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Connectedness Criteria for Graphs by Means of Omega Invariant

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Abstract. A realizable degree sequence can be realized in many ways as a graph. There are several tests for determining realizability of a degree sequence. Up to now, not much was known about the common properties of these realizations. Euler characteristic is a well-known characteristic of graphs and their underlying surfaces. It is used to determine several combinatorial properties of a surface and of all graphs embedded onto it. Recently, last two authors defined a number Ω which is invariant for all realizations of a given degree sequence. Ω is shown to be related to Euler characteristic and cyclomatic number. Several properties of Ω are obtained and some applications in extremal graph theory are done by authors. As already shown, the number Ω gives direct information compared with the Euler characteristic on the realizability, number of realizations, being acyclic or cyclic, number of components, chords, loops, pendant edges, faces, bridges etc.

In this paper, another important topological property of graphs which is connectedness is studied by means of Ω. It is shown that all graphs with $\Omega(G) \leq -4$ are disconnected, and if $\Omega(G) \geq -2$, then the graph could be connected or disconnected. It is also shown that if the realization is a connected graph and $\Omega(G) = -2$, then certainly the graph should be acyclic. Similarly, it is shown that if the realization is a connected graph G and $\Omega(G) \geq 0$, then certainly the graph should be cyclic. Also, the fact that when $\Omega(G) \leq -4$, the components of the disconnected graph could not all be cyclic, and that if all the components of a graph G are cyclic, then $\Omega(G) \geq 0$ are proven.

1. Introduction

Let $G = (V, E)$ be a graph with $\mid V(G) \mid= n$ vertices and $\mid E(G) \mid= m$ edges. For a vertex $v \in V(G)$, we denote the degree of v by d_v or $d_G(v)$. A vertex with degree one is called a pendant vertex. With slight abuse of language, we shall use the term "pendant edge" for an edge having a pendant vertex. If u and v are adjacent vertices of G, then the edge e connecting them will be denoted by $e = uv$. In such a case, the vertices u and v are called adjacent vertices and the edge e is said to be incident with u and v.

The degree sequence $DS(G)$ of a graph G which is a non-decreasing sequence of non-negative integers which are the degrees of the vertices of G. Written with multiplicities, a degree sequence in general is written as $DS(G) = \{{d_1}^{(a_1)}, {d_2}^{(a_2)}, {d_3}^{(a_3)}, \cdots, \Delta^{(a_\Delta)}\}$, where Δ denotes the biggest vertex degree and a_i's are positive

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integers. It is sometimes useful to state a degree sequence as $DS(G) = \{1^{(a_1)}, 2^{(a_2)}, 3^{(a_3)}, \cdots, \Delta^{(a_\Delta)}\}$, where some of a_i's could be zero.

Let $D = \{d_1, d_2, d_3, \cdots, \Delta\}$ be a set of non-decreasing non-negative integers. We say that a graph G is a realization of the set D if the degree sequence of G is equal to D. It is clear from the definition that for a realizable degree sequence, there is at least one graph having this degree sequence. For example, the completely different two graphs in Fig. 1 have the same degree sequence:

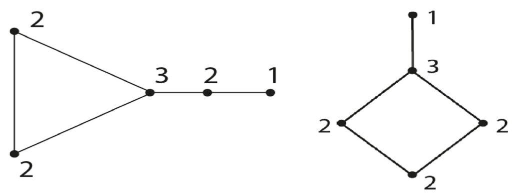

Figure 1 Graphs with the same DS

There are some tests for determining realizability of a given set such as Havel-Hakimi and Sierksma and Hoogeveen criteria, [5], [6], [9]. But these tests so far only help to determine the realizability of the given degree sequence and do not give any information on the topological and combinatorial properties of them.

A graph is called connected when there is a path between every pair of vertices. In a connected graph, there are no unreachable vertices. A graph that is not connected is disconnected. There are partial results for determining the connectedness of a given graph. In this paper, we shall give the criteria to determine the connectedness just by means of the degree sequence $DS(G)$ of a graph G. In fact we shall give results which help to determine the connectedness of all realizations of a given sequence.

It is well-known that the number a_1 of leaves of a tree T is given by $a_1 = 2+a_3+2a_4+3a_5+4a_6+\cdots+(\Delta-2)a_\Delta$, where Δ is the largest vertex degree in T and a_i denotes the number of vertices of degree i. Note that this equation can be rearranged as

$$a_3 + 2a_4 + 3a_5 + 4a_6 + \cdots + (\Delta - 2)a_\Delta - a_1 = -2. \tag{1}$$

The third and fourth authors recently realized that the left hand side of Eqn. (1) is taking other integer values as well resulting in numerous applications and gave the following definition:

Definition 1.1 ([2]). *Let $D = \{1^{(a_1)}, 2^{(a_2)}, 3^{(a_3)}, \cdots, \Delta^{(a_\Delta)}\}$ be the degree sequence of a graph G. The $\Omega(G)$ of the graph G is defined only in terms of the degree sequence as*

$$\Omega(G) = a_3 + 2a_4 + 3a_5 + \cdots + (\Delta - 2)a_\Delta - a_1 = \sum_{i=1}^{\Delta}(i-2)a_i.$$

With slight abuse of language, we shall use $\Omega(D)$ for the Ω of a given degree sequence. For several well-known graph classes the path P_n, cycle C_n, star S_n, complete K_n, tadpole $T_{r,s}$, complete bipartite $K_{r,s}$ with $n = r + s$, and tree T_n with n vertices, the Ω values are $\Omega(C_n) = 0$, $\Omega(P_n) = -2$, $\Omega(S_n) = -2$, $\Omega(T_n) = -2$, $\Omega(K_n) = n(n-3)$, $\Omega(K_{r,s}) = 2[rs - (r+s)]$ and $\Omega(T_{r,s}) = 0$. Note that the Ω of a path, star or tree is equal to -2. This is in fact true for all connected acyclic graphs as we shall see in Theorem 4.1.

2. Some Properties of Ω

We now recall some basic properties of Ω from [2]. The following relation is a very useful tool in calculating $\Omega(G)$ for a given graph G and will be used in the proofs of many results on Ω:

Theorem 2.1 ([2]). *For any graph G,*

$$\Omega(G) = 2(m - n).$$

The following important property of Ω can be used as another test to determine the realizability of a given degree sequence:

Theorem 2.2 ([2]). *For any graph G, $\Omega(G)$ is even.*

Therefore if $\Omega(G)$ is odd, then we have the following obvious result:

Corollary 2.3 ([2]). *Let D be a set of non-negative integers. If $\Omega(D)$ is odd, then D is not realizable.*

The following result giving the number of regions of a graph is also very useful in obtaining our results:

Theorem 2.4 ([2]). *Let $D = \{1^{(a_1)}, 2^{(a_2)}, 3^{(a_3)}, \cdots, \Delta^{(a_\Delta)}\}$. If D is realizable as a connected planar graph G, then the number r of closed regions in G is given by*

$$r = \frac{\Omega(G)}{2} + 1.$$

In many cases, we shall face with disconnected graphs. The following result shows the additivity of Ω on the set of the components of G:

Theorem 2.5 ([2]). *Let G be a disconnected graph with c components $G_1, G_2, \cdots, G_c$. Then*

$$\Omega(G) = \sum_{i=1}^{c} \Omega(G_i).$$

In the case of a disconnected graph, we obtain the following direct generalization of Theorem 2.4 to disconnected graphs:

Corollary 2.6 ([2]). *Let $D = \{1^{(a_1)}, 2^{(a_2)}, 3^{(a_3)}, \cdots, \Delta^{(a_\Delta)}\}$ be realizable as a graph G with c components. The number r of faces of G is given by*

$$r = \frac{\Omega(G)}{2} + c.$$

As $r \geq 0$, we reach the following very useful property:

Corollary 2.7. *For each graph G, we have*

$$c \geq -\frac{\Omega(G)}{2}.$$

Equivalently, for all graphs, we have $c \geq n - m$.

The following is a useful property of Ω:

Theorem 2.8. *Let $D = \{1^{(a_1)}, 2^{(a_2)}, 3^{(a_3)}, \cdots, \Delta^{(a_\Delta)}\}$ and let $\Omega(D) \geq 0$. If D is realizable as a connected graph G, then*

$$\frac{\Omega(G)}{2} = l + ch + e_m,$$

where l is the number of loops, ch is the number of chords and e_m is the number of multiple edges in G.

3. Relation Between Ω and Euler Characteristic

Compact orientable surfaces are classified according to their genus and the classification theorem states that any compact orientable surface is homeomorphic to one of the sphere ($g = 0$), torus ($g = 1$), double torus ($g = 2$), $\cdots$, n-holed torus ($g = n$). The Euler characteristic $\chi(S)$ of a compact orientable surface S of genus g is defined as the number $\chi(S) = n - m + r$ where n, m and r are the numbers of the vertices, edges and faces of a given graph embedded on the surface S. As it is a fixed number for all graphs embedded in some certain surface, it is a topological invariant. This invariant was independently discovered by Euler and Descartes, and therefore it is also known as the Descartes-Euler polyhedral formula. See [7] for the results concerning the Theory of Algebraic Topology of Surfaces. It is well-known that $\chi(S) = 2 - 2g$ for an orientable surface S of genus g. If G is a graph embedded in a surface S, then the number $n - m + r$ is called the Euler characteristic of the graph G and denoted by $\chi(G)$.

For non-planar graphs which are graphs embedded in a surface of positive genus, the outside region is a bounded region as the surface is compact and therefore we count the region outside the graph when calculationg the number r. For planar graphs, the region outside the graph is non bounded and therefore does not form a face of the graph. That is, for planar graphs, we take the number of regions surrounded by the edges of a given planar graph as $r = m - n + 1$. Note that a closed region could be bounded by any n-cycle (n-gon) where $n \geq 3$, a loop (1-gon) or a pair of multiple edges (2-gon). Therefore, the number r is equal to the so-called cyclomatic number of a graph which counts basically the number of independent (non-overlapping) cycles in a given graph.

In case of planar graphs where $g = 0$, we have $\chi(G) = n - m + r = 2$. That is, in such a graph, the number r of regions is equal to $m - n + 2$. This can be seen from the fact that $n = a_1 + a_2 + \cdots + a_\Delta$, $m = \frac{1 \cdot a_1 + 2 \cdot a_2 + \cdots + \Delta \cdot a_\Delta}{2}$ and $r = \frac{\Omega(G)}{2} + 2 = \frac{a_3 + 2 \cdot a_4 + \cdots + (\Delta - 2) a_\Delta}{2} + 2$, counting the region outside the graph as well. For example, in a tree, as $n = m + 1$, we conclude that $r = 1$. Indeed a tree does not divide the plane into two closed regions. There is only one region surrounding the tree which is the whole plane. The relation between χ and Ω was given in [2]:

Lemma 3.1. *For any graph, we have*

$$\Omega(G) = 2(r - \chi(G)).$$

4. Connectedness of Realizations

Given a set D of non-negative integers. If D is realizable, then the realization of it may not be unique in most cases. To be able to decide on the connectedness or disconnectedness of these realizations is an important problem. There are two notions which shall be mentioned here related to the connectedness. D is called forcibly connected if every realization of it is connected and potentially connected if at least one realization of it is connected. Although there is no complete result on the forcibly connectedness, there are some on potentially connectedness. The next two results are very important in the characterization of being cyclic or acyclic of a connected graph:

Theorem 4.1. *The necessary and sufficient condition for a simple connected planar graph G to be a tree is* $\Omega(G) = -2$.

Proof.

$$\begin{aligned} \Omega(G) = -2 \quad &\Longleftrightarrow 2(m - n) = -2 \\ &\Longleftrightarrow n - m = 1 \\ &\Longleftrightarrow \chi(G) = 1 + r = 1 \\ &\Longleftrightarrow r = 0 \\ &\Longleftrightarrow G \text{ is a tree.} \end{aligned}$$

□

Theorem 4.2. *Let G be a connected graph.* $\Omega(G) \geq 0$ *iff G is cyclic.*

Proof. By Theorem 2.4, we get

$$\begin{aligned} G \text{ is cyclic} &\Leftrightarrow r \geq 1 \\ &\Leftrightarrow \tfrac{\Omega(G)}{2} + 1 \geq 1 \\ &\Leftrightarrow \Omega(G) \geq 0. \end{aligned}$$

□

The following result says that any realization of a degree sequence with $\Omega \leq -4$ must be disconnected:

Theorem 4.3. *If $\Omega(D) \leq -4$, D cannot be forcibly or potentially connected.*

Proof. Let, on the contrary, at least one realization, say G, of D be connected. G is either cyclic or acyclic. If it is a connected acyclic graph, then by Theorem 4.1, we know that $\Omega(G) = -2$ which is impossible. If it is a connected cyclic graph, then by Theorem 4.2, we know that $\Omega(G) \geq 0$ which is impossible.

An alternative proof is as follows: By Corollary 2.7, we know that the number c of components of G satisfies the inequality

$$c \geq -\frac{\Omega(G)}{2}.$$

As $\Omega \leq -4$, these two inequalities give $c \geq 2$ giving the result. □

The following result is one of the main results in this paper which shows that every degree sequence D with $\Omega(D) \geq 0$ is potentially connected:

Theorem 4.4. *Let $D = \{1^{(a_1)}, 2^{(a_2)}, 3^{(a_3)}, \cdots, \Delta^{(a_\Delta)}\}$. If $\Omega(D) \geq 0$, then D is potentially connected and its connected realization with the longest cycle is the one having a cycle of length $a_2 + a_3 + a_4 + \cdots + a_\Delta$ together with loops, chords, multiple edges and a_1 pendant edges. Also when $\Omega(D) \geq 0$, every connected realization of D must be cyclic.*

Proof. To show the potential connectedness of D, we just need to find a connected realization of it. We construct the realization having the longest cycle by means of the following steps to obtain the required graph: Any cycle with this degree sequence could have maximum length $a_2 + a_3 + a_4 + \cdots + a_\Delta$ as every vertex of degree at least two can be used in constructing this cycle. Draw a cycle with $a_2 + a_3 + \cdots + a_\Delta$ edges. Then we have a graph with degree sequence $\{2^{(a_2+a_3+\cdots+a_\Delta)}\}$. If $a_1 > 0$, then add a_1 pendant edges to the vertices of the constructed cycle. Hence we have added maximum one pendant edge to each vertex of degree 3, two pendant edges to each vertex of degree 4, three pendant edges to each vertex of degree 5, $\cdots$, $\Delta - 2$ pendant edges to each vertex of degree Δ. If $\Omega(D) = 0$, then as $a_3 + 2a_4 + 3a_5 + \cdots + (\Delta - 2)a_\Delta - a_1 = 0$, we have $a_3 + 2a_4 + 3a_5 + \cdots + (\Delta - 2)a_\Delta = a_1$. That is, all pendant vertices are used. If $\Omega(D) > 0$, then we have used all a_1 pendant vertices and to get the required degrees for all vertices on the main cycle, we need to add total of $a_3 + 2a_4 + 3a_5 + \cdots + (\Delta - 2)a_\Delta - a_1$ degrees. As all pendant vertices are already used, we need to add chords, loops and multiple edges to get all the required degrees. As each such edge has two vertices and adding each of them reduces the total degree we should add by two, we need to add $(a_3 + 2a_4 + 3a_5 + \cdots + (\Delta - 2)a_\Delta - a_1)/2$ edges. By Theorem 2.8, this number is equal to $\Omega(D)/2$ and the required graph is obtained. If $\Omega \geq 0$, by Theorem 2.8, as $\Omega(D)/2$ chords, loops or multiple edges are added to the vertices of the main cycle, the realized graph will be cyclic. □

As we have already mentioned, there are some results on potentially connectedness of a given degree sequence. In [4], the following test is given:

Theorem 4.5. *Let $D = \{1^{(a_1)}, 2^{(a_2)}, 3^{(a_3)}, \cdots, \Delta^{(a_\Delta)}\}$ be a degree sequence. Then D is potentially connected if and only if $\sum_{i=1}^{n} d_i \geq 2(n-1)$.*

Note that the condition for being potentially connected can be restated in terms of Ω:

Theorem 4.6. *Let $D = \{1^{(a_1)}, 2^{(a_2)}, 3^{(a_3)}, \cdots, \Delta^{(a_\Delta)}\}$ be a degree sequence. Then D is potentially connected if and only if $\Omega(D) \geq -2$.*

Proof. Replacing $\sum_{i=1}^{n} d_i$ with $2m$ in Theorem 4.5 gives $2m \geq 2(n-1)$. As $\Omega(D) = 2(m-n)$ by Theorem 2.1, the result follows. □

We have seen that when $\Omega(D) \leq -4$, we can certainly say that the realization G of D is disconnected. When $\Omega(D) \geq -2$, we cannot decide about the connectedness of G that easily. G could be connected or disconnected. By the above results, the following are obvious:

Corollary 4.7.

i) *If all components of a graph G are cyclic, then* $\Omega(G) \geq 0$.

ii) *If all components of a graph G are acyclic, then* $\Omega(G) \leq -2$.

iii) *If the components of a graph G are both cyclic and acyclic, then* $\Omega(G)$ *could be any even integer.*

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