal{X})$ and that $\overline{\mathcal{R}(T)}$ and $\overline{\mathcal{R}(T^*)}$ are orthogonally complemented. Then T has the polar decomposition $T = U|T|$ and the following properties hold:*

(i) $U|T|^\alpha = |T^*|^\alpha U$;
(ii) $|T^*|^\alpha = U\,|T|^\alpha\,U^*$;
(iii) $|T|^\alpha = U^*\,|T^*|^\alpha\,U$;
(iv) $|TUT^*|^\alpha = U\big||T|U|T|\big|^\alpha U^* = U\big||T|T|\big|^\alpha U^*$;
(v) $|T^*UT|^\alpha = U^*\big||T^*|U|T^*|\big|^\alpha U$;
(vi) $|T^*U^*T|^\alpha = U^*\big||T^*|U^*|T^*|\big|^\alpha U = U^*\big||T^*|T^*\big|^\alpha U$.

Corollary 5.3. *Suppose that $T \in \mathcal{L}(\mathcal{X})$ is an MP-invertible operator. Then $T^\dagger = U^*|T^\dagger|$ is the polar decomposition of $T^\dagger$ and the following properties hold:*

(i) $U(|T|^\alpha)^\dagger = (|T^*|^\alpha)^\dagger U$;
(ii) $\big|T^\dagger\big|^\alpha = U|(T^*)^\dagger|^\alpha U^*$;
(iii) $|T^\dagger U^*(T^*)^\dagger|^\alpha = U^*\big||T^\dagger|U^*|T^\dagger|\big|^\alpha U = U^*\big||T^\dagger|T^\dagger\big|^\alpha U$;
(iv) $|T^\dagger U(T^*)^\dagger|^\alpha = U^*\big||T^\dagger|U|T^\dagger|\big|^\alpha U$;
(v) $|(T^*)^\dagger UT^\dagger|^\alpha = U^*\big||(T^*)^\dagger|U|(T^*)^\dagger|\big|^\alpha U = U^*\big||T|^\dagger U|T|^\dagger\big|^\alpha U$.

Proof. Theorem 2.4 and Lemma 3.1 imply that $T^\dagger = U^*|T^*|^\dagger = U^*|T^\dagger|$. Now, by applying Theorem 5.1, we conclude the implications (i)–(v).

□

6. Characterizations of binormal operators via the generalized Aluthge transformation

In this section, we present a new characterization of binormal operators via the generalized Aluthge transformation, $*$-generalized Aluthge transformation, and satisfy the triangle equality.

Theorem 6.1. *Let $T \in \mathcal{L}(\mathcal{X})$ be a binormal operator and let $\overline{\mathcal{R}(T)}$ and $\overline{\mathcal{R}(T^*)}$ be orthogonally complemented. Then $T = U|T|$ is a polar decomposition of T and the following properties hold:*

(i) $\left|T^*_{\alpha,\beta}\right| = |T|^\alpha |T^*|^\beta$;
(ii) *The generalized Aluthge transformation $T_{\alpha,\beta}$ accepts the polar decomposition and $T_{\alpha,\beta} = U|T_{\alpha,\beta}|$;*
(iii) *The solutions of the equation $|T + X| = |T| + |X|$ is the generalized Aluthge transformation of T;*
(iv) $T^* T_{\alpha,\beta} = |T|\,|T_{\alpha,\beta}|$;
(v) $\langle T, T_{\alpha,\beta}\rangle = T^* T_{\alpha,\beta}$;
(vi) *The equation $|T_{\alpha,\beta} + T_{\alpha',\beta'}| = |T_{\alpha,\beta}| + |T_{\alpha',\beta'}|$ holds for all $\alpha, \beta, \alpha', \beta' > 0$;*
(vii) $T^*_{\alpha,\beta} T_{\alpha',\beta'} = |T_{\alpha,\beta}|\,|T_{\alpha',\beta'}|$
(viii) $\langle T_{\alpha,\beta}, T_{\alpha',\beta'}\rangle = T^*_{\alpha,\beta} T_{\alpha',\beta'}$;
(ix) $T^{(*)}_{\alpha,\beta} = U|T^*_{\alpha,\beta}|$;
(x) $\left|T^{(*)}_{\alpha,\beta}\right| = |T|^\alpha\,|T^*|^\beta$;
(xi) $\left|T^{(*)}_{\alpha,\beta}\right| = \left|T^*_{\alpha,\beta}\right|$;
(xii) $T^{(*)}_{\alpha,\beta} = U|T^{(*)}_{\alpha,\beta}|$;
(xiii) *The solutions of the equation $|T + X| = |T| + |X|$ is the generalized $*$-Aluthge transformation of T;*
(xiv) $\left|T^*_{\alpha,\beta}\right|\left|T^{(*)}_{\alpha',\beta'}\right| = \left|T^*_{\alpha+\alpha',\beta+\beta'}\right| = |T|^{\alpha+\alpha'}\,|T^*|^{\beta+\beta'}$.

Proof. Since $\overline{\mathcal{R}(T)}$ and $\overline{\mathcal{R}(T^*)}$ are orthogonally complemented, then Theorem 5.1 ensures that $T = U|T|$ is a polar decomposition of T.

(i) Since T is a binormal operator, then Proposition 4.1 implies that $|T|^\alpha$ and $|T^*|^\beta$ commute. Hence, by the implication (ii) of Corollary 5.2, we have

$$\left|T^*_{\alpha,\beta}\right| = \left(|T|^\alpha U|T|^{2\beta}U^*|T|^\alpha\right)^{\frac{1}{2}} = \left(|T|^\alpha|T^*|^{2\beta}|T|^\alpha\right)^{\frac{1}{2}} = |T|^\alpha\,|T^*|^\beta. \tag{11}$$

(ii) We know that the generalized Aluthge transformation of $T_{\alpha,\beta}$ is $T_{\alpha,\beta} = |T|^\alpha U|T|^\beta$. Part (i) of Corollary 5.2 permits to assure $T_{\alpha,\beta} = |T|^\alpha|T^*|^\beta U$. Therefore (11) implies that

$$T_{\alpha,\beta} = |T|^\alpha|T^*|^\beta U = \left|T^*_{\alpha,\beta}\right| U. \tag{12}$$

Taking $*$-operation, we deduce $T^*_{\alpha,\beta} = U^*\left|T^*_{\alpha,\beta}\right|$. Then $T_{\alpha,\beta} = U\left|T_{\alpha,\beta}\right|$.

(iii) Since $T = U|T|$, part (ii) deduces $T_{\alpha,\beta} = U\left|T_{\alpha,\beta}\right|$. Then [2, Theorem 2.3] implies it.

(iv) We can use [18, Prposition 2.1] to prove it. This implies that $T^*T_{\alpha,\beta} = |T|\,|T_{\alpha,\beta}|$.

(v) By combining the implication (iv) and [18, Theorem 2.3], we obtain it.

(vi), (vii), and (viii) are evident.

(ix) The implication (i) and part (i) of Corollary 5.2 conclude that

$$T^{(*)}_{\alpha,\beta} = |T^*|^\alpha U|T^*|^\beta = U|T|^\alpha|T^*|^\beta = U|T^*_{\alpha,\beta}|. \tag{13}$$

(x) The implication (iii) of Corollary 5.2 yields that

$$\left|T^{(*)}_{\alpha,\beta}\right| = \left(|T^*|^\beta U^*|T^*|^{2\alpha}U|T^*|^\beta\right)^{\frac{1}{2}} = \left(|T^*|^\beta|T|^{2\alpha}|T^*|^\beta\right)^{\frac{1}{2}} = |T|^\alpha\,|T^*|^\beta. \tag{14}$$

(xi) − (xiv) are evident.

□

In [12, Theorem 3.1], Ito characterized binormal operators via the Aluthge transformation. We characterize binormal operators via the generalized Aluthge transformation.

Theorem 6.2. *Let $T = U|T|$ be the polar decomposition. Then the following statements are equivalent:*

(i) *T is binormal;*

(ii) *For all $\alpha > 0,\ \beta > 0, T_{\alpha,\beta} = U\left|T_{\alpha,\beta}\right|$ is the polar decomposition;*

(iii) *For all $\alpha > 0,\ \beta > 0, T^{(*)}_{\alpha,\beta} = U|T^{(*)}_{\alpha,\beta}|$ is the polar decomposition;*

(iv) *For all $\alpha > 0,\ \beta > 0, |T + T_{\alpha,\beta}| = |T| + |T_{\alpha,\beta}|$;*

(v) *For all $\alpha > 0, \beta > 0, |T + T^{(*)}_{\alpha,\beta}| = |T| + |T^{(*)}_{\alpha,\beta}|$.*

(vi) *There exist $\alpha > 0$ and $\beta > 0$ such that $T_{\alpha,\beta} = U\left|T_{\alpha,\beta}\right|$;*

(vii) *There exist $\alpha > 0$ and $\beta > 0$ such that $T^{(*)}_{\alpha,\beta} = U|T^{(*)}_{\alpha,\beta}|$;*

(viii) *There exist $\alpha > 0$ and $\beta > 0$ such that $|T + T_{\alpha,\beta}| = |T| + |T_{\alpha,\beta}|$;*

(ix) *There exist $\alpha > 0$ and $\beta > 0$ such that $|T + T^{(*)}_{\alpha,\beta}| = |T| + |T^{(*)}_{\alpha,\beta}|$.*

Proof. (i) ⇒ (ii) Since T is binormal, by Theorem 6.1 part (ii), the desired result follows.

(ii) ⇒ (i) By the assumption $T_{\alpha,\beta} = U|T_{\alpha,\beta}|$ and the implication (ii) of Corollary 5.2, we have

$$|T|^{\alpha}\,|T^*|^{\beta} = |T|^{\alpha}U\,|T|^{\beta}\,U^* = T_{\alpha,\beta}U^* = U|T_{\alpha,\beta}|U^* \geqslant 0.$$

Then $|T|^{\alpha}\,|T^*|^{\beta} = |T^*|^{\beta}\,|T|^{\alpha}$, that is, T is binormal.

(i) ⇒ (iii) Since T is binormal, Theorem 6.1 part (xii) follows it.

(iii) ⇒ (i) The assumption $T^{(*)}_{\alpha,\beta} = U|T^{(*)}_{\alpha,\beta}|$ and part (iii) of Corollary 5.2 yield that

$$\begin{aligned} |T^*|^{\alpha}\,U\,|T^*|^{\beta} &= U\left(|T^*|^{\beta}\,U^*\,|T^*|^{2\alpha}\,U\,|T^*|^{\beta}\right)^{\frac{1}{2}} \\ U|T|^{\alpha}\,|T^*|^{\beta} &= U\left(|T^*|^{\beta}\,|T|^{2\alpha}\,|T^*|^{\beta}\right)^{\frac{1}{2}}. \end{aligned}$$

Then

$$|T|^{\alpha}\,|T^*|^{\beta} = U^*U\left(|T^*|^{\beta}\,|T|^{2\alpha}\,|T^*|^{\beta}\right)^{\frac{1}{2}}. \tag{15}$$

Taking $*$-operation of (15) concludes that $|T^*|^{\beta}\,|T|^{\alpha} = \left(|T^*|^{\beta}\,|T|^{2\alpha}\,|T^*|^{\beta}\right)^{\frac{1}{2}}U^*U$. Hence,

$$|T^*|^{\beta}\,|T|^{2\alpha}\,|T^*|^{\beta} = \left(|T^*|^{\beta}\,|T|^{2\alpha}\,|T^*|^{\beta}\right)^{\frac{1}{2}}U^*U\left(|T^*|^{\beta}\,|T|^{2\alpha}\,|T^*|^{\beta}\right)^{\frac{1}{2}}.$$

It is implies that

$$\left(|T^*|^{\beta}\,|T|^{2\alpha}\,|T^*|^{\beta}\right)^{\frac{1}{2}}(1 - U^*U)\left(|T^*|^{\beta}\,|T|^{2\alpha}\,|T^*|^{\beta}\right)^{\frac{1}{2}} = 0.$$

Therefore, $(1 - U^*U)^{\frac{1}{2}}\left(|T^*|^{\beta}\,|T|^{2\alpha}\,|T^*|^{\beta}\right)^{\frac{1}{2}} = 0$, and thus

$$(1 - U^*U)\left(|T^*|^{\beta}\,|T|^{2\alpha}\,|T^*|^{\beta}\right)^{\frac{1}{2}} = 0. \tag{16}$$

Equations (15) and (16) imply

$$|T|^{\alpha}\,|T^*|^{\beta} = \left(|T^*|^{\beta}\,|T|^{2\alpha}\,|T^*|^{\beta}\right)^{\frac{1}{2}}.$$

That is, $|T|^{\alpha}\,|T^*|^{\beta}$ is positive. This means that T is binormal.

(ii) ⇔ (iv) Applying [2, Theorem 2.3] implies it.

(iii) ⇔ (iv) Again, applying [2, Theorem 2.3] concludes it.

Equivalently implications (i) ⇔ ⋯ ⇔ (v) and details of its proofs and Remark 4.3 conclude that (i) ⇔ (vi) ⇔ ⋯ ⇔ (ix). □

Corollary 6.3. *Let $T = U|T|$ be the polar decomposition and let T be a binormal operator. Then the following properties hold:*

(i) $U|T_{\alpha,\beta}|^r = |T^*_{\alpha,\beta}|^r U$;
(ii) $|T^*_{\alpha,\beta}|^r = U\left|T_{\alpha,\beta}\right|^r U^*$;
(iii) $|T_{\alpha,\beta}|^r = U^*|T^*_{\alpha,\beta}|^r U$;
(iv) $|T_{\alpha,\beta}UT^*_{\alpha,\beta}|^r = U\left||T_{\alpha,\beta}|U|T_{\alpha,\beta}|\right|^r U^* = U\left||T_{\alpha,\beta}|T_{\alpha,\beta}\right|^r U^*$;
(v) $|T_{\alpha,\beta}UT^*_{\alpha,\beta}|^r = U\left||T_{\alpha,\beta}|T_{\alpha,\beta}\right|^r U^*$;
(vi) $|T^*_{\alpha,\beta}UT_{\alpha,\beta}|^r = U^*\left||T^*_{\alpha,\beta}|U|T^*_{\alpha,\beta}|\right|^r U$;
(vii) $|T^*_{\alpha,\beta}U^*T_{\alpha,\beta}|^r = U^*\left||T^*_{\alpha,\beta}|U^*|T^*_{\alpha,\beta}|\right|^r U$;
(viii) $|T^*_{\alpha,\beta}U^*T_{\alpha,\beta}|^r = U^*\left||T^*_{\alpha,\beta}|T^*_{\alpha,\beta}\right|^r U$.

Corollary 6.4. *Let $T \in \mathcal{L}(\mathcal{X})$ be binormal MP-invertible with the polar decomposition $T = U|T|$. Then the following properties hold:*

(i) $(T_{\alpha,\beta}|T_{\alpha,\beta}|^r)^\dagger = (|T_{\alpha,\beta}|^r)^\dagger T^\dagger_{\alpha,\beta}$;
(ii) $(|T^*_{\alpha,\beta}|^r T_{\alpha,\beta})^\dagger = T^\dagger_{\alpha,\beta}(|T^*_{\alpha,\beta}|^r)^\dagger$;
(iii) $(|T^*_{\alpha,\beta}|^s T_{\alpha,\beta}|T_{\alpha,\beta}|^r)^\dagger = (|T_{\alpha,\beta}|^r)^\dagger T^\dagger_{\alpha,\beta}(|T^*_{\alpha,\beta}|^s)^\dagger$;
(iv) $(|T_{\alpha,\beta}|^r T^*_{\alpha,\beta})^\dagger = (T^*_{\alpha,\beta})^\dagger(|T_{\alpha,\beta}|^r)^\dagger$;
(v) $(T_{\alpha,\beta}|T_{\alpha,\beta}|^r T^*_{\alpha,\beta})^\dagger = (T^*_{\alpha,\beta})^\dagger(|T_{\alpha,\beta}|^r)^\dagger T^\dagger_{\alpha,\beta}$;
(vi) $(U|T_{\alpha,\beta}|^r)^\dagger = (|T_{\alpha,\beta}|^r)^\dagger U^*$;
(vii) $(|T^*_{\alpha,\beta}|^r U)^\dagger = U^*(|T^*_{\alpha,\beta}|^r)^\dagger$;
(viii) $(|T^*_{\alpha,\beta}|^s U|T_{\alpha,\beta}|^r)^\dagger = (|T_{\alpha,\beta}|^r)^\dagger U^*(|T^*_{\alpha,\beta}|^s)^\dagger$.

Proof. Applying Theorems 6.2, 3.2, and 4.4 conclude these results. □

Theorem 6.5. *Let $T = U|T|$ be the polar decomposition and let T be a binormal operator. Then the following conditions are mutually equivalent:*

(i) *For all $\alpha > 0, \beta > 0$, $\langle T, T_{\alpha,\beta}\rangle$ is self-adjoint;*
(ii) *For all $\alpha > 0$, $\beta > 0$, $\langle T, T_{\alpha,\beta}\rangle$ is positive;*
(iii) *For all $\alpha > 0$, $\beta > 0$, $T^*T_{\alpha,\beta}$ is self-adjoint;*
(iv) *For all $\alpha > 0$, $\beta > 0$, $T^*T_{\alpha,\beta}$ is positive;*
(v) *For all $\alpha > 0$, $\beta > 0$, $|T||T_{\alpha,\beta}| = |T_{\alpha,\beta}||T|$;*
(vi) *For all $\alpha > 0$, $\beta > 0$, $T|T_{\alpha,\beta}| = T_{\alpha,\beta}|T|$;*
(vii) *There exist $\alpha > 0$ and $\beta > 0$ such that $\langle T, T_{\alpha,\beta}\rangle$ is self-adjoint;*
(viii) *There exist $\alpha > 0$ and $\beta > 0$ such that $\langle T, T_{\alpha,\beta}\rangle$ is positive;*
(ix) *There exist $\alpha > 0$ and $\beta > 0$ such that $T^*T_{\alpha,\beta}$ is self-adjoint;*
(x) *There exist $\alpha > 0$ and $\beta > 0$ such that $T^*T_{\alpha,\beta}$ is positive;*
(xi) *There exist $\alpha > 0$ and $\beta > 0$ such that $|T||T_{\alpha,\beta}| = |T_{\alpha,\beta}||T|$;*
(xii) *There exist $\alpha > 0$ and $\beta > 0$ such that $T|T_{\alpha,\beta}| = T_{\alpha,\beta}|T|$.*

Proof. By combining Theorem 6.2, [18, Corollary 2.5], implications (iv) and (v) of Theorem 6.1, and Remark 4.3, the results are obtained. □

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