



Index, the Prime Ideal Factorization in Simplest Quartic Fields and Counting Their Discriminants

Abdelmejid Bayad^a, Mohammed Seddik^a

^aUniversité d'Évry Val d'Essonne, Université Paris-Saclay
Laboratoire de Mathématiques et Modélisation d'Évry (UMR 8071)
I.B.G.B.I., 23 Bd. de France, 91037 Évry Cedex, France

Abstract. We consider the simplest quartic number fields \mathbb{K}_m defined by the irreducible quartic polynomials $x^4 - mx^3 - 6x^2 + mx + 1$, where m runs over the positive rational integers such that the odd part of $m^2 + 16$ is square free. In this paper, we study the index $I(\mathbb{K}_m)$ and determine the explicit prime ideal factorization of rational primes in simplest quartic number fields \mathbb{K}_m . On the other hand, we establish an asymptotic formula for the number of simplest quartic fields with discriminant $\leq x$ and given index.

1. Introduction and motivation

Let \mathbb{K} be a number field. We start by giving most important known results on the index $I(\mathbb{K})$, the prime ideal factorization of rational primes and the asymptotic formula counting the number fields with discriminant $\leq x$. The efficient knowledge of the indices of the number field \mathbb{K} allows us to simplify these two problems considerably. In this paper, we are interested by those two important problems in the case of the simplest quartic fields. For other number theory problems related to the simplest fields, see [13, 19, 20] and references therein.

1.1. Index $I(\mathbb{K})$ for number fields \mathbb{K}

Let \mathbb{K} be a number field of degree n over \mathbb{Q} and let $\mathcal{O}_{\mathbb{K}}$ be its ring of integers. Denote by

$$\widehat{\mathcal{O}}_{\mathbb{K}} = \{\theta \in \mathcal{O}_{\mathbb{K}} : \mathbb{K} = \mathbb{Q}(\theta)\}$$

the set of primitive elements of $\mathcal{O}_{\mathbb{K}}$. For any $\theta \in \mathcal{O}_{\mathbb{K}}$ we denote $F_{\theta}(x)$ the characteristic polynomial of θ over \mathbb{Q} . Let $D(\mathbb{K})$ be the discriminant of \mathbb{K} . It is well known that if $\theta \in \widehat{\mathcal{O}}_{\mathbb{K}}$, the discriminant of $F_{\theta}(x)$ has the form

$$D(\theta) = I(\theta)^2 D(\mathbb{K}), \tag{1}$$

where $I(\theta) = (\mathcal{O}_{\mathbb{K}} : \mathbb{Z}[\theta])$ is called the index of θ . The index of the number field \mathbb{K} is given by

$$I(\mathbb{K}) = \gcd_{\theta \in \widehat{\mathcal{O}}_{\mathbb{K}}} I(\theta). \tag{2}$$

2010 *Mathematics Subject Classification.* 11R04, 11R16, 11R27, 11R29, 11R42

Keywords. Common index divisor, the prime ideal factorization of rational primes, simplest quartic fields, discriminant

Received: 13 February 2019; Accepted: 07 April 2019

Communicated by Yilmaz Simsek

Email addresses: abdelmejid.bayad@univ-evry.fr (Abdelmejid Bayad), seddik.mohamed2011@gmail.com (Mohammed Seddik)

A prime number p is called a common index divisor of \mathbb{K} if $p \mid I(\mathbb{K})$.

The knowledge of the $I(\mathbb{K})$ makes it possible to find the explicit decomposition of the prime numbers in \mathbb{K} : If $p \nmid I(\mathbb{K})$, by equation (2) there exists a primitive integer $\theta \in \widehat{\mathcal{O}}_{\mathbb{K}}$ where $p \nmid I(\theta)$ and by Dedekind's theorem [9§18] we explicitly have the factorization of p using the minimal polynomial of θ . However, if p is a common index divisor, the prime ideal factorization of p in $\mathcal{O}_{\mathbb{K}}$ is more difficult.

Let us recall the statement of Dedekind's theorem. Let $\mathbb{K} = \mathbb{Q}(\theta)$ be an algebraic number field with $\theta \in \mathcal{O}_{\mathbb{K}}$. Let p be a rational prime. Let

$$f(x) = \text{Irr}_{\mathbb{Q}}(x, \theta) \in \mathbb{Z}[x].$$

We consider the canonical surjection map $\mathbb{Z}[x] \rightarrow \mathbb{Z}/p\mathbb{Z}[x]$. We write

$$\bar{f}(x) = g_1(x)^{e_1} \cdots g_r(x)^{e_r},$$

where $g_1(x), \dots, g_r(x)$ are distinct monic irreducible polynomials in $\mathbb{Z}/p\mathbb{Z}[x]$ and e_1, \dots, e_r are positive integers.

For $i = 1, 2, \dots, r$ denote by $f_i(x)$ any monic polynomial of $\mathbb{Z}[x]$ such that $\bar{f}_i = g_i$. We then set

$$P_i = \langle p, f_i(\theta) \rangle.$$

If $I(\theta) \not\equiv 0 \pmod p$ then we have P_1, \dots, P_r are distinct prime ideals of $\mathcal{O}_{\mathbb{K}}$ with $p\mathcal{O}_{\mathbb{K}} = P_1^{e_1} \cdots P_r^{e_r}$, and $N(P_i) = p^{\deg(f_i)}$.

Let p be a prime number and $v_p(I(\mathbb{K}))$ denote the greatest exponent s such that p^s divides $I(\mathbb{K})$. In 1928 Ore [24] conjectured that in general $v_p(I(\mathbb{K})) = 0$ or > 0 don't depends only on the decomposition type of p in \mathbb{K} . In 1930, Engstrom [10] proved Ore's conjecture by exhibiting two number fields K_1 and K_2 of degree 8 over \mathbb{Q} , with the same factorization type of the prime number 3, but such that $v_3(I(\mathbb{K}_1)) \neq v_3(I(\mathbb{K}_2))$. In general Ore's conjecture still open. For more details on Ore's conjecture and its developments, see [8]. In recent paper [27] Śliwa proves that, when p is nonramified in \mathbb{K} , $v_p(I(\mathbb{K}))$ is determined by the decomposition type of p in \mathbb{K} . In the other words, if p not divide $D(\mathbb{K})$, $v_p(I(\mathbb{K}))$ is determined by the decomposition type of p in \mathbb{K} . For the computation of $I(\mathbb{K})$ we refer to [22].

In the following we review some known results on the computation of the index $I(\mathbb{K})$.

- If \mathbb{K} is a quadratic field ($n = 2$), by classical number theory on quadratic fields, one can show that $I(\mathbb{K}) = 1$.
- In case \mathbb{K} is a cubic field ($n = 3$), Engstrom [10] showed that $I(\mathbb{K}) = 1$ or 2 . Llorente and Nart in [18, Theorem 1] determine the type of decomposition of the rational primes and in [18, Theorem 4] give a necessary and sufficient condition for $I(\mathbb{K})$ to be 2. Moreover, in the paper [28] Spearman and Williams give the explicit prime ideal factorization of 2 in cubic fields with index 2.
- In the case \mathbb{K} is a quartic field ($n = 4$), Engstrom [10] showed that

$$I(\mathbb{K}) = 1, 2, 3, 4, 6 \text{ or } 12. \tag{3}$$

- Funakura [11, Theorem 5] showed that in the case of a pure quartic field, we have $I(\mathbb{K}) = 1$ or 2 . Recently, Spearman and Williams [30] determines the explicit prime ideal factorization of 2 when the index of pure quartic field \mathbb{K} is equal to 2.

1.2. Counting discriminants with given index

A general problem in algebraic number theory is that of counting the number fields by their discriminants. If we let $N_n(x)$ denote the number of fields of degree n over \mathbb{Q} whose discriminants do not exceed x in absolute value. Then there is a conjecture that $\lim_{x \rightarrow \infty} \frac{N_n(x)}{x}$ exist and is non-zero. This conjecture is

proved in many special cases:

- For quadratic field, the problem is simple we have

$$N_2(x) = \sum_{\substack{d \leq x \\ d \equiv 1 \pmod{4}}} 1 + \sum_{\substack{d \leq \frac{x}{4} \\ d \equiv 2,3 \pmod{4}}} 1 \sim \frac{1}{\zeta(2)}x, \text{ as } x \rightarrow \infty.$$

- For cubic fields, by [7], we have $N_3(x) \sim \frac{1}{3\zeta(3)}x$, as $x \rightarrow \infty$.
- For quartic and quintic fields, the conjecture is solved by Bhargava. See [3, 4].
- In addition, Kable and Yuki in [15] showed that the number of quintic number fields whose discriminants do not exceed x in absolute value is bounded by a constant times $x^{1+\epsilon}$ for any $\epsilon > 0$.

Next, we are interesting by the cyclic case. Denote by $N_n(C_n, x)$ is the number of cyclic fields of degree n with discriminants less than x .

- For cyclic cubic fields Cohn [6] and Cohen, Diaz y Diaz and Olivier in [5, p.577, §3] showed that

$$N_3(C_3, x) \sim c_3 x^{1/2}, \text{ as } x \rightarrow \infty \text{ where } c_3 = \frac{11\sqrt{3}}{36\pi} \prod_{\substack{p \equiv 1 \pmod{6} \\ p \text{ prime}}} \left(1 - \frac{2}{p(p+1)}\right).$$

- For cyclic quartic fields Baily [2, p. 209, Theorem 9] and its revised form by many authors see [21], [25] and [5, p.580, §5] by different methods, showed that

$$N_4(C_4, x) = c_4 x^{1/2} + c'_4 x^{1/3} + O(x^{1/6+\epsilon}), \forall \epsilon > 0 \tag{4}$$

where

$$c_4 = \frac{3}{\pi^2} \left(1 + \frac{\sqrt{2}}{24}\right) \prod_{\substack{p \equiv 1 \pmod{4} \\ p \text{ prime}}} \left(1 + \frac{2}{p^{3/2} + p^{1/2}}\right) - 1,$$

$$c'_4 = \frac{3 + 2^{-1/3} + 2^{-2/3}}{1 + 2^{-2/3}} \frac{\zeta(2/3)}{4\pi\zeta(4/3)} \prod_{\substack{p \equiv 1 \pmod{4} \\ p \text{ prime}}} \left(1 + \frac{2}{p + p^{1/3}}\right) \left(\frac{1 - 1/p}{1 + 1/p}\right).$$

- Spearman and Williams [29] showed that $I(\mathbb{K})$ where \mathbb{K} is a cyclic quartic field assumes all values $\{1, 2, 3, 4, 6, 12\}$ and they give necessary and sufficient conditions for each to occur and find an asymptotic formula for the number of cyclic quartic fields with discriminant $\leq x$ and $I(\mathbb{K}) = i$ for each $i \in \{1, 2, 3, 4, 6, 12\}$,

$$N_4(C_4, x, i) \sim \alpha_i x^{\frac{1}{2}}, \text{ as } x \rightarrow \infty,$$

where

$$\alpha_1 \approx 0.0970153, \quad \alpha_2 \approx 0.0067627, \quad \alpha_3 \approx 0.0101764,$$

$$\alpha_4 \approx 0.0067627, \quad \alpha_6 \approx 0.0006321, \quad \alpha_{12} \approx 0.0006321.$$

In this paper we consider the *simplest quartic fields* and study their index, the explicit prime ideal factorization of rational primes and establish asymptotic formula for the number of simplest quartic fields with discriminants less than x with given index.

1.3. Simplest quartic fields

The simplest quartic fields are the totally real cyclic number field of degree 4 having transformation of type $x \rightarrow \frac{ax+b}{cx+d}$ as a generator of the Galois group.

For any rational integer m , let θ_m be a root of the quartic polynomial

$$P_m(x) = x^4 - mx^3 - 6x^2 + mx + 1.$$

The discriminant of $P_m(x)$ is given by $D(\theta_m) = 4\Delta_m^3$, where $\Delta_m = m^2 + 16$.

Since $P_m(-x) = P_{-m}(x)$, we may and we will assume that $m \geq 0$. Gras in [13] proved that $P_m(x)$ is reducible precisely when $\Delta_m = m^2 + 16$ is a square, which occurs only for excluded cases $m = 0, 3$, and showed that the form $m^2 + 16$ represents infinitely many square-free integers.

Set $\alpha = \theta_m - \theta_m^{-1} > 0$. Then we have $\alpha^2 - m\alpha - 4 = \theta_m^2 - m\theta_m - 6 + m\theta_m^{-1} + \theta_m^{-2} = \theta_m^{-2}P_m(\theta_m) = 0$, and $\alpha = \frac{m + \sqrt{\Delta_m}}{2}$. In particular, $k_m = \mathbb{Q}(\sqrt{\Delta_m})$ is the quadratic subfield of the real quartic field $\mathbb{K}_m = \mathbb{Q}(\theta_m)$.

We can factorize P_m in the form, $P_m = \left(x^2 - \frac{m + \sqrt{\Delta_m}}{2}x - 1\right)\left(x^2 - \frac{m - \sqrt{\Delta_m}}{2}x - 1\right)$.

Since $\theta_m > 1$ and $\theta_m^2 - \alpha\theta_m - 1 = 0$, we obtain $\theta_m = \frac{1}{2} \left(\frac{m + \sqrt{\Delta_m}}{2} + \sqrt{\frac{\Delta_m + m\sqrt{\Delta_m}}{2}} \right)$.

In the same way, $\sigma(\alpha) = \frac{m - \sqrt{\Delta_m}}{2}$ and $\sigma(\theta_m) = \frac{1}{2} \left(\frac{m - \sqrt{\Delta_m}}{2} + \sqrt{\frac{\Delta_m - m\sqrt{\Delta_m}}{2}} \right)$.

Note also that $\mathbb{K}_m = \mathbb{Q}(\theta_m) = \mathbb{Q}\left(\sqrt{\frac{\Delta_m + m\sqrt{\Delta_m}}{2}}\right)$.

We will say that $\mathbb{K}_m = \mathbb{Q}(\theta_m)$ is a simplest quartic field if $m \geq 1$ is such that the odd part of Δ_m is square-free, which implies that $m \neq 3$. Let $f_{\mathbb{K}_m}$ and f_{k_m} denote the conductors of the simplest quartic field \mathbb{K}_m and of its real quadratic subfield k_m , then by the conductor-discriminant formula, we have $D(\mathbb{K}_m) = f_{\mathbb{K}_m}^2 f_{k_m}$.

Lemma 1.1. Assume that $m \geq 1$ and that the odd part of $\Delta_m = m^2 + 16$ is square-free. Then we have

$$(f_{\mathbb{K}_m}, f_{k_m}) = \begin{cases} (\Delta_m, \Delta_m) & \text{if } m \equiv 1 \pmod{2}, \\ (\Delta_m, \Delta_m/4) & \text{if } m \equiv 2 \pmod{4}, \\ (\Delta_m/2, \Delta_m/4) & \text{if } m \equiv 4 \pmod{8}, \\ (\Delta_m/2, \Delta_m/16) & \text{if } m \equiv 0 \pmod{8}. \end{cases} \tag{5}$$

An integral basis of \mathbb{K}_m is given by

$$\mathcal{O}_{\mathbb{K}_m} = \begin{cases} \mathbb{Z}[1, \theta_m, \theta_m^2, \frac{1+\theta_m^3}{2}] & \text{if } v_2(m) = 0, \\ \mathbb{Z}[1, \theta_m, \frac{1+\theta_m^2}{2}, \frac{\theta_m+\theta_m^3}{2}] & \text{if } v_2(m) = 1, \\ \mathbb{Z}[1, \theta_m, \frac{1+\theta_m^2}{2}, \frac{1+\theta_m+\theta_m^2+\theta_m^3}{4}] & \text{if } v_2(m) = 2, \\ \mathbb{Z}[1, \theta_m, \frac{1+2\theta_m-\theta_m^2}{4}, \frac{1+\theta_m+\theta_m^2+\theta_m^3}{4}] & \text{if } v_2(m) \geq 3. \end{cases} \tag{6}$$

For the proof of the above lemma see [13, 16].

Note that Olajos [23] proved that \mathbb{K}_m admits power integral bases only for $m = 2, 4$ and he gave all generators of power integral bases. Moreover, Gaál and Pentrányi [12] compute the minimal index $m(\mathbb{K}_m) = \min_{\theta \in \widehat{\mathcal{O}}_{\mathbb{K}_m}} I(\theta)$ of the simplest quartic fields.

The purpose of this paper :

- We study the index $I(\mathbb{K}_m)$ and explicit prime ideal factorization of rational primes,
- We establish an asymptotic formula for the number of the simplest quartic fields \mathbb{K}_m with given index and the discriminants less than x .

2. Statement of the main results

We state our main results.

2.1. *Computation of the index $I(\mathbb{K}_m)$ and the explicit prime ideal factorization of rational primes*

Theorem 2.1. *Assume that $m \geq 1$ and that the odd part of $\Delta_m = m^2 + 16$ is square-free. Then we have*

1.

$$I(\mathbb{K}_m) = \begin{cases} 2 & \text{if } m \text{ odd,} \\ 1 & \text{if } m \text{ even.} \end{cases}$$

2. *The prime ideal factorization of 2 in \mathbb{K}_m with index 2 is*

$$2\mathcal{O}_{\mathbb{K}_m} = \left\langle 2, \frac{1 + \sqrt{m^2 + 16}}{2} \right\rangle \left\langle 2, \frac{1 - \sqrt{m^2 + 16}}{2} \right\rangle .$$

3. *The prime ideal factorization of 2 in \mathbb{K}_m with index 1 is as follows:*

1) *If $v_2(m) = 1$, then,*

$$2\mathcal{O}_{\mathbb{K}_m} = \left\langle 2, \frac{m\theta_m^3 + 10\theta_m^2 - m\theta_m + 6}{4} \right\rangle^2 .$$

2) *If $v_2(m) = 2$, then,*

$$2\mathcal{O}_{\mathbb{K}_m} = \left\langle 2, \frac{5 + \theta_m + \theta_m^2 + \theta_m^3}{4} \right\rangle^4 .$$

3) *If $v_2(m) = 3$, then,*

$$2\mathcal{O}_{\mathbb{K}_m} = \left\langle 2, \frac{(m^3 + 25m + 4)\theta_m^3 + (5m^2 + 168)\theta_m^2 - (m^3 + 21m - 56)\theta_m - m^2 + 8}{16} \right\rangle^2 .$$

4) *If $v_2(m) \geq 4$, then,*

$$2\mathcal{O}_{\mathbb{K}_m} = \left\langle 2, \frac{2 + 7\theta_m + \theta_m^3}{4} \right\rangle^2 \left\langle 2, \frac{6 + 7\theta_m + \theta_m^3}{4} \right\rangle^2 .$$

Proposition 2.2. *Assume that $m \geq 1$ and that the odd part of $\Delta_m = m^2 + 16$ is square-free. Let $\mathbb{K}_m = \mathbb{Q}(\theta_m)$ where θ_m be a root of P_m . Then*

$$I(\theta_m) = 2^{\min(v_2(m)+1, 4)} .$$

Remark 2.3. *Note that for any p odd prime, we can give explicitly the prime ideal factorization of p using the minimal polynomial of θ_m . To do it, we need to factorize $P_m \pmod p$ and we use Dedekind's theorem.*

2.2. *Asymptotic formula for the number of simplest quartic fields with discriminant $\leq x$ and given index*

Assume that $m \geq 1$ and that the odd part of $\Delta_m = m^2 + 16$ is square-free. Let $\mathbb{K}_m = \mathbb{Q}(\theta_m)$ where θ_m be a root of P_m .

We define for a positive integer i

$$N(x, i) = \text{number of } \mathbb{K}_m \text{ with } D(\mathbb{K}_m) \leq x \text{ and } I(\mathbb{K}_m) = i. \tag{7}$$

We state our second main result.

Theorem 2.4. *We have*

1. $N(x, i) = 0$ for $i \neq 1, 2$.

2. For $i = 1, 2$, we have the asymptotic formulas $N(x, i) \sim C_i x^{\frac{1}{6}}$, as $x \rightarrow \infty$ where

$$C_1 = \left(\frac{1}{2\sqrt[3]{4}} + \frac{1}{8\sqrt[3]{2}} + \frac{1}{4} \right) \prod_{\substack{p \equiv 1 \pmod{4} \\ p \text{ prime}}} \left(1 - \frac{2}{p^2} \right) \approx 0.5943471641 \quad \text{and} \quad C_2 = \frac{1}{2} \prod_{\substack{p \equiv 1 \pmod{4} \\ p \text{ prime}}} \left(1 - \frac{2}{p^2} \right) \approx 0.4474206426.$$

Remark 2.5. From Theorem 2.4, the number $N(x)$ of simplest quartic fields \mathbb{K}_m with discriminant $D(\mathbb{K}_m) \leq x$ is given by

$$N(x) = N(x, 1) + N(x, 2) \sim (C_1 + C_2) x^{\frac{1}{6}}, \text{ as } x \rightarrow \infty.$$

Is it possible, from the result of Theorem 2.4, to prove the existence of term in $x^{1/6}$ in the formula (4) conjectured by Cohen, Diaz y Diaz and Olivier in [5, p.580, §3].

3. Proof of main results.

3.1. Proof of Theorem 2.1

Assume that $m \geq 1$ and that the odd part of $\Delta_m = m^2 + 16$ is square-free. Let $\mathbb{K}_m = \mathbb{Q}(\theta_m)$ where θ_m be a root of P_m .

1) if $v_2(m) = 0$, by (5), $D(\mathbb{K}_m) = f_{\mathbb{K}_m}^2 f_{k_m} = \Delta_m^3$ and from (1), we obtain $I(\theta_m) = 2$. We have also $2 \nmid D(\mathbb{K}_m)$, and then 2 is non-ramified in \mathbb{K}_m . That is $2\mathcal{O}_{\mathbb{K}_m} = P, P_1P_2$ or $P_1P_2P_3P_4$. Denote $k_m = \mathbb{Q}(\sqrt{\Delta_m})$ be a real quadratic subfield of \mathbb{K}_m of discriminant $D(k_m) = \Delta_m \equiv 1 \pmod{8}$. Then, the splitting type of 2 in k_m is P_1P_2 , so the splitting of 2 is P_1P_2 or $P_1P_2P_3P_4$ in \mathbb{K}_m . We claim that $2\mathcal{O}_{\mathbb{K}_m} = P_1P_2$ in \mathbb{K}_m . To prove this claim, we proceed by contradiction. Assume that the splitting of 2 is $P_1P_2P_3P_4$. By use of Engstrom result [10, p.234], we obtain $v_2(I(\mathbb{K}_m)) = 2$. This is a contradiction with $I(\theta_m) = 2$, and this claim is proved. Therefore, the prime ideal factorization of 2 in $\mathcal{O}_{\mathbb{K}_m}$ is given by $2\mathcal{O}_{\mathbb{K}_m} = P_1P_2$ and by Engstrom [10], we have $v_2(I(\mathbb{K}_m)) = 1$. We have also, the prime ideal factorization of 2 in \mathcal{O}_{k_m} is given by

$$2\mathcal{O}_{k_m} = \left\langle 2, \frac{1 + \sqrt{m^2 + 16}}{2} \right\rangle \left\langle 2, \frac{1 - \sqrt{m^2 + 16}}{2} \right\rangle,$$

so is the same prime ideal factorization of 2 in $\mathcal{O}_{\mathbb{K}_m}$.

2) If $v_2(m) \geq 1$, by (5), we have $D(\mathbb{K}_m) = f_{\mathbb{K}_m}^2 f_{k_m}$ is divisible by 2. Then 2 is ramified in $\mathcal{O}_{\mathbb{K}_m}$. By Engstrom [10], we have $v_2(I(\mathbb{K}_m)) = 0$. Let us consider all cases.

(a) if $v_2(m) = 1$, take $\varphi = \frac{1 + \theta_m^2}{2}$, the minimal polynomial of φ is given by

$$x^4 - \left(\frac{m^2}{2} + 8\right)x^3 + \left(\frac{5m^2}{4} + 20\right)x^2 - (m^2 + 16)x + \frac{m^2}{4} + 4,$$

with $D(\varphi) = \frac{m^4(m^2+16)^3}{2^6}$. By (5), $D(\mathbb{K}_m) = \frac{\Delta_m^3}{4}$. From (1), we get $I(\varphi) = \frac{m^2}{4} \neq 0$. Hence φ is a primitive integer. We then obtain $2 \nmid I(\varphi)$. Dedekind's theorem gives explicitly the prime ideal factorization of 2 using the minimal polynomial of φ . We give here the details of this fact. From the congruence

$$x^4 - \left(\frac{m^2}{2} + 8\right)x^3 + \left(\frac{5m^2}{4} + 20\right)x^2 - (m^2 + 16)x + \frac{m^2}{4} + 4 \equiv (x^2 + x + 1)^2 \pmod{2},$$

we have $2\mathcal{O}_{\mathbb{K}_m} = \left\langle 2, \varphi^2 + \varphi + 1 \right\rangle^2$, so we get $2\mathcal{O}_{\mathbb{K}_m} = \left\langle 2, \frac{m\theta_m^3 + 10\theta_m^2 - m\theta_m + 6}{4} \right\rangle^2$.

(b) if $v_2(m) = 2$, take $\varphi = \frac{1 + \theta_m + \theta_m^2 + \theta_m^3}{4}$, the minimal polynomial of φ is given by

$$P_\varphi(x) = x^4 - \left(\frac{m^3}{4} + \frac{m^2}{4} + 4m + 4\right)x^3 + \left(\frac{5m^3}{16} - 3\frac{m^2}{4} + 5m - 12\right)x^2 + \left(-\frac{m^3}{16} + \frac{m^2}{2} - m + 8\right)x - \frac{m^2}{16} - 1,$$

with $D(\varphi) = \frac{m^2(m^2+16)^3}{2^8}$. By (5), we have $D(\mathbb{K}_m) = \frac{\Delta_m^3}{16}$ and from (1), we obtain $I(\varphi) = \frac{m^2}{2^4} \neq 0$. We then have φ is a primitive integer. Hence, we get $2 \nmid I(\varphi)$. Dedekind's theorem gives explicitly the prime ideal factorization of 2 using the minimal polynomial of φ , then we have

$$P_\varphi(x) \equiv (x + 1)^4 \pmod{2}.$$

In fact, we have $2\mathcal{O}_{\mathbb{K}_m} = \langle 2, \varphi + 1 \rangle^4$, so we get $2\mathcal{O}_{\mathbb{K}_m} = \langle 2, \frac{5+\theta_m+\theta_m^2+\theta_m^3}{4} \rangle^4$.

(c) if $v_2(m) \geq 3$, take $\varphi = \frac{2+7\theta_m+\theta_m^3}{4}$, the minimal polynomial of φ is given by

$$P_\varphi(x) = x^4 - \left(\frac{1}{4}m^3 + \frac{11}{2}m + 2\right)x^3 + \left(\frac{3}{8}m^3 - \frac{47}{16}m^2 + \frac{33}{4}m - 59\right)x^2 + \left(\frac{13}{16}m^3 + \frac{47}{16}m^2 + \frac{61}{4}m + 60\right)x - \frac{15}{32}m^3 + \frac{65}{64}m^2 - 9m + 18,$$

with $D(\varphi) = \frac{1}{2^{16}}(64m^4 + 2135m^2 + 18032)^2(m^2 + 16)^3(m^2 + 18)^2$. By (5), $D(\mathbb{K}_m) = \frac{\Delta_m^3}{2^6}$ and from (1), we obtain $I(\varphi) = \frac{1}{2^5}(64m^4 + 2135m^2 + 18032)(m^2 + 18)$. Therefore, we have

$$\frac{1}{2^5}(64m^4 + 2135m^2 + 18032)(m^2 + 18) \equiv 1 \pmod{2}.$$

This implies that φ is primitive integer and $2 \nmid I(\varphi)$. Dedekind's theorem gives explicitly the prime ideal factorization of 2 using the minimal polynomial of φ . In fact,

i. if $v_2(m) = 3$, we get $P_\varphi(x) \equiv (x^2 + x + 1)^2 \pmod{2}$ and then we have $2\mathcal{O}_{\mathbb{K}_m} = \langle 2, \varphi^2 + \varphi + 1 \rangle^2$, so we get

$$2\mathcal{O}_{\mathbb{K}_m} = \langle 2, \frac{(m^3 + 25m + 4)\theta_m^3 + (5m^2 + 168)\theta_m^2 - (m^3 + 21m - 56)\theta_m - m^2 + 8}{16} \rangle^2.$$

ii. if $v_2(m) \geq 4$, we get

$$P_\varphi(x) \equiv x^2(x + 1)^2 \pmod{2},$$

so, $2\mathcal{O}_{\mathbb{K}_m} = \langle 2, \varphi \rangle^2 \langle 2, \varphi + 1 \rangle^2$ and $2\mathcal{O}_{\mathbb{K}_m} = \langle 2, \frac{2+7\theta_m+\theta_m^3}{4} \rangle^2 \langle 2, \frac{6+7\theta_m+\theta_m^3}{4} \rangle^2$.

3) In the following we prove that $3 \nmid I(\mathbb{K}_m)$. Let $f_{\mathbb{K}_m}$ and f_{k_m} denote the conductors of the simplest quartic field \mathbb{K}_m and of its real quadratic subfield k_m . So we have $D(\theta_m) = 4 \cdot \Delta_m^3$, and $D(\mathbb{K}_m) = f_{\mathbb{K}_m} \cdot f_{k_m}^2$. By (5) and (1), $3 \nmid I(\theta_m)$, Then $3 \nmid I(\mathbb{K}_m)$. This completes the proof of Theorem 2.1.

3.2. Proof of Proposition 2.2

Assume that $m \geq 1$ and that the odd part of $\Delta_m = m^2 + 16$ is square-free. Let $\mathbb{K}_m = \mathbb{Q}(\theta_m)$ where θ_m be a root of P_m . Let $f_{\mathbb{K}_m}$ and f_{k_m} denote the conductors of the simplest quartic field \mathbb{K}_m and of its real quadratic subfield k_m . So $D(\mathbb{K}_m) = f_{\mathbb{K}_m} \cdot f_{k_m}^2$. By (5) and (1) we deduce the result.

3.3. Proof of Theorem 2.4

Before proving Theorem 2.4 we state the next Lemma.

Lemma 3.1. For d be an integer ≥ 2 we consider the functions

$$\begin{aligned} \chi_0(d) &= \#\{\text{solutions of the congruence } t^2 + 16 \equiv 0 \pmod{d^2}\}, \\ \chi_1(d) &= \#\{\text{solutions of the congruence } t^2 + 4 \equiv 0 \pmod{d^2}\}, \\ \chi_3(d) &= \#\{\text{solutions of the congruence } 4t^2 + 1 \equiv 0 \pmod{d^2}\}. \end{aligned}$$

Then we have the χ_k 's are multiplicative functions, $\chi_k(p) = 1 + \left(\frac{-1}{p}\right)$ for odd prime p and $\chi_0(2) = \chi_1(2) = 2$ and $\chi_3(2) = 0$.

Proof of Lemma 3.1:

Let $d_1 \geq 2$ and $d_2 \geq 2$ two coprime integers, we have $\chi_{k=0,1,3}(d_1 d_2) = \chi_{k=0,1,3}(d_1) \times \chi_{k=0,1,3}(d_2)$. Let $p \geq 3$ prime number and $a \in \mathbb{Z}$. By Hensel's theorem, the congruence $x^2 \equiv a \pmod{p^2}$ has a solution if and only if $\left(\frac{a}{p}\right) = 1$ and in this case there exist exactly two solutions modulo p^2 . So for p odd prime we then obtain have $\chi_{k=0,1,3}(p) = 1 + \left(\frac{-1}{p}\right)$.

Proof of Theorem 2.4:

1. By Theorem 2.1, we have, $N(x, i) = 0$ for $i \neq 1, 2$.
2. Let $N(x, 1) = N_1(x, 1) + N_2(x, 1) + N_3(x, 1)$, where

$$\begin{aligned} N_1(x, 1) &= \#\left\{\mathbb{K}_m : v_2(m) = 1, \frac{\Delta_m^3}{4} \leq x, \text{ odd part of } \Delta_m \text{ is square free}\right\}, \\ N_2(x, 1) &= \#\left\{\mathbb{K}_m : v_2(m) = 2, \frac{\Delta_m^3}{4} \leq 4x, \text{ odd part of } \Delta_m \text{ is square free}\right\}, \\ N_3(x, 1) &= \#\left\{\mathbb{K}_m : v_2(m) \geq 3, \frac{\Delta_m^3}{4} \leq 16x, \text{ odd part of } \Delta_m \text{ is square free}\right\}. \end{aligned}$$

So,

$$\begin{aligned} N_1(x, 1) &= \#\left\{m \geq 2 : v_2(m) = 1, m^2 + 16 \leq (4x)^{\frac{1}{3}}, \text{ odd part of } m^2 + 16 \text{ is square free}\right\}, \\ &= \#\left\{1 \leq m' \leq \sqrt{\left(\frac{x}{2^4}\right)^{\frac{1}{3}} - 4} : m'^2 + 4 \text{ is square free}\right\}, \text{ by writing } m = 2m'. \end{aligned}$$

We then get by [26], $N_1(x, 1) \sim \left(\frac{x}{2^4}\right)^{\frac{1}{6}} \prod_{p \text{ prime}} \left(1 - \frac{\chi_1(p)}{p^2}\right)$, as $x \rightarrow \infty$. By Lemma 3.1, we obtain

$$N_1(x, 1) \sim \frac{1}{2\sqrt[3]{4}} \prod_{\substack{p \equiv 1 \pmod{4} \\ p \text{ prime}}} \left(1 - \frac{2}{p^2}\right) x^{\frac{1}{6}}, \text{ as } x \rightarrow \infty.$$

Secondly, we have

$$\begin{aligned} N_2(x, 1) &= \#\left\{m \geq 4 : v_2(m) = 2, m^2 + 16 \leq (16x)^{\frac{1}{3}}, \text{ odd part of } m^2 + 16 \text{ is square free}\right\}, \\ &= \#\left\{0 \leq m' \leq \frac{1}{2} \sqrt{\left(\frac{x}{2^8}\right)^{\frac{1}{3}} - 1 - \frac{1}{2}} : 2m'^2 + 2m' + 1 \text{ is square free}\right\}, \text{ by writing } m = 4m'. \end{aligned}$$

So by [26], we get $N_2(x, 1) \sim \left(\frac{x}{2^{14}}\right)^{\frac{1}{6}} \prod_{p \text{ prime}} \left(1 - \frac{\chi(p)}{p^2}\right)$, as $x \rightarrow \infty$

where $\chi(p) = \#\{\text{solutions of the congruence } 2m'^2 + 2m' + 1 \equiv 0 \pmod{p^2}\}$. We have $\chi(2) = 0$ and for p odd prime, $2m'^2 + 2m' + 1 \equiv 0 \pmod{p^2}$ if and only if $(4m' + 1)^2 + 4 \equiv 0 \pmod{p^2}$, so we obtain, for p odd prime $\chi(p) = \chi_1(p)$.

So we get $N_2(x, 1) \sim \left(\frac{x}{2^{14}}\right)^{\frac{1}{6}} \prod_{p \text{ odd prime}} \left(1 - \frac{\chi_1(p)}{p^2}\right)$, as $x \rightarrow \infty$. By Lemma 3.1, we obtain

$$N_2(x, 1) \sim \frac{1}{8\sqrt[3]{2}} \prod_{\substack{p \equiv 1 \pmod{4} \\ p \text{ prime}}} \left(1 - \frac{2}{p^2}\right) x^{\frac{1}{6}}, \text{ as } x \rightarrow \infty.$$

Thirdly,

$$N_3(x, 1) = \#\{m \geq 8 : v_2(m) = 3, m^2 + 16 \leq (64x)^{\frac{1}{3}}, \text{ odd part of } m^2 + 16 \text{ is square free}\},$$

$$= \#\left\{1 \leq m' \leq \frac{1}{2} \sqrt{\left(\frac{x}{2^6}\right)^{\frac{1}{3}} - 1} : 4m'^2 + 1 \text{ is square free}\right\}, \text{ by writing } m = 8m'.$$

Then by [26], we get $N_3(x, 1) \sim \left(\frac{x}{2^{12}}\right)^{\frac{1}{6}} \prod_{p \text{ prime}} \left(1 - \frac{\chi_3(p)}{p^2}\right)$, as $x \rightarrow \infty$.

By Lemma 3.1, we obtain $N_3(x, 1) \sim \frac{1}{4} \prod_{\substack{p \equiv 1 \pmod{4} \\ p \text{ prime}}} \left(1 - \frac{2}{p^2}\right) x^{\frac{1}{6}}$, as $x \rightarrow \infty$.

Finally, we have

$$N(x, 2) = \#\{\mathbb{K}_m : v_2(m) = 0, \Delta_m^3 \leq x, \text{ odd part of } \Delta_m \text{ is square free}\},$$

$$= \#\left\{1 \leq m \leq \sqrt{x^{\frac{1}{3}} - 16} : m^2 + 16 \text{ is square free}\right\},$$

so by [26], we get $N(x, 2) \sim x^{\frac{1}{6}} \prod_{p \text{ prime}} \left(1 - \frac{\chi_0(p)}{p^2}\right)$, as $x \rightarrow \infty$. By Lemma 3.1, we obtain $N(x, 2) \sim \frac{1}{2} \prod_{\substack{p \equiv 1 \pmod{4} \\ p \text{ prime}}} \left(1 - \frac{2}{p^2}\right) x^{\frac{1}{6}}$, as $x \rightarrow \infty$. This completes the proof of Theorem 2.4.

4. Acknowledgements

This article is dedicated to Professor Gradimir V. Milovanovic on the Occasion of his 70th anniversary.

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