Some Recurrence Formulas for the q-Bernoulli and q-Euler Polynomials

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Abstract. The recurrence relations have a very important place for the special polynomials such as q-Appell polynomials. In this paper, we give some recurrence formulas that allow us a better understanding of q-Appell polynomials. We investigate the q-Bernoulli polynomials and q-Euler polynomials, which are q-Appell polynomials, and we obtain their recurrence formulas by using the methods of the q-umbral calculus and the quantum calculus. Our methods include some operators which are quite handy for obtaining relations for the q-Appell polynomials. Especially, some applications of q-derivative operator are used in this work.

1. Introduction

Throughout of this paper, we use the notation

$$[x]_q = \begin{cases} \frac{1-q^x}{1-q}, q \neq 1 \\ x, q = 1, \end{cases}$$

where $0 < q < 1$ when $q \in \mathbb{R}$ and $|q| < 1$ when $q \in \mathbb{C}$.

Let P be the algebra of polynomials in the single variable x over the field of complex numbers $\mathbb{C}$. Derivative operator t is defined by

$$tp(x) = \frac{p(x) - p(qx)}{x - qx}, \tag{1}$$

for all $p(x) \in P$. Specially,

$$tx^n = \frac{x^n - (qx)^n}{x - qx} = [n]_q x^{n-1}.$$

The q-analogue of the exponential series is defined by

$$\varepsilon_q(yt) = \sum_{k=0}^{\infty} \frac{(yt)^k}{[k]_q!}.$$

One must notice that $\varepsilon_q (yt)$ is well defined for all $|yt| < \frac{1}{|1-q|}$ if $|q| < 1$ and for all $yt \in \mathbb{C}$ if $|q| > 1$ or $q = 1$.

For detailed information about the q-calculus, see [9], [10], [13], [14].

Let $\langle L \mid p(x) \rangle$ be the action of a linear functional L on a polynomial $p(x)$. Let $\mathfrak{F}$ denote the algebra of formal power series

$$f(t) = \sum_{k=0}^{\infty} \frac{a_k}{[k]_q!} t^k.$$

This algebra is called *q-umbral algebra*. Each $f \in \mathfrak{F}$ defines a linear functional on P and for all $k \geq 0$, $a_k = \left\langle f(t) \mid x^k \right\rangle$.

In the special case,

$$\left\langle t^k \mid x^n \right\rangle = [n]_q! \delta_{n,k},$$

where

$$\delta_{n,k} = \left\{ \begin{array}{l} 0 \text{ if } n \neq k \\ 1 \text{ if } n = k. \end{array} \right.$$

Let $f(t)$, $g(t)$ be in $\mathfrak{F}$, we have

$$\langle f(t)g(t) \mid p(x) \rangle = \langle f(t) \mid g(t)p(x) \rangle.$$

The *order* $o(f(t))$ of a power series $f(t)$ is the smallest integer k for which the coefficient of t^k does not vanish. A series $f(t)$ for which $o(f(t)) = 1$ is called a *delta series*. And a series $f(t)$ for which $o(f(t)) = 0$ is called a *invertible series*.

Let $f(t)$ be a delta series and let $g(t)$ be an invertible series. Then there exist a unique sequence $S_n(x)$ of polynomials satisfying the orthogonality conditions

$$\left\langle g(t)f(t)^k \mid S_n(x) \right\rangle = [n]_q! \delta_{n,k} \tag{2}$$

for all $n, k \geq 0$.

The sequence $S_n(x)$ in (2) is the q-Sheffer polynomials for pair $(g(t), f(t))$, where $g(t)$ must be invertible series and $f(t)$ must be delta series. In particular, the q-Sheffer polynomials for pair $(g(t), t)$ is the q-Appell polynomial for $g(t)$.

Because of (1), we have the following recurrence formula for every q-Sheffer polynomials:

$$S_n(x) - S_n(qx) = (1 - q^n)xS_{n-1}(x), \tag{3}$$

q-Sheffer polynomials (cf. [13]).

Every q-Appell polynomials satisfy the identities listed below:

The polynomial $S_n(x)$ is q-Appell for $g(t)$ if and only if

$$\frac{1}{g(t)} \varepsilon_q (yt) = \sum_{k=0}^{\infty} \frac{S_k(y)}{[k]_q!} t^k \tag{4}$$

for all constants $y \in \mathbb{C}$.

The polynomial $S_n(x)$ is q-Appell for $g(t)$ if and only if

$$S_n(x) = g(t)^{-1} x^n \tag{5}$$

The polynomial $S_n(x)$ is q-Appell for $g(t)$ if and only if

$$tS_n(x) = [n]_q S_{n-1}(x). \tag{6}$$

By using (6), one obtain

$$\frac{1}{t} S_n(x) = \frac{1}{[n+1]_q} S_{n+1}(x).$$ (7)

For detailed information about the q-Appell polynomials, see [14], [13], [4], [5], [6]
q-derivative operator is defined by $D_{t,q} : t^n \longrightarrow [n]_q \, t^{n-1}$,

$$D_{t,q} f(t) = \frac{f(t) - f(qt)}{t - qt},$$ (8)

where $q \neq 1$ (cf. [14]).

Remark 1.1. *There are two kind of notation of q-derivative operator in the q-calculus theory. Operator t, which is defined by (1), can be aplicable for only all $p(x) \in \mathbf{P}$. On the other hand, operator $D_{t,q}$, which is defined by (8), is usable for all arbitrary functions. Of course, one can see that the action of these two operators are exactly same on any polynomials. But to avoid any confusion, we use operator $D_{t,q}$ on all $f(t) \in \mathfrak{F}$ and we use operator t on any polynomials in this paper.*

We have

$$D_{t,q}^n (f(t) g(t)) = \sum_{k=0}^{n} \binom{n}{k}_q q^{-k(n-k)} D_{t,q}^k f(t) D_{t,q}^{n-k} g\left(q^k t\right),$$ (9)

(cf. [14]),

$$D_{t,q}\left(\frac{f(t)}{g(t)}\right) = \frac{g(t) D_{t,q} f(t) - f(t) D_{t,q} g(t)}{g(t) g(qt)},$$ (10)

(cf. [10]).

Higher-order q-Bernoulli polynomials are defined by means of the following generating function:

$$\sum_{k=0}^{\infty} B_{k,q}^{(\alpha)}(x) \frac{t^k}{[k]_q!} = \left(\frac{t}{\varepsilon_q(t) - 1}\right)^{\alpha} \varepsilon_q(xt),$$

where $\alpha \in \mathbb{N}$ (cf. [8]).

For $\alpha = 1$, the higher-order q-Bernoulli polynomials reduced to q-Bernoulli polynomials by

$$B_{n,q}^{(1)}(x) = B_{n,q}(x).$$

Higher-order q-Bernoulli polynomials are q-Appell polynomials for $g(t) = \left(\frac{\varepsilon_q(t)-1}{t}\right)^{\alpha}$. Then, by (5), the following relationship holds true:

$$B_{n,q}^{(\alpha)}(x) = \left(\frac{t}{\varepsilon_q(t) - 1}\right)^{\alpha} x^n,$$ (11)

(cf. [11]).

Higher-order q-Euler polynomials are defined by means of the following generating function:

$$\sum_{k=0}^{\infty} E_{k,q}^{(\alpha)}(x) \frac{t^k}{[k]_q!} = \left(\frac{2}{\varepsilon_q(t) + 1}\right)^{\alpha} \varepsilon_q(xt),$$

where $\alpha \in \mathbb{N}$ (cf. [8]).

For $\alpha = 1$, the higher-order q-Euler polynomials reduced to q-Euler polynomials by

$$E_{n,q}^{(1)}(x) = E_{n,q}(x).$$

Higher-order q-Euler polynomials are also q-Appell polynomials for $g(t) = \left(\frac{\varepsilon_q(t)+1}{2}\right)^\alpha$. Then, by (5), the following relationship holds true:

$$E_{n,q}^{(\alpha)}(x) = \left(\frac{2}{\varepsilon_q(t)+1}\right)^\alpha x^n, \tag{12}$$

(cf. [12]).

2. Some Relations for q-Appell polynomials

Roman [14] gave some introduction about the nonclassical umbral calculi. His results include some identities for generalized Sheffer polynomials which are the generalization of q-Appell polynomials. In this section, we obtain a recurrence formula for q-Appell polynomials by using the identities of generalized Sheffer polynomials which are given by Roman [14]. θ operator is defined by

$$\theta : x^n \longrightarrow \frac{n+1}{[n+1]_q} x^{n+1}.$$

Observe that

$$\theta t x^n = [n]_q \theta x^{n-1} = n x^n,$$

and so

$$\theta t = xD$$

where D is the ordinary derivative (cf. [14]).

If we investigate the relationship between operators θ and $D_{x,q}$, we get

$$\theta t = \frac{n}{[n]_q} x D_{x,q}. \tag{13}$$

Lemma 2.1. *Let $S_n(x)$ be a q-Appell polynomial. Then*

$$\theta S_n(x) = \frac{n}{[n]_q} x S_n(x). \tag{14}$$

Proof. It follows from the equation (6) that

$$\theta S_n(x) = \theta \frac{1}{[n+1]_q} t S_{n+1}(x).$$

By using (13) and (6), we get

$$\theta S_n(x) \quad = \quad \frac{n}{[n]_q} x S_n(x),$$

which gives the desired result (cf. [7]). □

Let c_n be a sequence of nonzero constants. Roman ([14]) gave a recurrence formula for generalized (c_n) Sheffer polynomials. By taking

$$c_n = \frac{(1-q)\left(1-q^2\right)\cdots(1-q^n)}{(1-q)^n},$$

one can get the following result:

Let $S_n(x)$ be a q-Appell polynomials for $g(t)$. Then

$$(n+1)S_{n+1}(x) = [n+1]_q \left(\theta - \frac{D_{t,q}(g(t))}{g(t)}\right) S_n(x), \tag{15}$$

[7].

3. q-Bernoulli Polynomials Case

In this section, we investigate some operator action on the higher-order q-Bernoulli polynomials. Then we obtain some recurrence formulas for these polynomials.

By using (11)and (6), we get

$$\left(\varepsilon_q(t) - 1\right) B_{n,q}^{(\alpha)}(x) = [n]_q B_{n-1,q}^{(\alpha-1)}(x).$$

By linearity, one can easily have

$$\varepsilon_q(t) B_{n,q}^{(a)}(x) = [n]_q B_{n-1,q}^{(a-1)}(x) + B_{n,q}^{(a)}(x). \tag{16}$$

Using (11), (7) and (16), we obtain the following equation:

$$\frac{\varepsilon_q(t)}{\varepsilon_q(t)-1} B_{n,q}^{(a)}(x) = B_{n,q}^{(a)}(x) + \frac{1}{[n+1]_q} B_{n+1,q}^{(a+1)}(x). \tag{17}$$

A recurrence formula for $B_{n,q}^{(\alpha)}(x)$ is obtained by using (3):

$$B_{n,q}^{(\alpha)}(x) = B_{n,q}^{(\alpha)}(qx) + (1-q^n)\,x B_{n-1,q}^{(\alpha)}(x).$$

We give a recurrence formula for the q-Bernoulli polynomials by the following theorem:

Theorem 3.1.

$$(q(n+1) - 1) B_{n+1,q}(x) = [n+1]_q \left(\frac{qnx}{[n]_q} - 1\right) B_{n,q}(x) - B_{n+1,q}^{(2)}(x).$$

Proof. Firstly, by using (10), we calculate the q-derivative of

$$g(t) = \frac{\varepsilon_q(t) - 1}{t}.$$

$$D_{t,q}g(t) = \frac{t\varepsilon_q(t) - \left(\varepsilon_q(t) - 1\right)}{t^2 q}.$$

Hence, we obtain

$$\frac{D_{t,q}g(t)}{g(t)} = \frac{1}{q}\left(\frac{\varepsilon_q(t)}{\left(\varepsilon_q(t) - 1\right)} - \frac{1}{t}\right).$$

From (15), we get

$$(n + 1) B_{n+1,q} (x) = [n + 1]_q \left(\theta - \frac{1}{q} \left(\frac{\varepsilon_q (t)}{\varepsilon_q (t) - 1} - \frac{1}{t} \right) \right) B_{n,q} (x) .$$

By using (14), (17) and (7), we complete the proof. $\square$

4. *q*-Euler Polynomials Case

Similar to the previous section, we study some operator action on the higher-order *q*-Euler polynomials. Then we get some recurrence formulas.

By using (12), we have

$$\left(\varepsilon_q (t) + 1 \right) E_{n,q}^{(\alpha)} (x) = 2 E_{n,q}^{(\alpha-1)} (x) .$$

By linearity, we get the action of $\varepsilon_q (t)$ on $E_{n,q}^{(\alpha)} (x)$:

$$\varepsilon_q (t) E_{n,q}^{(\alpha)} (x) = 2 E_{n,q}^{(\alpha-1)} (x) - E_{n,q}^{(\alpha)} (x) . \tag{18}$$

Then, by using (12) and (18), we obtain

$$\frac{\varepsilon_q (t)}{\varepsilon_q (t) + 1} E_{n,q}^{(a)} (x) = E_{n,q}^{(a)} (x) + \frac{1}{2} E_{n,q}^{(a+1)} (x) . \tag{19}$$

A recurrence formula for $E_{n,q}^{(\alpha)} (x)$ is obtained by using (3):

$$E_{n,q}^{(\alpha)} (x) = E_{n,q}^{(\alpha)} (qx) + (1 - q^n) x E_{n-1,q}^{(\alpha)} (x) .$$

We have a recurrence formula for the *q*-Euler polynomials by the following theorem:

Theorem 4.1.

$$\frac{(n + 1)}{[n + 1]_q} E_{n+1,q} (x) = \left(\frac{nx}{[n]_q} - 1 \right) E_{n,q} (x) - \frac{1}{2} E_{n,q}^{(2)} (x) .$$

Proof. We calculate the *q*-derivative of

$$g (t) = \frac{\varepsilon_q (t) + 1}{2} ,$$

and we get

$$\frac{D_{t,q} g (t)}{g (t)} = \frac{\varepsilon_q (t)}{\varepsilon_q (t) + 1}$$

By using (15), we obtain

$$(n + 1) E_{n+1,q} (x) = [n + 1]_q \left(\theta - \frac{\varepsilon_q (t)}{\varepsilon_q (t) + 1} \right) E_{n,q} (x)$$

$$= [n + 1]_q \left(\theta E_{n,q} (x) - \frac{\varepsilon_q (t)}{\varepsilon_q (t) + 1} E_{n,q} (x) \right) .$$

By using (14) and (19), we arrive the desired result. $\square$

5. Acknowledgements

A part of this paper was presented in the Mediterranean International Conference of Pure & Applied Mathematics and related areas (MICOPAM2018) hold in Antalya, Turkey on October 26-29, 2018.

This article is dedicated to Professor Gradimir V. Milovanovic on the Occasion of his 70th anniversary.

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