



## On Relations for the Partitions of Numbers

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**Abstract.** In this paper, we present some interrelations among some restricted and unrestricted partitions of integers. Mainly, we derive new effective formulas for the partition function and compare our partition formula with well known recurrence formulas. Moreover, as the number increase, we observe how effective the new recurrence formulas are.

### 1. Introduction

Over the centuries, there has been widespread interest in the partition of an integer which is one of the most fundamental problems in number theory since the rich history of partition of a number has been gone back to cornerstones mathematicians Euler, Jacobi and S. Ramanujan. Also many useful applications of the partition of an integer has investigated in many branch of mathematics ([1], [7], [10], [11], [16]). The partition function to be studied is the number of ways  $n$  can be written as a sum of positive integers  $n$ . Euler investigated the generating function of the number of partitions of an integer  $n$  as follows;

$$F(x) = \prod_{n=1}^{\infty} \frac{1}{(1-x^n)} = \sum_{n=0}^{\infty} p(n)x^n \quad (1)$$

where  $0 < x < 1$  ([5, Theorem 14.4]). But using Euler's recurrence, for the value of  $p(200)$ , we need to compute all values of  $p(n)$  where  $1 \leq n \leq 199$ . The main problems are both numbers of steps and larger of numbers founded in each step since increasing of  $p(m)$  is faster than increasing of an integer  $m$  and so the recurrence formulas that compute the values of the partition with the help of the smaller integers are much useful and effective. It is naturally to ask whether we may work out the value of  $p(\cdot)$  with the help of the smaller integers. After Euler's recurrence for the partition of an integer, at recent years recurrence formulas have been improved by J.A.Ewell and M. Merca ([8, Theorem 1.2], [12, Theorem 1]).

In the literature, the restricted partitions are substantial as unrestricted partition of an integer ([3], [6], [13], [14], [15]). The generating function of the number of partitions of an integer  $n$  into odd part is

$$\prod_{n=1}^{\infty} \frac{1}{1-x^{2n-1}} = \sum_{n=0}^{\infty} Q(n)x^n.$$

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The generating function of the number of partitions of an integer  $n$  into distinct odd parts is

$$\prod_{n=1}^{\infty} (1 + x^{2n-1}) = \sum_{n=0}^{\infty} Z(n)x^n.$$

In this paper, we study some relations among  $p(\cdot)$ ,  $Q(\cdot)$  and  $Z(\cdot)$  and obtain some new recurrence relations for  $p(\cdot)$ ,  $Q(\cdot)$  and  $Z(\cdot)$ . Mainly, we define a new function for recurrence relations and so we present formulas for the partition function. Then we compare our partition formula with well known recurrence formulas to show efficient of new ones.

In this paper, we use the notation  $\mathbb{Q}$  ( $\mathbb{N}, \mathbb{Z}$ ) for the rational numbers, (natural numbers, integers) and  $a\mathbb{Z} = \{at : t \in \mathbb{Z}\}$ . We also adopt the convention that  $p(0) = Q(0) = Z(0) = 1$ ,  $p(n) = Q(n) = Z(n) = 0$  whenever  $n \in \mathbb{Q} - \mathbb{N}$  and  $0 < x < 1$ . We recall the following special cases of Jacobi’s identity which are used later frequently.

$$\prod_{n=0}^{\infty} ((1 - x^{2kn+k-t})(1 - x^{2kn+k+t})(1 - x^{2kn+2k})) = \sum_{m=-\infty}^{\infty} (-1)^m x^{m(km+t)}. \tag{2}$$

$$\prod_{n=0}^{\infty} ((1 + x^{2kn+k-t})(1 + x^{2kn+k+t})(1 - x^{2kn+2k})) = \sum x^{m(km+t)}. \tag{3}$$

### 2. The relations among partitions

In this section, we study some interrelations among some restricted and unrestricted partitions of integers. We start a well known identities as follows:

$$\prod_{n=1}^{\infty} (1 + x^n)(1 - x^{2n-1}) = 1, \tag{4}$$

for a positive integer  $n > 1$ , we also have a well known identities

$$Q(n) = \left( \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} p(i)Q(n - 2i) \right) - p(n). \tag{5}$$

Moreover, using Equation 4, we get the connection between the number of partitions of an integer  $n$  into distinct odd parts and the number of partitions of an integer  $n$  into odd part.

**Proposition 2.1.** For a positive integer  $n \geq 1$ , we have

$$Z(n) = \sum_{j=1}^n (-1)^j Q(j)Z(n - j). \tag{6}$$

*Proof.* Substituting  $x$  by  $(-x)$  in Equation 4, it follows that

$$1 = \sum_{n=0}^{\infty} \left( \sum_{j=0}^n (-1)^j Q(j)Z(n - j) \right) x^n.$$

Therefore, equating coefficients of same powers of  $x$ , we obtain the desired conclusion.  $\square$

To get new recurrences for the partitions of an integer, we focus on the number of partitions of an integer  $n$  into odd part to find relation between  $p(\cdot)$  and  $Q(\cdot)$ .

**Theorem 2.1.** For a positive integer  $n > 1$ , we have

$$Q(2n) = p(n) + \sum_{\substack{t=1 \\ t, (t-3) \in 4\mathbb{Z}}}^{2n-1} p\left(n - \frac{1}{4}t(t+1)\right).$$

$$Q(2n-1) = \sum_{\substack{t=1 \\ (t-1), (t-2) \in 4\mathbb{Z}}}^{2n-2} p\left(\frac{1}{2}(2n-1) - \frac{1}{4}t(t+1)\right).$$

*Proof.* If  $k = t = \frac{1}{2}$  in Equation 3, then we get

$$2 \prod_{n=1}^{\infty} (1+x^n) \prod_{n=1}^{\infty} (1-x^{2n}) = \sum_{m=(-\infty)}^{\infty} x^{\frac{1}{2}m(m+1)}.$$

Then the series product leads to the following identities

$$\begin{aligned} \sum_{n=0}^{\infty} Q(n)x^n &= \frac{1}{2} \left( \sum_{n=0}^{\infty} p(n)x^{2n} \right) \left( 1 + \sum_{m=1}^{\infty} x^{\frac{1}{2}m(m+1)} + \sum_{m=1}^{\infty} x^{\frac{1}{2}m(m-1)} \right) \\ &= \frac{1}{2} \sum_{n=0}^{\infty} p(n)x^{2n} + \frac{1}{2} \sum_{n=0}^{\infty} \left( \sum_{t=1}^n p\left(\frac{1}{2}n - \frac{1}{4}t(t \mp 1)\right) \right) x^n \\ &= \frac{1}{2} \sum_{n=0}^{\infty} \left( p\left(\frac{n}{2}\right) + \left( \sum_{t=1}^n p\left(\frac{1}{2}n - \frac{1}{4}t(t \mp 1)\right) \right) \right) x^n. \end{aligned}$$

Now we can observe that

$$p\left(\frac{n}{2} - \frac{1}{4}t(t+1)\right) = p\left(\frac{n}{2} - \frac{1}{4}l(l-1)\right)$$

when  $l = t + 1$  and so we get

$$\sum_{t=1}^n \left( p\left(\frac{n}{2} - \frac{1}{4}t(t+1)\right) + p\left(\frac{n}{2} - \frac{1}{4}t(t-1)\right) \right) = 2 \sum_{t=1}^{n-1} p\left(\frac{n}{2} - \frac{1}{4}t(t+1)\right).$$

Therefore, we obtain the equation

$$Q(n) = p\left(\frac{n}{2}\right) + \sum_{t=1}^{n-1} p\left(\frac{n}{2} - \frac{1}{4}t(t+1)\right).$$

On the other hand, it is easy to check that

i) if  $n$  is an even number then  $\frac{1}{4}t(t+1) \in \mathbb{Z}$  if and only if  $t$  or  $(t-3) \in 4\mathbb{Z}$ . Then we get  $p\left(\frac{n}{2} - \frac{1}{4}t(t+1)\right) = 0$  if and only if 4 divide  $(t-1)$  or  $(t-2)$ .

ii) if  $n$  is an odd number then  $\frac{1}{4}t(t+1) \in \mathbb{Z}$  if and only if  $(t-1), (t-2) \in 4\mathbb{Z}$ . Thus it follows that  $p\left(\frac{n}{2} - \frac{1}{4}t(t+1)\right) = 0$  if and only if 4 divides  $t, (t-3)$ .

The conclusion follows upon equating coefficients of the same powers of  $x$ .  $\square$

**Theorem 2.2.** For each positive integer  $n$ ,

$$Q\left(\frac{n}{2}\right) = p(n) + \sum_{k=1}^n (-1)^k \left[ p(n - (2k^2 \mp k)) \right].$$

*Proof.* If  $t = 2, l = 1$  in Equation 2, we have

$$\prod_{n=1}^{\infty} (1 - x^{4n})(1 - x^{4n-1})(1 - x^{4n-3}) = \sum_{n=-\infty}^{\infty} (-1)^n x^{2m^2+m}$$

and then it follows that

$$\prod_{n=1}^{\infty} \frac{1 - x^n}{1 - x^{4n-2}} = \sum_{n=-\infty}^{\infty} (-1)^n x^{2m^2+m}.$$

By the product, the foregoing identity yields

$$\begin{aligned} \sum_{n=0}^{\infty} Q\left(\frac{n}{2}\right)x^n &= \sum_{n=0}^{\infty} p(n)x^n \left( 1 + \sum_{m=1}^{\infty} (-1)^m x^{2m^2-m} + \sum_{m=1}^{\infty} (-1)^m x^{2m^2+m} \right) \\ &= \sum_{n=0}^{\infty} \left( p(n) + \sum_{k=1}^n (-1)^k p(n - (2k^2 \mp k)) \right) x^n. \end{aligned}$$

Upon equating coefficients of the same power of  $x$ , we prove our desired conclusion.  $\square$

Now we define a notation for the product of generating functions and also investigate some fundamental properties of this function with infinite product on the series representations.

Let  $H(x) = \sum_{n=0}^{\infty} h(n)x^n$  be a generating function. Then  $H^2(x) = H(x)H(x)$  has a series form and  $H^2(x) = \sum_{n=0}^{\infty} h(2, n)x^n$ . Then by Cauchy product, it follows that

$$h(2, m) = \sum_{i=0}^m h(i)h(m - i).$$

Now we focus on whether the integer  $m$  is even or not. Thus we have

$$h(2, 2t) = (h(t))^2 + 2 \sum_{i=0}^{t-1} h(i)h(2t - i),$$

$$h(2, 2t + 1) = 2 \sum_{i=0}^t h(i)h(2t + 1 - i).$$

**Theorem 2.3.** For a positive integer  $n$ , we have

$$Q(2, n) = \sum_{k=0}^n p\left(n - \frac{k^2 + k}{2}\right).$$

*Proof.* If  $k = t = \frac{1}{2}$  in Equation 3, then we get

$$\prod_{n=1}^{\infty} (1 - x^n)(1 + x^n)^2 = \sum_{k \in \mathbb{N}} x^{\frac{k(k+1)}{2}}$$

and so

$$\sum_{k \in \mathbb{N}} x^{\frac{k(k+1)}{2}} \sum_{m=0}^{\infty} p(m)x^m = \left( \sum_{n=0}^{\infty} Q(n)x^n \right)^2.$$

Then the product leads to the following identities

$$\sum_{n=0}^{\infty} \left( \sum_{k=0}^n p\left(n - \frac{k^2 + k}{2}\right) \right) x^n = \sum_{n=0}^{\infty} \left( \sum_{j=0}^n Q(j)Q(n - j) \right) x^n,$$

and comparing the coefficient of  $x^n$  for an integer  $n$ , we have that

$$\sum_{k=0}^n p\left(n - \frac{k^2 + k}{2}\right) = Q(2, n).$$

□

Now, we focus on new recurrence formulas for the partition function. By Theorem 2.2, we get the following result:

**Corollary 2.1.** *For any odd positive integer  $n$ , we have*

$$p(n) = \sum_{k=1}^n (-1)^{k+1} p(n - (2k^2 \mp k)).$$

Now using this notation, we can obtain recurrence formulas for the partition function as follows;

**Theorem 2.4.** *For a positive integer  $m$ , we have*

$$p(2m) = p(2, m) + \sum_{t=0}^{\lfloor \frac{\sqrt{m}}{4} \rfloor} \left[ p(2, m - 4t^2 - 7t - 3) + p(2, m - 4t^2 - 9t - 5) \right]$$

$$p(2m + 1) = \sum_{t=0}^{\lfloor \frac{\sqrt{m}}{4} \rfloor} \left[ p(2, m - 4t^2 - 3t) + p(2, m - 4t^2 - 5t - 1) \right].$$

*Proof.* Combining Equation 5 with Theorem 2.1, for any integer  $n$ , we get that

$$p(n) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} p(i) \left[ p\left(\frac{n}{2} - i\right) + \sum_{t=1}^{\lfloor \frac{\sqrt{n+1}}{4} \rfloor} p\left(\frac{n}{2} - i - \frac{1}{4}t(t+1)\right) \right].$$

Then using division algorithm, we can rewrite the second sum on the right equation as

$$\begin{aligned} \sum_{t=1}^{\lfloor \frac{\sqrt{n+1}}{4} \rfloor} p\left(\frac{n}{2} - i - \frac{1}{4}t(t+1)\right) &= \sum_{d=0}^4 \sum_{r=1}^4 p\left(\frac{n}{2} - i - \frac{1}{4}(4d+r)(4d+r+1)\right) \\ &= \sum_{d=0}^4 \sum_{r=1}^4 p\left(\frac{n}{2} - i - d - 2dr - 4d^2 - \frac{r^2+r}{4}\right). \end{aligned}$$

If  $n = 2m$  then it follows that

$$\sum_{r=1}^4 p\left(m - i - d - 2dr - 4d^2 - \frac{r(r+1)}{4}\right) = p(m - i - 7d - 4d^2 - 3) + p(m - i - 9d - 4d^2 - 5)$$

since  $p(x) = 0$  whenever  $x$  is not positive integer. Then we have

$$\sum_{t=1}^{\lfloor \sqrt{n+1} \rfloor} p\left(\frac{n}{2} - i - \frac{1}{4}t(t+1)\right) = \sum_{d=0} p(m - i - 7d - 4d^2 - 3) + p(m - i - 9d - 4d^2 - 5)$$

and so

$$\begin{aligned} p(n) &= \sum_{i=0}^{\lfloor n/2 \rfloor} p(i) \left[ p(m - i) + \sum_{d=0} \left\{ p(m - i - 7d - 4d^2 - 3) + p(m - i - 9d - 4d^2 - 5) \right\} \right] \\ &= p^2(m) + \sum_{d=0} \left\{ p(2, m - 7d - 4d^2 - 3) + p(2, m - 9d - 4d^2 - 5) \right\}. \end{aligned}$$

If  $n = 2m + 1$  then using the same argument, we have

$$\sum_{r=1}^4 p\left(\frac{2m+1}{2} - i - d - 2dr - 4d^2 - \frac{r(r+1)}{4}\right) = p(m - i - 3d - 4d^2) + p(m - i - 7d - 4d^2 - 5).$$

Thus the we complete the proof.  $\square$

From [5, page 309], we recall that the number of partitions of  $k$  into parts not exceeding  $m$  is denoted by  $p_m(k)$  for integers  $m, k$ . Then  $p_m(k) = p(k)$  for  $m \geq k$ . It is clear that  $p_m(k)$  is less than  $p(k)$  and the computation of  $p_m(k)$  is simpler for integers  $m, k$ . The generating function for the number of partitions of  $k$  into parts not exceeding  $m$  is defined as

$$F_m(x) = \prod_{i=1}^m \frac{1}{1 - x^i} = 1 + \sum_{i=1}^{\infty} p_m(i)x^i.$$

**Theorem 2.5.** For any positive integer  $m > 2$ , we have

$$p(m) = \sum_{i=1}^{\lfloor \sqrt{m} \rfloor} \left( p_i(m - i(i+1)) + (-1)^{i+1} p\left(m - \frac{5i^2 \mp 3i}{2}\right) \right).$$

*Proof.* By [10, Theorem 363], we recall that

$$\prod_{n=1}^{\infty} \frac{1}{(1 - x^{5n-2})(1 - x^{5n-3})} = 1 + \sum_{m=1}^{\infty} \prod_{i=1}^m \frac{x^{m(m+1)}}{(1 - x^i)}.$$

Let  $k = \frac{5}{2}$  and  $t = \frac{3}{2}$  in Equation 2. Then we get

$$\sum_{m=-\infty}^{\infty} (-1)^m x^{\frac{5m^2+3m}{2}} = \prod_{n=1}^{\infty} \frac{1 - x^n}{(1 - x^{5n-2})(1 - x^{5n-3})}.$$

Hence it follows that

$$\left( \sum_{n=0}^{\infty} p(n)x^n \right) \left( 1 + \sum_{k=1}^{\infty} (-1)^k \left\{ x^{\frac{5k^2+3k}{2}} + x^{\frac{5k^2-3k}{2}} \right\} \right) = \left( 1 + \sum_{m=1}^{\infty} \prod_{i=1}^m \frac{x^{m(m+1)}}{(1 - x^i)} \right). \tag{7}$$

Now rearranging the left side of Equation 7, we get

$$1 + \sum_{m=1}^{\infty} \prod_{i=1}^m \frac{x^{m(m+1)}}{(1 - x^i)} = \sum_{n=0}^{\infty} p(n)x^n + \sum_{n=0}^{\infty} p(n)x^n \sum_{k=1}^{\infty} (-1)^k x^{\frac{5k^2 \mp 3k}{2}}$$

$$\begin{aligned}
 &= \sum_{n=0}^{\infty} p(n)x^n + \sum_{n=1}^{\infty} \left( \sum_{k=1}^n (-1)^k p\left(n - \frac{5k^2 \mp 3k}{2}\right) \right) x^n \\
 &= \sum_{n=1}^{\infty} \left[ p(n) + \sum_{k=1}^n (-1)^k p\left(n - \frac{5k^2 \mp 3k}{2}\right) \right] x^n.
 \end{aligned}$$

Now rearranging the right side of Equation 7, we get

$$\begin{aligned}
 1 + \sum_{m=1}^{\infty} \left( \prod_{i=1}^m \frac{x^{m(m+1)}}{(1-x^i)} \right) &= 1 + \sum_{m=1}^{\infty} \left( x^{m(m+1)} \left( 1 + \sum_{i=1}^{\infty} p_m(i)x^i \right) \right) \\
 &= 1 + \sum_{n=1}^{\infty} x^{n(n+1)} + \sum_{m=1}^{\infty} \sum_{i=1}^{\infty} p_m(i)x^{m(m+1)+i} \\
 &= 1 + x^2 + \sum_{m=3}^{\infty} \sum_{i=1}^m p_i(m - i(i+1))x^m.
 \end{aligned}$$

Comparing the coefficient of  $x^n$  in Equation 7 on both sides, we complete the proof.  $\square$

### 3. Further Remarks and Observations

In the last section, we compare the recurrence formulas in Theorem 2.4 and Theorem 2.5 with recurrence formulas of both Euler and Ewell.

i) At each step of Euler’s recurrence formula, we need to calculate the values of the partition of  $p\left(n - \frac{3i^2 \mp i}{2}\right)$  for all  $0 \leq i \leq n$  and so requires  $\left\lceil \sqrt{\frac{8n}{3}} \right\rceil$  of the values partitions. Then to compute  $p(n)$ , we must reckon all elements of the set

$$ES = \{p(i) : 0 \leq i < n\}.$$

We shall illustrate this with the numerical examples  $n = 21$ ;

$$\begin{aligned}
 p(21) &= p(20) + p(19) - p(16) - p(14) + p(9) + p(6) \\
 &= 627 + 490 - 231 - 135 + 30 + 11 = 792.
 \end{aligned}$$

ii) At each step of Ewell’s recurrence formula, computation of  $p(n)$  requires  $\left\lceil \sqrt{2n} \right\rceil$  of the values  $p(m)$ ,  $0 \leq m < n$  which is less than Euler’s recurrence but to compute  $p(n)$ , we must reckon all element of the set

$$EWS = \{p(i - 2t^2) : 0 \leq i, t \leq n\}.$$

To compute  $p(21)$ , we must investigate every elements of the set

$$EWR = \{p(19), p(17), p(15), p(13), p(11), p(9), p(7), p(5), p(4), p(3), p(2), p(1)\}$$

and so we obtain

$$p(21) = 2(p(19) - p(13) + p(3)) + p(5) + p(0) = 792.$$

iii) The number of steps which are need to calculate the partition of an odd integer  $n$  in Corollary 2.1 is less than that of Euler’s recurrence and the largeness of the numbers founded at each step is less. By Corollary 2.1, it follows that

$$p(21) = p(20) + p(18) - p(15) - p(11) + p(0) + p(6) = 792.$$

iv) At the first step of recurrence formula in Theorem 2.4, we need to figure out the values of partitions of  $p(\frac{n}{2} - i)$  for all  $0 \leq i \leq \lfloor \frac{n}{2} \rfloor$  and so find  $\lfloor \frac{n}{2} \rfloor$  of the values partitions. To compute  $p(n)$ , we must reckon every elements of the set  $NR = \{p(i) : 0 \leq i \leq \lfloor \frac{n}{2} \rfloor\}$ . Therefore the recurrence formulas in Theorem 2.4 that compute the values of the partition with the help of the smaller integers are much useful and effective. Since computation of  $p(n)$  requires the set  $NR = \{p(i) : 0 \leq i \leq \lfloor \frac{n}{2} \rfloor\}$ , the recurrence formula in Theorem 2.4 are more compact and effective. We may demonstrate this by computing  $p(2, 10), p(21), p(100)$ ;

$$p(2, 10) = (p(5))^2 + 2(p(0)p(10) + p(1)p(9) + p(2)p(8) + p(3)p(7) + p(4)p(6)) = 481.$$

$$p(21) = p(2, 0) + p(2, 3) + p(2, 9) + p(2, 10) = 1 + 10 + 300 + 481 = 792.$$

$$\begin{aligned} p(100) &= p(2, 11) + p(2, 17) + p(2, 32) + p(2, 36) + p(2, 45) + p(2, 47) + p(2, 50) \\ &= 190569292. \end{aligned}$$

The biggest value in the set  $NR$  is  $p(10) = 42$  but the biggest value in the set  $ES$  (in the set  $EWR$ )  $p(20) = 627$  ( $p(19) = 490$ ).

v) At each step of the recurrence formula in Theorem 2.5, we need to compute both the values of the partition of  $p(n - \frac{5i^2 \pm 3i}{2})$  and the number of partitions of  $k$  into parts not exceeding  $i$  for all  $0 \leq i \leq n$ . Hence we maybe reckon both  $\lfloor \sqrt{\frac{8n}{5}} \rfloor$  value of partition  $n$  and the number of partitions of  $k$  into parts not exceeding  $i$  for all  $0 \leq i \leq n$ . Therefore, the computation of  $p(n)$  by Theorem 2.5 requires less than value of partition by Euler's recurrence and since  $p_m(k)$  is less than  $p(k)$  for integers  $m, k$ , we calculate the smaller value and so the recurrence formula in Theorem 2.5 is more effective with smaller integers. By the recurrence formula in Theorem 2.5, it follows that

$$\begin{aligned} p(21) &= p(20) + p(17) - p(14) - p(8) + p_3(9) + p_2(15) + p(3) + p_1(19) + p_4(1) \\ &= 627 + 297 - 135 - 22 + 12 + 8 + 3 + 1 + 1 = 792. \end{aligned}$$

Since the increasing of  $p(m)$  is faster than the increasing of an integer  $m$ , the new recurrence formulas that compute the values of the partition with the help of the smaller integers are much useful and effective. When we compare the size of numbers in the recurrence formulas, the new recurrence formulas maybe more influential. Therefore one might be led to see that the more largeness of the number is, the more effective the new recurrence formulas is.

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