Filomat 34:2 (2020), 559–565
https://doi.org/10.2298/FIL2002559S

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On Boole-Type Combinatorial Numbers and Polynomials

Yilmaz Simsek[a]

[a]*Department of Mathematics, Faculty of Science, University of Akdeniz, TR-07058 Antalya, Turkey.*

Abstract. The aim of this paper is to construct generating functions for Boole-type combinatorial numbers and polynomials. Using these generating functions, we derive not only fundamental properties of these numbers and polynomials, but also some identities and formulas. Finally, we present a brief historical remarks and observations on our generating functions and Peters and Boole-type numbers and polynomials.

1. Introduction, Definitions and Notations

Throughout this paper, we need the following definitions and notations. $\mathbb{N} = \{1, 2, 3, \ldots\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ denotes the set of integers, the set of real numbers and the set of complex numbers, respectively.

Generating functions for some special numbers and polynomials are given below:

The Apostol-Bernoulli polynomials and the Apostol-Euler polynomials are defined by, respectively:

$$F_B(t, x; \lambda) = \frac{te^{xt}}{\lambda e^t - 1} = \sum_{n=0}^{\infty} \mathcal{B}_n(x; \lambda) \frac{t^n}{n!} \tag{1}$$

and

$$F_E(t, x; \lambda) = \frac{2e^{xt}}{\lambda e^t + 1} = \sum_{n=0}^{\infty} \mathcal{E}_n(x; \lambda) \frac{t^n}{n!}, \tag{2}$$

which, for $x = 0$, are reduced to the Apostol-Bernoulli numbers $\mathcal{B}_n(\lambda) = \mathcal{B}_n(0; \lambda)$ and the Apostol-Euler numbers $\mathcal{E}_n(\lambda) = \mathcal{E}_n(0; \lambda)$. For special value of the parameter λ, we also get the Bernoulli numbers and the Euler numbers (*cf.* [2], [7], [8]).

The Stirling numbers of the first kind and the second kind are defined by, respectively:

$$F_{S_1}(t, k) = \frac{(\log(1+t))^k}{k!} = \sum_{n=0}^{\infty} s(n, k) \frac{t^n}{n!}, \tag{3}$$

2010 *Mathematics Subject Classification*. 05A10, 05A15, 11B83, 26C05, 30D05, 35A99.

Keywords. Functional equation, Generating functions, Binomial coefficient, Combinatorial numbers, Apostol-Bernoulli and Apostol-Euler polynomials, Stirling numbers, Cauchy numbers, Peters polynomials, Boole polynomials, Changhee numbers, Special numbers and polynomials.

Received: 25 January 2019; Revised: 12 March 2019; Accepted: 15 March 2019

Communicated by Miodrag Spalević

This article is dedicated to Professor Gradimir V. Milovanović on the Occasion of his 70th anniversary.

Email address: ysimsek@akdeniz.edu.tr (Yilmaz Simsek)

$$(x)_n = x(x-1)\ldots(x-n+1) = \sum_{j=0}^{n} x^j s(n,j) \tag{4}$$

and

$$F_{S_2}(t,k) = \frac{(e^t-1)^k}{k!} = \sum_{n=0}^{\infty} S(n,k)\frac{t^n}{n!} \tag{5}$$

(*cf.* [1], [2], [5], [8]).

Integral representation of the Cauchy numbers C_n is given by

$$C_n = \int_0^1 (x)_n dx \tag{6}$$

(*cf.* [5]).

The Peters polynomials are defined by

$$F_P(t,x;\lambda,\mu) = \frac{(1+t)^x}{\left(1+(1+t)^{\lambda}\right)^{\mu}} = \sum_{n=0}^{\infty} s_n(x;\lambda,\mu)\frac{t^n}{n!} \tag{7}$$

(*cf.* [3], [4], [5]). Setting $x = 0$ in (7), the polynomials $s_n(x;\lambda,\mu)$ are reduced to the Peters numbers $s_n(\lambda,\mu) = s_n(0;\lambda,\mu)$. When $\mu = 1$, equation (7) is reduced to the generating function for the Boole polynomials $\xi_n(x;\lambda) = s_n(x;\lambda,1)$ (*cf.* [3], [5]). Substituting $x = 0$ and $\mu = 1$ into (7), the Peters polynomials are reduced to the Boole numbers $\xi_n(\lambda) = s_n(0;\lambda,1)$ (*cf.* [3]) and also $Ch_n = 2\xi_n(1) = 2s_n(0;1,1)$ denotes the Changhee numbers (*cf.* [4]).

We [7] defined the following combinatorial numbers and polynomials:

The numbers $Y_n(\lambda)$ and the polynomials $Y_n(x;\lambda)$ are defined by, respectively:

$$\mathcal{F}(t;\lambda) = \frac{2}{\lambda(1+\lambda t)-1} = \sum_{n=0}^{\infty} Y_n(\lambda)\frac{t^n}{n!} \tag{8}$$

and

$$\mathcal{F}(t,x;\lambda) = (1+\lambda t)^x \mathcal{F}(t;\lambda) = \sum_{n=0}^{\infty} Y_n(x;\lambda)\frac{t^n}{n!} \tag{9}$$

(*cf.* [7]).

Observe that

$$Y_n(-1) = (-1)^{n+1} Ch_n$$

(*cf.* [9, Lemma 2]).

We [7, Eq. (2.3)] constructed the following p-adic integral representation for the p-adic meromorphic function as follows:

$$\int_{\mathbb{X}} \lambda^x (1+\lambda t)^x \chi(x) d\mu_{-q}(x) = \frac{[2]}{(\lambda q)^d (1+\lambda t)^d + 1} \sum_{j=0}^{d-1} (-1)^j \chi(j) (\lambda q)^j (1+\lambda t)^j,$$

where $\mu_q(x) = \mu_q(x+p^N\mathbb{Z}_p) = \frac{q^x}{[p^N]}$, $[x] = [x]_q = [x:q] = \begin{cases} \frac{1-q^x}{1-q}, & q \neq 1 \\ x, & q = 1 \end{cases}$, $\mathbb{X} = \mathbb{X}_d = \lim_{\overleftarrow{N}} \mathbb{Z}/dp^N\mathbb{Z}$, $\mathbb{X}_1 = \mathbb{Z}_p$ denotes the set of p-adic integers and d is an odd positive integer and $\lambda \in \mathbb{Z}_p$ with $\lambda \neq 1$, χ is the Dirichlet character with odd conductor d.

By the above p-adic integral representation, we constructed the following generating function for the so-called generalized Apostol-Changhee numbers and polynomials, respectively:

$$F_{\mathfrak{C}}(t;\lambda,q,\chi)=\frac{[2]\sum_{j=0}^{d-1}(-1)^{j}\chi(j)(\lambda q)^{j}(1+\lambda t)^{j}}{(\lambda q)^{d}(1+\lambda t)^{d}+1}=\sum_{n=0}^{\infty}\mathfrak{Ch}_{n,\chi}(\lambda,q)\frac{t^{n}}{n!}$$

and

$$F_{\mathfrak{C}}(t,z;\lambda,q,\chi)=F_{\mathfrak{C}}(t;\lambda,q,\chi)(1+\lambda t)^{z}=\sum_{n=0}^{\infty}\mathfrak{Ch}_{n,\chi}(z;\lambda,q)\frac{t^{n}}{n!}. \tag{10}$$

2. Generating functions for combinatorial type numbers

In this section, with the aid of (10), we derive the following generating function

$$G_{y_7}(t,\lambda,q)=\frac{[2]}{(\lambda q)(1+\lambda t)+1}=\sum_{n=0}^{\infty}y_{7,n}(\lambda,q)\frac{t^{n}}{n!} \tag{11}$$

and

$$F_{y_7}(t,z;\lambda,q)=G_{y_7}(t,\lambda,q)(1+\lambda t)^{z}=\frac{[2](1+\lambda t)^{z}}{(\lambda q)(1+\lambda t)+1}=\sum_{n=0}^{\infty}y_{7,n}(z;\lambda,q)\frac{t^{n}}{n!}. \tag{12}$$

By using the above equations, we derive various kind of identities, relations and formulas for the polynomials $y_{7,n}(z;\lambda,q)$ and the numbers $y_{7,n}(\lambda,q)$.

2.1. *Identities and relations for the numbers $y_{7,n}(\lambda,q)$*

In this section, by using equation (11) with its functional equations, we provide some identities and relations for not only the numbers $y_{7,n}(\lambda,q)$, but also the Stirling numbers, Apostol-type numbers and also combinatorial numbers.

Theorem 2.1. *Let $n\in\mathbb{N}_0$. Then we have*

$$y_{7,n}(\lambda,q)=[2](-1)^{n}\frac{(\lambda^{2}q)^{n}n!}{(\lambda q+1)^{n+1}}. \tag{13}$$

Proof. By (11), we have

$$[2]\sum_{n=0}^{\infty}(-1)^{n}\frac{(\lambda^{2}q)^{n}n!}{(\lambda q+1)^{n+1}}\frac{t^{n}}{n!}=\sum_{n=0}^{\infty}y_{7,n}(\lambda,q)\frac{t^{n}}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on the both sides of the above equation, we get the derived result. □

Theorem 2.2 (Recurrence Relation). *Let*

$$y_{7,0}(\lambda,q)=\frac{1+q}{1+\lambda q}.$$

Then we have

$$y_{7,n}(\lambda,q)=-\frac{n\lambda^{2}q}{\lambda q+1}y_{7,n-1}(\lambda,q) \tag{14}$$

where $n\in\mathbb{N}$.

Proof. By (11), we have

$$[2] = (\lambda q + 1) \sum_{n=0}^{\infty} y_{7,n}(\lambda, q) \frac{t^n}{n!} + q\lambda^2 \sum_{n=0}^{\infty} n y_{7,n-1}(\lambda, q) \frac{t^n}{n!}.$$

Therefore we get

$$[2] = (\lambda q + 1) y_{7,0}(\lambda, q)$$

and

$$0 = (\lambda q + 1) y_{7,n}(\lambda, q) + q\lambda^2 n y_{7,n-1}(\lambda, q).$$

Thus we get the result of theorem. □

Theorem 2.3. *Let $m \geq 1$. Then we have*

$$\mathcal{B}_m \left(\frac{q\lambda^2}{q\lambda^2 - q\lambda - 1} \right) = \frac{m}{[2]} (q\lambda^2 - q\lambda - 1) \sum_{n=0}^{m-1} S(m-1, n) y_{7,n}(\lambda, q). \tag{15}$$

Proof. Replacing t by $e^t - 1$ in (11), and by using (1), we get

$$\sum_{n=0}^{\infty} y_{7,n}(\lambda, q) \frac{(e^t - 1)^n}{n!} = \frac{[2]}{t(q\lambda^2 - q\lambda - 1)} \sum_{m=0}^{\infty} \mathcal{B}_m \left(\frac{q\lambda^2}{q\lambda^2 - q\lambda - 1} \right) \frac{t^m}{m!}.$$

After some elementary calculations, we get

$$\frac{(q\lambda^2 - q\lambda - 1)}{[2]} \sum_{m=0}^{\infty} m \sum_{n=0}^{m-1} S(m-1, n) y_{7,n}(\lambda, q) \frac{t^m}{m!} = \sum_{m=0}^{\infty} \mathcal{B}_m \left(\frac{q\lambda^2}{q\lambda^2 - q\lambda - 1} \right) \frac{t^m}{m!}.$$

Comparing the coefficients of $\frac{t^m}{m!}$ on the both sides of the above equation, we get the derived result. □

By using same computation of equation (15), we also derive the following thoerem:

Theorem 2.4. *Let $m \in \mathbb{N}_0$. Then we have*

$$\mathcal{E}_m \left(\frac{-q\lambda^2}{q\lambda^2 - q\lambda - 1} \right) = -\frac{2}{[2]} (q\lambda^2 - q\lambda - 1) \sum_{n=0}^{m} S(m, n) y_{7,n}(\lambda, q).$$

2.2. Logarithm functions associated with integral representation of the numbers

Here, we give integral representation of the numbers $y_{7,n}(\lambda, q)$ and also give some integral formulas. Integrating equation (11) with repect to t from 0 to 1, we get

$$\sum_{n=0}^{\infty} \int_0^1 y_{7,n}(\lambda, q) \frac{t^n}{n!} dt = \frac{[2]}{\lambda q + 1} \int_0^1 \frac{dt}{\frac{\lambda^2 q}{\lambda q + 1} t + 1}.$$

Hence, we have

$$\sum_{n=0}^{\infty} \frac{y_{7,n}(\lambda, q)}{(n+1)n!} = \frac{[2]}{\lambda^2 q} \ln \left(\frac{\lambda^2 q}{\lambda q + 1} + 1 \right).$$

Thus, we arrive at the following theorem:

Theorem 2.5.

$$\sum_{n=0}^{\infty}\frac{y_{7,n}(\lambda,q)}{(n+1)!}=\frac{[2]}{\lambda^2 q}\ln\left(\frac{\lambda^2 q+\lambda q+1}{\lambda q+1}\right). \tag{16}$$

Combining (16) with (13), we get

$$\sum_{n=0}^{\infty}\frac{[2](-1)^n(\lambda^2 q)^n n!}{(n+1)!(\lambda q+1)^{n+1}}=\frac{[2]}{\lambda^2 q}\ln\left(\frac{\lambda^2 q+\lambda q+1}{\lambda q+1}\right).$$

Therefore, after some elementary calculations, we arrive at a series representation of ln function by the following corollary:

Corollary 2.6.

$$\ln\left(\frac{\lambda^2 q}{\lambda q+1}+1\right)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)}\left(\frac{\lambda^2 q}{\lambda q+1}\right)^{n+1}.$$

We remark that the above series has also been studied in [6].

3. A new polynomials $y_{7,n}(x;\lambda,q)$

In this section, we give some properties of the polynomials $y_{7,n}(x;\lambda,q)$. By using equation (12), we derive formulas for these polynomials.

By (12), we get

$$\sum_{n=0}^{\infty}y_{7,n}(x;\lambda,q)\frac{t^n}{n!}=\sum_{n=0}^{\infty}(x)_n\frac{(\lambda t)^n}{n!}\sum_{n=0}^{\infty}y_{7,n}(\lambda,q)\frac{t^n}{n!}.$$

Therefore

$$\sum_{n=0}^{\infty}y_{7,n}(x;\lambda,q)\frac{t^n}{n!}=\sum_{n=0}^{\infty}\sum_{j=0}^{n}\binom{n}{j}(x)_{n-j}\lambda^{n-j}y_{7,j}(\lambda,q)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on the both sides of the above equation, we get the following theorem.

Theorem 3.1.

$$y_{7,n}(x;\lambda,q)=\sum_{j=0}^{n}\binom{n}{j}(x)_{n-j}\lambda^{n-j}y_{7,j}(\lambda,q). \tag{17}$$

By (17), we see that

$$y_{7,n}(x;\lambda,q)=[2]\sum_{j=0}^{n}(-1)^j\binom{n}{j}j!\frac{\lambda^{n+j}q^j}{(\lambda q+1)^{j+1}}(x)_{n-j}.$$

Integrating the above equation from 0 to 1 with respect to x, we get

$$\int_0^1 y_{7,n}(x;\lambda,q)dx=[2]\sum_{j=0}^{n}(-1)^j\binom{n}{j}j!\frac{\lambda^{n+j}q^j}{(\lambda q+1)^{j+1}}\int_0^1(x)_{n-j}dx.$$

By (6), we derive

$$\int_0^1 y_{7,n}(x;\lambda,q)dx = [2]\sum_{j=0}^{n}(-1)^j\binom{n}{j}j!\frac{\lambda^{n+j}q^j}{(\lambda q+1)^{j+1}}C_{n-j}. \tag{18}$$

By (4), we also derive

$$\int_0^1 y_{7,n}(x;\lambda,q)dx = [2]\sum_{j=0}^{n}(-1)^j\binom{n}{j}j!\frac{\lambda^{n+j}q^j}{(\lambda q+1)^{j+1}}\sum_{k=0}^{n-j}\frac{s(n-j,k)}{k+1}. \tag{19}$$

Combining (18) and (19), we get the following theorem:

Theorem 3.2.

$$\sum_{j=0}^{n}(-1)^j\binom{n}{j}j!\frac{\lambda^{n+j}q^j}{(\lambda q+1)^{j+1}}C_{n-j} = \sum_{j=0}^{n}(-1)^j\binom{n}{j}j!\frac{\lambda^{n+j}q^j}{(\lambda q+1)^{j+1}}\sum_{k=0}^{n-j}\frac{s(n-j,k)}{k+1}. \tag{20}$$

By using (20), we have

$$\sum_{j=0}^{n}(-1)^j\binom{n}{j}j!\frac{\lambda^{n+j}q^j}{(\lambda q+1)^{j+1}}\left(C_{n-j}-\sum_{k=0}^{n-j}\frac{s(n-j,k)}{k+1}\right)=0.$$

We observe from the above equation that the well-known formula for the Cauchy numbers is given by

$$C_{n-j}=\sum_{k=0}^{n-j}\frac{s(n-j,k)}{k+1}$$

(*cf.* [5]).

4. Further remarks and observations

Motivation of the numbers $y_{7,n}(\lambda,q)$ is briefly given by

$$y_{7,n}(\lambda,q)=(-1)^{n+1}\frac{q+1}{2q^n}Y_n(-q\lambda). \tag{21}$$

In addition, substituting $q=1$ into (12), we have

$$\frac{2(1+\lambda t)^x}{\lambda(1+\lambda t)+1}=\sum_{n=0}^{\infty}y_{7,n}(x;\lambda,1)\frac{t^n}{n!}.$$

When $\lambda=1$, we obtain

$$\frac{2(1+t)^x}{t+2}=\sum_{n=0}^{\infty}y_{7,n}(x;1,1)\frac{t^n}{n!}$$

Hence, we have the following relations between the polynomials $y_{7,n}(x;\lambda,q)$, Peters polynomials and Boole polynomials:

$$s_n(x;1,1)=\frac{1}{2}y_{7,n}(x;1,1).$$

Substituting $x=0$, $\lambda=q=1$, we see that

$$s_n(0;1,1)=\xi_n(1)=\frac{1}{2}Ch_n=\frac{1}{2}y_{7,n}(0;1,1).$$

References

[1] N. P. Cakic, G. V. Milovanovic, On generalized Stirling numbers and polynomials, Mathematica Balkanica 18 (2004) 241–248.

[2] G. B. Djordjevic, G. V. Milovanovic, Special classes of polynomials, University of Nis, Faculty of Technology Leskovac, 2014.

[3] C. Jordan, Calculus of Finite Differences (2nd ed), Chelsea Publishing Company, New York, 1950.

[4] D. S. Kim, T. Kim, J. Seo, A note on Changhee numbers and polynomials, Advanced Studies in Theoretical Physics 7 (2013) 993–1003.

[5] S. Roman, The Umbral Calculus, Dover Publ. Inc., New York, 2005.

[6] Y. Simsek, Integral representations of generating functions for combinatorial numbers and polynomials, The Proceedings Book of The Mediterranean International Conference of Pure&Applied Mathematics and Related Areas (MICOPAM 2018) Dedicated to Professor Gradimir V. Milovanović on the Occasion of his 70th Anniversary, (Edited by Y. Simsek), ISBN 978-86-6016-036-4, October 26-29, 2018, Antalya, TURKEY, pp. 217–220.

[7] Y. Simsek, Construction of some new families of Apostol-type numbers and polynomials via Dirichlet character and p-adic q-integrals, Turkish Journal of Mathematics 42 (2018) 557–577.

[8] H. M. Srivastava, J. Choi, Series Associated with the Zeta and Related Functions, Kluwer Academic Publishers, Dordrecht, Boston, London, 2001.

[9] H. M. Srivastava, I. Kucukoglu, Y. Simsek, Partial differential equations for a new family of numbers and polynomials unifying the Apostol-type numbers and the Apostol-type polynomials, Journal of Number Theory 181 (2017) 117–146.